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- Please mark the box to the left which lists your section.

- Do not detach pages from this exam packet or unstaple the packet.

- Show your work. Answers without reasoning can not be given credit except for the True/False and multiple choice problems.

- Please write neatly.

- Do not use notes, books, calculators, computers, or other electronic aids.

- Unspecified functions are assumed to be smooth and defined everywhere unless stated otherwise.

- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10

12A		10
13A		10
14A		10

12B		10
13B		10
14B		10

Total:		140

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) **T** | F   The length of the curve $\vec{r}(t) = \langle \sin(t), t^4 + t, \cos(t) \rangle$ on $t \in [0, 1]$ is the same as the length of the curve $\vec{r}(t) = \langle \sin(t^2), t^8 + t^2, \cos(t^2) \rangle$ on $[0, 1]$.

Solution:
This is a consequence of the chain rule: $\int_a^b |r(s(t))'| \ dt = \int_a^b |r'(s(t))||s'(t)|dt = \int_{s(a)}^{s(b)} |r'(s)|ds$.

2) **T** | F   The parametric surface $\vec{r}(u, v) = (5u - 3v, u - v - 1, 5u - v - 7)$ is a plane.

Solution:
Yes, the coordinate functions $r(u, v) = (x(u, v), y(u, v), z(u, v))$ are all linear.

3) **T** | F   Any function $u(x, y)$ that obeys the differential equation $u_{xx} + u_x - u_y = 1$ has no local maxima.

Solution:
If $\nabla u = (u_x, u_y) = (0, 0)$, then $u_{xx} = 1$ which is incompatible with a local maximum, where $u_{xx} > 0$ by the second derivative test.

4) **T** | F   The scalar projection of a vector $\vec{a}$ onto a vector $\vec{b}$ is the length of the vector projection of $\vec{a}$ onto $\vec{b}$.

Solution:
By definition.

5) T | **F**   If $f(x, y)$ is a function such that $f_x - f_y = 0$ then f is conservative.

Solution:
The notion of conservative applies to vector fields and not to functions.

6) T | **F** $(\vec{u} \times \vec{v}) \cdot \vec{w} = (\vec{u} \times \vec{w}) \cdot \vec{v}$ for all vectors $\vec{u}, \vec{v}, \vec{w}$.

Solution:
While $|(\vec{u} \times \vec{w}) \cdot \vec{v}|$ and $|(\vec{u} \times \vec{w}) \cdot \vec{v}|$ are both equal to the volume of the parallelepiped determined by $\vec{u}, \vec{v}$ and $\vec{w}$, the sign is different. An example: for $\vec{u} = \langle 1, 0, 0 \rangle, \vec{v} = \langle 0, 1, 0 \rangle, \vec{w} = \langle 0, 0, 1 \rangle$, we have $(\vec{u} \times \vec{v}) \cdot \vec{w} = 1$ and $(\vec{u} \times \vec{w}) \cdot \vec{v} = -1$.

7) | T | **F** | The equation $\rho = \phi/4$ in spherical coordinates is half a cone.

Solution:
The equation $\rho = \phi/4$ defines a heart shaped rotational symmetric surface. The surface $\phi = c = const$ would define half a cone for $c \in [0, \pi]$.

8) | T | **F** | The function $f(x,y) = \begin{cases} \frac{x}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ is continuous at every point in the plane.

Solution:
Taking $y = 0$, we get $f(x,0) = 1/x$ which is discontinuous.

9) | **T** | F | $\int_0^1 \int_0^x 1 \, dy dx = 1/2$.

Solution:
This is the area of half of the unit square.

10) | **T** | F | Let $\vec{a}$ and $\vec{b}$ be two vectors which are perpendicular to a given plane Σ. Then $\vec{a} + \vec{b}$ is also perpendicular to Σ.

Solution:
If $\vec{v}$ is a vector in the plane, then $\vec{a} \cdot \vec{v} = 0$ and $\vec{b} \cdot \vec{v} = 0$ then also $(\vec{a} + \vec{b}) \cdot \vec{v} = 0$.

11) | **T** | F | If $g(x,t) = f(x-vt)$ for some function f of one variable $f(z)$ then g satisfies the differential equation $g_{tt} - v^2 g_{xx} = 0$.

Solution:
Actually one could show that $g(x,t) = f(x-vt) + h(x+vt)$ is the general solution of the wave equation $g_{tt} - v^2 g_{xx} = 0$.

12) [T] [F] If $f(x,y)$ is a continuous function on $\mathbf{R}^2$ such that $\int \int_D f \; dA \geq 0$ for any region D then $f(x,y) \geq 0$ for all (x,y).

