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- Please mark the box to the left which lists your section.
- Do not detach pages from this exam packet or unstaple the packet.
- Show your work! Answers without reasoning can not be given credit, except for the TF and multiple choice problems.
- Please write neatly. Answers which the grader can not read will not receive credit.
- No notes, books, calculators, computers, or other electronic aids can be used.
- All unspecified functions mentioned in this exam are assumed to be smooth: you can differentiate as many times as you want with respect to any variables.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) | T | F |

Any function of three variables $f(x, y, z)$ satisfies the partial differential equation $f_{xyz} + f_{yzx} = 2f_{zxy}$.

Solution:
By Clairot's theorem

2) | T | F |

If $f_x(x, y) = f_y(x, y)$ for all x, y, then $f(x, y)$ is a constant.

Solution:
$f_x = f_y$ is an example of a PDE called a transport equation. It has solutions like for example $f(x, y) = x + y$. Any function which stays invariant by replacing x with y is a solution: like $f(x, y) = \sin(xy) + x^5y^5$.

3) | T | F |

$(1, 1)$ is a local maximum of the function $f(x, y) = x^2y - x + \cos(y)$.

Solution:
$(1, 1)$ is not even a critical point.

4) | T | F |

If f is a smooth function of two variables, then the number of critical points of f inside the unit disc is finite.

Solution:
Take $f(x, y) = x^2$ for example. Every point on the y axes $\{x = 0\}$ is a critical point.

5) | T | F |

The value of the function $f(x, y) = \sin(-x + 2y)$ at $(0.001, -0.002)$ can by linear approximation be estimated as -0.003.

Solution:
The correct approximation would be $f(0, 0) + 0.001(-1) - 0.002(2) = -0.005$.

6) | T | F |

If $(1, 1)$ is a critical point for the function $f(x, y)$ then $(1, 1)$ is also a critical point for the function $g(x, y) = f(x^2, y^2)$.

Solution:
If $\nabla f(1,1) = (f_x(1,1), f_y(1,1)) = (0,0)$ then also $\nabla g(1,1) = (f_x(1,1)2x, f_y(1,1)2y) = (0,0)$.

7) T **F**

There is no function $f(x,y,z)$ of three variables, for which every point on the unit sphere is a critical point.

Solution:
Take a function like $g(t) = te^{-t}$ with a maximum at $t = 1$ and define $f(x,y,z) = g(x^2 + y^2 + z^2)$.

8) T **F**

If (x_0, y_0) is a maximum of $f(x,y)$ under the constraint $g(x,y) = g(x_0, y_0)$, then (x_0, y_0) is a maximum of $g(x,y)$ under the constraint $f(x,y) = f(x_0, y_0)$.

Solution:
Assume you have a situation f, g, where this is true and where the constraint is $g = 0$, produce a new situation $f, h = -g$, where the first statement is still true but where the extrema of h under the constraint of f is a minimum.

9) **T** F

If $\vec{u}$ is a unit vector tangent at (x,y,z) to the level surface of $f(x,y,z)$ then $D_u f(x,y,z) = 0$.

Solution:
The directional derivative measures the rate of change of f in the direction of u. On a level surface, in the direction of the surface, the function does not change (because f is constant by definition on the surface).

10) T **F**

In cylindrical coordinates (r, θ, z), the surface $z = r^2$ describes a cone.

Solution:
$z = x^2 + y^2$ is a cone.

11) **T** F

The function $u(x,t) = x^2/2 + t$ satisfies the heat equation $u_t = u_{xx}$.

Solution:
Just differentiate.

12) T F The vector $\vec{r}_u - \vec{r}_v$ is tangent to the surface parameterized by $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$.

Solution:
Both vectors $\vec{r}_u$ and $\vec{r}_v$ are tangent to the surface. So also their difference.

