

Name:

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MWF9 Ivan Petrakiev
MWF10 Oliver Knill
MWF10 Thomas Lam
MWF10 Michael Schein
MWF10 Teru Yoshida
MWF11 Andrew Dittmer
MWF11 Chen-Yu Chi
MWF12 Kathy Paur
TTh10 Valentino Tosatti
TTh11.5 Kai-Wen Lan
TTh11.5 Jeng-Daw Yu

- Please mark the box to the left which lists your section.
- Do not detach pages from this exam packet or unstaple the packet.
- Show your work! Answers without reasoning can not be given credit, except for the TF and multiple choice problems.
- Please write neatly. Answers which the grader can not read will not receive credit.
- No notes, books, calculators, computers, or other electronic aids can be used.
- All unspecified functions mentioned in this exam are assumed to be smooth: you can differentiate as many times as you want with respect to any variables.
- You have 90 minutes time to complete your work.

1		30
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		110

Problem 1) TF questions (30 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

- 1) T F $f(x, y)$ and $g(x, y) = f(x^2, y^2)$ have the same critical points.

Solution:

The function g has always $(0, 0)$ as a critical point, even if f has not.

- 2) T F If a function $f(x, y) = ax + by$ has a critical point, then $f(x, y) = 0$ for all (x, y) .

Solution:

At a critical point the gradient is $(a, b) = (0, 0)$, which implies $f = 0$.

- 3) T F $f_{xyxyx} = f_{yyxxx}$ for $f(x, y) = \sin(\cos(y + x^{14}) + \cos(x))$.

Solution:

Follows from Clairot's theorem.

- 4) T F Given 2 arbitrary points in the plane, there is a function $f(x, y)$ which has these points as critical points and no other critical points.

Solution:

Connect the two points with a line and take this height as the x-axes, centered at the midpoint and with units such that the two points have coordinates $(-1, 0), (1, 0)$. The function $f(x, y) = -y^2(x^3 - 1)$ has the two points as critical points. One is a local max, the other is a saddle point.

- 5) T F It is possible that for some unit vector u , the directional derivative $D_u f(x, y)$ is zero even though the gradient $\nabla f(x, y)$ is nonzero.

Solution:

Just go in to the direction tangent to the level curve.

- 6) T F If (x_0, y_0) is the maximum of $f(x, y)$ on the disc $x^2 + y^2 \leq 1$ then $x_0^2 + y_0^2 < 1$.

Solution:

The maximum could be on the boundary.

- 7) T F The linear approximation $L(x, y, z)$ of the function $f(x, y, z) = 3x + 5y - 7z$ at $(0, 0, 0)$ satisfies $L(x, y, z) = f(x, y, z)$.

Solution:

$f(0, 0, 0) = 0$ and $\nabla f(0, 0, 0) = (3, 5, -7)$.

- 8) T F If $f(x, y) = \sin(x) + \sin(y)$, then $-\sqrt{2} \leq D_u f(x, y) \leq \sqrt{2}$.

Solution:

$|D_u f| \leq \|\nabla f\| \leq \sqrt{2}$.

- 9) T F There are no functions $f(x, y)$ for which every point on the unit circle is a critical point.

Solution:

There are many rotationally symmetric functions with this property.

- 10) T F An absolute maximum (x_0, y_0) of $f(x, y)$ is also an absolute maximum of $f(x, y)$ constrained to a curve $g(x, y) = c$ that goes through the point (x_0, y_0) .

Solution:

The Lagrange multiplier vanishes in this case.

- 11) T F If $f(x, y)$ has two local maxima on the plane, then f must have a local minimum on the plane.

Solution:

Look at a camel type surface. It has a saddle between the local maxima.

- 12) T F The acceleration of the curve $\vec{r}(t) = (\cos(t), \sin(t), t)$ at time $t = 0$ is 1.

Solution:

The acceleration is a vector.