Solution:
Assume $f(a,b) < 0$ at some point (a,b), then $f(x,y) < 0$ in a small neighborhood D of (a,b) and also $\int \int_D f \; dA < 0$ contradicting the assumption.

13) [T] [F] Assume the two functions $f(x,y)$ and $g(x,y)$ have both the critical point $(0,0)$ which are saddle points, then $f + g$ has a saddle point at $(0,0)$.

Solution:
Example $f(x,y) = x^2 - y^2/2$, $g(x,y) = -x^2/2 + y^2$ have both a saddle point at $(0,0)$ but $f + g = x^2/2 + y^2/2$ has a minimum at $(0,0)$.

14) [T] [F] If $f(x,y)$ is a function of two variables and if $h(x,y) = f(g(y), g(x))$, then $h_x(x,y) = f_y(g(y), g(x))g'(y)$.

Solution:
The correct identity would be $h_x(x,y) = f_y(g(y), g(x))g'(x)$ according to the chain rule.

15) [T] [F] If we rotate a line around the z axis, we obtain a cylinder.

Solution:
The surface could also be a one-sheeted hyperboloid or a cone.

16) [T] [F] If $u(x,y)$ satisfies the transport equation $u_x = u_y$, then the vector field $\vec{F}(x,y) = \langle u(x,y), u(x,y) \rangle$ is a gradient field.

Solution:
$\vec{F} = (P,Q) = (u,u)$. From $u_x = u_y$ we get $Q_x = P_y$ which implies that F is a gradient field.

17) [T] [F] $3 \operatorname{grad}(f) = \frac{d}{dt} f(x+t, y+t, z+t)$.

Solution:
The left hand side is a vector field, the right hand side a function.

18) | T | F | If a vector field $\vec{F}$ is defined at all points in three-space except the origin and curl($\vec{F}$) = $\vec{0}$ everywhere, then the line integral of $\vec{F}$ around any closed path not passing through the origin is zero.

Solution:
The region without the origin is simply connected.

TF Problems for regular sections:

19) | T | **F** | If $\vec{F}$ is a vector field in space and f is equal to the line integral of $\vec{F}$ along the straight line C from $(0,0,0)$ to (x,y,z), then $\nabla f = \vec{F}$.

Solution:
This would be true if $\vec{F}$ were a conservative vector field. In that case, f would be a potential. In general this is false: for example if $\vec{F}(x,y,z) = \langle 0, x, 0 \rangle$, then $\int_C \vec{F} \cdot d\vec{r} = x^2/2$ and $\nabla f(x,y,z) = \langle x, 0, 0 \rangle$ which is different from $\vec{F}$.

20) | **T** | F | The line integral of $\vec{F}(x,y) = (x,y)$ along an ellipse $x^2 + 2y^2 = 1$ is zero.

Solution:
The curl $Q_x - P_y$ of the vector field $\vec{F}(x,y) = (P,Q)$ is 0. By Green's theorem, the line integral is zero. An other way to see this is that F is a gradient field $F = \nabla f$ with $f(x,y) = (x^2 + y^2)/2$. Therefore F is conservative: the line integral along any closed curve in the plane is zero.

21) | T | **F** | The identity $\operatorname{div}(\operatorname{grad}(f)) = 0$ is always true.

Solution:
$\operatorname{div}(\operatorname{grad}(f)) = \Delta f$.

TF Problem for probability theory sections:

22) | T | **F** | Inside a bag is are two coins, one coin has both sides heads and one coin is normal (one head and one tail). I randomly pick one of the coins and randomly look at one side, seeing a head. Is the probability that the other side of the same coin is a tail equal to $1/2$?

Solution:
The probability space has 4 elements $\{(1,H),(1,H),(2,H),(2,T)\}$ each experiment drawing a coin occurring with the same probability $1/4$. The event A that we see head by drawing a coin is $\{(1,H),(1,H),(2,H)\}$. The event B that the other side of the same coin is tail is $B = \{(2,T)\}$. The problem asks, whether the conditional probability $P(B|A)$ is $1/2$. But $P(B|A) = P(B \cap A)/P(A) = (1/4)/(3/4) = 1/3$. This problem could also be solved with common sense: the knowledge that we have seen a head first has increased the probability that we have a coin with two heads. Without that knowledge, the probability would be $1/2$.

23) [**T**] [F] If X and Y are independent random variables, then $D(X + Y) = D(X) + D(Y)$.

