Review Exercises

1. A factory produces items in boxes of 2. Over the long run:

92% of boxes contain 0 defective items;

5% of boxes contain 1 defective item; and

3% of boxes contain 2 defective items.

A box is picked at random from production, then an item is picked at random from the box. Given that the item is defective, what is the chance that the second item in the box is defective?

2. A box contains 1 black ball and 1 white ball. A ball is drawn at random, then replaced in the box with an additional ball of the same color. Then a second ball is drawn at random from the three balls in the box. What is the probability that the first ball drawn was white, given that at least one of the two balls drawn was white?

3. Suppose I toss three coins. Two of them at least must land the same way. No matter whether they land heads or tails, the third coin is equally likely to land either the same way or oppositely. So the chance that all three coins land the same way is $1/2$. True or False? Explain!

4. There are two boxes.
Box 1 contains 2 red balls and 3 black balls.
Box 2 contains 8 red balls and 12 black balls.
One of the two boxes is picked at random, and then a ball is picked at random from the box.

a) Is the color of the ball independent of which box is chosen?

b) What if there were 10 black balls rather than 12 in Box 2, but the other numbers were the same?

5. To pass a test you have to perform successfully two consecutive tasks, one easy and one hard. The easy task you think you can perform with probability z, and the hard task you think you can perform with probability h, where $h < z$. You are allowed three attempts, either in the order (easy, hard, easy) or in the order (hard, easy, hard). Whichever order, you must be successful twice in a row to pass. Assuming that your attempts are independent, in what order should you choose to take the tasks in order to maximize your probability of passing the test?

6. Show that if A and B are independent, then so are A^c and B, A and B^c, and A^c and B^c.

7. A population of 50 registered voters contains 30 in favor of Proposition 134 and 20 opposed. An opinion survey selects a random sample of 4 voters from this population, as follows. One person is picked at random from the 50 voters, then another at random from the remaining 49, and so on, till 4 people have been picked.

a) What is the probability that there will be no one in favor of 134 in the sample?

b) What is the probability that there will be at least one person in favor?

c) What is the probability that exactly one pro 134 person will appear in the sample?

d) What is the probability that the majority of the sample will be pro 134? (Majority means strictly more than half.)

8. Cards are dealt from a well-shuffled standard deck until the first heart appears.

a) What is the probability that exactly 5 deals are required?

b) What is the probability that 5 or fewer deals are required?

c) What is the probability that exactly 3 deals were required, given that 5 or fewer were required?

9. Suppose events A, B, and C are independent with probabilities $1/5$, $1/4$, and $1/3$, respectively. Write down numerical expressions for the following probabilities:

a) $P(A \text{ and } B \text{ and } C)$

b) $P(A \text{ or } B \text{ or } C)$

c) $P(\text{exactly one of the three events occurs})$

10. The four major blood types are present in approximately the following proportions in the population of the U.S.A.

Type	A	B	AB	0
proportion	42%	10%	4%	44%

Note that each person's blood is exactly one of these four types. Type AB is a separate type, not the intersection of type A and type B.

a) If two people are picked at random from this population, what is the chance that their blood is of the same type? Of different types?

b) If four people are picked at random, let $P(k)$ be the chance that there are exactly k different blood types among them. Find $P(k)$ for $k = 1, 2, 3, 4$.

11. A hat contains n coins, f of which are fair, and b of which are biased to land heads with probability $2/3$. A coin is drawn from the hat and tossed twice. The first time it lands heads, and the second time it lands tails. Given this information, what is the probability that it is a fair coin?

12. Suppose n ordinary dice are rolled.

a) What is the chance that the dice show n different faces?

b) What is the chance that at least one number appears more than once?

