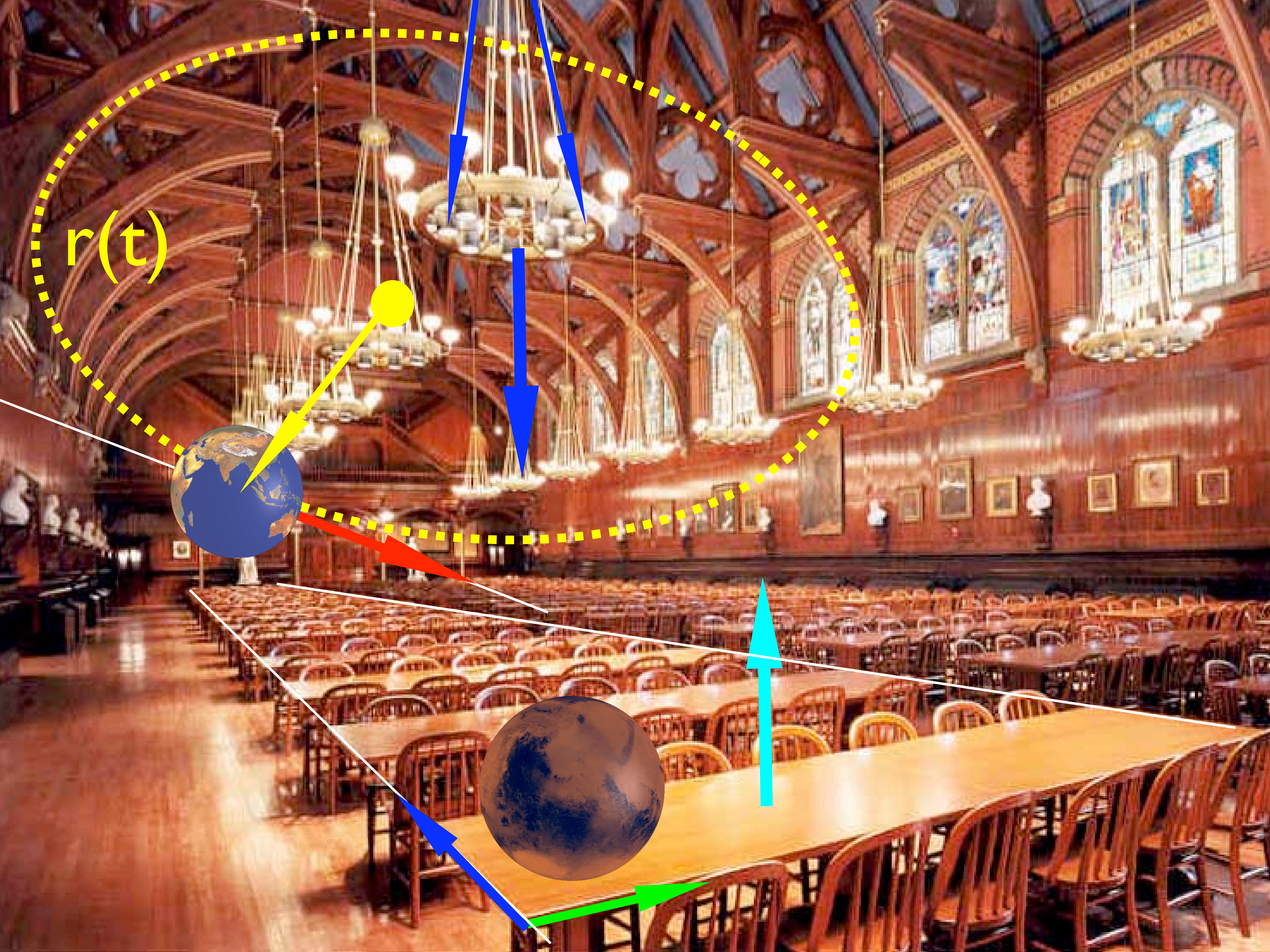
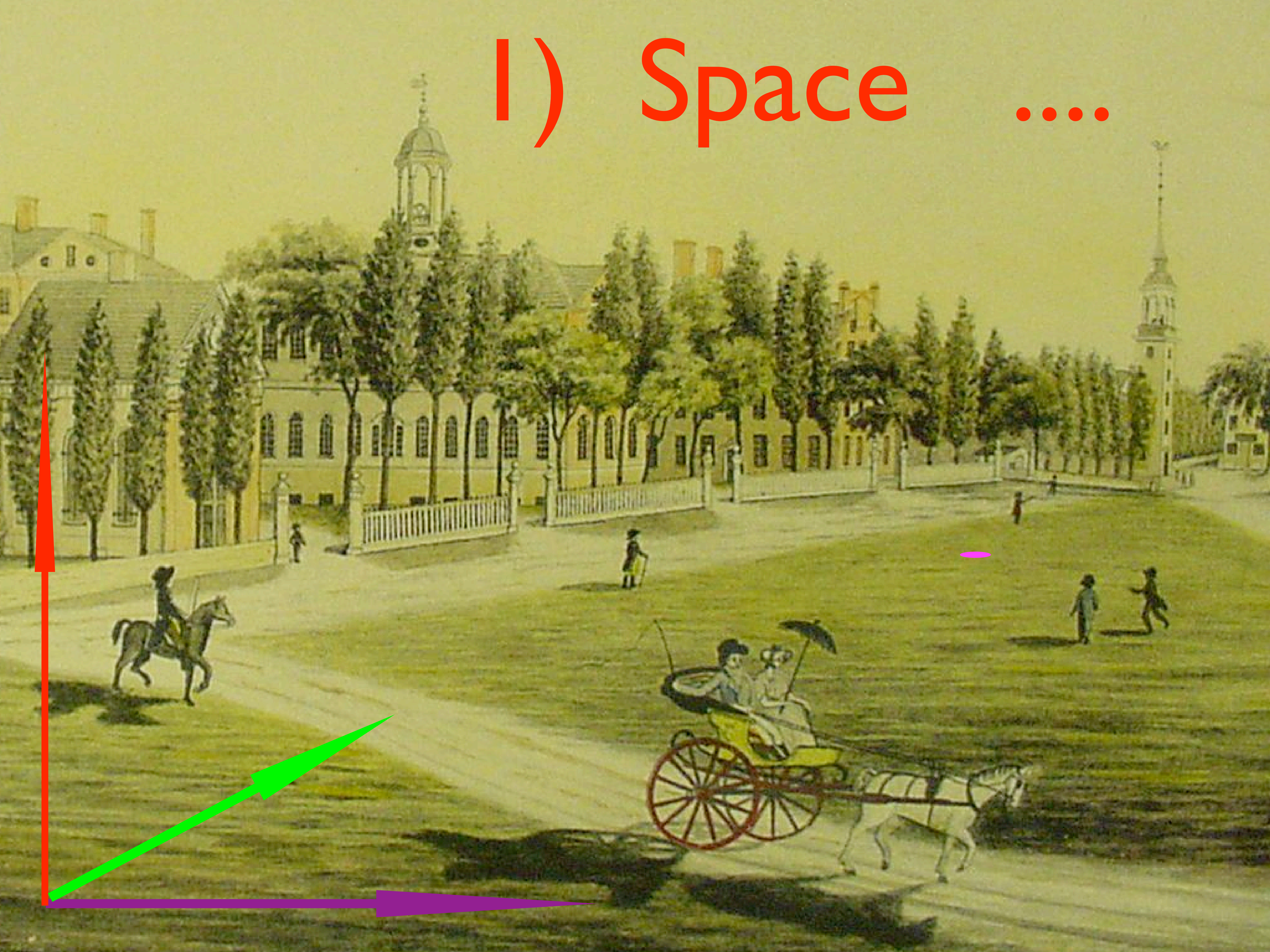
Math21a, Review Fall 2005, Part I

Oliver Knill, 1/10-11/2006

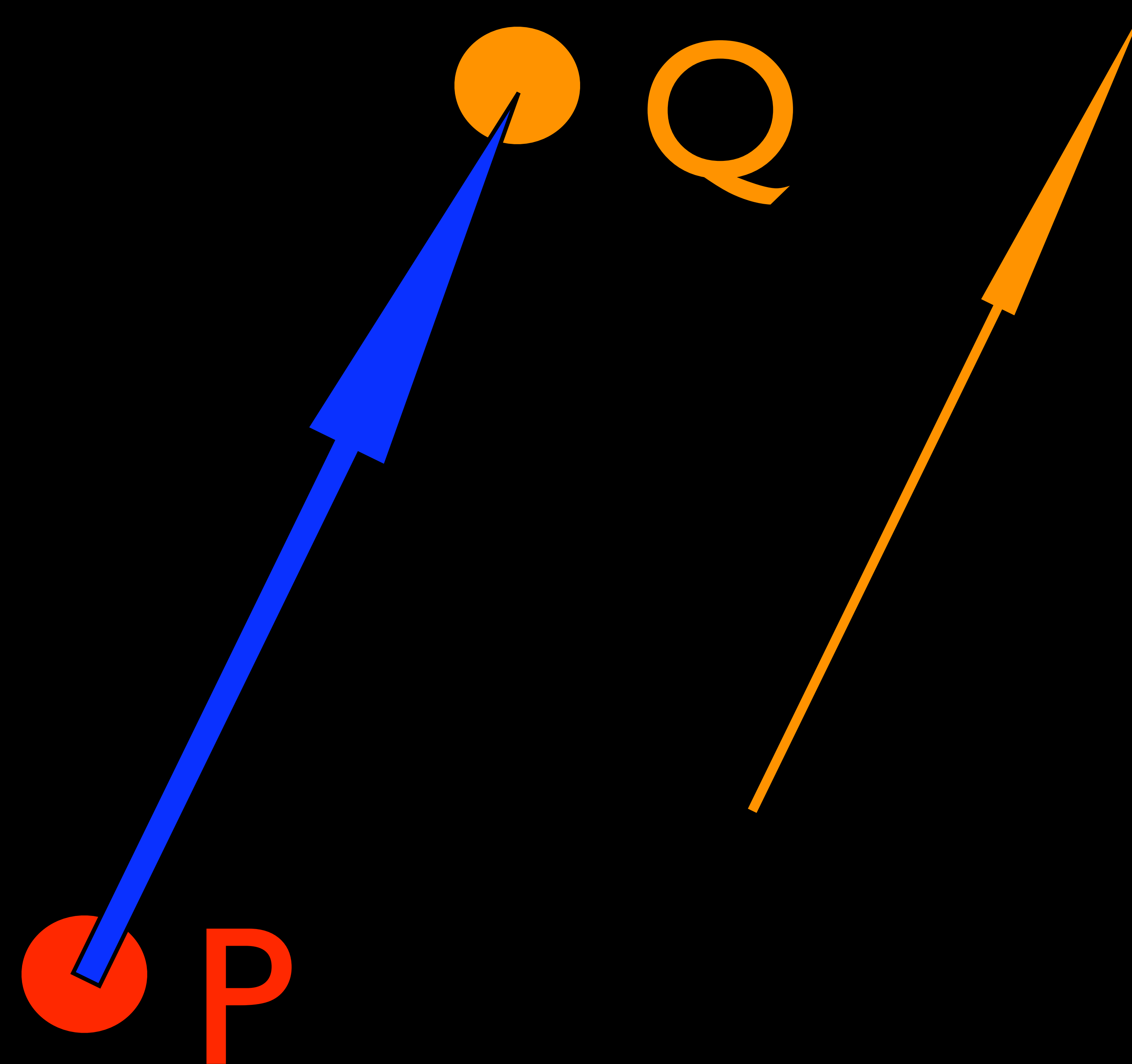

r(t)

1) Space ....

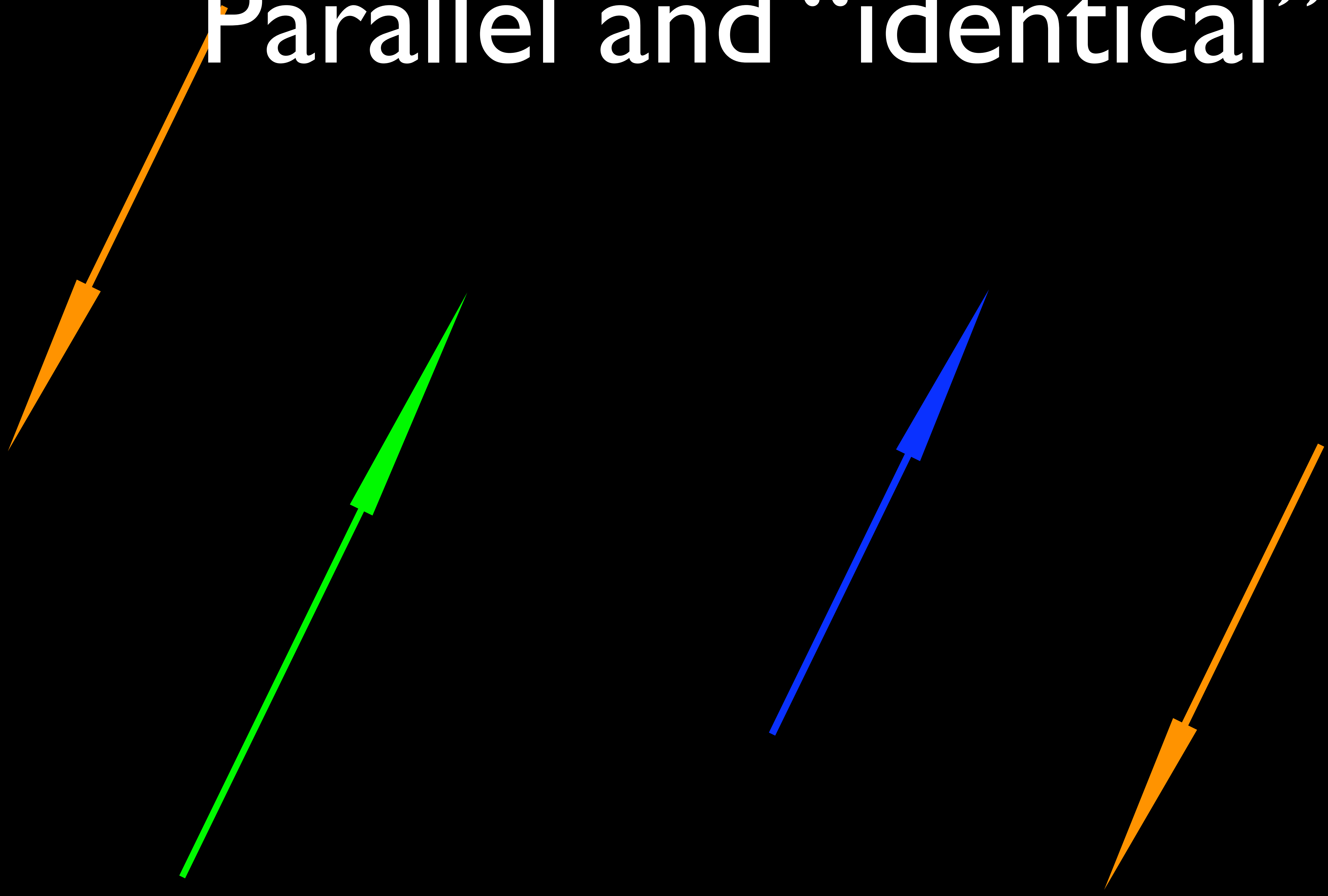


Vectors

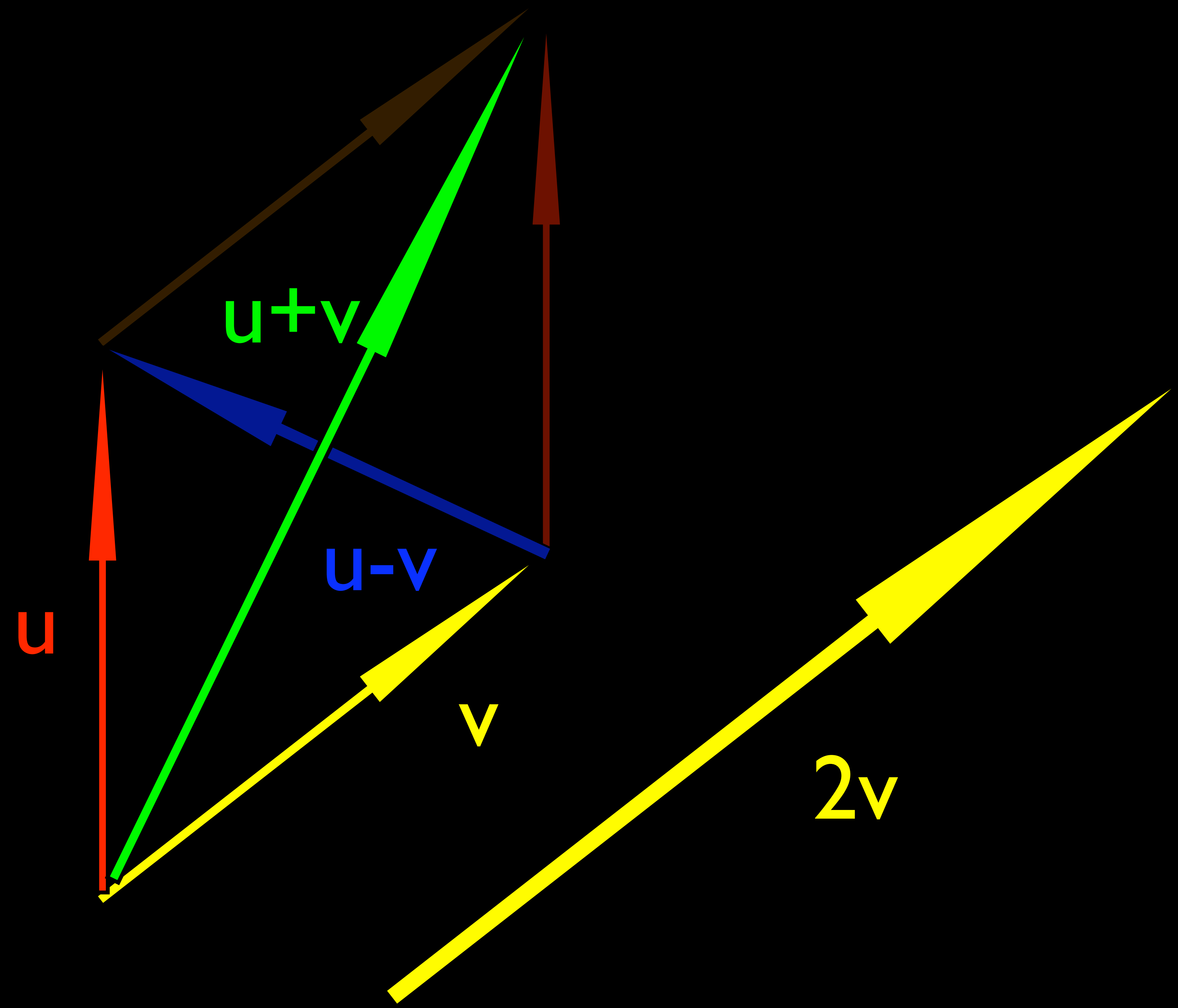
Determined by two points P,Q.
Can be placed anywhere in space but translated vectors are considered "equal"



Parallel and “identical”