Solution:
If X and Y are independent, then $E(XY) = E(X)E(Y)$ and $E((X + Y)^2) - E(X^2 + 2XY + Y^2) = E(X^2) + E(X)E(Y) + E(Y^2) = (E(X) + E(Y))^2$.
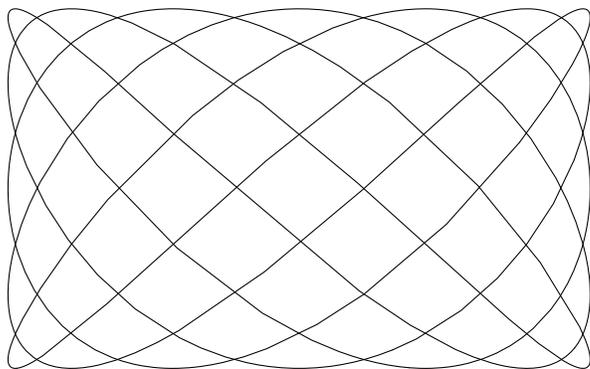
24) [T] [**F**] The expectation of the product of two random variables is always the product of the expectations.
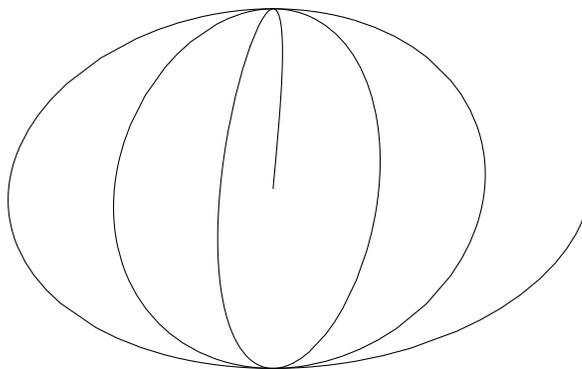
Solution:
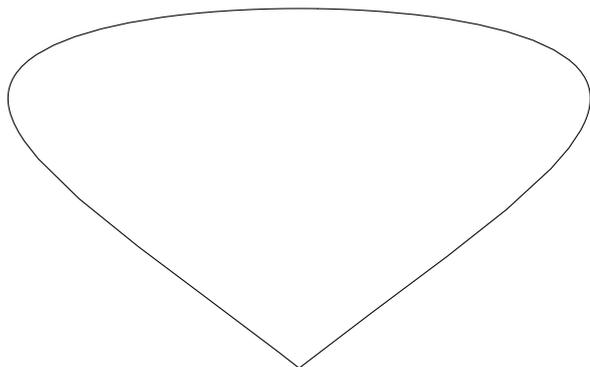This is only true for independent random variables

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Problem 2) (10 points)

Match the equations with the curves. No justifications are needed.

I

II

III
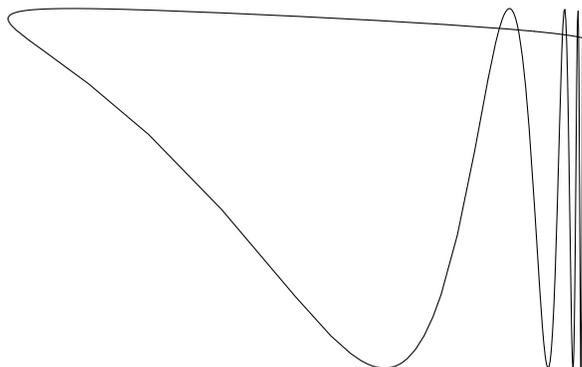
IV

Enter I,II,III,IV here	Equation
	$\vec{r}(t) = (\sin(t), t(2\pi - t))$
	$\vec{r}(t) = (\cos(5t), \sin(7t))$
	$\vec{r}(t) = (t\cos(t), \sin(t))$
	$\vec{r}(t) = (\cos(t), \sin(6/t))$

Solution:

Enter I,II,III,IV here	Equation
III	$\vec{r}(t) = (\sin(t), t(2\pi - t))$
I	$\vec{r}(t) = (\cos(5t), \sin(7t))$
II	$\vec{r}(t) = (t\cos(t), \sin(t))$
IV	$\vec{r}(t) = (\cos(t), \sin(6/t))$

Problem 3) (10 points)

In this problem, vector fields F are written as $F = \langle P, Q \rangle$. We use abbreviations $\mathrm{curl}(F) = Q_x - P_y$ and $\mathrm{div}(F) = P_x + Q_y$. When stating $\mathrm{curl}(F)(x,y) = 0$ we mean that $\mathrm{curl}(F)(x,y) = 0$ vanishes for **all** (x,y). The statement $\mathrm{curl}(F) \neq 0$ means that $\mathrm{curl}(F)(x,y)$ does not vanish for at least one point (x,y). The same remark applies if curl is replaced by div.

Check the box which match the formulas of the vector fields with the corresponding picture I,II,III or IV. Mark also the places, indicating the vanishing or not vanishing of curl and div. In each of the four lines, you should finally have circled three boxes. No justifications are needed.