13. **Formula for $P(A \mid B)$ by conditioning on cases of B.** Show if $B_1, \ldots, B_n$ is a partition of B, then

$$P(A|B) = P(A|B_1)P(B_1|B) + \cdots + P(A|B_n)P(B_n|B)$$

14. There are 100 boxes, and for each $i = 1, 2, \ldots, 100$, box i contains proportion $i/100$ of gold coins (the rest are silver). One box is chosen at random, then a coin is drawn at random from this box.

a) If the coin drawn is gold, which box would you guess was chosen? Why?

b) Suppose the boxes were not picked at random, but according to the following scheme. All the even-numbered boxes are equally likely, all the odd-numbered boxes are equally likely, but the chance of drawing an odd-numbered box is twice the chance of drawing an even-numbered box. If the coin drawn is gold, which box would you guess was chosen? [*Hint:* Write down the prior odds.]

15. There are three boxes, each with two drawers. Box 1 has a gold coin in each drawer, and box 2 has a silver coin in each drawer. Box 3 has a silver coin in one drawer and a gold coin in the other. One box is chosen at random, then a drawer is chosen at random from the box. Find the probability that box 1 is chosen, given that the chosen drawer yields a gold coin.

16. A dormitory has n students, all of whom like to gossip. One of the students hears a rumor, and tells it to one of the other $n-1$ students picked at random. Subsequently, each student who hears the rumor tells it to a student picked at random from the dormitory (excluding, of course, himself/herself and the person from whom he/she heard the rumor). Let p_r be the probability that the rumor is told r times without coming back to a student who has already heard it from a dormitory-mate. So $p_1 = p_2 = 1$, and $p_n = 0$.

a) Find a formula for p_r for r between 3 and $n-1$.

b) Estimate this probability for $n = 300$ and $r = 30$.

17. Some time ago I received the following letter:

"You may have previously received a letter notifying you that you had been a selectee in a recent sweepstake that we were conducting. According to our records, you have not claimed your gift.
We are always pleased when our bigger gifts are awarded because it's good publicity for our company. However, last year there were thousands of dollars worth of unclaimed gifts simply because the selectees failed to respond.
This letter is to inform you that one of the following people has won a New Datsun Sentra:

Collin Andrus	Oklahoma City, OK
James W. Pitman	Berkeley, CA
Larry Abbott	Burbank, CA

In compliance with the rules of the sweepstake, you are hereby notified that you are a selectee in Category I, which means you will receive one of the following:

1. R.C.A. Color TV;
2. 5 FT. Grandfather Clock;
3. Datsun Nissan Sentra.

To claim your gift, all you have to do is call toll free 1-800-643-3249 for an available time and date for you and your spouse to visit Heavenly Valley Townhouses and attend a sales representation tour on the many advantages that interval ownership has to offer."

According to small print on the back of the letter:

"The retail values and odds of receiving each gift are No. 1—$1/10,000$ ($329.95), No. 2—$9998/10,000$ ($249.95), No. 3—$1/10,000$($5,995.00)."

Let us assume that this is an honestly conducted sweepstake, and that each of the three individuals named above had originally a 1 in $10,000$ chance of winning the new Datsun. Now I know that the winner is one of these individuals, the rules of conditional probability imply that I have a one in three chance of winning the Datsun. True or false? Explain!

18. Suppose there are m equally likely possibilities for one stage, and n equally likely possibilities for another. Show that the two stages are independent if and only if all mn possible joint outcomes are equally likely.

19. A box contains 5 tickets numbered $1, 2, 3, 4$, and 5. Two tickets are drawn at random from the box. Find the chance that the numbers on the two tickets differ by two or more if the draws are made:

a) with replacement;

b) without replacement.

Repeat the problem with n tickets numbered $1, 2, \ldots, n$.

1.rev.11. $\frac{9f}{9f+8b}$

1.rev.13. *Hint:* Write $P(A|B) = \sum_{i=1}^{n} P(AB_i|B)$.

1.rev.15. 2/3

1.rev.17. False.

1.rev.1. 6/11

1.rev.3. False

1.rev.5. The chance of passing when you use the first order is $zh(2-z)$. With the second order, it's $hz(2-h)$.