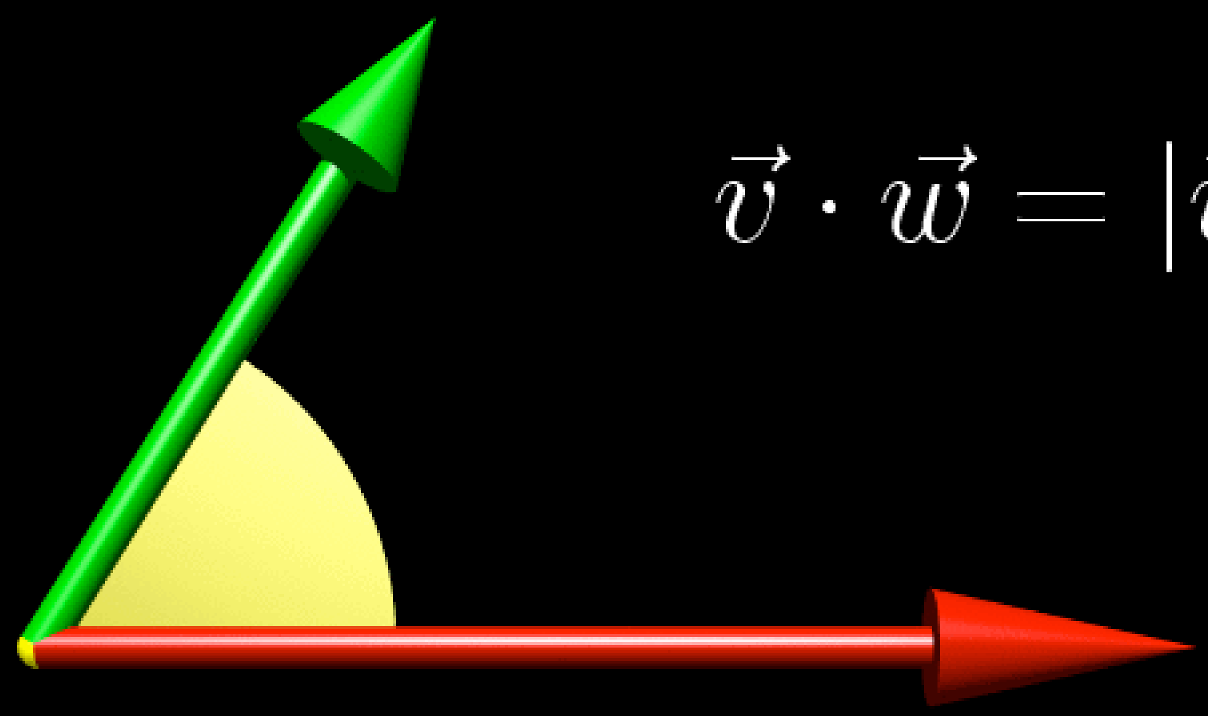


Addition, Scaling



Dot Product

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

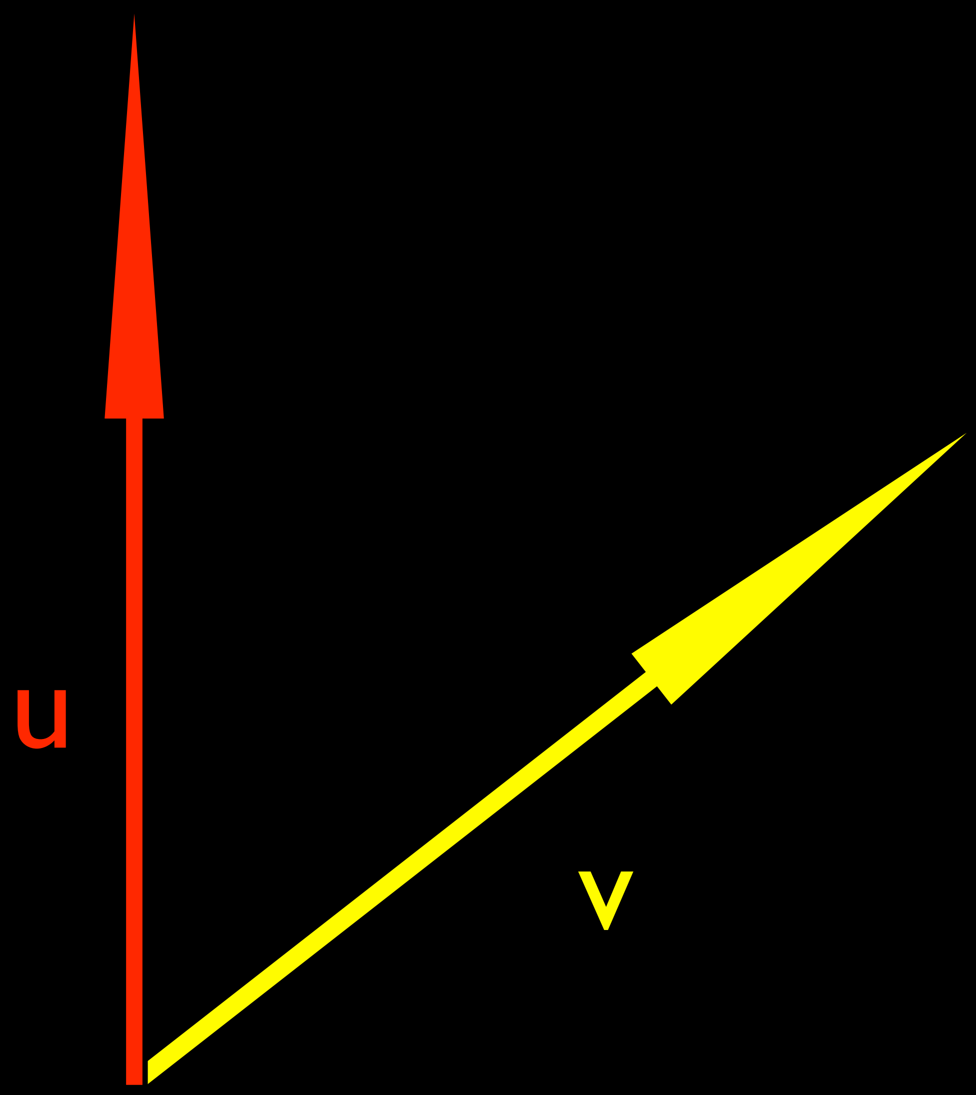


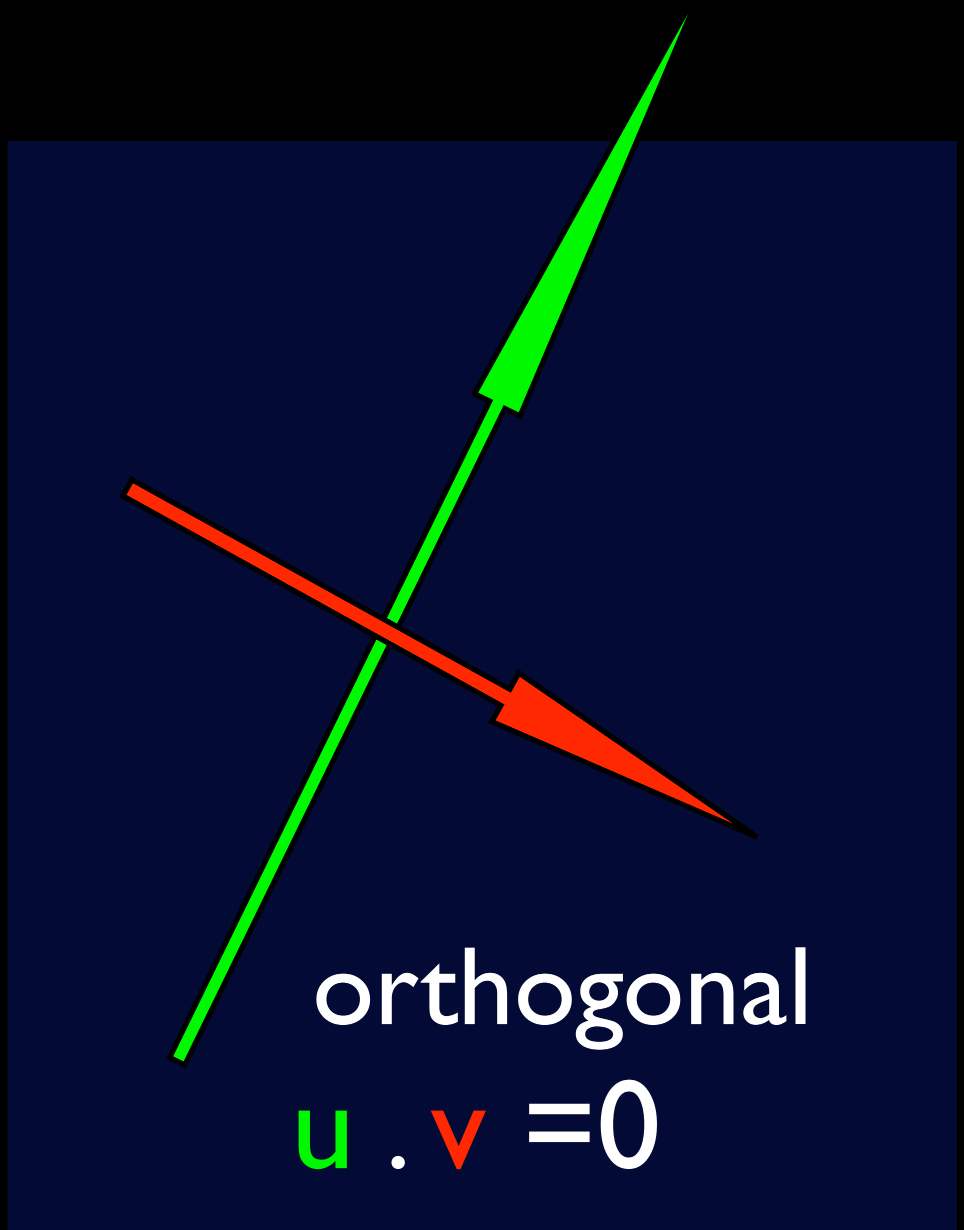
$$\vec{v} \cdot \vec{w} = |\vec{v}|\ |\vec{w}|\ \cos(\theta)$$

Angle, Length

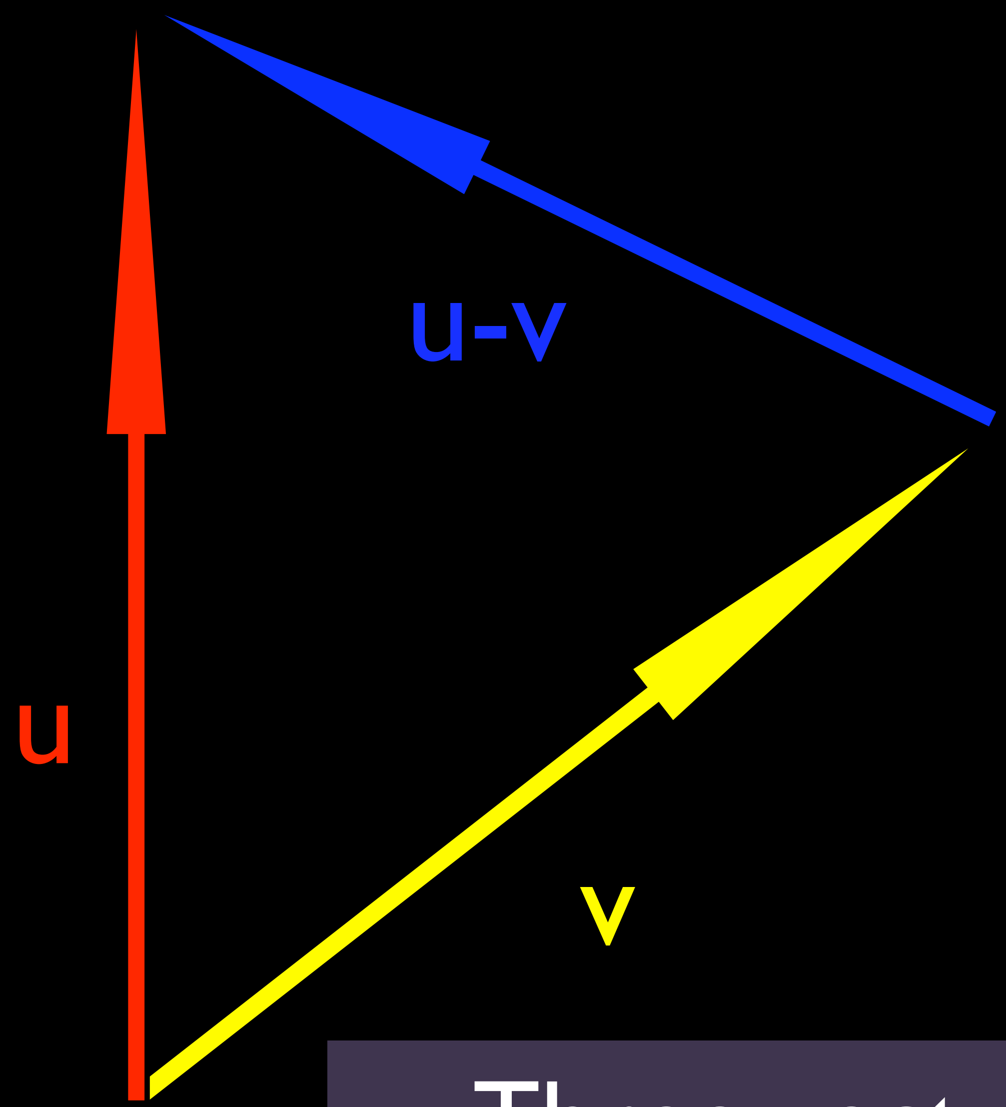
$\cos(t) = u \cdot v / (|u|\,|v|)$

$u.u = |u|^2$



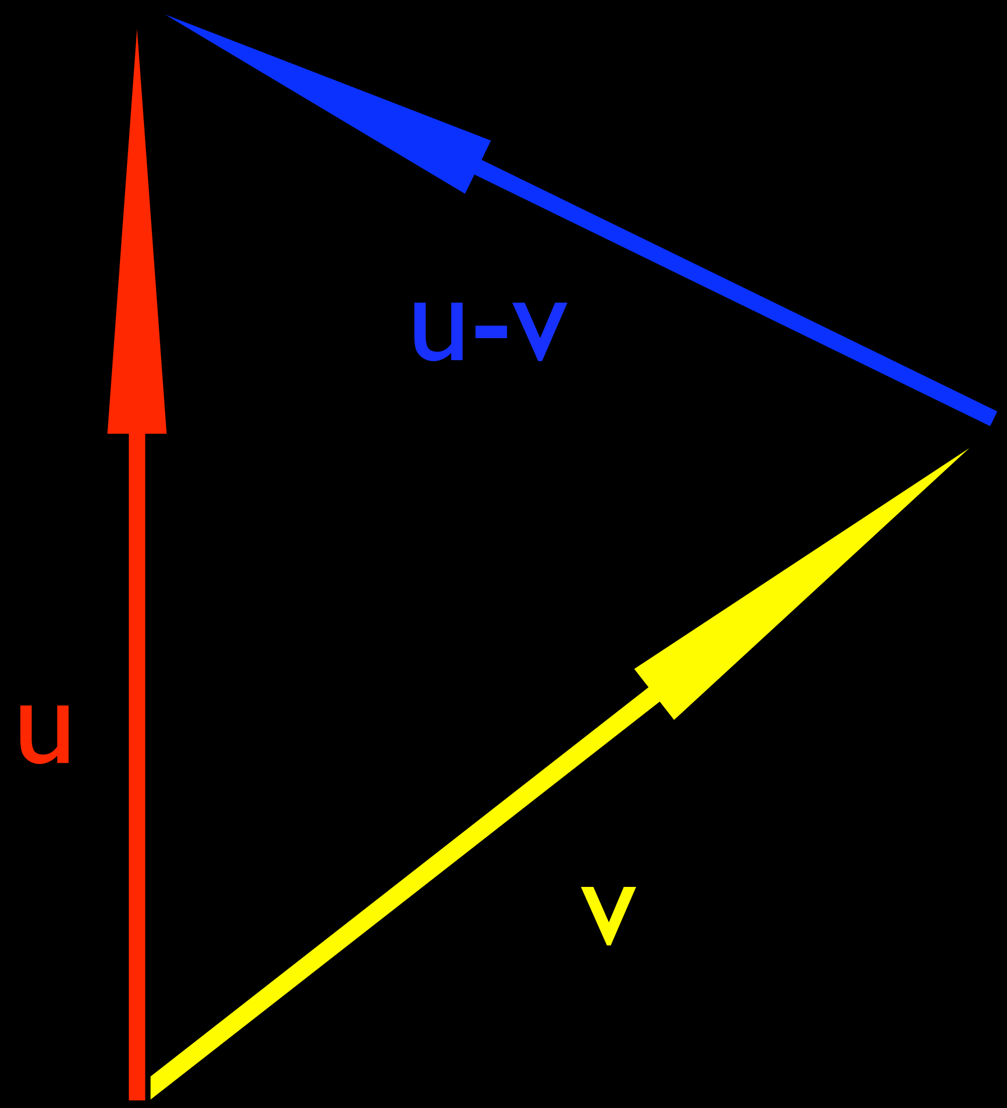


From vectors to angles



Three vectors u,v,u-v form a triangle.
We know |u|,|v|,|u-v|.
This determines the angles of the triangle.

From



$$|(u-v)|^2 = (u-v).(u-v)$$

$$=|u|^2+|v|^2 - 2(u.v)$$

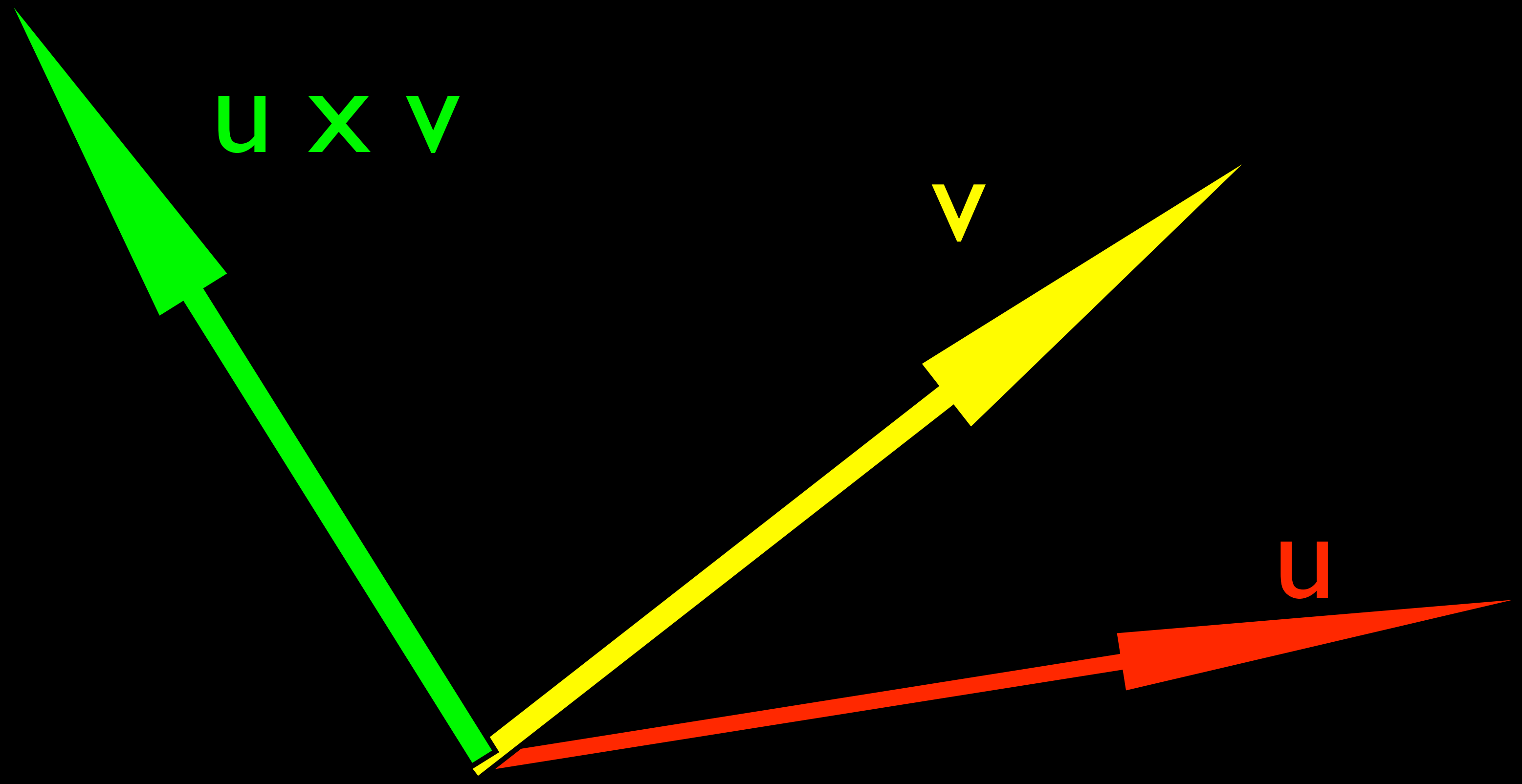
we know the dot product.

and so the angle.

$$\cos(t)=\frac{|(u-v)|^2 - |u|^2-|v|^2}{-2\,|u|\,|v|}$$

Cross Product

$$\vec{v} \cdot \vec{w} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$



Cross Product

$$\vec{v} \cdot \vec{w} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} =$$