Vectorfield	I	II	III	IV	$\mathrm{curl}(F) = 0$	$\mathrm{curl}(F) \neq 0$	$\mathrm{div}(F) = 0$	$\mathrm{div}(F) \neq 0$
$\vec{F}(x,y) = (0,5)$								
$\vec{F}(x,y) = (y,-x)$								
$\vec{F}(x,y) = (x,y)$								
$\vec{F}(x,y) = (2,x)$								

I
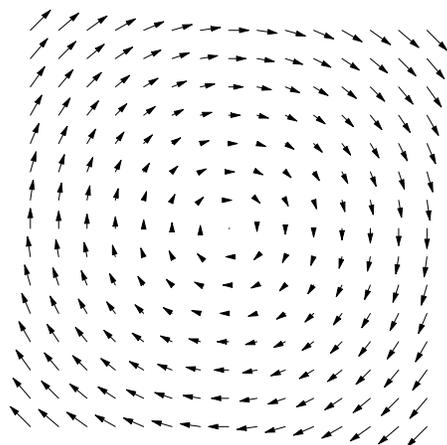
II
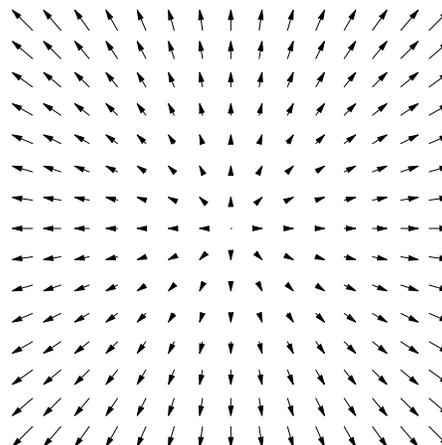
III

IV
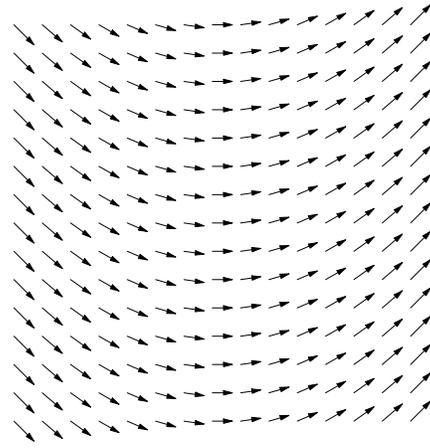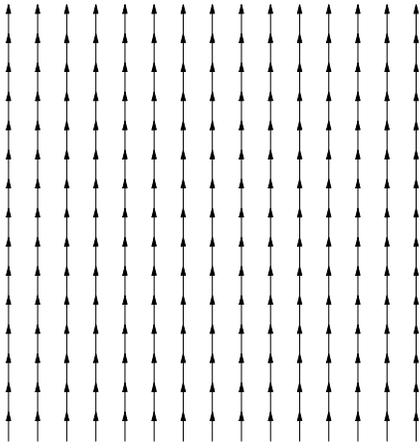
Solution:

Vectorfield	I	II	III	IV	curl$(F) = 0$	curl$(F) \neq 0$	div$(F) = 0$	div$(F) \neq 0$
$\vec{F}(x, y) = (0, 5)$			X		X		X	
$\vec{F}(x, y) = (y, -x)$	X					X	X	
$\vec{F}(x, y) = (x, y)$		X			X			X
$\vec{F}(x, y) = (2, x)$				X		X	X	

Problem 4) (10 points)

a) Find the scalar projection of the vector $\vec{v} = (3, 4, 5)$ onto the vector $\vec{w} = (2, 2, 1)$.

b) Find the equation of a plane which contains the vectors $\langle 1, 1, 0 \rangle$ and $\langle 0, 1, 1 \rangle$ and contains the point $(0, 1, 0)$.

Solution:
a) $|\vec{v} \cdot \vec{w}|/|\vec{w}| = (6 + 8 + 5)/3 = \boxed{19/3}$
b) $(1, 1, 0) \times (0, 1, 1) = (1, -1, 1)$. The plane has the form $x - y + z = d$ and $d = -1$ is obtained by plugging in the point $(0, 1, 0)$. The solution is $\boxed{x - y + z = -1}$

Problem 5) (10 points)

Find the surface area of the ellipse cut from the plane $z = 2x + 2y + 1$ by the cylinder $x^2 + y^2 = 1$.