1.rev.7. a) $\frac{20}{50} \cdot \frac{19}{49} \cdot \frac{18}{48} \cdot \frac{17}{47} = .021$ b) $1 -$ answer to a) c) $4 \cdot \frac{30}{50} \cdot \frac{20}{49} \cdot \frac{19}{48} \cdot \frac{18}{47}$
d) $4 \cdot \frac{30}{50} \cdot \frac{29}{49} \cdot \frac{28}{48} \cdot \frac{20}{47} + \frac{30}{50} \cdot \frac{29}{49} \cdot \frac{28}{48} \cdot \frac{27}{47}$

1.rev.9. a) 1/60 b) 3/5 c) 13/30

Review Exercises

1. Ten dice are rolled. Write down numerical expressions for

a) the probability that exactly 4 dice are sixes.

b) the probability that exactly 4 dice are sixes given that none of the dice is a five.

c) the probability of 4 sixes, 3 fives, 2 fours, and a three.

d) the probability that none of the first three dice is a six given 4 sixes among the ten dice.

2. A fair die is rolled 36 times. Approximate the probability that 12 or more sixes appear.

3. Suppose I roll a fair die, then toss as many coins as there are spots on the die.

a) What is the probability that exactly three heads appear among the coins?

b) Given three heads appear, what is the probability that the die showed 4?

4. A fair coin is tossed 10 times. Given that at least 9 of the tosses resulted in tails, what is the probability that exactly 9 of the tosses resulted in tails?

5. A thumb tack was tossed 100 times, and landed point up on 40 tosses and point down on 60 tosses. Given this information, what is the probability that the first three tosses landed point down?

6. Four numbers are drawn at random from a box of ten numbers $0, 1, \ldots, 9$. Find the probability that the largest number drawn is a six:

a) if the draws are made with replacement;

b) if the draws are made without replacement.

8. Suppose you and I each roll ten dice. What is the probability that we each roll the same number of sixes?

9. In a certain town, 10% of the families have no children, 10% have one child, 40% have two children, 30% have three children, and 10% have four children. Assume that births are independent of each other, and equally likely to produce male or female.

a) One family is picked at random from all of the families in this town. What is the probability that there are at least two children in the family?

b) One family is picked at random from all of the families in this town. Guess the size of the family, given that it has at least two girls. Give reasons for your guess.

c) A family is picked at random from among the families with four children. Then a child is picked at random from the selected family. What is the chance that the child picked is a girl with at least one brother?

10. Lie detectors. According to a newspaper report, in 2 million lie detector tests, 300,000 were estimated to have produced erroneous results. Assuming these figures to be correct, answer the following:

a) If ten tests were picked at random from these 2 million tests, what would be the chance that at least one of them produced an erroneous result? Sketch the histogram of the distribution of the number of erroneous results among these ten tests.

b) Suppose these 2 million tests were done on a variety of machines. If a machine were picked at random, then ten tests picked at random from these tests performed on that machine, would it be reasonable to suppose that the chance that at least one of them produced an erroneous result would be the same as in a)? Explain.

11. Consider two machines, A and B, each producing the same items. Each machine produces a large number of these items every day. However, production per day from machine B, being newer, is twice that of A. Further the rate of defectives is 1% for B and 2% for A. The daily output of the machines is combined and then a random sample of size 12 taken. Find the probability that the sample contains 2 defective items. What assumptions are you making?

13. A seed manufacturer sells seeds in packets of 50. Assume that each seed germinates with a chance of 99%, independently of all others. The manufacturer promises to replace, at no cost to the buyer, any packet that has 3 or more seeds that do not germinate. What is the chance that the manufacturer has to replace more than 40 of the next 4000 packets sold?

14. a) If Ted and Jim are among 10 people arranged randomly in a line, what is the chance that they stand next to each other?

b) What if the ten people are arranged at random in a circle?

c) Generalize to find the chance of k particular people ending up all together if n people are arranged at random in a line or a circle.

15. Draws are made at random with replacement from a box of colored balls with the following composition:

color	red	blue	green	yellow
proportion	0.1	0.2	0.3	0.4

Write down and justify unsimplified expressions for the probabilities of the following events:

a) exactly 5 yellow balls appear in 20 draws;

b) exactly 2 red, 4 blue, 6 green and 8 yellow balls appear in 20 draws;

c) the number of draws required to produce 3 red balls is 25.