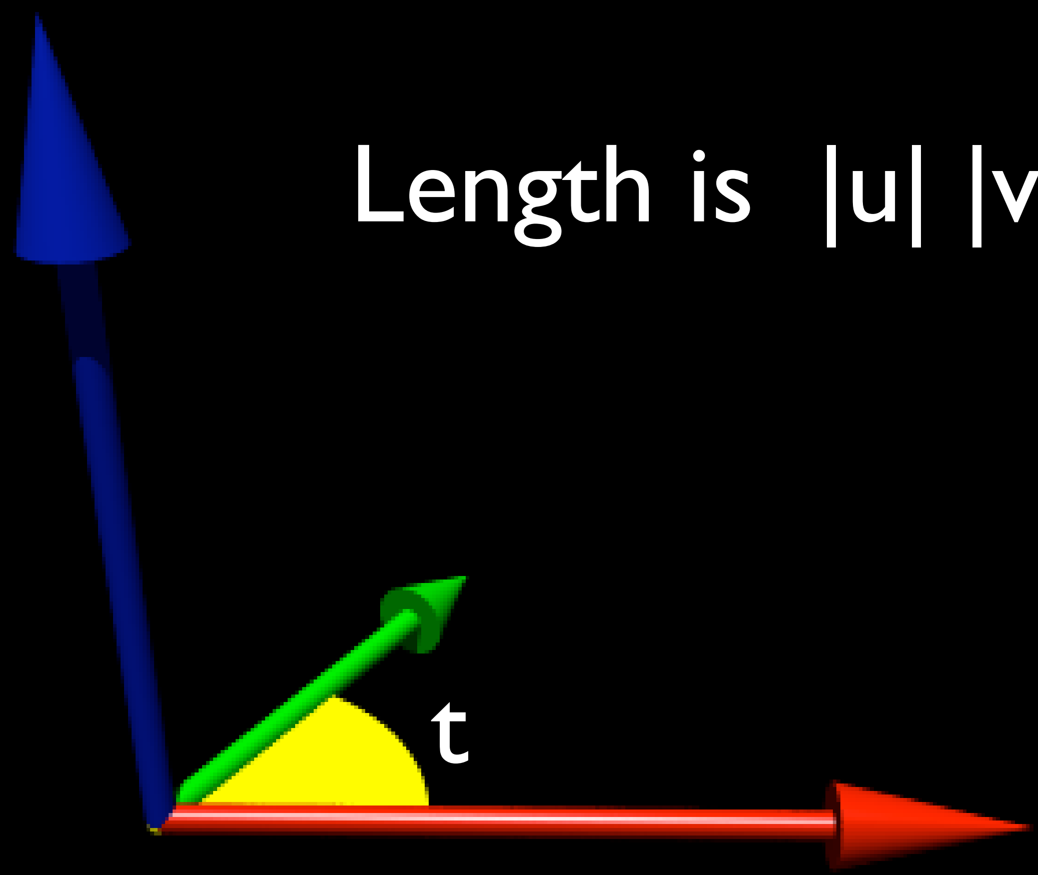
$$+ \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

$$- \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

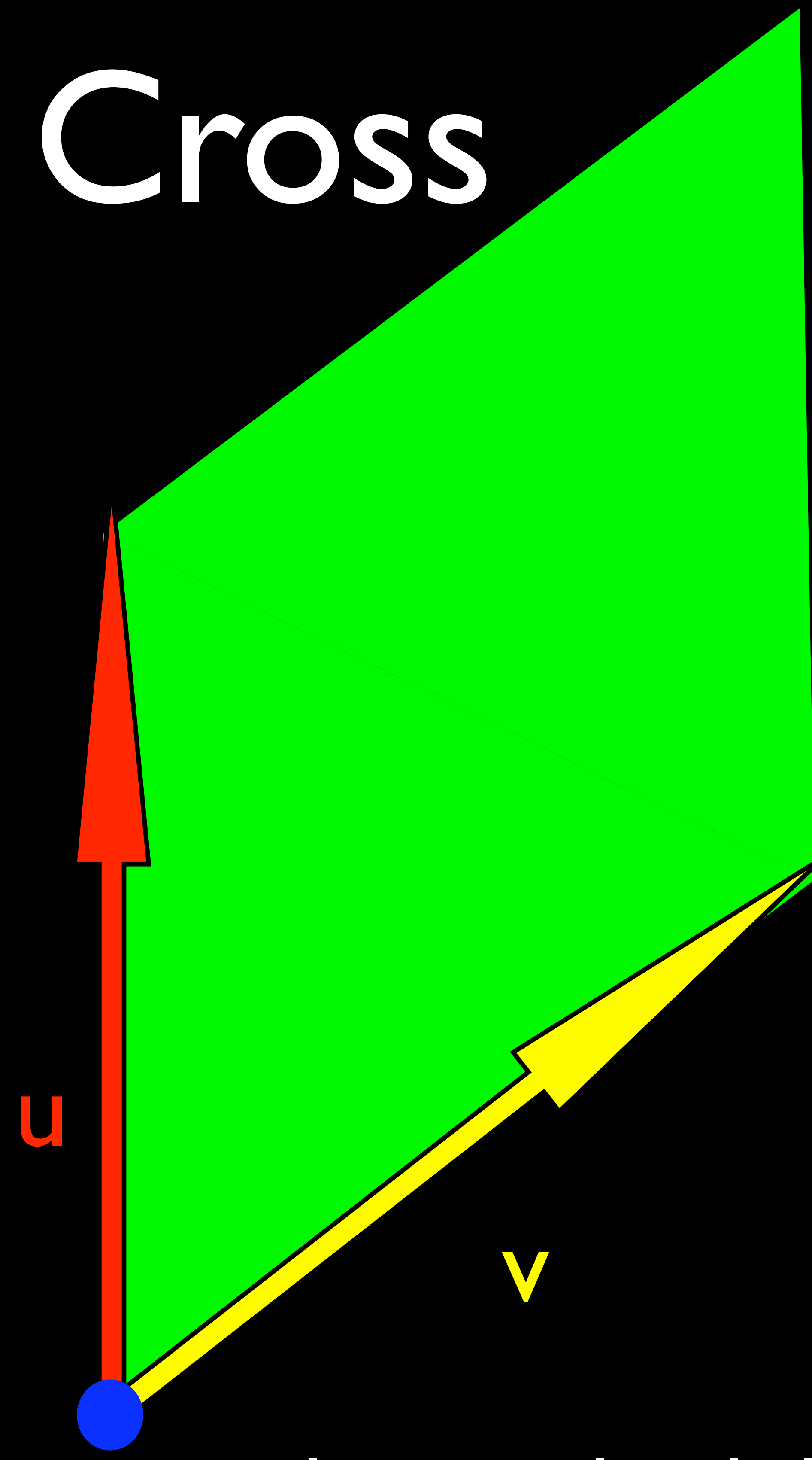
$$+ \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

Length of Cross Product

Length is |u| |v| sin(t)



Cross Product and area.

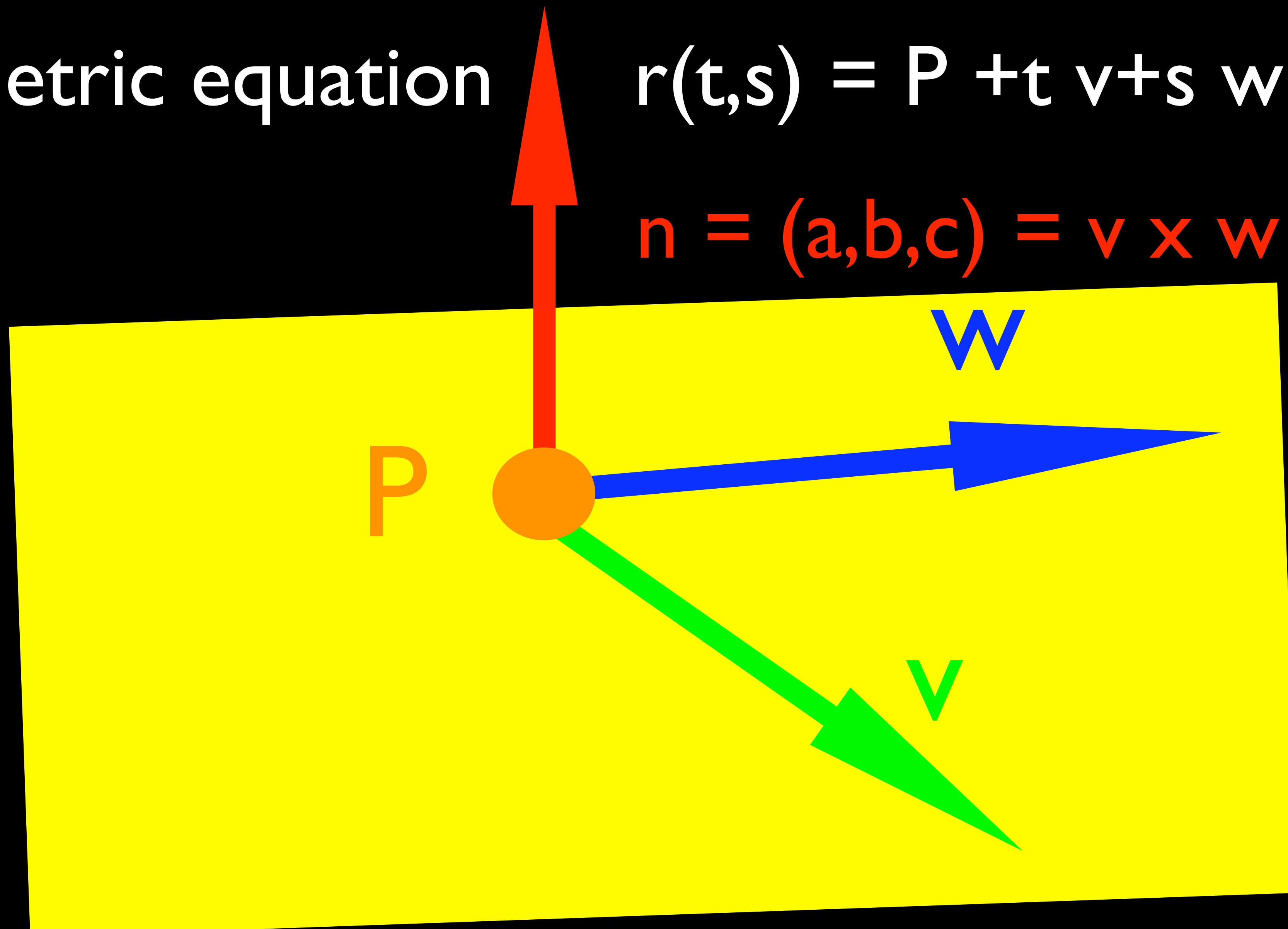


$$| u \times v | = |u| \, |v| \sin(t)$$

Planes

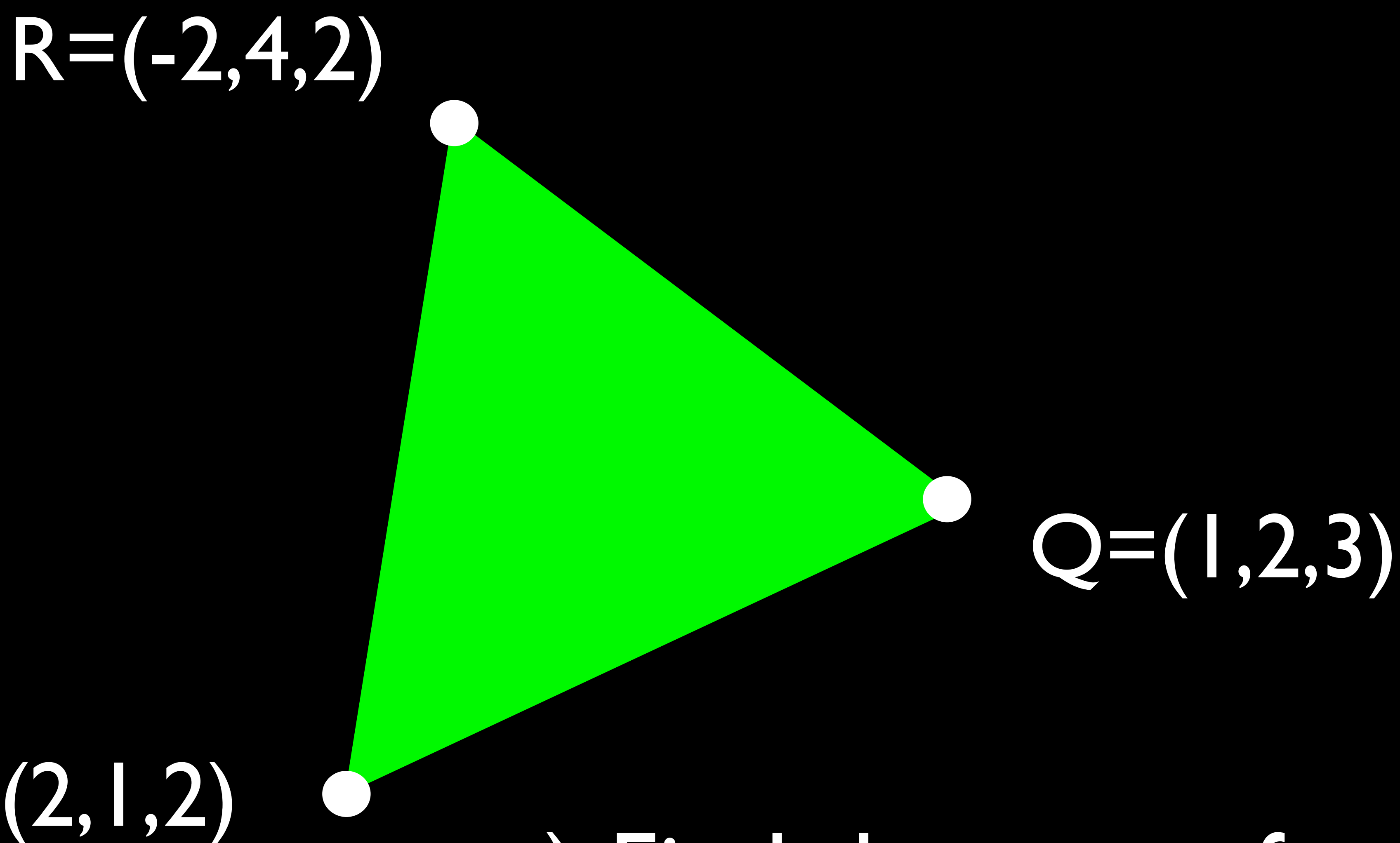
Parametric equation $r(t,s) = P + t\,v + s\,w$