Solution:
Parameterize the surface $r(u, v) = \langle u, v, 2u + 2v + 1 \rangle$ on the disc $R = \{u^2 + y^2 \leq 1\}$. We get $r_u \times r_v = \langle 1, 0, 2 \rangle \times \langle 0, 1, 2 \rangle = \langle -2, -2, 1 \rangle$ and $|r_u \times r_v| = 3$. The surface integral $\int \int_R |r_u \times r_v| \, dudv = \int \int_R 3 \, dudv = 3 \int \int_R \, dudv$ which is 3 times the area of the disc R:
Solution: $\boxed{3\pi}$.

Problem 6) (10 points)

Sketch the plane curve $\vec{r}(t) = (\sin(t)e^t, \cos(t)e^t)$ for $t \in [0, 2\pi]$ and find its length.

Solution:

$\vec{r}\,'(t) = \langle \cos(t)e^t + \sin(t)e^t, -\sin(t)e^t + \cos(t)e^t \rangle$ satisfies
$|\vec{r}\,'(t)| = \sqrt{2}e^t$ so that $\int_0^{2\pi} |\vec{r}\,'(t)|\, dt = \boxed{\sqrt{2}(e^{2\pi} - 1)}$.
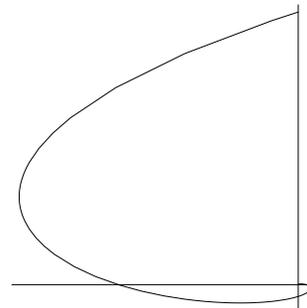
Problem 7) (10 points)

Let $f(x, y, z) = 2x^2 + 3xy + 2y^2 + z^2$ and let R denote the region in $\mathbf{R}^3$, where $2x^2 + 2y^2 + z^2 \leq 1$. Find the maximum and minimum values of f on the region R and list all points, where said maximum and minimum values are achieved. Distinguish between local extrema in the interior and extrema on the boundary.

Solution:
a) Extrema in the interior of the ellipsoid $2x^2 + 2y^2 + z^2 < 1$.
$\nabla f(x, y, z) = \langle 4x + 3y, 3x + 4y, 2z \rangle = \langle 0, 0, 0 \rangle$ for $(x, y, z) = (0, 0, 0)$. One has $f(0, 0, 0) = 0$ as a critical point. It is a candidate for the minimum.
b) To get the extrema on the boundary $g(x, y, z) = 2x^2 + 2y^2 + z^2 - 1 = 0$ we solve the Lagrange equations $\nabla f = \lambda \nabla g, g = 0$. They are

$$
\begin{aligned}
4x + 3y &= \lambda 4x \\
3x + 4y &= \lambda 4y \\
2z &= \lambda 2z \\
2x^2 + 2y^2 + z^2 &= 1
\end{aligned}
$$

We obtain $z = 0, x = \pm y$ or $z = \pm 1, x = y = 0$ giving 6 critical points
$(1/2, 1/2, 0), (-1/2, 1/2, 0), (1/2, -1/2, 0), (-1/2, -1/2, 0), (0, 0, 1), (0, 0, -1)$.
c) Comparing the values $f(0, 0, 0) = 0$, $f(1/2, 1/2, 0) = f(-1/2, -1/2, 0) = 7/4$ and $f(1/2, -1/2, 0) = f(-1/2, 1/2, 0) = 1/2$ and $f(0, 0, \pm 1) = 1$ shows that
$\boxed{(1/2, 1/2, 0) \text{ and } (-1/2, -1/2, 0) \text{ are maxima}}$ and that $\boxed{(0, 0, 0) \text{ is the minimum.}}$

Problem 8) (10 points)

Sketch the region of integration of the following iterated integral and then evaluate the integral:

$$\int_0^\pi \left(\int_{\sqrt{z}}^{\sqrt{\pi}} \left(\int_0^x \sin(xy) dy \right) dx \right) dz .$$

Solution:
The region is contained inside the cube $[0, \sqrt{\pi}] \times [0, \sqrt{\pi}] \times [0, \pi]$.
It is bounded by the surfaces $x = \sqrt{z}, x = y, z = 0, y = \sqrt{\pi}$
(see picture). The integral can not be solved in the given
order. Using the picture as a guide, we write the integral as

$$\int_0^{\sqrt{\pi}} \int_0^{x^2} \int_0^x \sin(xy) \, dydzdx$$

Solve the most inner integral:

$$\int_0^{\sqrt{\pi}} \int_0^{x^2} -\frac{\cos(xy)}{x}\Big|_0^x \, dzdx = \int_0^{\sqrt{\pi}} \int_0^{x^2} (1 - \cos(x^2))/x \, dzdx$$