- 13) T F There exists a function $f(x, y)$ of two variables which has no critical points at all.

Solution:

True. Every non-constant linear function for example.

- 14) T F If $f_x(x, y) = f_y(x, y) = 0$ for all (x, y) then $f(x, y) = 0$ for all (x, y) .

Solution:

False, f could be constant.

- 15) T F $(0, 0)$ is a local maximum of the function $f(x, y) = x^2 - y^2 + x^4 + y^4$.

Solution:

$(0, 0)$ is a saddle point.

- 16) T F If $f(x, y)$ has a local maximum at the point $(0, 0)$ with discriminant $D > 0$ then $g(x, y) = f(x, y) - x^4 + y^3$ has a local maximum at the point $(0, 0)$ too.

Solution:

Adding $x^4 + y^3$ does not change the first and second derivatives.

- 17) T F The value of the function $f(x, y) = \sqrt{1 + 3x + 5y}$ at $(-0.002, 0.01)$ can by linear approximation be estimated as $1 - (3/2) \cdot 0.002 + (5/2) \cdot 0.01$.

Solution:

Use formula for $L(x, y)$.

- 18) T F The gradient of f at a point (x_0, y_0, z_0) is tangent to the level surface of f which contains (x_0, y_0, z_0) .

Solution:

It is a basic and important fact that ∇f is **perpendicular** to the level surface.

- 19)

T	F
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 If $D_{\vec{v}}f(1, 1) = 0$ for all vectors \vec{v} , then $(1, 1)$ is a critical point of $f(x, y)$.

Solution:

Especially, $D_{\nabla f}(f) = |\nabla f|^2 = 0$ so that $\nabla f = (0, 0, 0)$.

- 20)

T	F
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 The function $u(x, t) = x^3 + t^3$ satisfies the wave equation $u_{tt} = u_{xx}$.

Solution:

Just differentiate.

- 21)

T	F
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 Every critical point (x, y) of a function $f(x, y)$ for which the discriminant D is not zero is either a local maximum or a local minimum.

Solution:

The second derivative test give for negative D that we have a saddle point.

- 22)

T	F
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 The function $f(x, y) = e^y x^2 \sin(y^2)$ satisfies the partial differential equation $f_{xxyyyxyy} = 0$.

Solution:

By Clairots theorem, we can have all three x derivatives at the beginning.

- 23)

T	F
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 If $(0, 0)$ is a critical point of $f(x, y)$ and the discriminant D is zero but $f_{xx}(0, 0) < 0$ then $(0, 0)$ can not be a local minimum.

Solution:

If $f_{xx}(0, 0) < 0$ then on the x-axis the function $g(x) = f(x, 0)$ has a local maximum. This means that there are points close to $(0, 0)$ where the value of f is larger.

- 24)

T	F
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 In the second derivative test, one can replace the condition $D > 0, f_{xx} > 0$ with $D > 0, f_{yy} > 0$ to check whether a point is a local minimum.

Solution:

True. If $f_{xx}f_{yy} - f_{xy}^2 > 0$, then f_{xx} and f_{yy} must have the same signs.

- 25) T F The gradient $\langle 2x, 2y \rangle$ is perpendicular to the surface $z = x^2 + y^2$.

Solution:

The surface is the graph of a function $f(x, y)$. While the gradient of f is perpendicular to the level curve of f , it is only the projection of the gradient to the function $g(x, y, z) = f(x, y) - z$. The latter is perpendicular to the surface.

- 26) T F If $f(x, t)$ satisfies the Laplace equation $f_{xx} + f_{tt} = 0$ and simultaneously the wave equation $f_{xx} = f_{tt}$, then $f(x, t) = ax + bt + c$.

Solution:

Take $f(x, t) = xt$. (Here is how we get the general solution: From the two equations, we get $f_{xx} = 0$ and $f_{tt} = 0$. From $f_{xx} = 0$, we obtain that $f(x, t) = a(t)x + c(t)$. From $f_{tt} = 0$, we obtain $a(t)$ and $c(t)$ are linear in t . Therefore the general solution is $f(x, t) = atx + bt + cx + e$.)