13) T F If $(1,1,1)$ is a maximum of f under the constraints $g(x,y,z) = c, h(x,y,z) = d$, and the Lagrange multipliers satisfy $\lambda = 0, \mu = 0$, then $(1,1,1)$ is a critical point of f.

Solution:
Look at the Lagrange equations. If $\lambda = \mu = 0$, then $\nabla f = (0,0,0)$.

14) T F If $(0,0)$ is a critical point of $f(x,y)$ and the discriminant D is zero but $f_{xx}(0,0) > 0$ then $(0,0)$ can not be a local maximum.

Solution:
If $f_{xx}(0,0) > 0$ then on the x-axes the function $g(x) = f(x,0)$ has a local minimum. This means that there are points close to $(0,0)$ where the value of f is larger.

15) T F Let (x_0, y_0) be a saddle point of $f(x,y)$. For any unit vector $\vec{u}$, there are points arbitrarily close to (x_0, y_0) for which ∇f is parallel to $\vec{u}$.

Solution:
Just look at the level curves near a saddle point. The gradient vectors are orthogonal to the level curves which are hyperbola. You see that they point in any direction except 4 directions. To see this better, take a pen and draw a circle around the saddle point between two of your knuckles on your fist. At each point of the circle, now draw the direction of steepest increase (this is the gradient direction).

16) T F A function $f(x,y)$ on the plane for which the absolute minimum and the absolute maximum are the same must be constant.

Solution:
This would not be true if "absolute" would be replaced by "local".

17) | T | **F** |

The sign of the Lagrange multiplier tells whether the critical point of $f(x, y)$ constrained to $g(x, y) = 0$ is a local maximum or a local minimum.

Solution:
We would get the same Lagrange equations when replacing g with $-g$ and λ with $-\lambda$.

18) | T | **F** |

The gradient of a function $f(x, y, z)$ is tangent to the level surfaces of f

Solution:
The gradient is normal to the level surface.

19) | T | **F** |

The point $(0, 1)$ is a local minimum of the function $x^3 + (\sin(y - 1))^2$.

Solution:
While the gradient is $(3x^2, 2\sin(y - 1)\cos(y - 1))$, the critical point is not a minimum.

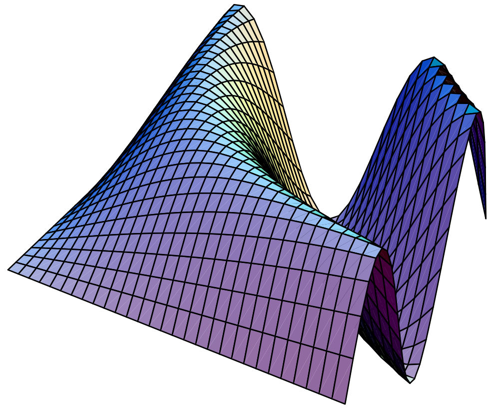
20) | **T** | F |

If $D_u f(x, y, z) = 0$ for all unit vectors u, then (x, y, z) is a critical point.

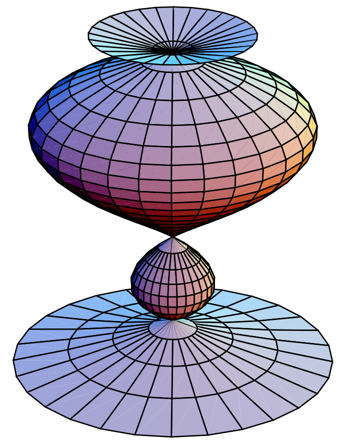
Solution:
If (x, y, z) is not a critical point, then the gradient vector $n = \nabla f(x, y, z)$ would have positive length and taking $u = n/||n||$ would give $D_u f(x, y, z) = ||n||^2 \neq 0$.