Now solve the z integral:

$$= \int_0^{\sqrt{\pi}} x^2 (1 - \cos(x^2))/x \, dx = \int_0^{\sqrt{\pi}} x(1 - \cos(x^2)) \, dx$$

to finally get

$$= (-\frac{\sin(x^2)}{2} + \frac{x^2}{2})\Big|_0^{\sqrt{\pi}} = \frac{\pi}{2} .$$

The answer is $\boxed{\frac{\pi}{2}}$.
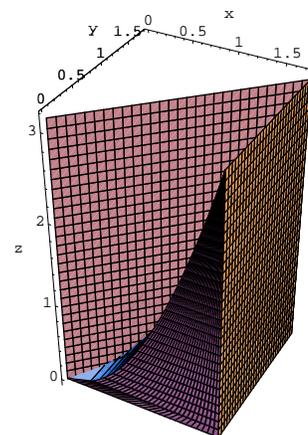
Problem 9) (10 points)

Evaluate the line integral

$$\int_C \vec{F} \cdot \vec{dr} ,$$

where C is the planar curve $\vec{r}(t) = (t^2, t/\sqrt{t+2}), t \in [0,2]$ and $\vec{F}$ is the vector field $\vec{F}(x, y) = (2xy, x^2 + y)$. Do this in two different ways:

a) by verifying that $\vec{F}$ is conservative and replacing the path with a different path connecting $(0,0)$ with $(4,1)$,

b) by finding a potential U satisfying $\nabla U = \vec{F}$.

Solution:
a) To verify that $\vec{F} = (P, Q)$ is conservative, it is enough to verify that $\text{curl}(\vec{F}) = Q_x - P_y = 0$. This is actually the case. To calculate the line integral, we therefore can replace the path with a straight line $\gamma : \vec{r}(t) = (4t, t)$ and calculate $\int_\gamma \vec{F} \, d\vec{r} = \int_0^1 (8t^2, 16t^2 + t) \cdot (4, 1) \, dt = (48t^2 + t)|_0^1 = \boxed{16 + 1/2}$.
b) A potential is $f(x, y) = x^2 y + y^2/2$. The value of f at $(4, 1)$ is $16 + 1/2$. The value of f at $(0, 0)$ is 0. The difference between the potential values is $\boxed{16 + 1/2}$ again.

Problem 10) (10 points)

Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = (x + e^x \sin(y), x + e^x \cos(y))$ and C is the right handed loop of the lemniscate described in polar coordinates as $r^2 = \cos(2\theta)$.
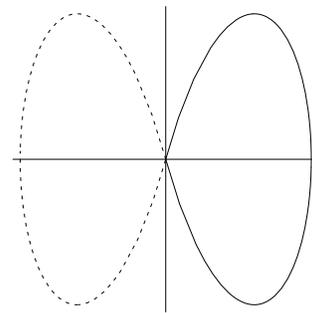
Solution:
By Green's theorem, the integral is $\int \int_R \text{curl}(F) \, dA$, where R is the region enclosed by C. From $\vec{F} = (P, Q) = (x + e^x \sin(y), x + e^x \cos(y))$ we calculate $\text{curl}(\vec{F}) = Q_x - P_y = 1 + e^x \cos(y) - \cos(y)e^x = 1$ so that the result is the area of R which is $\int_{-\pi/4}^{\pi/4} \cos(2\theta)/2 \, d\theta = \frac{\sin(2\theta)}{4}|_{-\pi/4}^{\pi/4} = \boxed{1/2}$.

Problem 11) (10 points)

a) Find the extremal points of $f(x, y) = (x - y)^4$ on the unit circle $g(x, y) = x^2 + y^2 = 1$.
b) Find the extremal points of $g(x, y)$ subject to the constraint $f(x, y) = 4$.

Solution:
a) $\nabla f = \langle 4(x-y)^3, -4(x-y)^3 \rangle$.
$\nabla g = \langle 2x, 2y \rangle$.
The Lagrange equations $\nabla f = \lambda \nabla g, g = 1$ have the four solutions $(\sqrt{2}/2, \sqrt{2}/2), (\sqrt{2}/2, \sqrt{2}/2), (-\sqrt{2}/2, \sqrt{2}/2), (-\sqrt{2}/2, -\sqrt{2}/2)$.
b) The Lagrange equations are the same except the last equation. There are two solutions $(\sqrt{2}/2, -\sqrt{2}/2)$ and $(-\sqrt{2}/2, \sqrt{2}/2)$.

Problem 12A) (10 points)

a) Find the line integral $\int_C \vec{F} \cdot d\vec{r}$ of the vector field $\vec{F}(x,y) = (xy, x)$ along the unit circle $C : t \mapsto \vec{r}(t) = (\cos(t), \sin(t)), t \in [0, 2\pi]$ by doing the actual line integral.

b) Find the value of the line integral obtained in a) by evaluating a double integral.