- 27) T F The function $f(x, y) = (x^4 - y^4)$ has neither a local maximum nor a local minimum at $(0, 0)$.

Solution:

The function is both smaller and bigger than $f(0, 0)$ for points near $(0, 0)$.

- 28) T F It is possible to find a function of two variables which has no maximum and no minimum.

Solution:

There are many linear functions like that.

- 29) T F The value of the function $f(x, y) = e^xy$ at $(0.001, -0.001)$ can by linear approximation be estimated as -0.001 .

Solution:

Because the gradient at $(0,0)$ is $(0,1)$ and $f(0,0) = 0$, the linear approximation is $L(x,y) = y$.

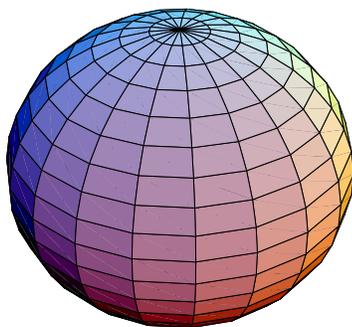
- 30) T F For any function $f(x,y,z)$ and any unit vectors u,v , one has the identity $D_{u \times v} f(x,y,z) = D_u f(x,y,z) D_v f(x,y,z)$.

Solution:

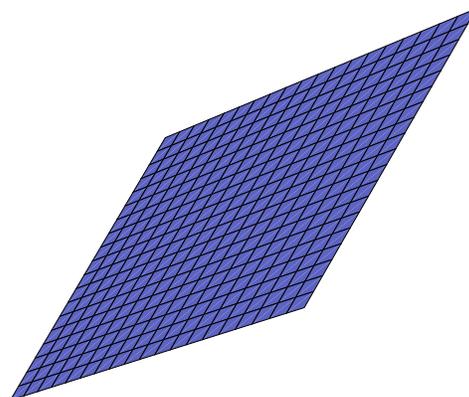
The directional derivative in the $u \times v$ direction has nothing to do with the directional derivatives into the other directions. An example, $u = (1,0,0), v = (0,1,0), f(x,y,z) = x + y$ is an example, where $D_{u \times v} f(x,y,z) = 0$ but $D_u f = 1, D_v f = 1$.

Problem 2) (10 points)

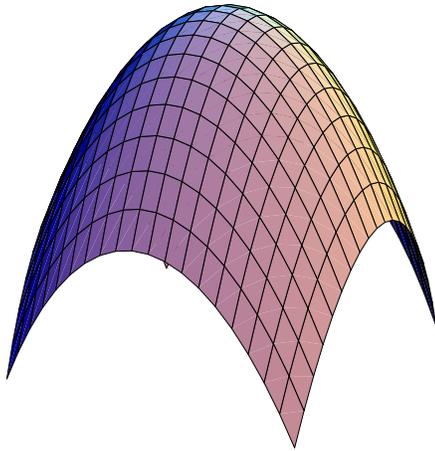
Match the parametric surfaces with their parameterization. No justification is needed.



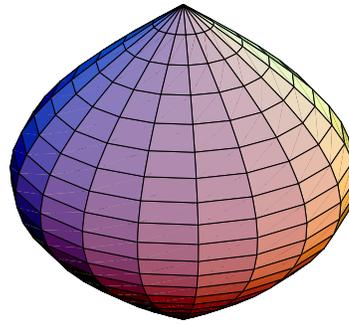
I



II



III



IV

Enter I,II,III,IV here	Parameterization
	$(u, v) \mapsto (\cos(u) \sin(v), \sin(u) \sin(v), \cos(v))$
	$(u, v) \mapsto (u - 1, v + 3, u + v)$
	$(u, v) \mapsto (u, v, 1 - u^2 - v^2)$
	$(u, v) \mapsto (\sin(v) \cos(u), \sin(v) \sin(u), v)$