Problem 2) (10 points)

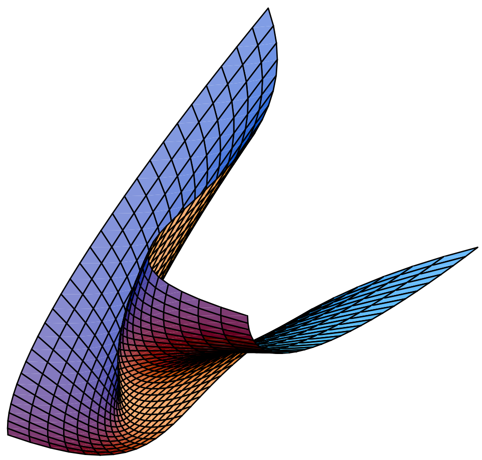
Match the parametric surfaces with their parameterization. No justification is needed.



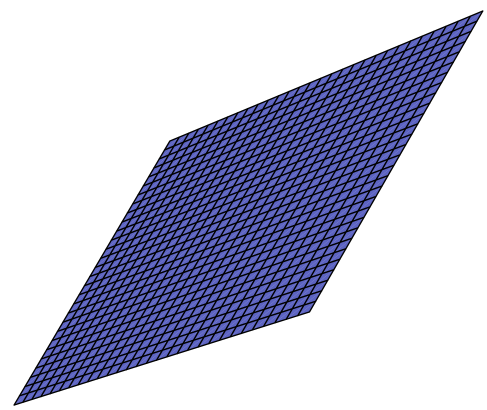
I



II



III



IV

Enter I,II,III,IV here	Parameterization
	$(u, v) \mapsto (u, v, u + v)$
	$(u, v) \mapsto (u, v, \sin(uv))$
	$(u, v) \mapsto (0.2 + u(1 - u^2)) \cos(v), (0.2 + u(1 - u^2)) \sin(v), u)$
	$(u, v) \mapsto (u^3, (u - v)^2, v)$

Solution:

Enter I,II,III,IV here	Parameterization
IV	$(u, v) \mapsto (u, v, u + v)$
I	$(u, v) \mapsto (u, v, \sin(uv))$
II	$(u, v) \mapsto (0.2 + u(1 - u^2)) \cos(v), (0.2 + u(1 - u^2)) \sin(v), u)$
III	$(u, v) \mapsto (u^3, (u - v)^2, v)$

Surface I is a graph.
Surface II is a surface of revolution.
Surface III is algebraic. One of the traces is (u^3, u^2), an other trace is the parabola (v^2, v).
Surface IV is a plane.

Problem 3) (10 points)

Use the technique of linear approximation to estimate $f(\log(2)+0.001, 0.006)$ for $f(x,y) = e^{2x-y}$. (Here, log means the natural logarithm).

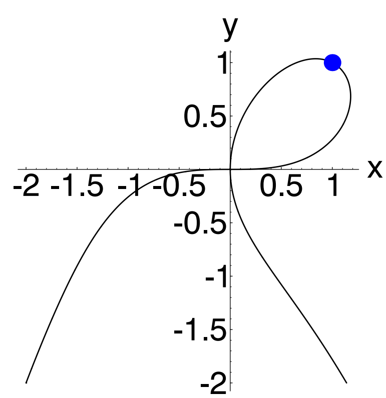
Solution:
$L(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$
$f(x_0, y_0) = e^{2\log 2} = 4$
$f_x(x_0, y_0) = 8$
$f_y(x_0, y_0) = -4$
$L(x,y) = 4 + 0.001 \cdot 8 - 4 \cdot 0.006 = \boxed{3.984}$.

Problem 4) (10 points)

Consider the equation

$$f(x,y) = 2y^3 + x^2y^2 - 4xy + x^4 = 0$$

It defines a curve, which you can see in the picture. Near the point $x = 1, y = 1$, the function can be written as a graph $y = y(x)$. Find the slope of that graph at the point $(1,1)$.