Solution:
a) $\int_0^{2\pi}(\cos(t)\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t))\ dt = \int_0^{2\pi} \cos^2(t) - \sin^2(t)\cos(t)\ dt = \pi + \sin^3(t)/3|_0^{2\pi} = \boxed{\pi}$
b) $\text{curl}(F) = Q_x - P_y = 1 - x$. By Green's theorem, the line integral is a double integral which we evaluate using Polar coordinates
$\int \int_D (1-x)\ dA = \int_0^1 \int_0^{2\pi}(1 - \cos(\theta)r)r\ d\theta dr = 2\pi/2 = \boxed{\pi}$.

Problem 13A) (10 points)

Consider the surface given by the graph of the function
$z = f(x,y) = \frac{100}{1+x^2+y^2} \sin\left(\frac{\pi}{8}(x^2 + y^2)\right)$ in the region $x^2 + y^2 \leq 16$. The surface is pictured to the right.
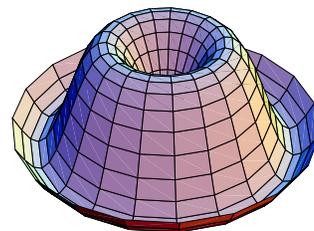
A magnetic field $\vec{B}$ is given by the curl of a vector potential $\vec{A}$. That is, $\vec{B} = \nabla \times \vec{A} = \text{curl}(A)$ and $\vec{A}$ is a vector field too. Suppose

$$\vec{A} = \left(z\sin(x^3), x\left(1 - z^2\right), \log(1 + e^{x+y+z})\right) .$$

Compute the flux of the magnetic field through this surface. (The surface has an upward pointing normal vector.)

Solution:
The surface S is bounded by the curve $\gamma : \vec{r}(t) = (4\cos(t), 4\sin(t), 0)$. By Stokes theorem, the flux of the curl of $\vec{A}$ through the surface S is the line integral of A along γ:

$$\int_\gamma \vec{A} \cdot d\vec{r} = \int_0^{2\pi} (0, 4\cos(t), \log(1 + e^{x+y})) \cdot (-4\sin(t), 4\cos(t), 0) \, dt = \int_0^{2\pi} 16\cos^2(t) \, dt =$$

$\boxed{16\pi}$.

$\boxed{\text{Problem 14A) (10 points)}}$

Let S be the surface given by the equations $z = x^2 - y^2$, $x^2 + y^2 \le 4$, with the upward pointing normal. If the vector field $\vec{F}$ is given by the formula $\vec{F}(x, y, z) = \langle -x, y, \sqrt{x^2 + y^2} \rangle$, find the flux of $\vec{F}$ through S.

Solution:
Parameterize the surface by $\vec{r}(u, v) = \langle u, v, u^2 - v^2 \rangle$. Then $r_u \times r_v = \langle 1, 0, 2u \rangle \times \langle 0, 1, -2v \rangle = \langle -2u, 2v, 1 \rangle$. The flux integral is

$$
\begin{aligned}
\int\int_D \vec{F} \cdot dS &= \int\int_D \langle -u, v, \sqrt{u^2 + v^2} \rangle \cdot \langle -2u, 2v, 1 \rangle \, dxdy \\
&= \int\int_D 2(u^2 + v^2) + \sqrt{u^2 + v^2} \, dA \\
&= \int_0^{2\pi} \int_0^2 (2r^2 + r)r \, drd\theta \\
&= 2\pi(2 \cdot 2^4/4 + 2^3/3) = 64\pi/3 .
\end{aligned}
$$

The answer is $\boxed{64\pi/3}$.

PROBLEMS TO THE BIO-CHEM SECTIONS.

$\boxed{\text{Problem 12B) (10 points)}}$

Bob will arrive at the bus station randomly between 3:15 and 3:45 and Chuck will arrive at random between 3:00 and 4:00 (independently of Bob). Each agrees to wait up to five

minutes for the other before leaving.

(a) What is the probability that they meet?
(b) Find the probability that Chuck arrives first.