Solution:

Enter I,II,III,IV here	Parameterization
I	$(u, v) \mapsto (\cos(u) \sin(v), \sin(u) \sin(v), \cos(v))$
II	$(u, v) \mapsto (u - 1, v + 3, u + v)$
III	$(u, v) \mapsto (u, v, 1 - u^2 - v^2)$
IV	$(u, v) \mapsto (\sin(v) \cos(u), \sin(v) \sin(u), v)$

Problem 3) (10 points)

a) Show that for any differentiable function $g(x)$, the function $u(x, y) = g(x^2 + y^2)$ satisfies the partial differential equation $yu_x = xu_y$.

b) Assuming $g'(5) \neq 0$, let u be the function defined in a). Find the unit vector \vec{v} in the direction of maximal increase at the point $(x, y) = (2, 1)$.

Solution:

a) Just differentiate:

$$yu_x = yg'(x^2 + y^2)2x = 2xyg'(x^2 + y^2)$$

$$xu_y = xg'(x^2 + y^2)2y = 2yxg'(x^2 + y^2)$$

These two expressions are the same.

b) The direction of maximal increase points into the direction of the gradient of u which is $\nabla u(x, y) = (g'(x^2 + y^2)2x, g'(x^2 + y^2)2y)$.

At the point $(x, y) = (2, 1)$ we have $(g'(5)4, g'(5)2)$. If we normalize that, we obtain

$$\boxed{\vec{v} = (4, 2)/\sqrt{20}}.$$

Problem 4) (10 points)

Which point on the surface $g(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{8}{z} = 1$ is closest to the origin?

Solution:

This is a Lagrange problem. One wants to minimize $f(x, y, z) = x^2 + y^2 + z^2$ under the constraint $g(x, y, z) = 1$. The Lagrange equations are

$$\begin{aligned} \frac{-1}{x^2} &= 2\lambda x \\ \frac{-1}{y^2} &= 2\lambda y \\ \frac{-8}{z^2} &= 2\lambda z \\ \frac{1}{x} + \frac{1}{y} + \frac{8}{z} &= 1 \end{aligned}$$

The first two equations show $x = y$, the first and third equations show $8/z^3 = 1/x^3$ or $z = 2x$. Plugging this into the last equation gives $2/x + 8/(2x) = 1$ or $x = 6, y = 6, z = 12$.

$$\boxed{(x, y, z) = (6, 6, 12)}.$$

There is an interesting twist to this problem (as noted by one of the students Jacob Aptekar): consider the points $(x, y, z) = (1, -1/n, 8/n)$, where n is a large integer, One can check that these points lie on the surface $g(x, y, z) = 1$. Their distance to the origin however decreases to 1 if n goes to infinity. So the point $(6, 6, 12)$, while a local minimum is not a global minimum.

Problem 5) (10 points)

Find all extrema of the function $f(x, y) = x^3 + y^3 - 3x - 12y + 20$ on the plane and

characterize them. Do you find a absolute maximum or absolute minimum among them?

Solution:

The critical points satisfy $\nabla f(x, y) = (0, 0)$ or $(3x^2 - 3, 3y^2 - 12) = (0, 0)$. There are 4 critical points $(x, y) = (\pm 1, \pm 2)$. The discriminant is $D = f_{xx}f_{yy} - f_{xy}^2 = 36xy$ and $f_{xx} = 6x$.

point	D	f_{xx}	classification	value
(-1,-2)	72	-6	maximum	38
(-1, 2)	-72	-6	saddle	6
(1, -2)	-72	6	saddle	34
(1, 2)	72	6	minimum	2

Note that there are no global (= absolute) maxima nor global minima because the function takes arbitrarily large and small values. For $y = 0$ the function is $g(x) = f(x, 0) = x^3 - 3x + 20$ which satisfies $\lim_{x \rightarrow \pm\infty} g(x) = \pm\infty$.