$n = (a,b,c) = v \times w$



$$a x + b y + c z = d$$

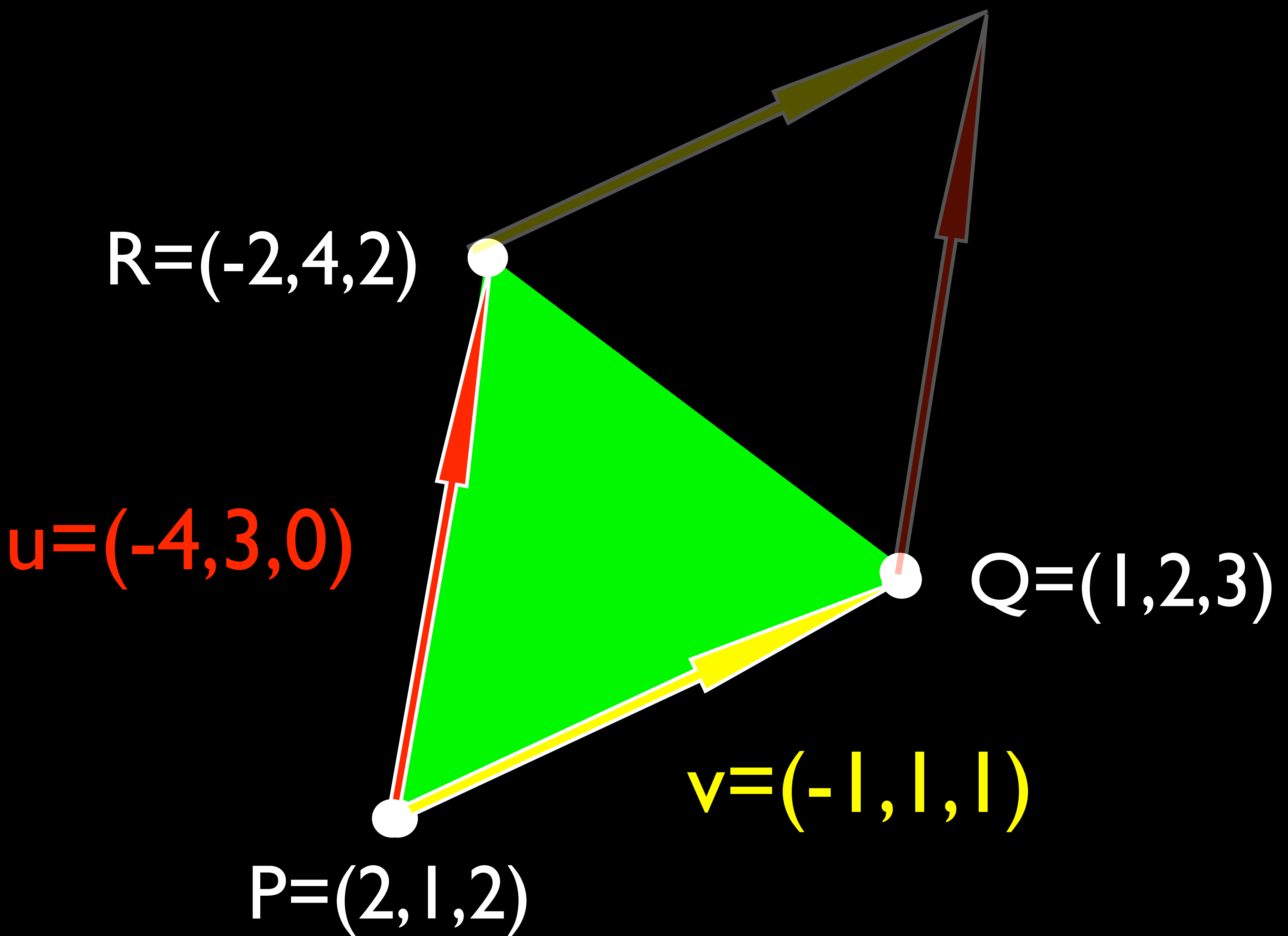
Problem 1



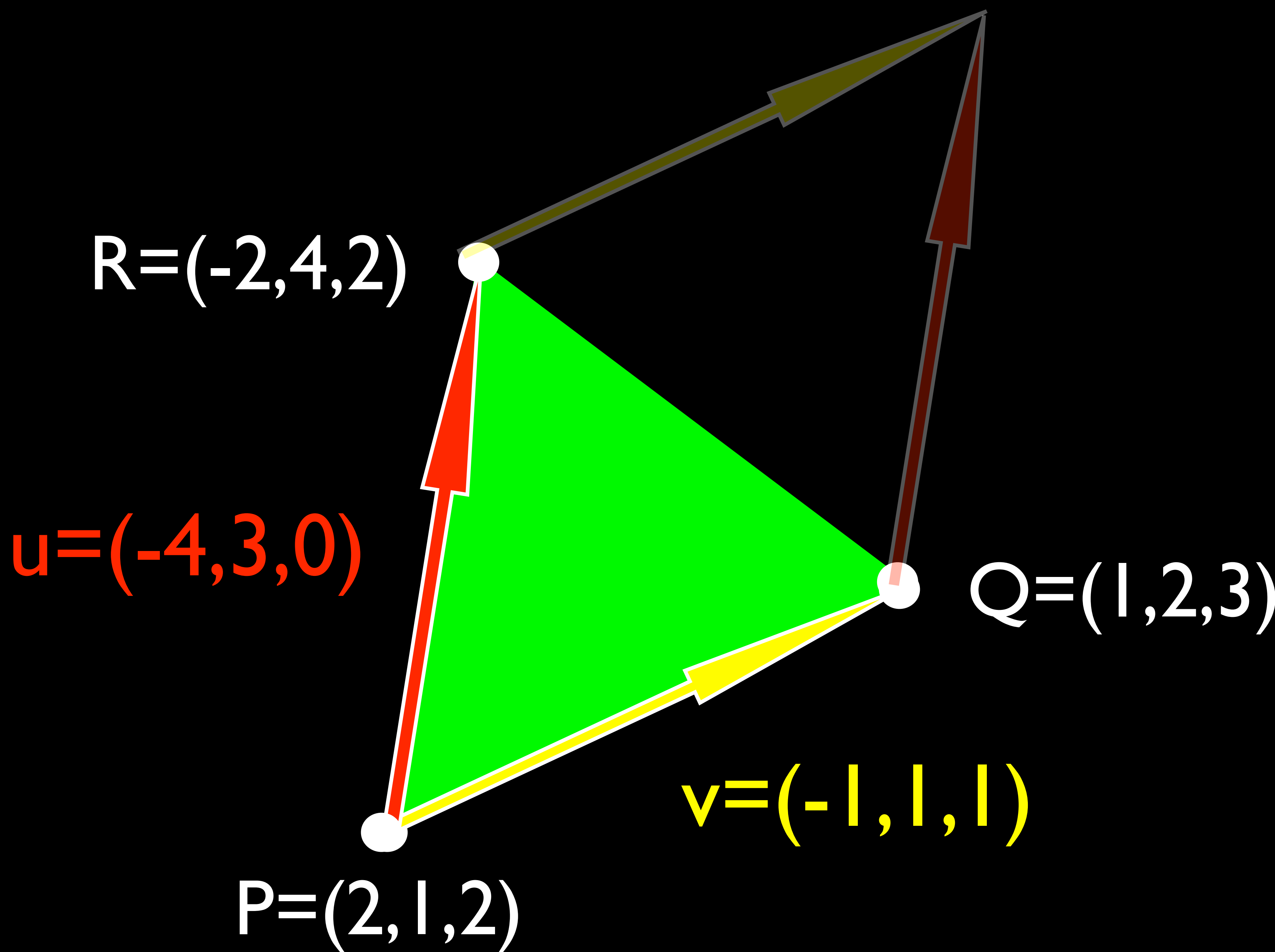
a) Find the area of the triangle PQR

b) Find the equation of the plane through PQR

Problem 1



Problem

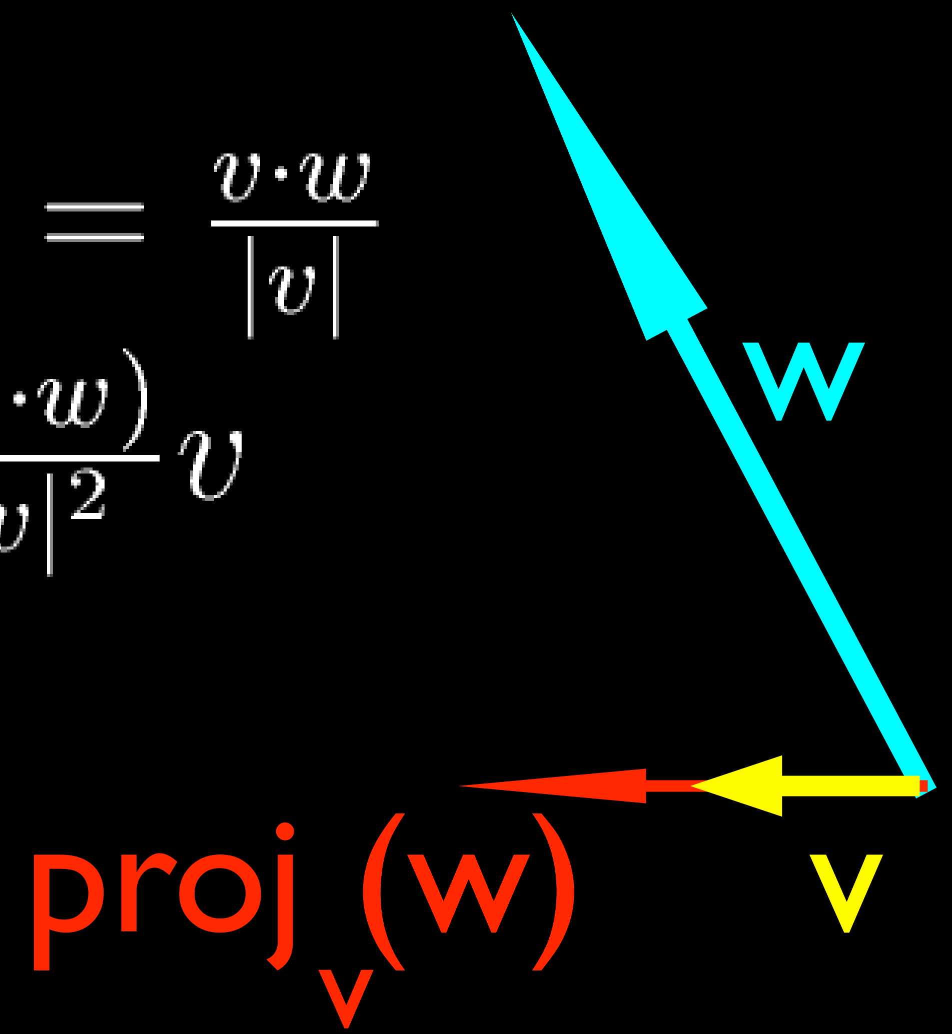


u x v=(3,4,-1)   Answer: $26^{1/2}/2$

Vector Projection

$$\mathrm{comp}_v(w) = \frac{v \cdot w}{|v|}$$

$$\mathrm{proj}_v(w) = \frac{(v \cdot w)}{|v|^2} v$$



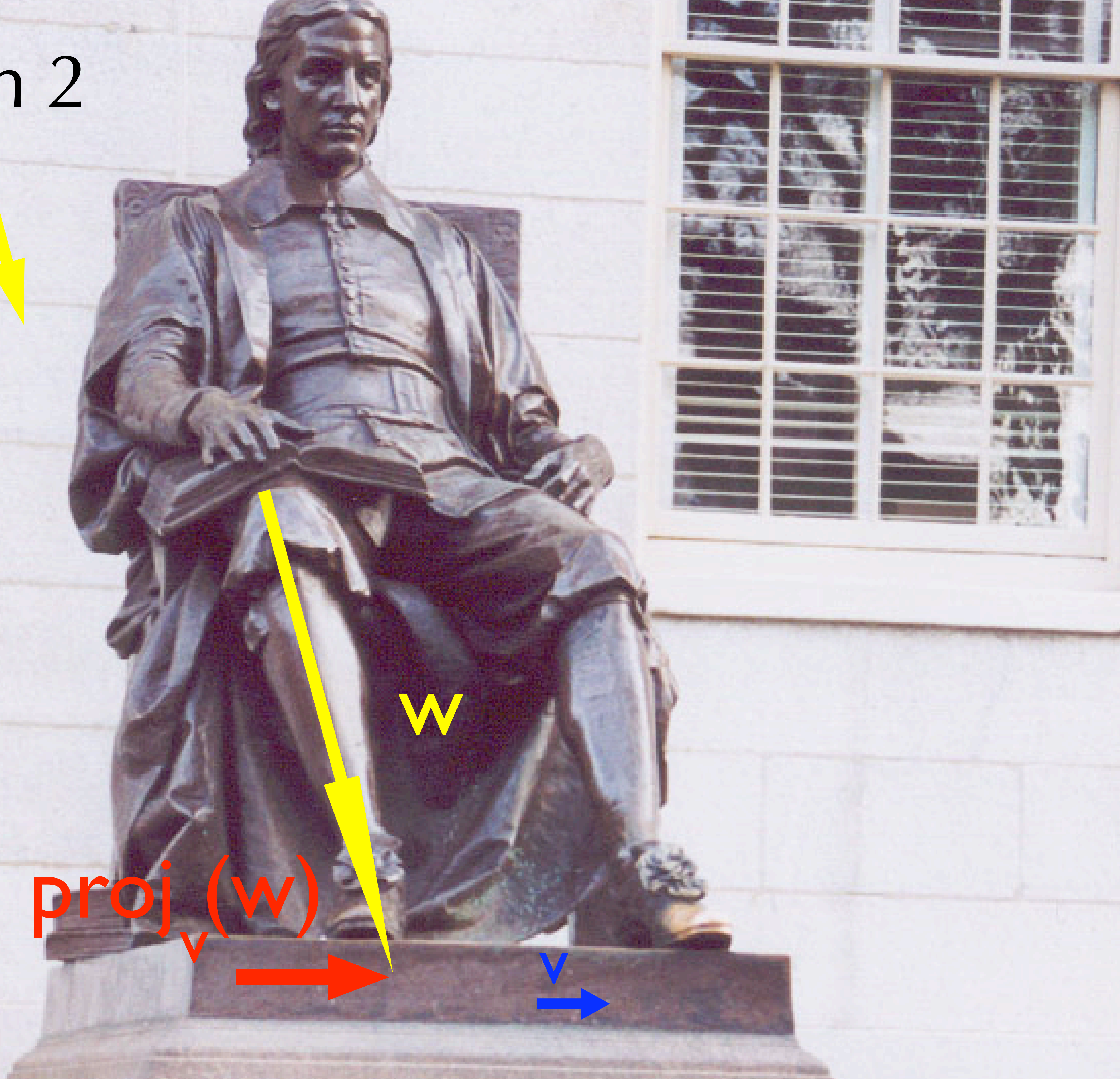
Problem 2

w=(3,1,-5)

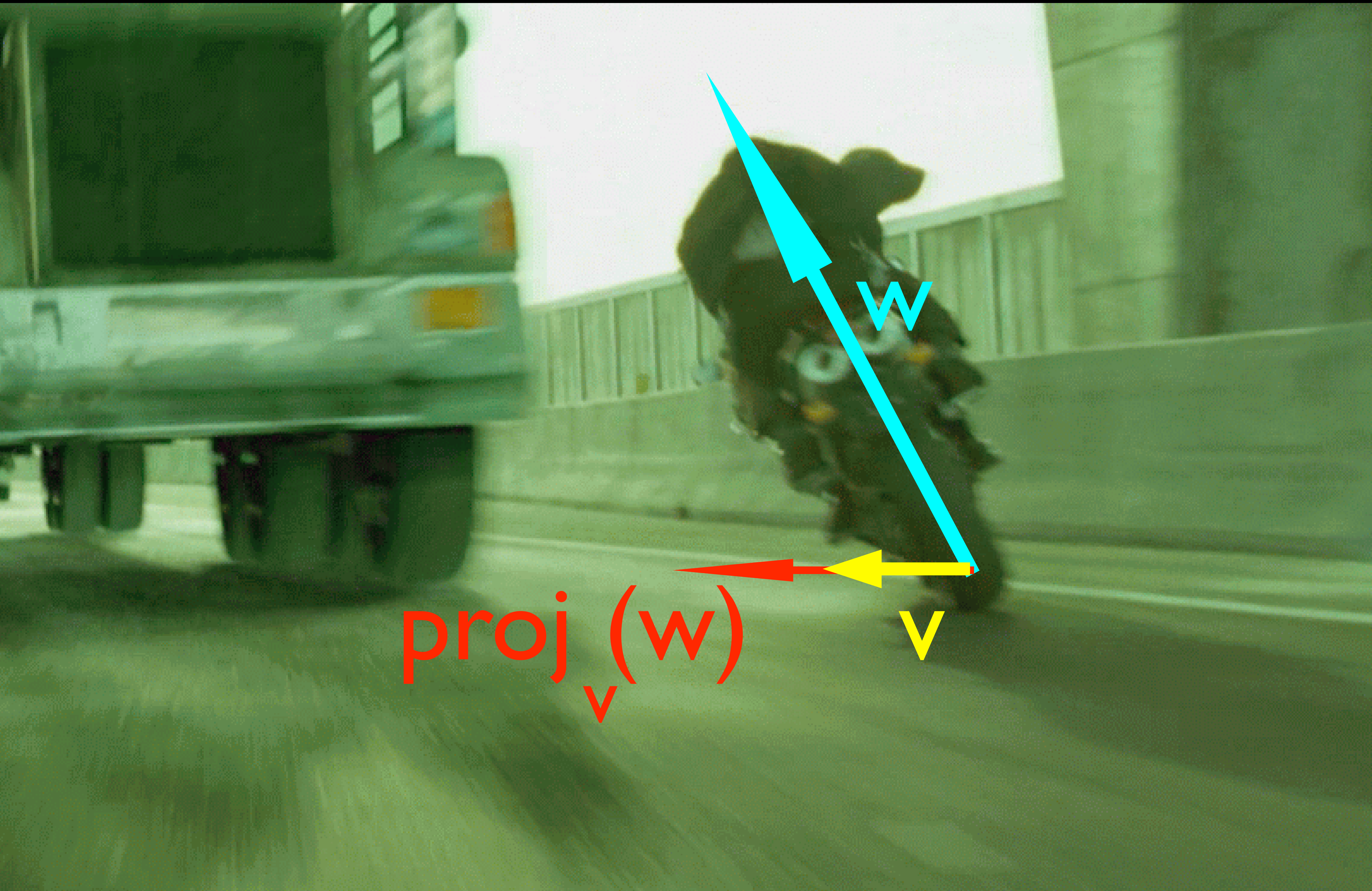
v=(1,1,0)

The vector projection of w onto v is

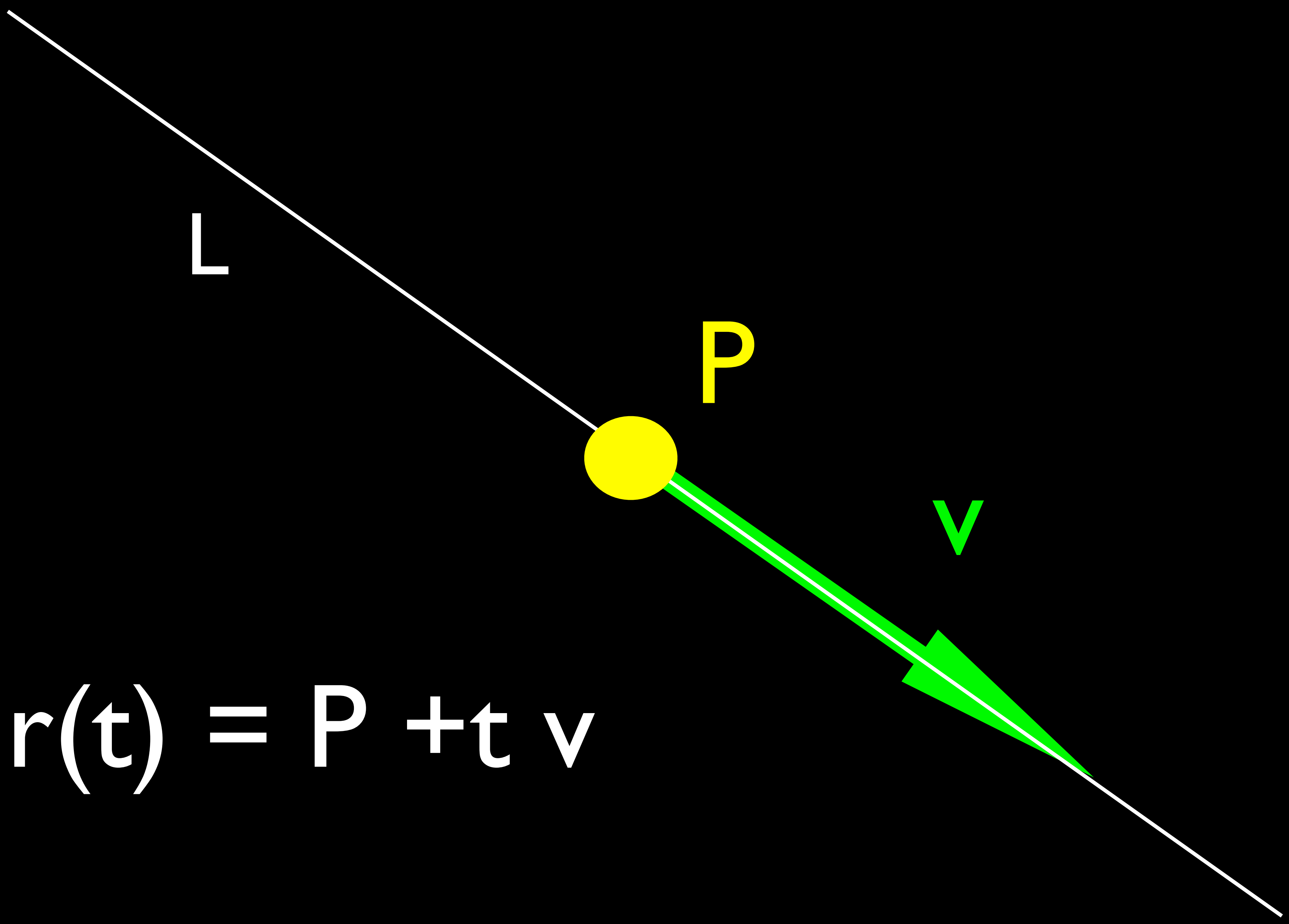
4v/2 = 2v



Projections



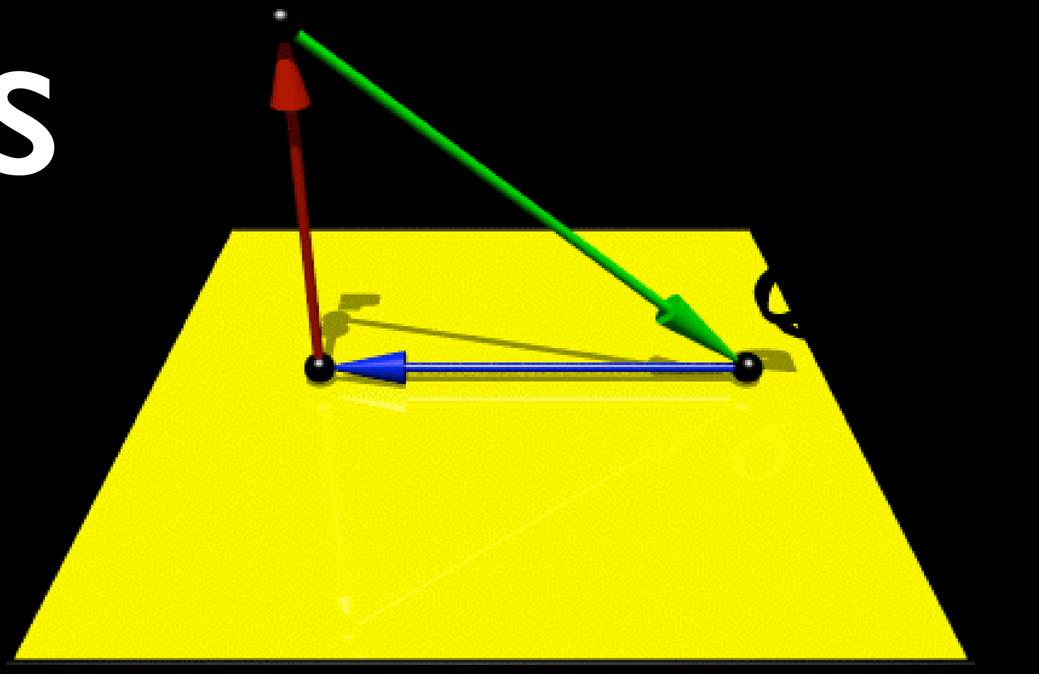
Parametrized Lines



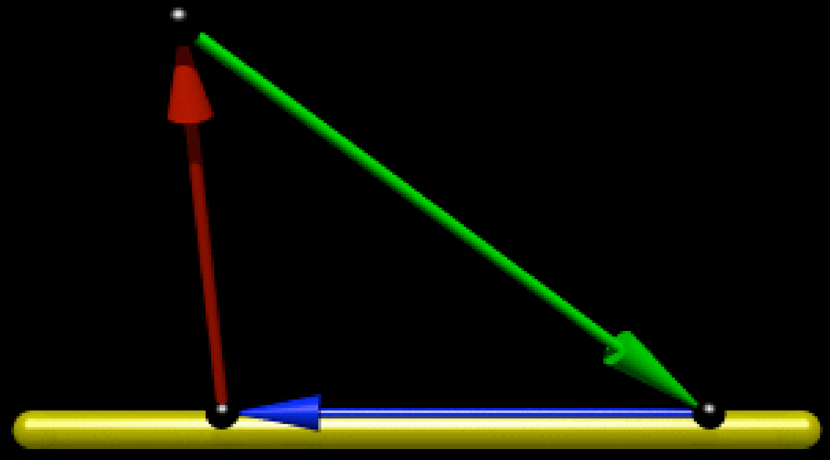
$r(t) = P + t\,v$

Distance Formulas

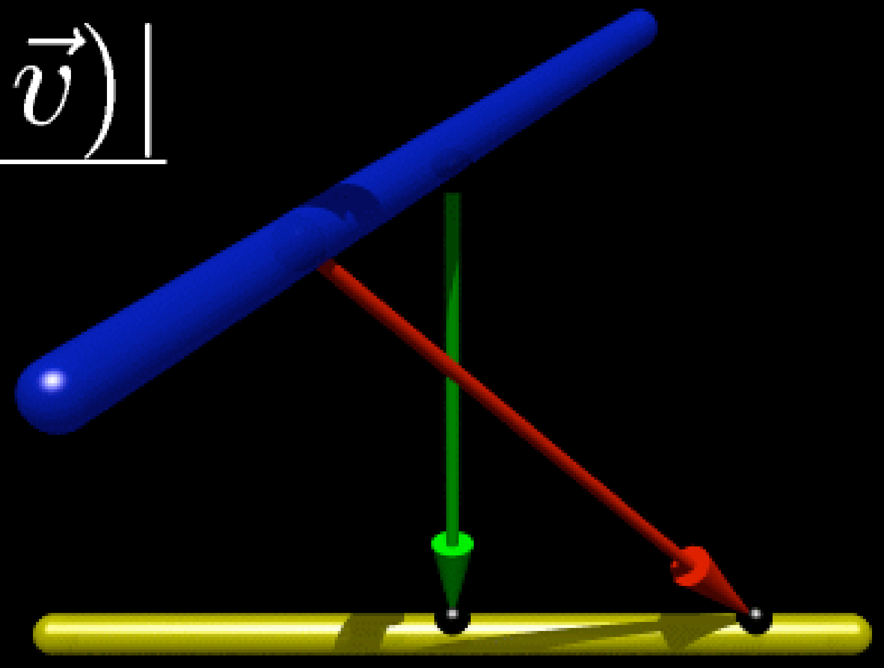
$$d(P,\Sigma) = \frac{|(\vec{PQ})\cdot\vec{n}|}{|\vec{n}|}$$



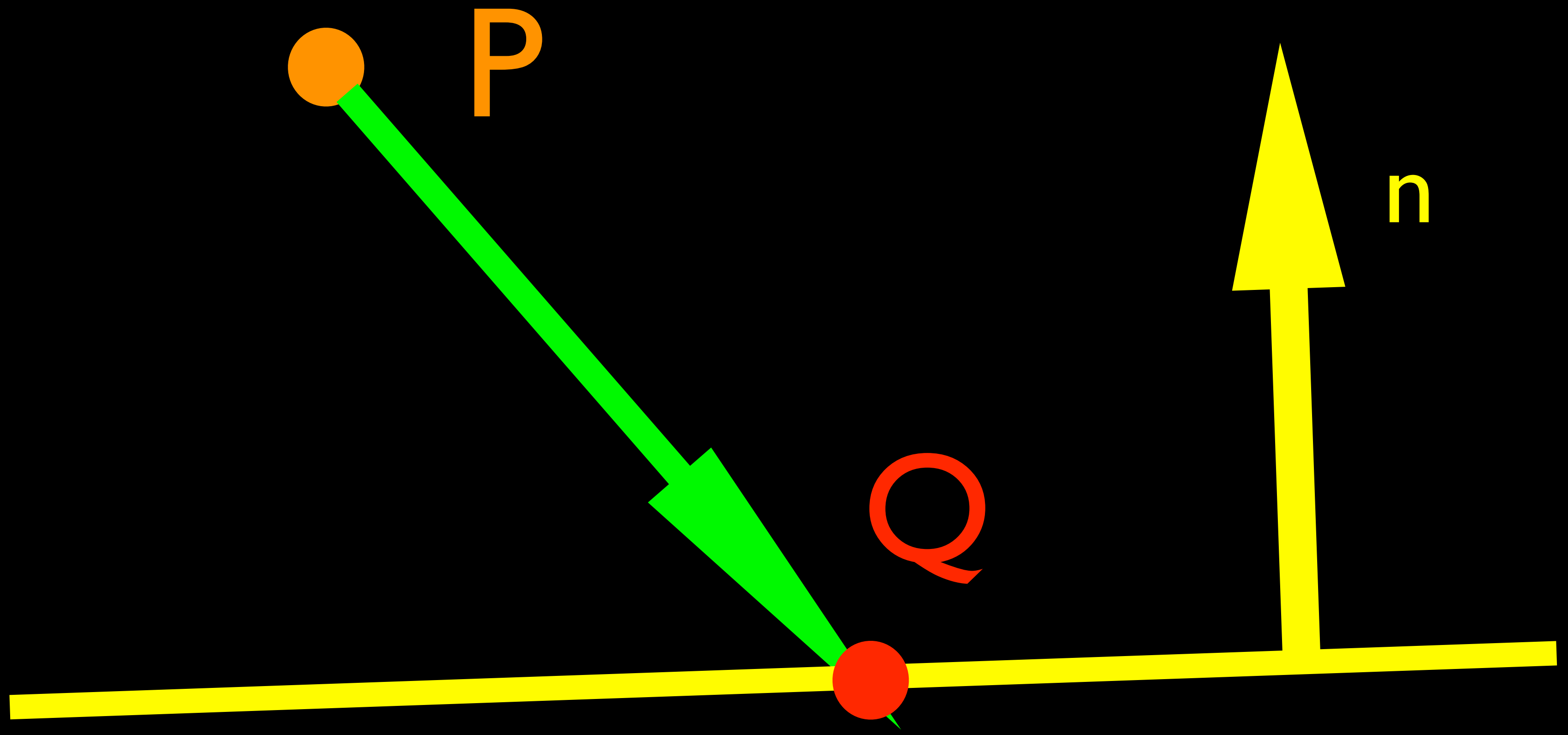
$$d(P,L) = \frac{|(\vec{PQ})\times\vec{u}|}{|\vec{u}|}$$