Solution:
The probability space is a rectangle $[3:15, 3:45] \times [3,4]$. Since the two events of arrival are assumed to be independent, the probability of an event in that rectangle is the area normalized so that the entire rectangle has area 1. In both cases a) and b), we have to find the area of the event. In a), this event is a strip of width $5min = 1/12$ and height $5 = 1/12$ around the line $x = y$. This strip has area $1/6 \times 1/2$. The entire rectangle has area $1/2$. The probability in a) is therefore $(1/6 \times 1/2)/(1/2) = \boxed{1/6}$. The probability that Chuck arrives first is the area of the region below the diagonal divided by the area of the entire rectangle, which is $(1/4)/(1/2) = \boxed{1/2}$.
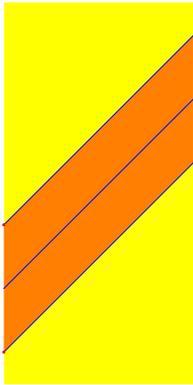
The event that the two arrive within 5 minutes of the other arrival.
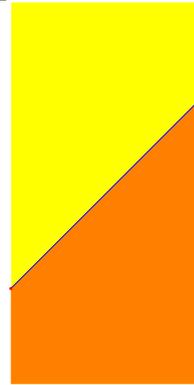
The evnet that Chuck arrives first is the normalized area blow the diagonal $x = y$.

Problem 13B) (10 points)

An urn contains 10 blue balls, 8 green balls and 5 red balls. Find the probability of a blue ball being drawn before a green ball if

a) No balls are replaced after each draw.
b) If each ball is replaced after being drawn.

Solution:
See problem 3.18 in Rozanov.
a) We can ignore the red balls and just look at the draws, when we draw a blue or green ball. There are 18 cases: 10 good cases: $bg, bbg, bbbg, bbbbg, ..., bbbbbbbbbbg$ and 8 bad cases $gb, ggb, gggb, ..., gggggggggb$. The probability is $10/18 = 5/9$.
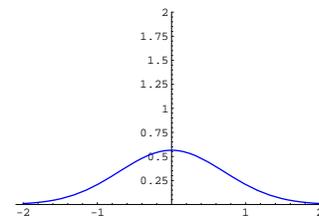b) The probability is $10/(10 + 8) = 5/9$.

Problem 14B) (10 points)
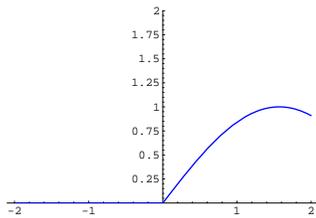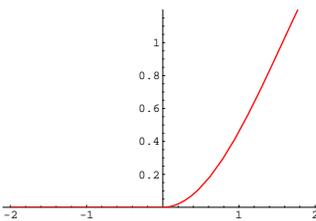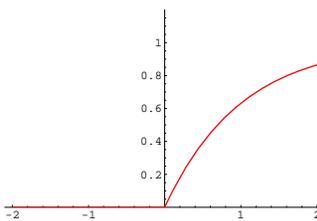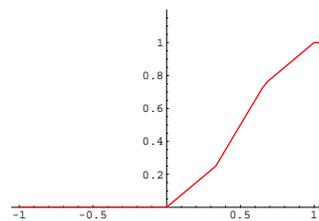

Match the following density functions
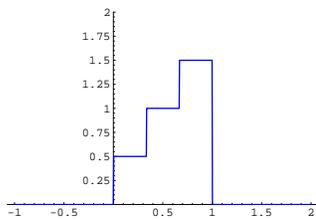
I
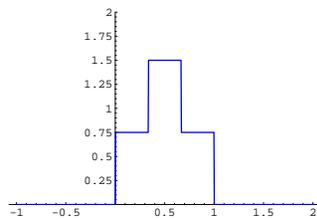
II

III

IV
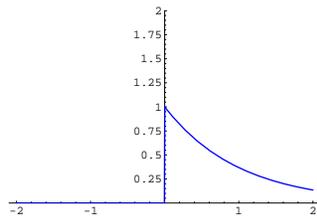
V
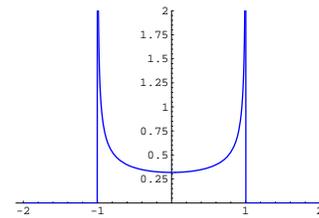
VI
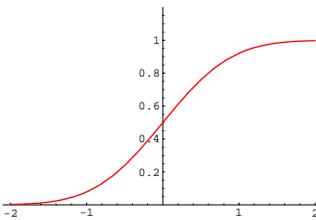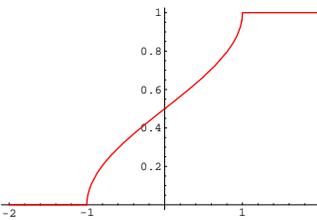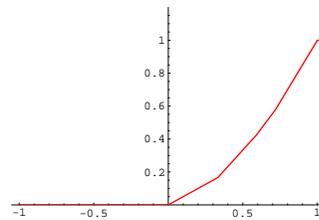
with these distribution functions

a

b

c

d

e

f

and give reasons for your choice.

Solution:
I) f)
II) c)
III) d)
IV) a)
V) b)
VI) e)