Problem 6) (10 points)

Find all the critical points of $f(x, y) = \frac{x^5}{5} - \frac{x^2}{2} + \frac{y^3}{3} - y$ and indicate whether they are local maxima, local minima or saddle points.

Solution:

$\nabla f(x, y) = (x^4 - x, (y^2 - 1)) = (0, 0)$ so that the critical points are $(0, 1), (0, -1), (1, 1), (1, -1)$. We have $D = (4x^3 - 1)2y$ and $f_{xx} = 4x^3 - 1$.

Point	D	f_{xx}	type
(0, 1)	$D = -2$	-	saddle
(0, -1)	$D = 2$	-1	local max
(1, 1)	$D = 6$	3	local min
(1, -1)	$D = -6$	-	saddle

Problem 7) (10 points)

Use the technique of linear approximation to estimate $f(0.003, -0.0001, \pi/2 + 0.01)$ for

$$f(x, y, z) = \cos(xy + z) + x + 2z .$$

Solution:

$$L(x, y) = f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0)$$

$$f(x_0, y_0, z_0) = \cos(\pi/2) + \pi = \pi$$

$$a = f_x(x_0, y_0, z_0) = -0 \sin(\pi/2) + 1 = 1$$

$$b = f_y(x_0, y_0, z_0) = -0 \sin(\pi/2) = 0$$

$$c = f_z(x_0, y_0, z_0) = -\sin(\pi/2) + 2 = 1$$

$$L(x, y) = \pi + 0.003 \cdot 1 + -0.0001 \cdot 0 + 0.01 \cdot 1 = \pi + 0.013.$$

Problem 8) (10 points)

Find the equation $ax + by + cz = d$ for the tangent plane to the level surface of

$$f(x, y, z) = \cos(xy + z) + x + 2z$$

(same function as in last problem) which contains the point $(0, 0, \pi/2)$.

Solution:

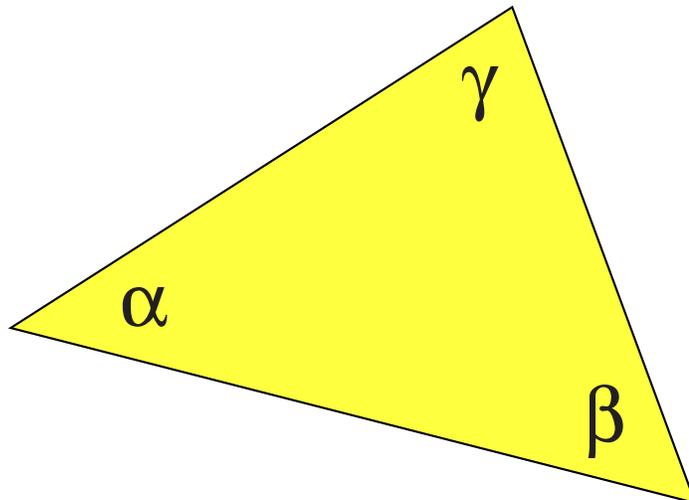
We have $\nabla f(0, 0, \pi/2) = (1, 0, 1)$ so that the plane is $x + z = \pi/2$

Problem 9) (10 points)

What is the shape of the triangle with angles α, β, γ for which

$$f(\alpha, \beta, \gamma) = \log(\sin(\alpha) \sin(\beta) \sin(\gamma))$$

is maximal?



Solution:

The Lagrange equations are $\cot(\alpha) = \lambda$, $\cot(\beta) = \lambda$, $\cot(\gamma) = \lambda$. Because α, β, γ are all in $[0, \pi]$, we conclude that all are the same. From the last equation follows $\alpha = \beta = \gamma = \pi/3$ and $\sin(\alpha) \sin(\beta) \sin(\gamma) = (\sqrt{3}/2)^3$.