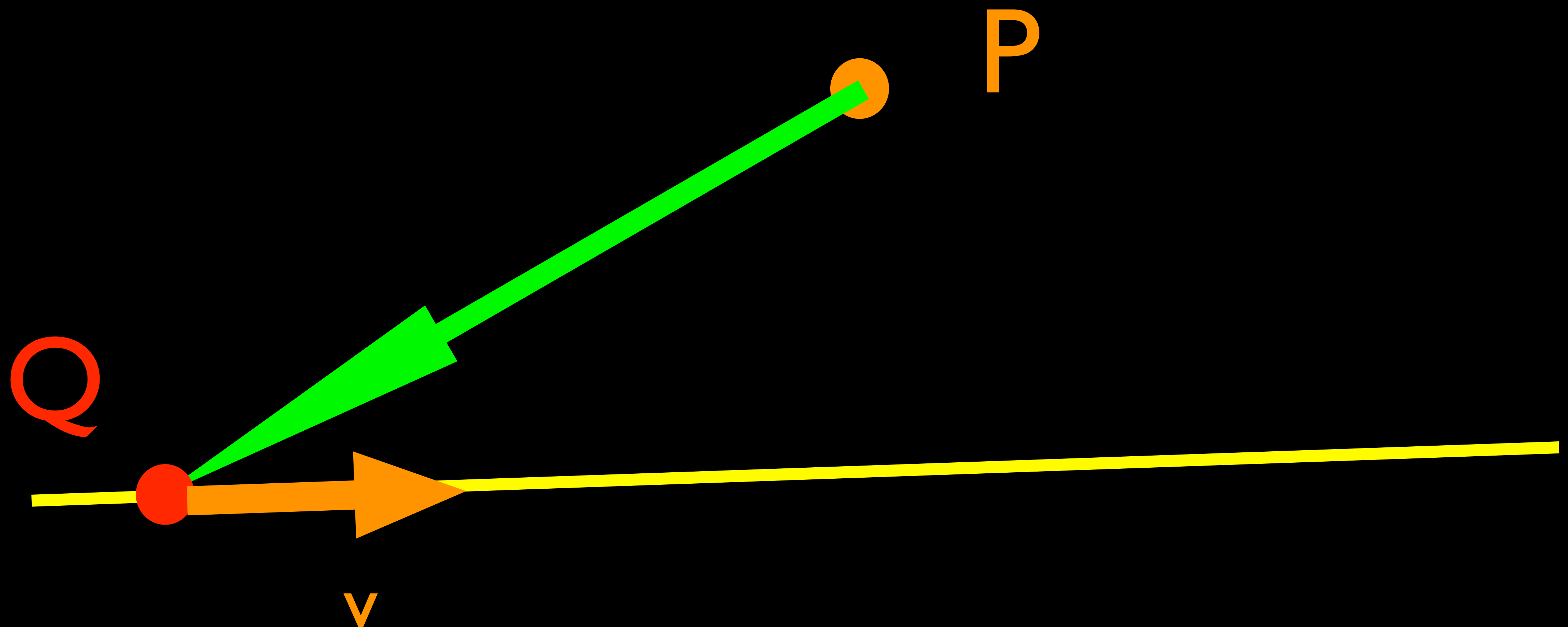
$$d(L,M) = \frac{|(\vec{PQ})\cdot(\vec{u}\times\vec{v})|}{|\vec{u}\times\vec{v}|}$$



Distance Point-Plane

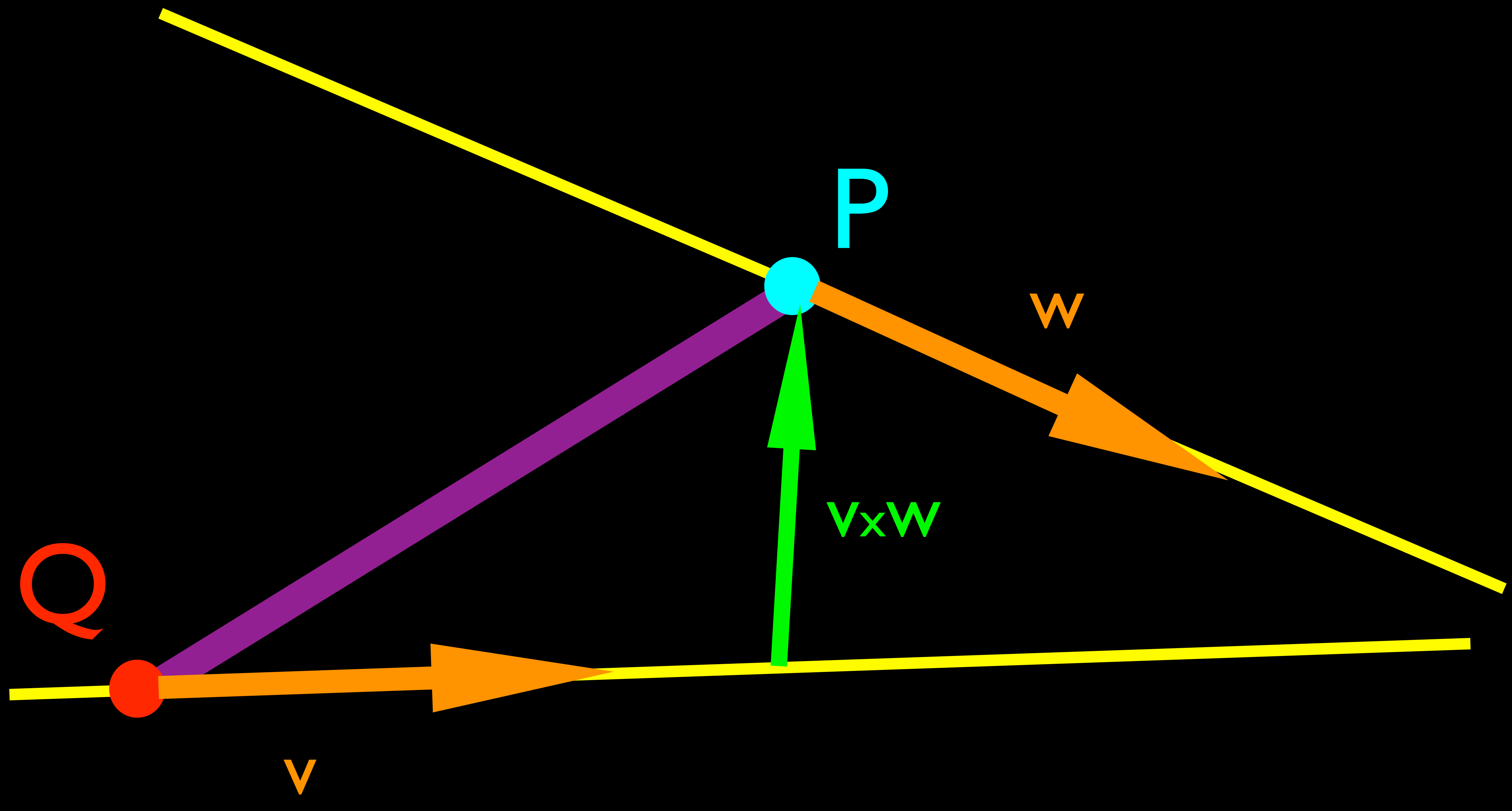


Distance Point-Line



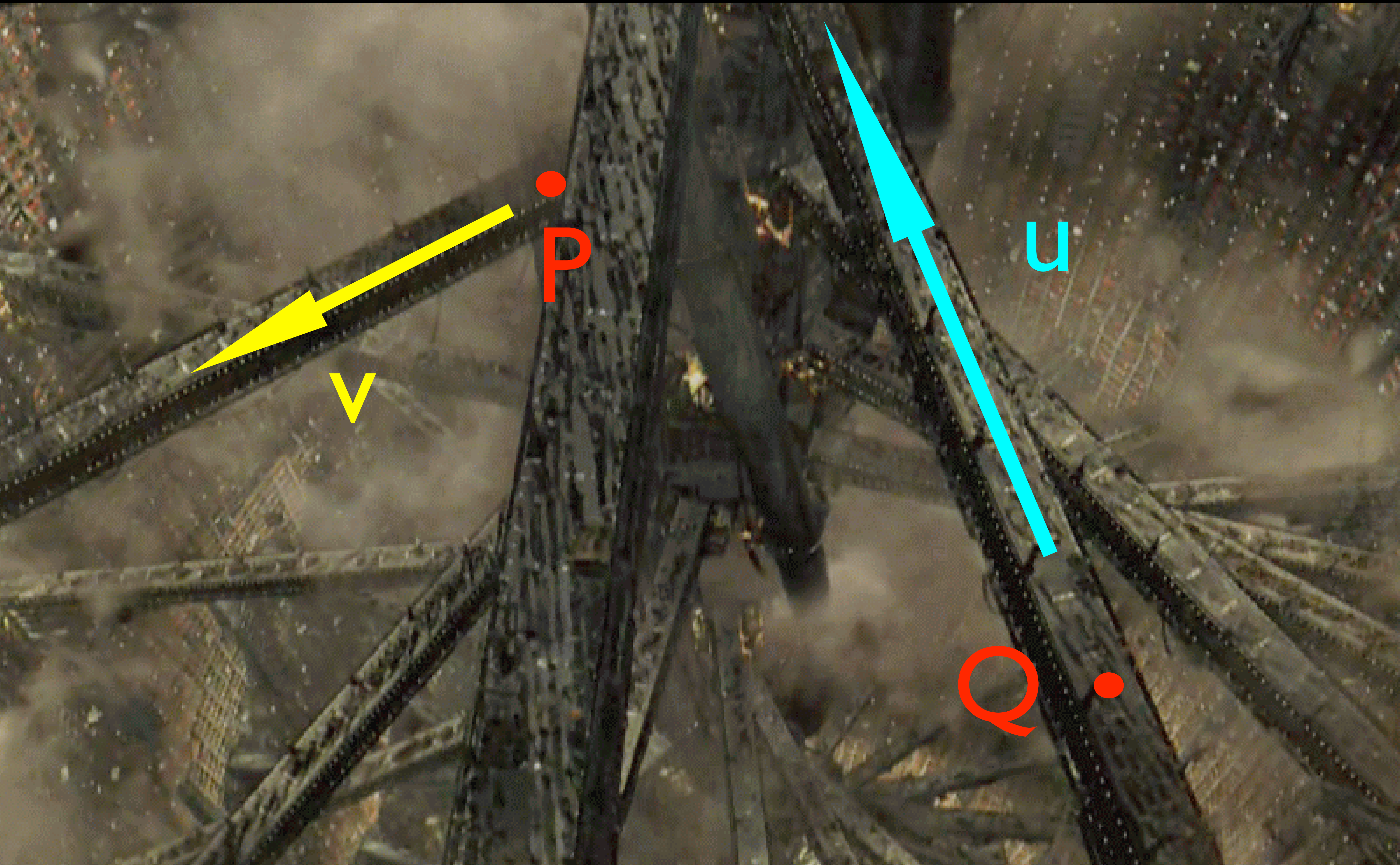
$$d = |\vec{PQ}|\sin(\theta) = \frac{|\vec{PQ}||\vec{v}|\sin(\theta)}{|\vec{v}|} = \frac{|\vec{PQ} \times \vec{v}|}{|\vec{v}|}$$

Distance Line-Line



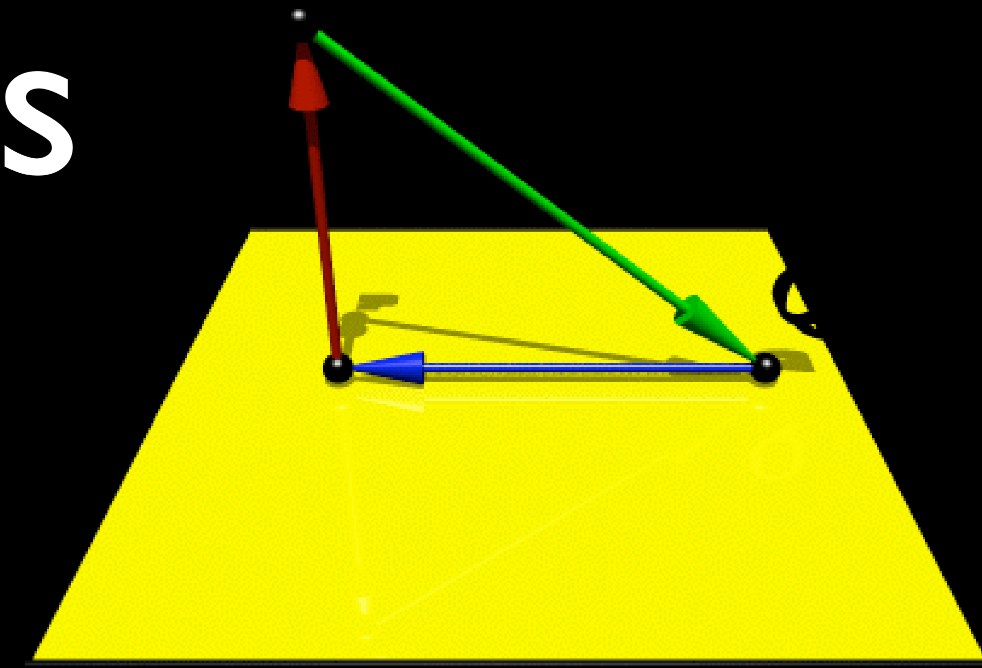
Project Q-P onto vxw

Distance Formulas

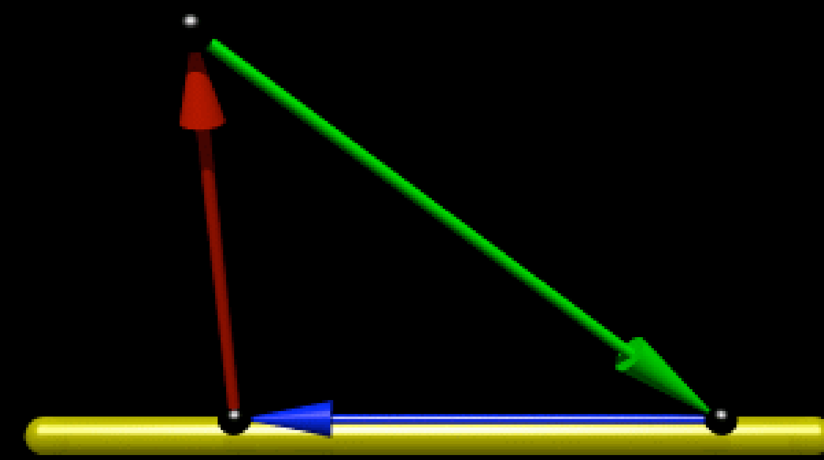


Distance Formulas

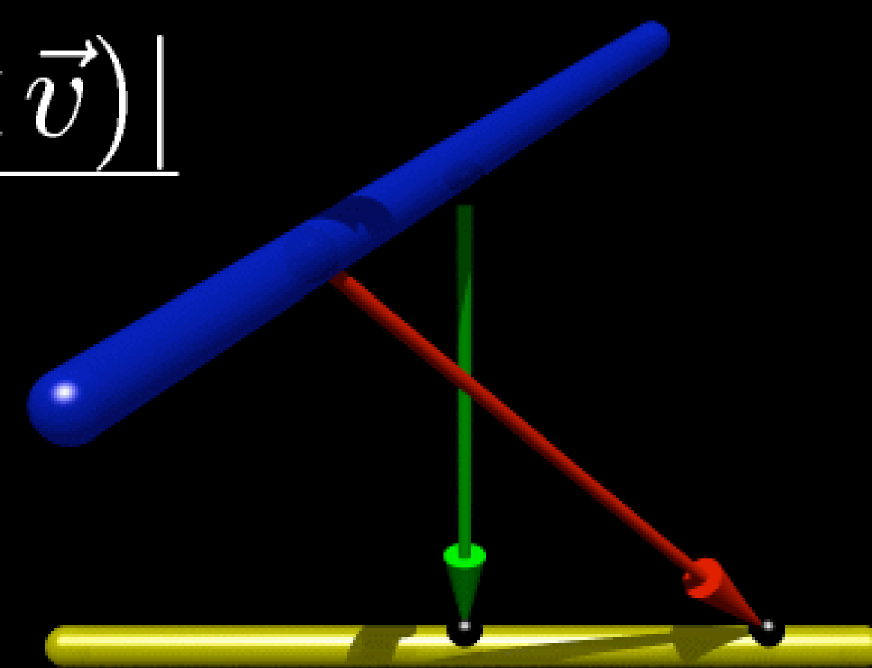
$$d(P,\Sigma) = \frac{|(\vec{PQ})\cdot\vec{n}|}{|\vec{n}|}$$



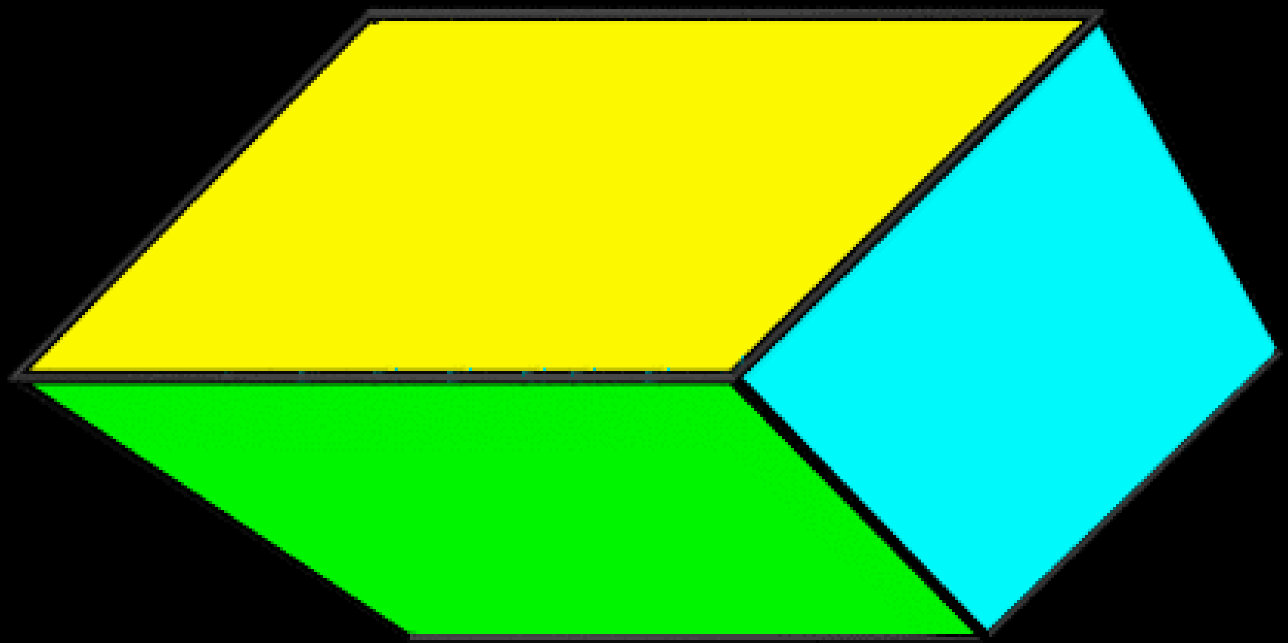
$$d(P,L) = \frac{|(\vec{PQ})\times\vec{u}|}{|\vec{u}|}$$



$$d(L,M) = \frac{|(\vec{PQ})\cdot(\vec{u}\times\vec{v})|}{|\vec{u}\times\vec{v}|}$$