16. Eight cards are drawn from a well-shuffled deck of 52 cards. What is the probability that the 8 cards contain: a) 4 aces; b) 4 aces and 4 kings;

c) 4 of a kind (any kind, including the possibility of 4 of two kinds).

17. If four dice are rolled, what is the probability of:

a) four of a kind; b) three of a kind; c) two pairs?

18. Seven dice are rolled. Write down unsimplified expressions for the probabilities of each of the following events:

a) exactly three sixes;

b) three of one kind and four of another;

c) two fours, two fives, and three sixes; d) each number appears;

e) the sum of the dice is 9 or more.

19. In a World Series, teams A and B play until one team wins four games. Suppose all games are independent, and that on each game, the probability that team A beats team B is $2/3$.

a) What is the probability that team A wins the series in four games?

b) What is the probability that team A wins the series, given team B won games 1 and 2?

20. A computer communication channel transmits words of n bits using an error-correcting code which is capable of correcting errors in up to k bits. Here each bit is either a 0 or a 1. Assume each bit is transmitted correctly with probability p and incorrectly with probability q independently of all other bits.

a) Find a formula for the probability that a word is correctly transmitted.

b) Calculate the probability of correct transmission for $n = 8$, $k = 2$, and $q = 0.01$.

21. Suppose a single bit is transmitted by repeating it n times and the message is interpreted by majority decoding. For example, for $n = 5$, if the message received is 10010, it is concluded that a 0 was sent. Assuming n is odd and each bit in the message is transmitted correctly with probability p, independently of the other bits, find a formula for the probability that the message is correctly received.

22. Suppose that, on average, 3% of the purchasers of airline tickets do not appear for the departure of their flight. Determine how many tickets should be sold for a flight on an airplane which has 400 seats, such that with probability 0.95 everybody who appears for the departure of the flight will have a seat. What assumptions are you making?

23. Ten percent of the families in a town have no children, twenty percent have one child, forty percent have two children, twenty percent have three, and ten percent have four. Assume each child in a family is equally likely to be a boy or a girl, independently of all the others. A family is picked at random from this town. Given that there is at least one boy in the family, what is the chance that there is also at least one girl?

24. In a large population, the distribution of the number of children per family is as follows:

Number of children n	0	1	2	3	4	5
Proportion families with n children	0.15	0.2	0.3	0.2	0.1	0.05

Assume that each child in a family is a boy or a girl with probability $1/2$, independently.

a) If a family is picked at random, what is the chance that it contains exactly two girls?

b) If a child is picked at random from the children of this population, what is the chance that the child comes from a family with exactly two girls?

25. At Wimbledon, men's singles matches are played on a "best of five sets" basis, that is, players A and B play until one of them has won 3 sets. Suppose each set is won by A with probability p, independently of all previous sets.

a) For each $i = 3, 4, 5$, find a formula in terms of p and $q = 1 - p$ that player A wins in exactly i sets.

b) In terms of p and q, what is the probability that player A wins the match?

c) Given that player A won the match, what is the probability (in terms of p and q) that the match lasted only three sets?

d) Compute the probability in c) for the case $p = 2/3$.

e) Do you think the assumption of independence made above is reasonable?

26. Suppose 3 points are picked at random from 10 points equally spaced around the circumference of a circle.

a) What is the probability that two particular adjacent points, say A and B, are both among the 3 points picked at random?

b) What is the probability that among the 3 points picked at random there is least one pair of adjacent points?

27. A university schedules its final examinations in 18 "examination groups", so that courses held at different times are in different examination groups. The examination times are spread over 6 days, with 3 examinations each day. Suppose all students take 4 examinations. About what proportion of students will have their 4 examinations on different days? [You need to make some assumptions—state what the assumptions are.]