Solution:
Use the formula for implicit differentiation which is derived from the chain rule $f_x(x, y(x)) \cdot 1 + f_y(x, y(x)) \cdot y'(x) = 0$. The slope is $y'(x) = -f_x(x,y)/f_y(x,y)_{(x,y)} = (1,1) = -1/2$.

Problem 5) (10 points)

Find and classify all the critical points of the function $f(x,y) = xy(4 - x^2 - y^2)$.

Solution:
The gradient of $f(x,y) = 4xy - x^3y - xy^3$ is $\nabla f(x,y) = \langle 4y - 3x^2y - y^3, 4x - x^3 - 3xy^2 \rangle$. We solve the system

$$\begin{aligned} y(4 - 3x^2 - y^2) &= 0 \quad (1) \\ x(4 - x^2 - 3y^2) &= 0 \quad (2) \end{aligned}$$

There are four possibilities:
1) $y = 0, x = 0$
2) $4 - 3x^2 - y^2 = 0, x = 0$
3) $4 - x^2 - 3y^2 = 0, y = 0$
4) $4 - 3x^2 - y^2 = 0, 4 - x^2 - 3y^2 = 0$.
This gives 7 critical points in total
1) gives the critical point $(0,0)$.
2) gives the critical points $(0,2), (0,-2)$.
3) gives the critical points $(2,0), (-2,0)$.
4) (subtract 3 times the second equation from the first): $(1,1), (-1,1)$.
The discriminant $D(x,y) = f_{xx}f_{yy} - f_{xy}^2$ at a general point is $-9(x^4 + y^4) - 16 + 24(x^2 + y^2) + 18x^2y^2$ and $f_{xx}(x,y) = -6xy$.
Applying the second derivative test gives

Critical point	(-2, 0)	(-1, -1)	(-1, 1)	(0, 0)	(1, -1)	(1, 1)	(2, 0)
Discriminant	-64	32	32	-16	32	32	-64
f_{xx}	0	-6	6	0	6	-6	0
Analysis	saddle	max	min	saddle	min	max	saddle

Problem 6) (10 points)

Let $f(x,y) = e^{(x-y)}$ so that $f(\log(2), \log(2)) = 1$. Find the equation for the tangent plane to the graph of f at the point $(\log(2), \log(2))$ and use it to estimate $f(\log(2) + 0.1, \log(2) + 0.004)$.

Remark: As usual on the college life as well as in galaxy life, log denotes the natural logarithm. There is no particular reason (except maybe at high school) to think, an other base is natural, except for the fact that we earthlings have 10 fingers. I'm sure you can read this word of wisdom somewhere in a "manual to the galaxy".

Solution:
The graph of f is a level curve of the function $g(x, y, z) = z - f(x, y)$. The gradient at the point $(x_0, y_0, f(x_0, y_0) = (\log(2), \log(2), 1)$ is $(a, b, c) = (-1, 1, 1)$, so that the tangent plane has an equation $ax + by + cz = -x + y + z = d$. and the constant d is obtained from $d = -x_0 + y_0 + z_0 = 1$. Therefore

$$-x + y + z - 1 = 0$$

At the point $(\log(2), \log(2), 1)$, the level surface $g = 0$ is close to the level surface $L(x, y, z) = x - y - z + 1 = 0$. If we plug in $x = 0.1, y = 0.004, z = 0$, we get 1.06.
Remark. We could have worked in two dimensions and estimate $f(x_0 + dx, f(y_0 + dx)$ by $f(x_0, y_0) + (1, -1) \cdot (dx, dy) = 1 + dx - dy$ which is for $dx = 0.1, dy = 0.004$ equal to $1 + 0.1 - 0.004 = \boxed{1.096}$.

Problem 7) (10 points)

Consider the graph of the function $h(x, y) = e^{-3x-y} + 4$.

a) (2 points) Find a function $g(x, y, z)$ of three variables such that this surface is the level set of g.
b) (2 points) Find a vector normal to the tangent plane of this surface at (x, y, z).
c) (2 points) Is this tangent plane ever horizontal? Why or why not?
d) (3 points) Give an equation for the tangent plane at $(0, 0)$.