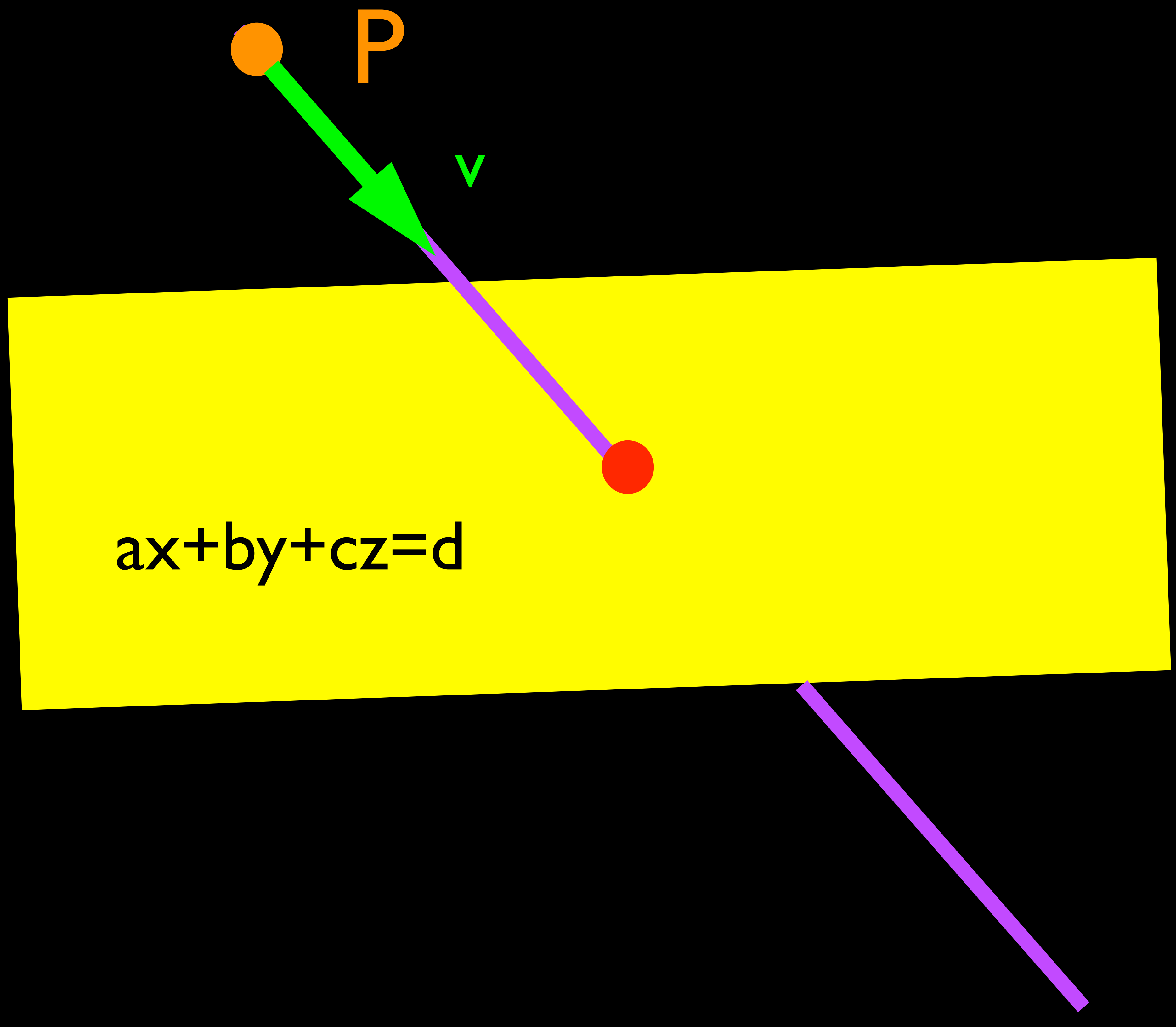


Parallelepiped

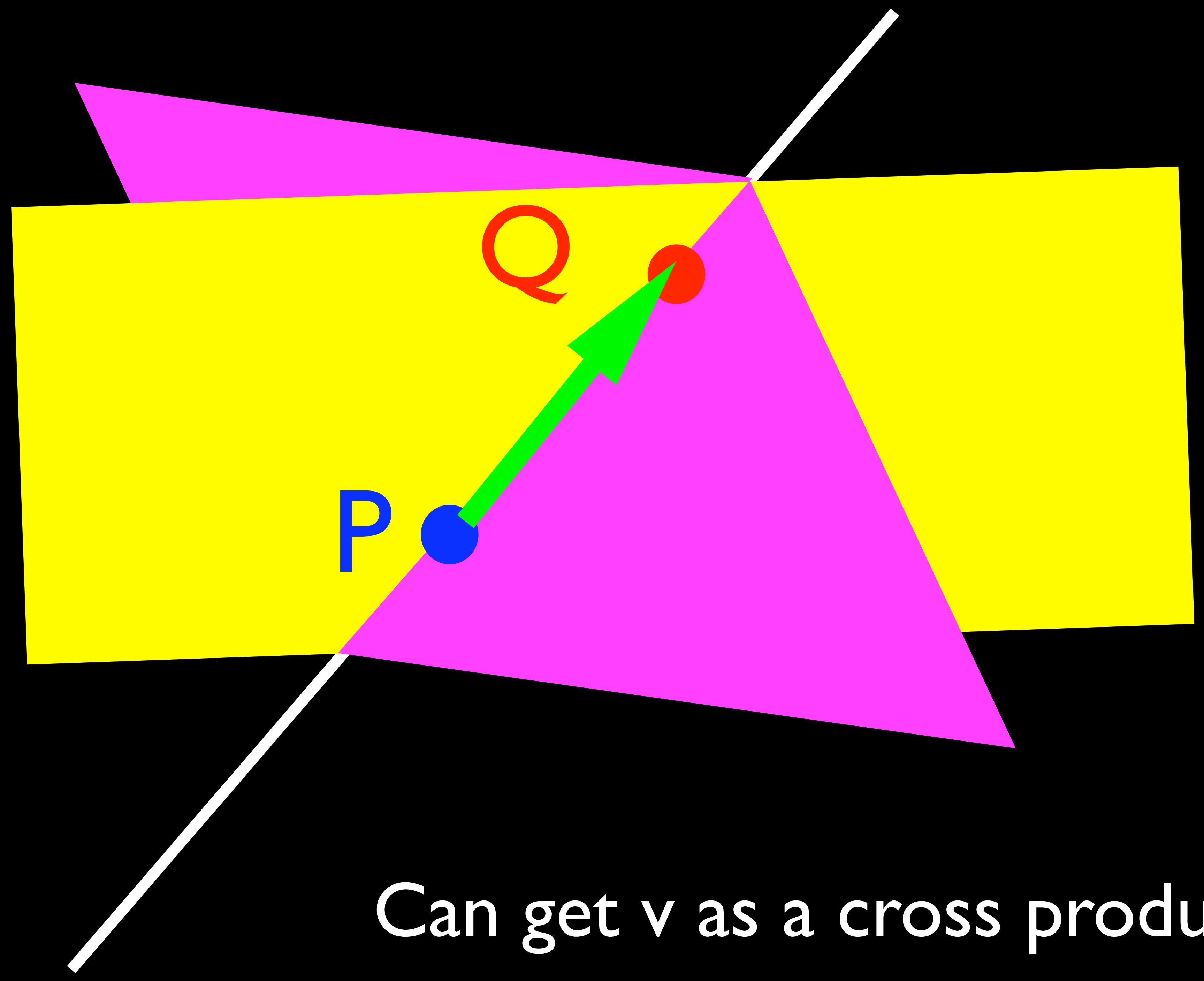


$$|(\vec{u} \times \vec{v}) \cdot \vec{w}| \quad = \quad \text{Volume}$$

Plane and Line



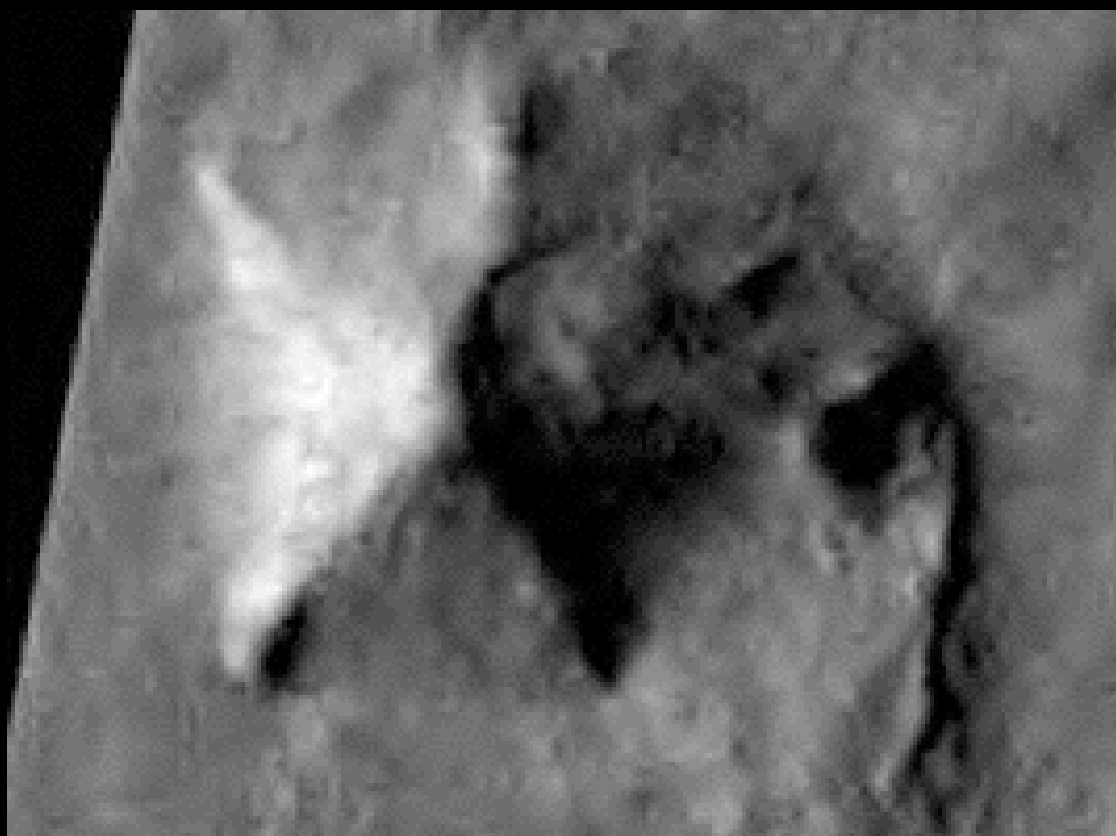
Plane-Plane



Can get v as a cross product!


One Mars year = 586 days
Spirit, Dec 2005

Cydonia pyramide
Problem 3



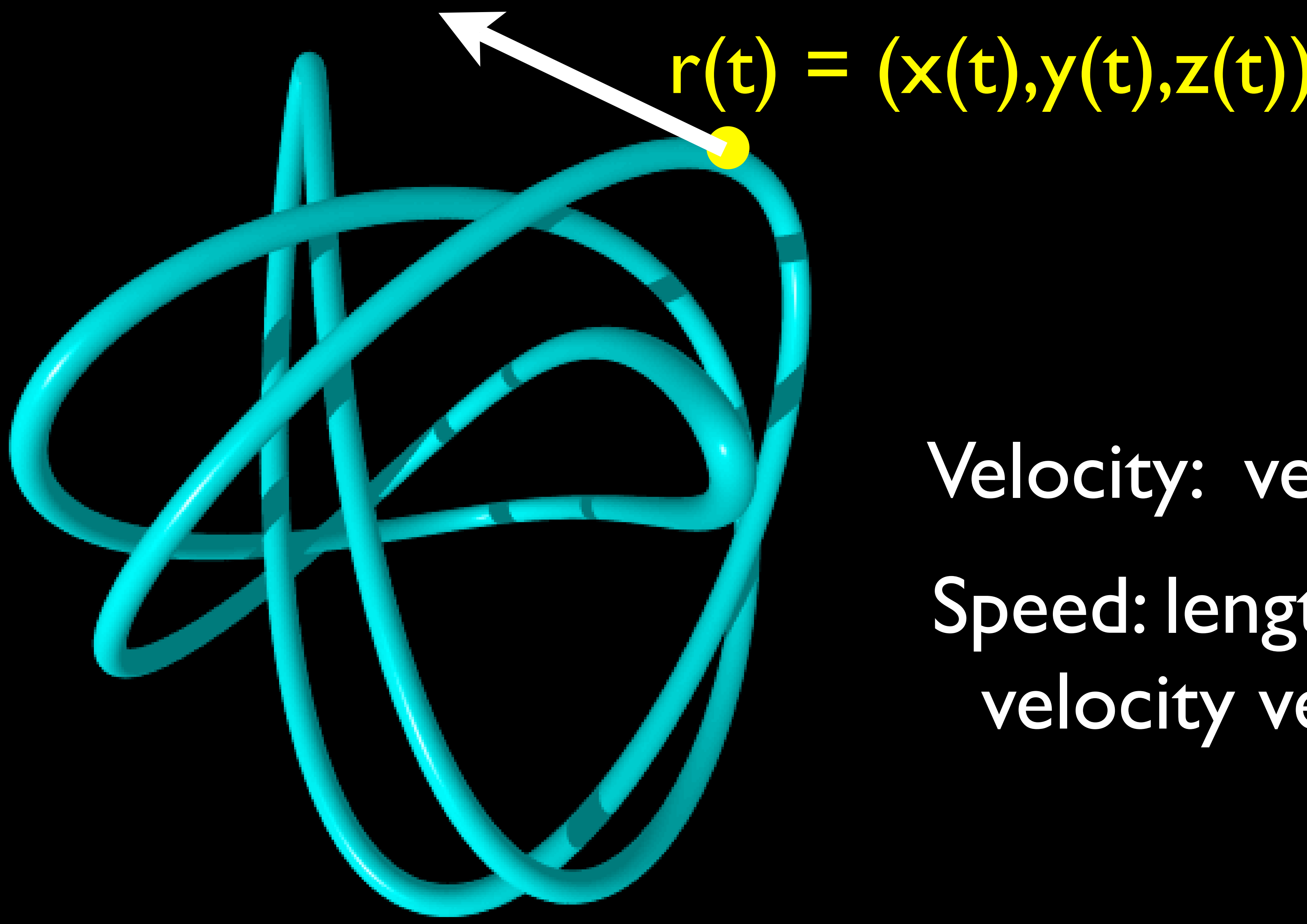
Find the distance from the tip of the cydonia pyramide on Mars with coordinates
$(1,-1,3)$ to
the surface modeled as the plane $x+2y+2z=1$.

Thats all I have to say about that.

2) Curves

Curves

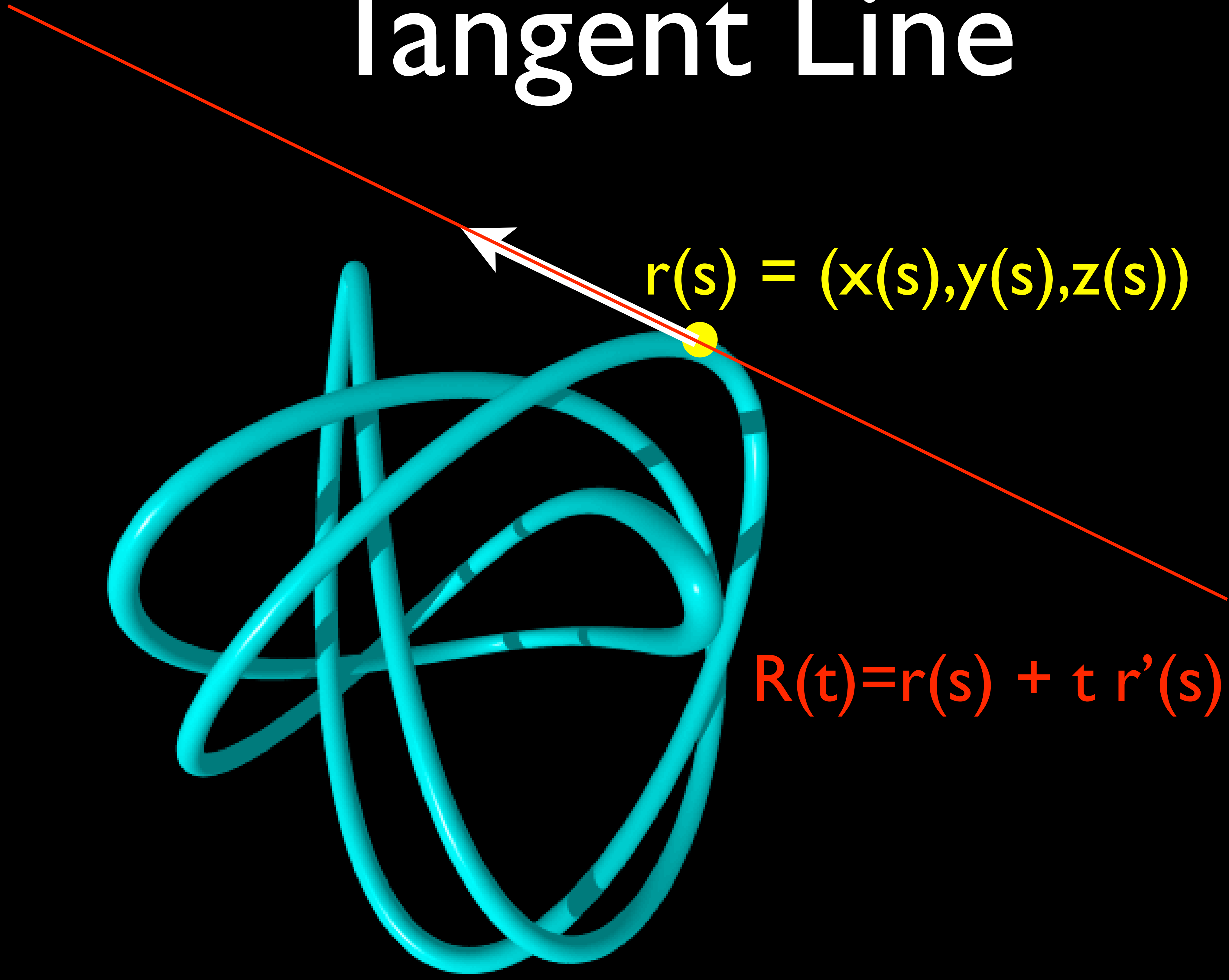
Parametric Curves



Velocity: vector

Speed: length of velocity vector

Tangent Line



Integratation

r ''(t) known at all times, r'(0) known, r(0) known, then r(t)= r(0)+r'(0) t + r''(t) t /2 is known

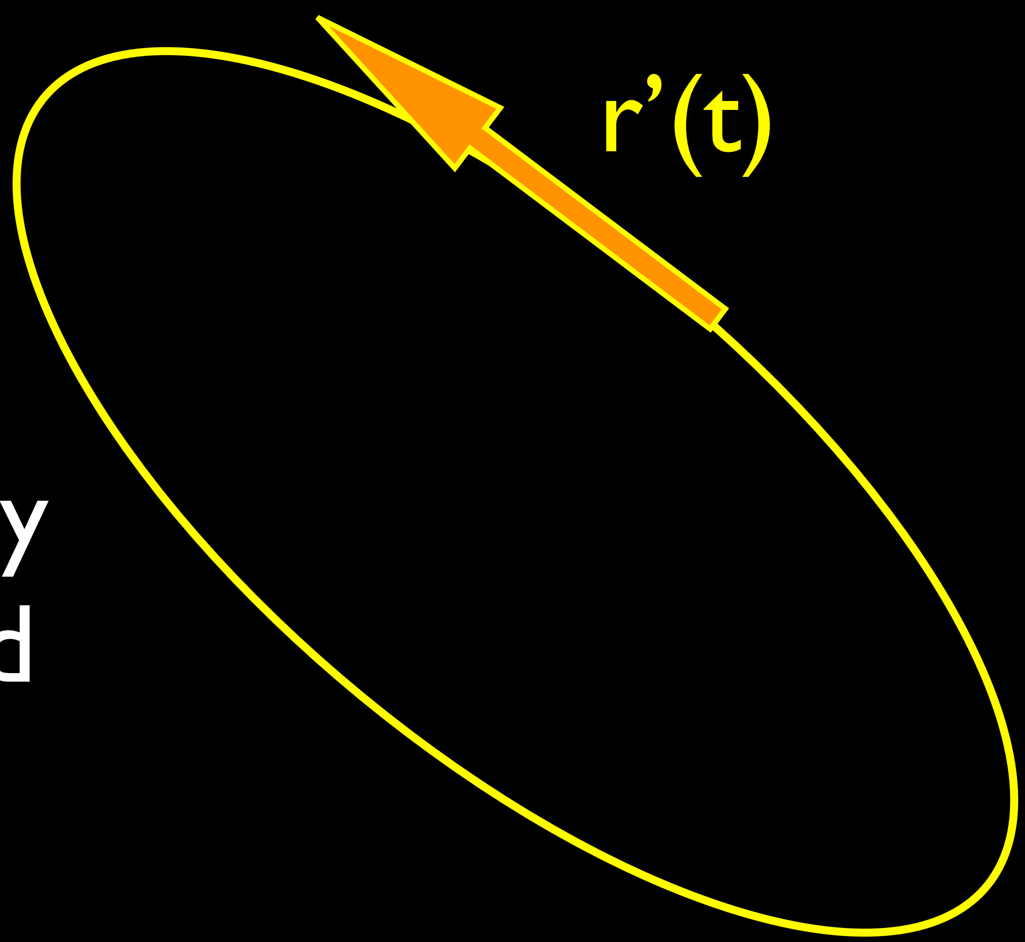
Even MS research is aware of this principle and built a prototype which knows where you are:

$$\vec{r}(t) = \vec{r}(0) + t\vec{r}'(0) + \vec{R}(t)$$

$$\vec{v}(t) = \int_0^t \vec{r}''(s)\, ds, \vec{R}(t) = \int_0^t \vec{v}(s)\, ds$$

Arc Length

r’(t) velocity
|r’(t)| speed



Integrate speed over parameter interval to obtain arc length.