28. **The matching problem.** There are n letters addressed to n people at n different addresses. The n addresses are typed on n envelopes. A disgruntled secretary shuffles the letters and puts them in the envelopes in random order, one letter per envelope.

a) Find the probability that at least one letter is put in a correctly addressed envelope. [*Hint*: Use the inclusion-exclusion formula of Exercise 1.3.12]

b) What is this probability approximately, for large n?

29. **Cosmic wimpout.** In this game five dice are rolled. Four of the dice have the same set of symbols and numbers on their faces. The numbers are 5 and 10, and let us call the symbols A, B, C, and D. The fifth die is the same, except symbol D is replaced by a different symbol W, indicating a wild roll. In one version of the game, the following kinds of rolls count for a score:

- any roll that shows one or more numbers;
- any roll that shows a triple of symbols, where the wild symbol W can count as any symbol you like, e.g., WAABC scores a triple, the W counting as A;
- a roll that shows W together with one of each of the other symbols A, B, C, and D.

Any other combination fails to score, and is called a wimpout. Calculate the probability of a wimpout.

32. Call a card hand of h cards a *straight* if the denominations can be arranged as d, $d+1$, ..., $d+h-1$, for some $1 \leq d \leq 13-h+1$, where $d=1$ represents ace, $d=11$ for jack, 12 for queen, and 13 for king (so aces only count low). Call the hand a *flush* if all h cards are of the same suit. Assume for simplicity that a straight flush counts both as a straight and as a flush. For which h is a straight more likely than a flush?

33. a) How could you simulate a biased coin landing heads with probability $p = 1/3$ if you only had available a fair coin?

b) How could you simulate fair coin tossing if you only had available a coin with unknown bias p strictly between 0 and 1?

34. a) Explain why if you and I each toss m fair coins, the chance that we both get the same (unspecified) number of heads equals the chance that we get exactly m heads and m tails between us.

b) If I toss m fair coins, and you toss $m+1$ fair coins, what is the chance that you get strictly more heads than I do?

2.rev.1. a) $\binom{10}{4}(1/6)^4(5/6)^6$ b) $\binom{10}{4}(1/5)^4(4/5)^6$ c) $\frac{10!}{4!3!2!}/6^{10}$ d) $\frac{\binom{7}{4}}{\binom{10}{4}}$

2.rev.3. a) 1/6 b) 1/4

2.rev.5. $\frac{\binom{97}{57}}{\binom{100}{60}}$

2.rev.7. $k \approx 1025$

2.rev.9. a) 0.8 b) guess 3 c) 0.4375

2.rev.11. 0.0102

2.rev.13. 0.99; the chance that any particular packet needs to be replaced is about 0.0144.

2.rev.15. a) $\binom{20}{5}(0.4)^5(0.6)^15$ b) $\frac{20!}{2!4!6!8!}(0.1)^2(0.2)^4(0.3)^6(0.4)^8$ c) $\binom{24}{2}(0.1)^3(0.9)^{22}$

2.rev.17. a) $\frac{\binom{6}{1}}{6^4}$ b) $\binom{6}{1} \times \binom{5}{1} \times \frac{\binom{4}{1}}{6^4}$ c) $\binom{6}{2} \times \frac{\binom{4}{2}}{6^4}$

2.rev.19. a) $(2/3)^4$ b) $\binom{4}{1}(2/3)^4(1/3) + (2/3)^4$

2.rev.21. $\sum_{x=0}^{(n-1)/2} \binom{n}{x} q^x p^{n-x}$.

2.rev.23. $\frac{(.4\times1/2)+(.2\times6/8)+(.1\times14/16)}{(.2\times1/2)+(.4\times3/4)+(.2\times7/8)+(.1\times15/16)}$

2.rev.25. a) p^3, $3p^3q$, $6p^3q^2$ b) $p^3 + 3p^3q + 6p^3q^2$ c) $\frac{1}{1+3q+6q^2}$ d) 0.375 e) no.

2.rev.27. 0.3971

2.rev.29. 0.0579

2.rev.31. *Hint*: $np \geq npq$

2.rev.33. *Hint*: Think about conditional probabilities.