Solution:

1. $g(x, y, z) = e^{-3x-y} + 4 - z$.
2. $\nabla g(x, y, z) = (3e^{3x_0-y_0}, -e^{3x_0-y_0}, -1)$. At the point (x_0, y_0, z_0), we have the gradient $(a, b, c) = (-3e^{-3x_0-y_0}, -e^{3x_0-y_0}, -1)$ and so the plane $ax + by + cz = d$, where $d = ax_0 + by_0 + cz_0$.
3. Horizontal would mean $a = b = 0$ which is not possible because $-e^{3x_0-y_0}$ is always negative.
4. The tangent plane which goes through the point $(0, 0, h(0, 0)) = (0, 0, 5) = (x_0, y_0, z_0)$ is $-3x - y - z = d$, where $d = 30 - 10 - 15 = -5$. $\boxed{3x + y + z = 5}$.

Problem 8) (10 points)

Minimize the function $E(x,y,z) = \frac{k^2}{8m}(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2})$ under the constraint $xyz = 8$, where k^2 and m are constants.

Remark. In quantum mechanics, E is the ground state energy of a particle in a box with dimensions x, y, z. The constant k is usually denoted by $\hbar$ and called the Planck constant.

Solution:
Write $C = k^2/(8m)$ to save typing. $\nabla E(x,y,z) = -2C(1/x^3, 1/y^3, 1/z^3)$. The constraint is $G(x,y,z) = xyz - 8 = 0$. We have $\nabla G(x,y,z) = (yz, xz, xy)$. The Lagrange equations are

$$\begin{aligned} 2C &= \lambda x^3 yz \\ 2C &= \lambda x y^3 z \\ 2C &= \lambda xyz^3 \\ xyz &= 8 \end{aligned}$$

Eliminating λ gives $x^2 = y^2 = z^2$ and $x = y = z = 2$ and the minimal energy is $\boxed{3C/4 = 3k^2/(32m)}$.

Problem 9) (10 points)

Assume $F(x,y) = g(x^2 + y^2)$, where g is a function of one variable. Find $F_{xx}(1,2) + F_{yy}(1,2)$, given that $g'(5) = 3$ and $g''(5) = 7$.

Solution:
$F_x = g'(x^2 + y^2)2x$.
$F_{xx}(x,y) = g''(x^2 + y^2)4x^2 + g'(x^2 + y^2)2$.
$F_y = g'(x^2 + y^2)2y$.
$F_{yy}(x,y) = g''(x^2 + y^2)4y^2 + g'(x^2 + y^2)2$.
$F_{xx} + F_{yy}(1,2) = 7 \cdot 4 \cdot 5 + 3(2+2) = \boxed{152}$.

Problem 10) (10 points)

Let $g(x, y)$ be the distance to the curve $x^2 + 2y^2 + y^4/10 = 1$.. Show that g is a solution of the partial differential equation

$$f_x^2 + f_y^2 = 1$$

outside the curve.

Hint: here no computations are needed. The shape of the curve is pretty much irrelevant. What does the PDE say about the gradient ∇f?

Remark: This example just needs thought. Use it as a "pillow problem" that is think about it before going to sleep. By the way, the PDE is called **eiconal equation**. It describes wave fronts in optics.

Solution:
The level curves of g are curves, for which the distance to the curve is constant. Lets look at the level curves of f, if f is the solution to the PDE. The PDE tells us $|\nabla f|^2 = 1$. Which means that the gradient of f is a unit vector everywhere. This means that the directional derivative in the gradient direction is 1 everywhere. This implies that the level curves of f are equidistributed too. The level curves of f and g are the same. Because f and g are both zero at the curve, the two functions are the same.