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

Frisbie Problem 4

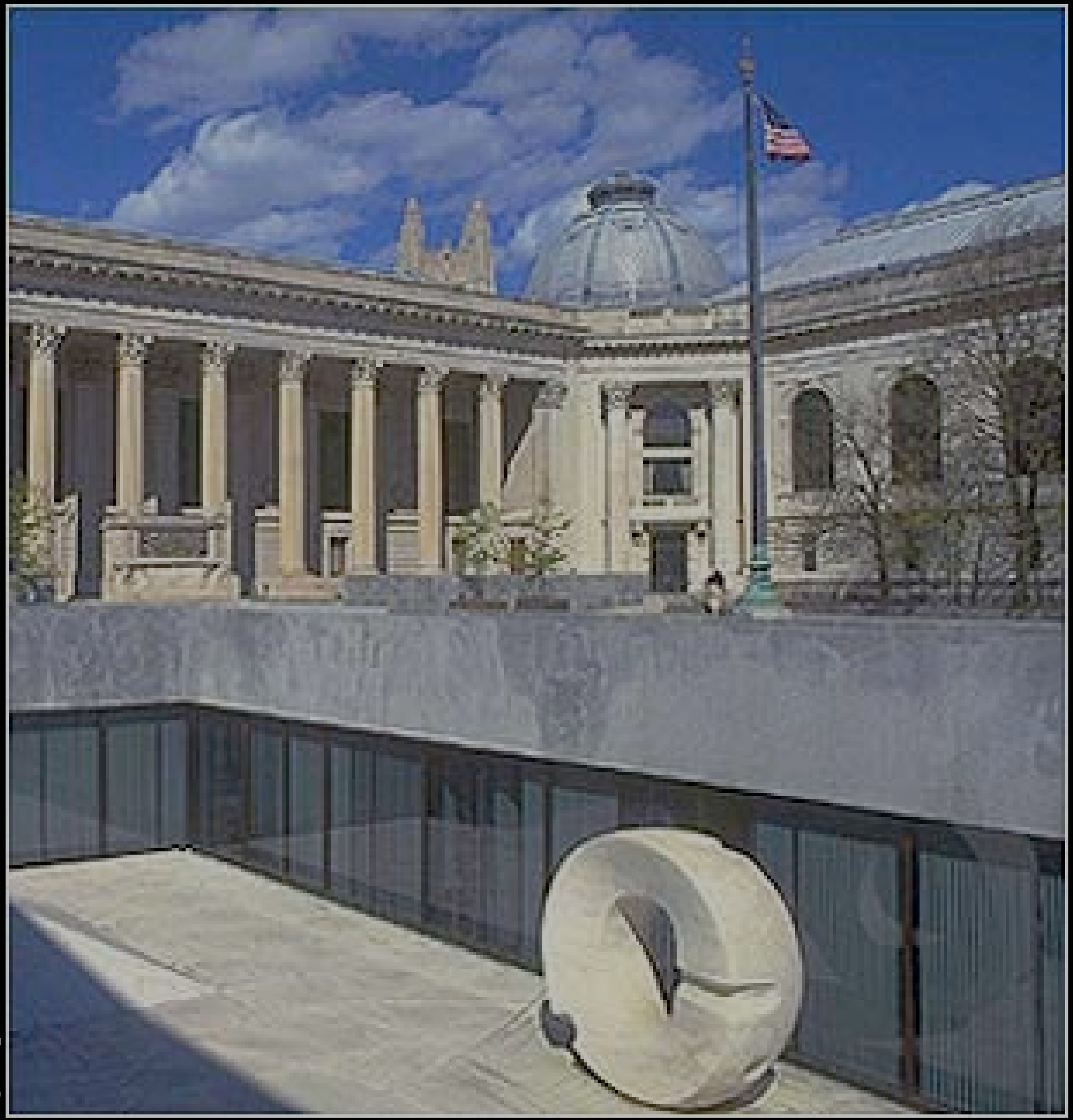
A frisbee flies on the following curve:

$$\vec{r}(t) = \langle \cos(e^t), e^t, \sin(e^t) \rangle$$

Find the length of the curve
from t=0 to t=1.

By the way...

Yale college has claimed to be the place where the frisbie was invented. The school has argued that a Yale undergraduate named Elihu Frisbie grabbed a passing collection tray from the chapel and flung it out into the campus, thereby becoming the true inventor of the Frisbie and winning glory for Yale.



But ...

evenso Yale has “Lux and Veritas”
in their emblem, it
is no accident that
“lux” is above “veritas....

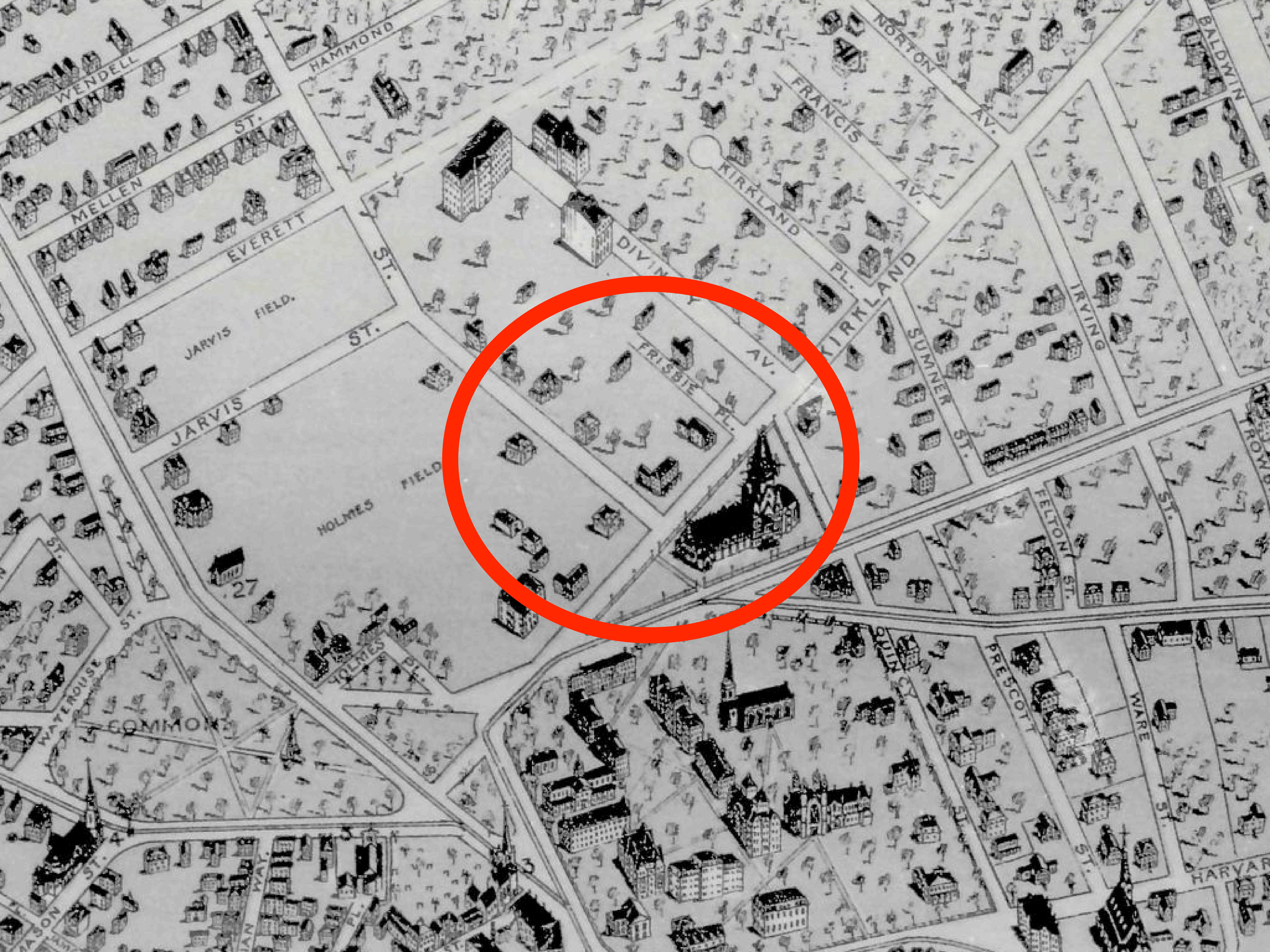
this story is not true! The frisbie was invented at Harvard

Lets go back in time

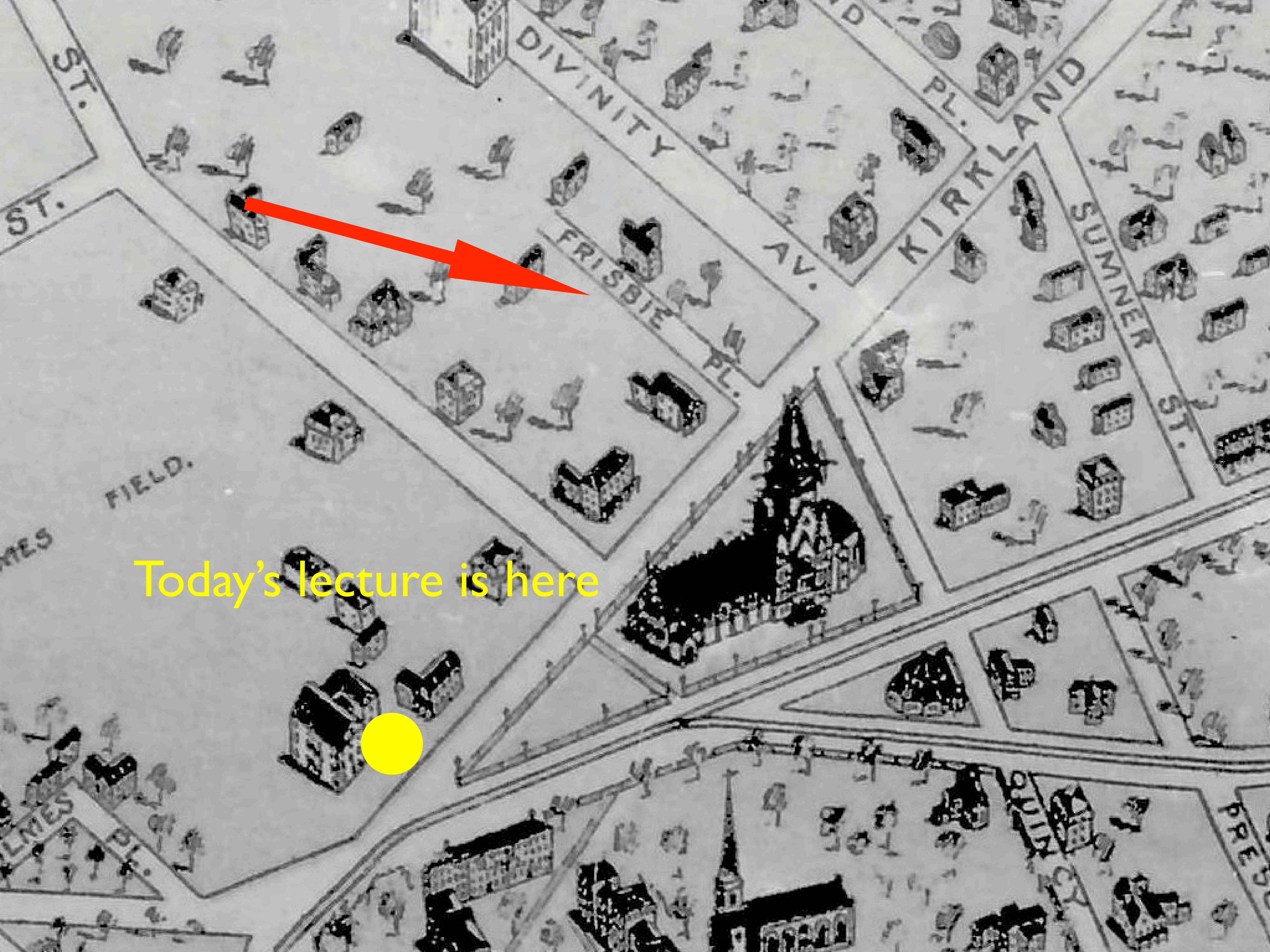
Cambridge 1775 ...

1854....
Station
R. Morse
Mrs Holmes
Keith
Baptst Church
Lawrence Scientific School
C.C. Foster
Ames
Dr. E. Howe
OXFORD
Prof. J. Farrar
Upham
KIRKLAND
DELTA
STREET
DIVINITY
J. Sparks
Rice
CAMBRIDG
Holworthy Hall
Stoughton Hall
Holden Hall
Hollis Hall
Harvard Hall
University Hall
Massachusetts Hall
HARVARD
Prof. C.C. Felton
Lieut Davis
Unitarian Church
STREET
Buckingham
Parker

Cambridge 1877 ...



WENDELL
HAMMOND
NORTON AV.
BALDWIN
MELLEN
ST.
EVERETT
FRANCIS AV.
KIRKLAND PL.
KIRKLAND
JARVIS FIELD
ST.
JARVIS ST.
FRISBIE PL.
AV.
SUMNER ST.
IRVING
HOLMES FIELD
FELTON ST.
ST.
27
COMMON
WATERHOUSE ST.
HOLMES PL.
PRESCOTT ST.
WARE ST.
ST.



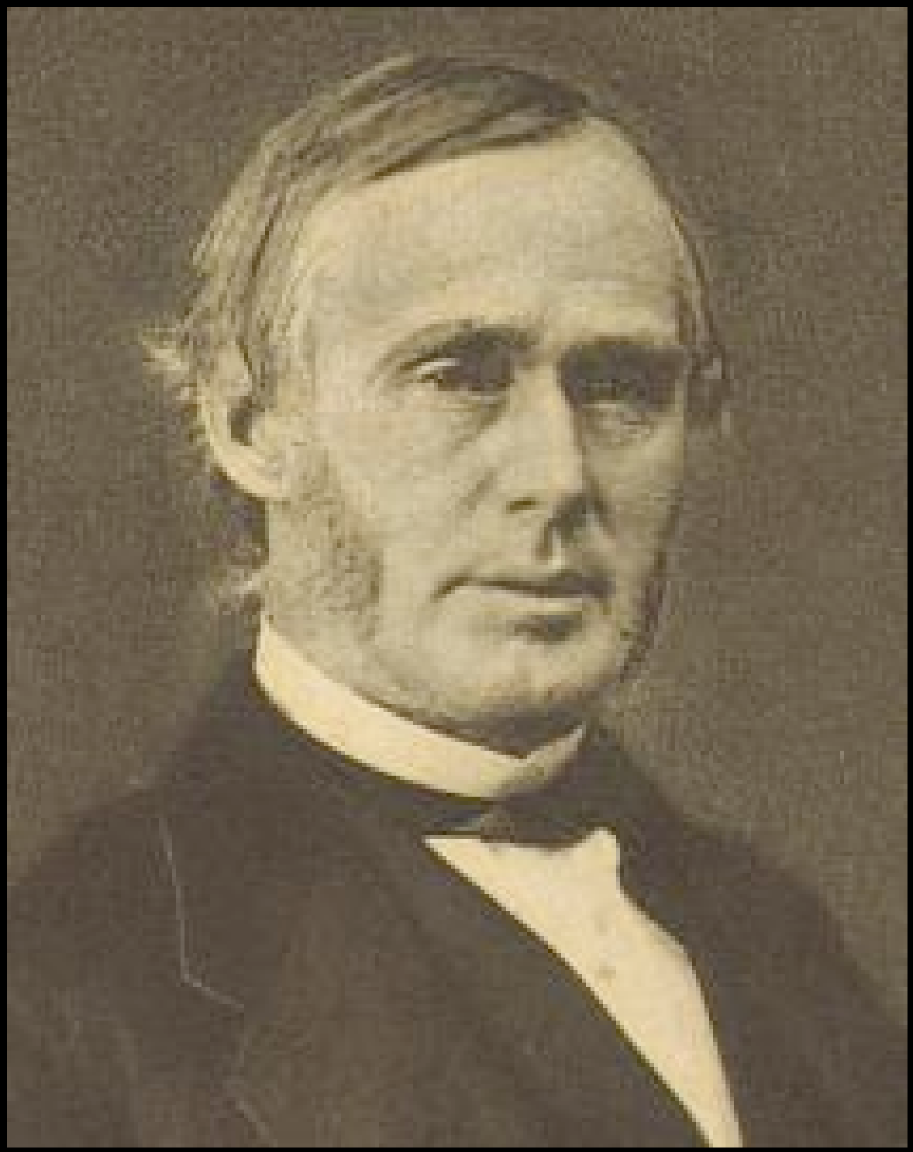
ST.
ST.
DIVINITY AV.
FRISBIE PL.
KIRKLAND
PL.
SUMNER ST.
FIELD.
Today's lecture is here
QUINCY

The fact is:

From 1871 to 1958, the Frisbie Baking company made pies that were sold to many New England colleges. Hungry Harvard students soon discovered that the empty pie tins could be tossed and cought, providing endless hours of game and sport.

One of the students was

George Frisbie Hoar graduated from Harvard University in 1846. Frisbie was often teased because his name could be found on every pie.

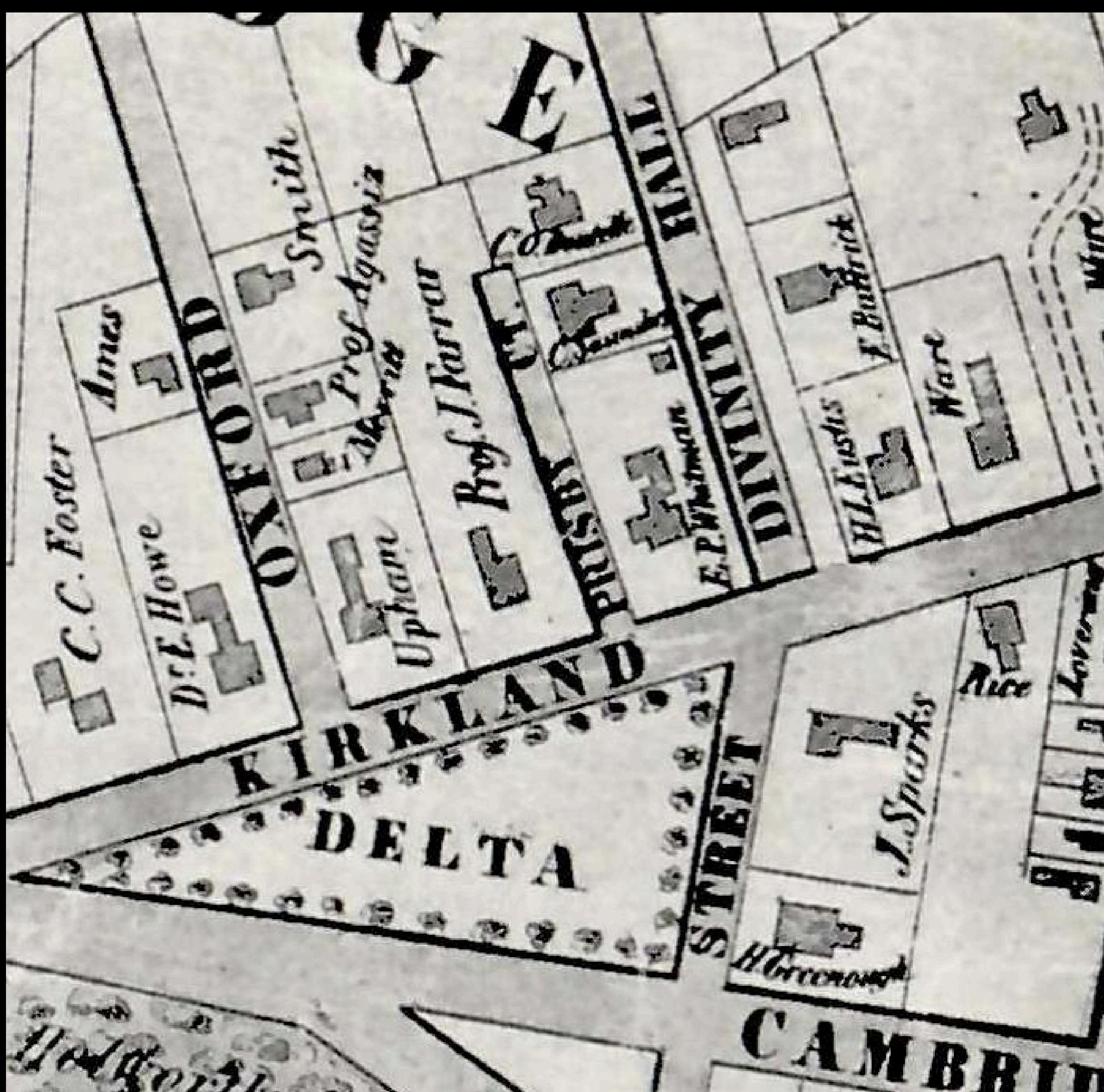


One beautiful march day as today,
at the exactly same spot as you sit now, a plate was thrown at him with the words: Frisbie, catch the Frisbie!

The frisbie was invented

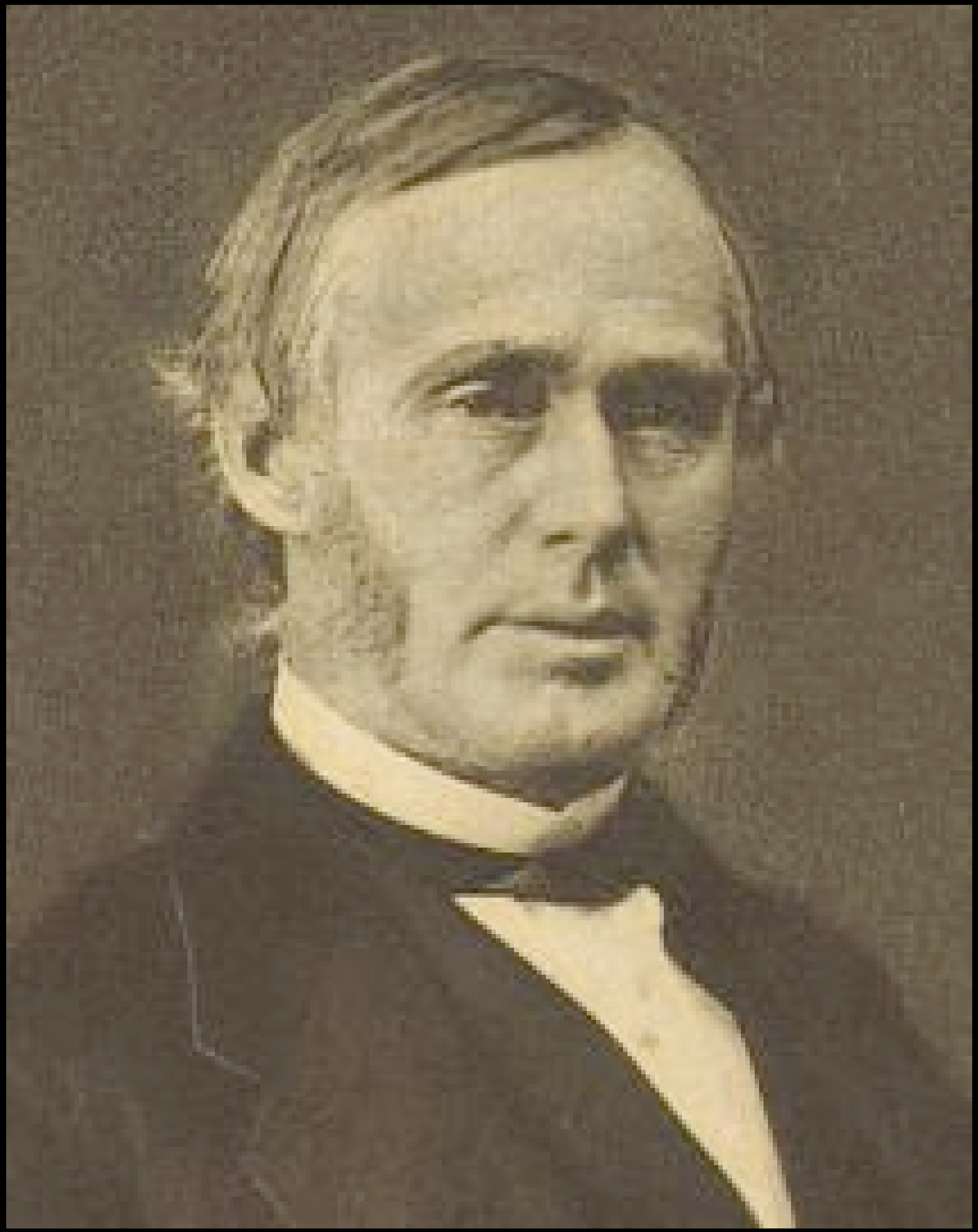
Note the transition from PRISBY to FRISBIE

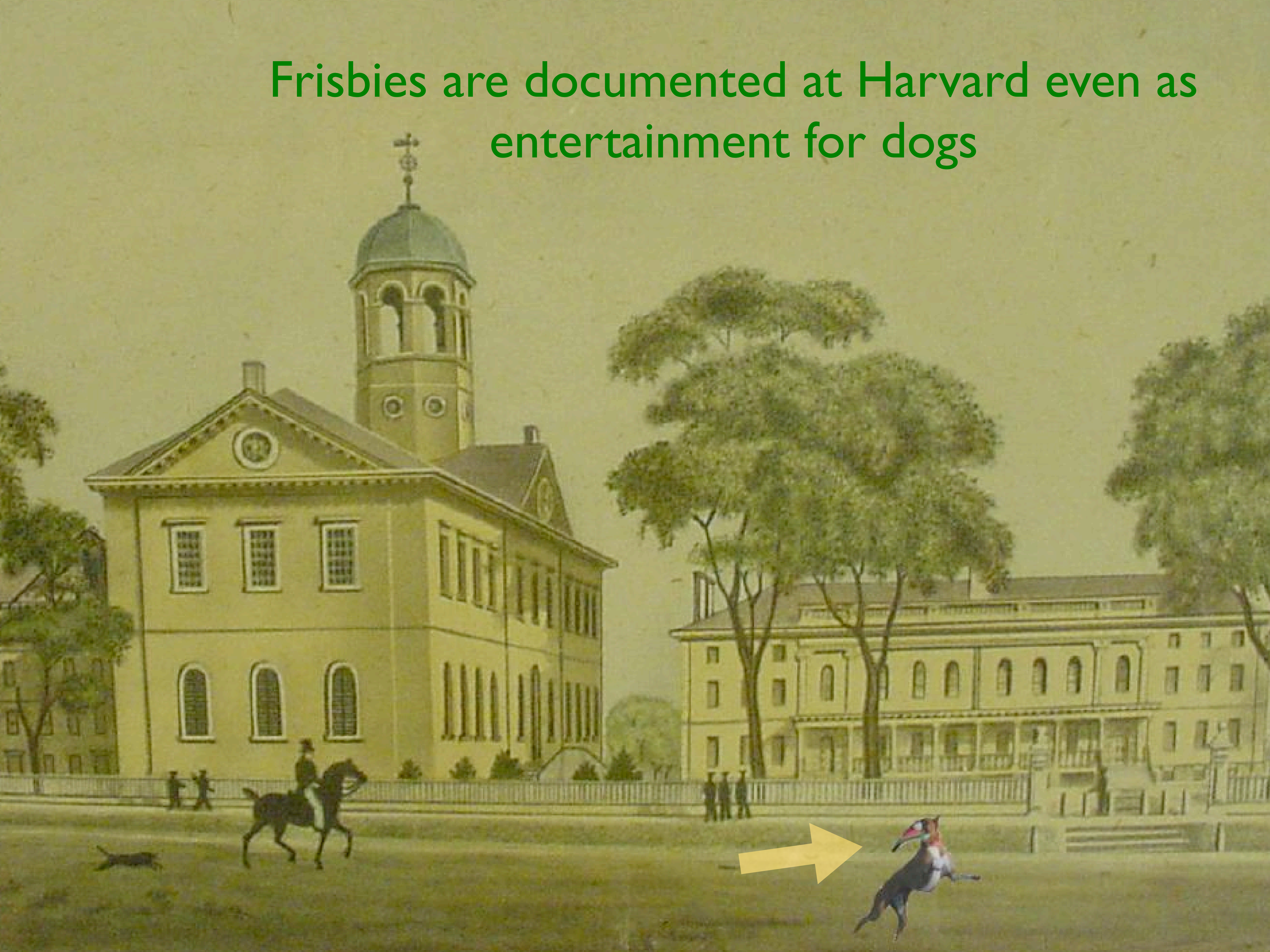
1854

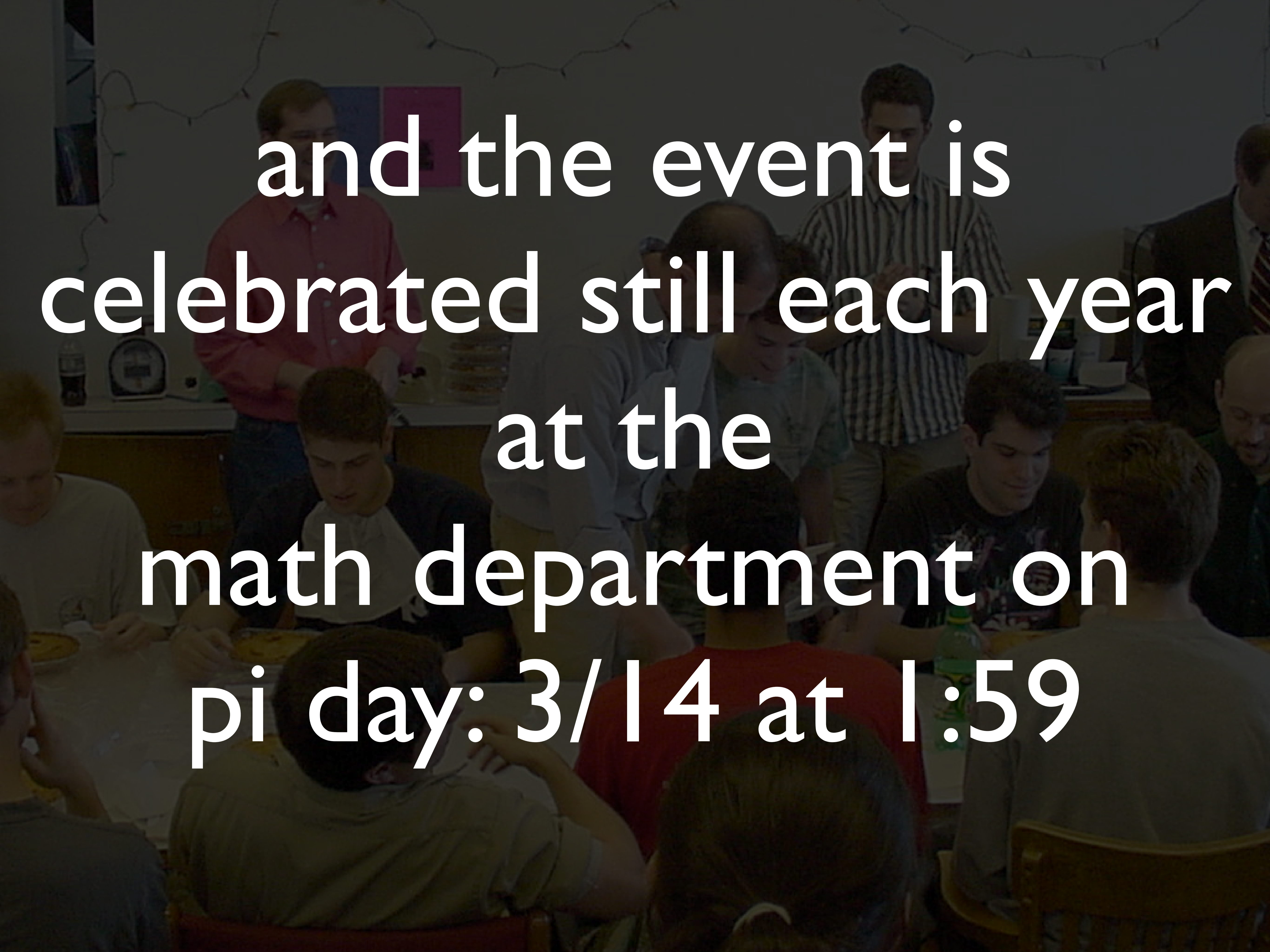


1877

George Frisbie Hoar (1826-1904), studied later law at Harvard Law School and served on the Mass state senate and the United States House of Representatives.




Frisbies are documented at Harvard even as
entertainment for dogs


and the event is
celebrated still each year
at the
math department on
pi day: 3/14 at 1:59

Back to the problem

$$\vec{r}(t) = \langle \cos(e^t), e^t, \sin(e^t) \rangle$$

Find the length of the curve
from t=0 to t=1.

Product Rules

$$\frac{d}{dt}\left(\vec{v}(t)\cdot\vec{w}(t)\right)=\vec{v}\,'(t)\cdot\vec{w}(t)+\vec{v}(t)\cdot\vec{w}\,'(t)$$

$$\frac{d}{dt}\left(\vec{v}(t)\times\vec{w}(t)\right)=\vec{v}\,'(t)\times\vec{w}(t)+\vec{v}(t)\times\vec{w}\,'(t)$$

Application:

$$\frac{d}{dt}\left(\vec{r}(t)\times\vec{r}\,'(t)\right)=0$$

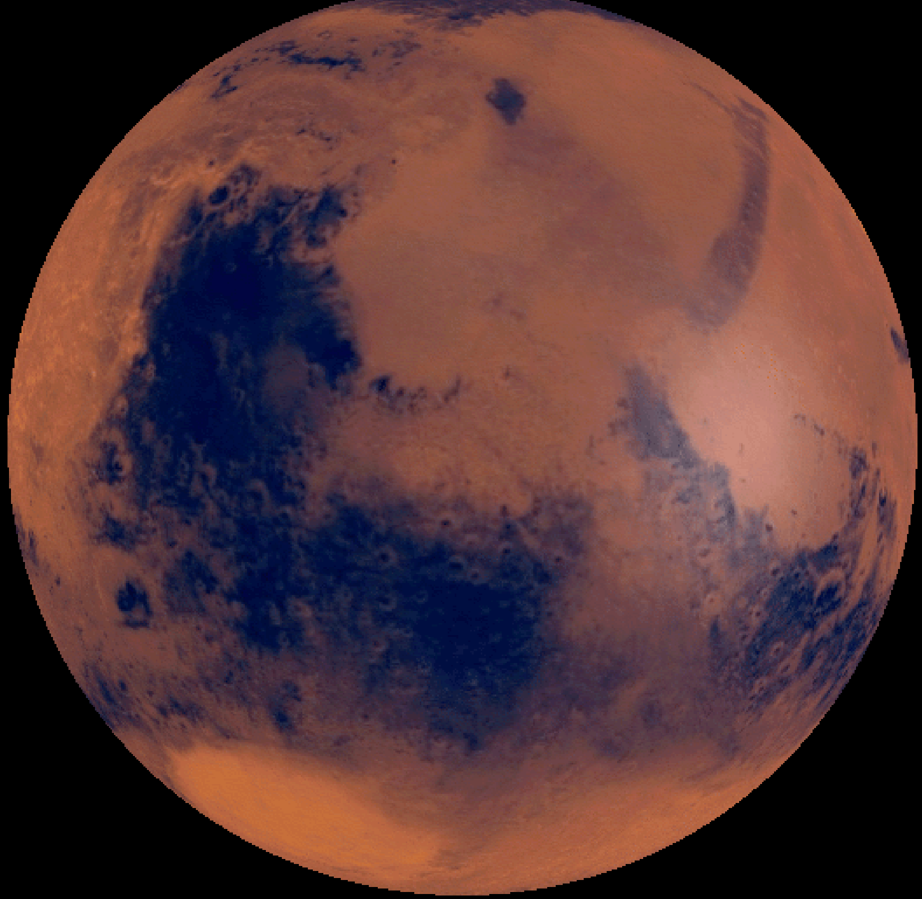
Angular momentum conservation

The Mars-bug riddle

Opportunity has photographed a "bunny"-shaped yellow object of about 4-5 centimeters diameter.

Curvature Problem 5

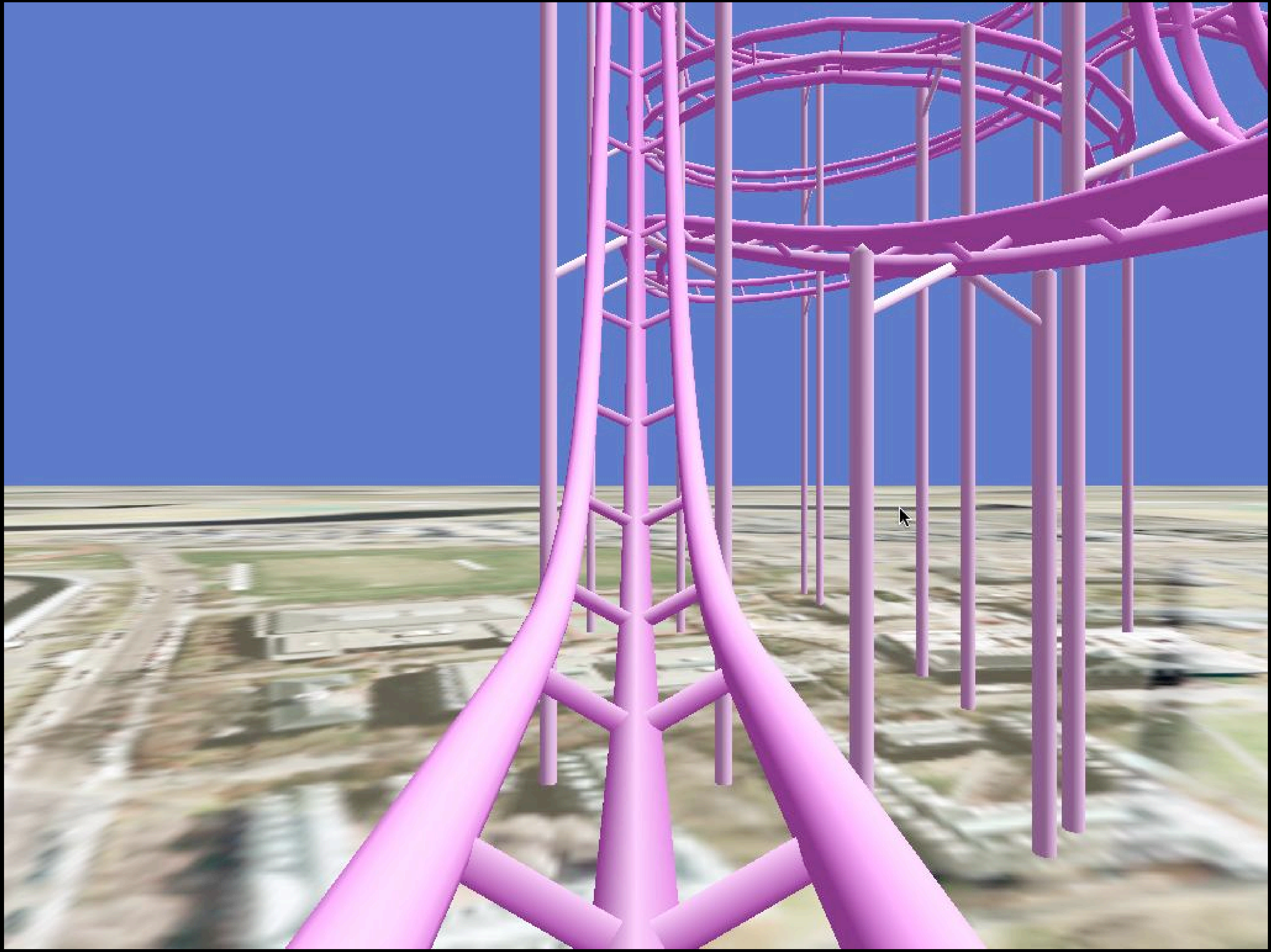
An mars bug flies along the path
$r(t) = (50\cos(t), 50\sin(t), 10)$

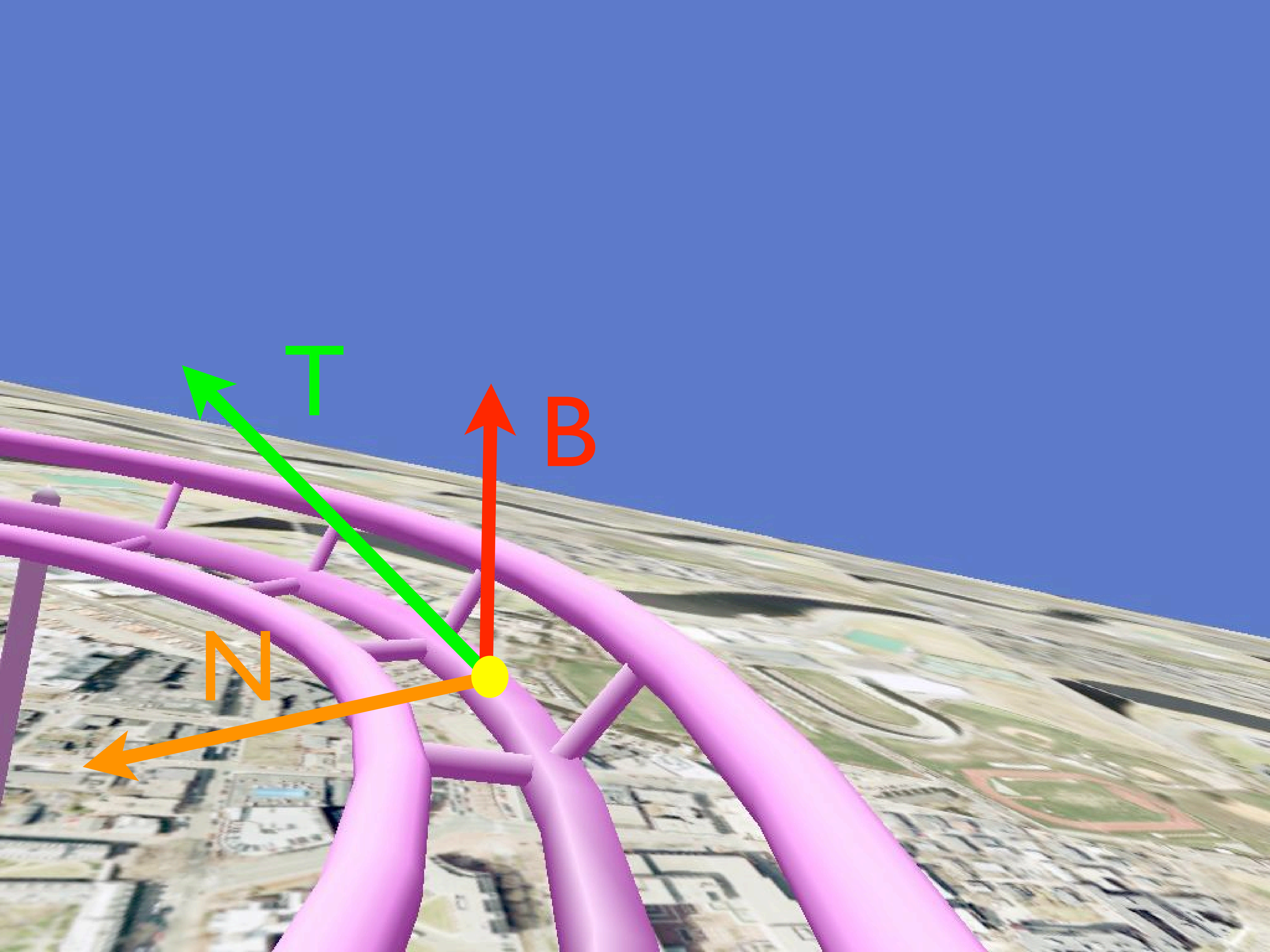


The mars rover travels along the path
$r(t) = (2\cos(t), 2\sin(t), 10)$

Which path has larger curvature?

Unit tangent vectors etc




T
B
N

Formulas

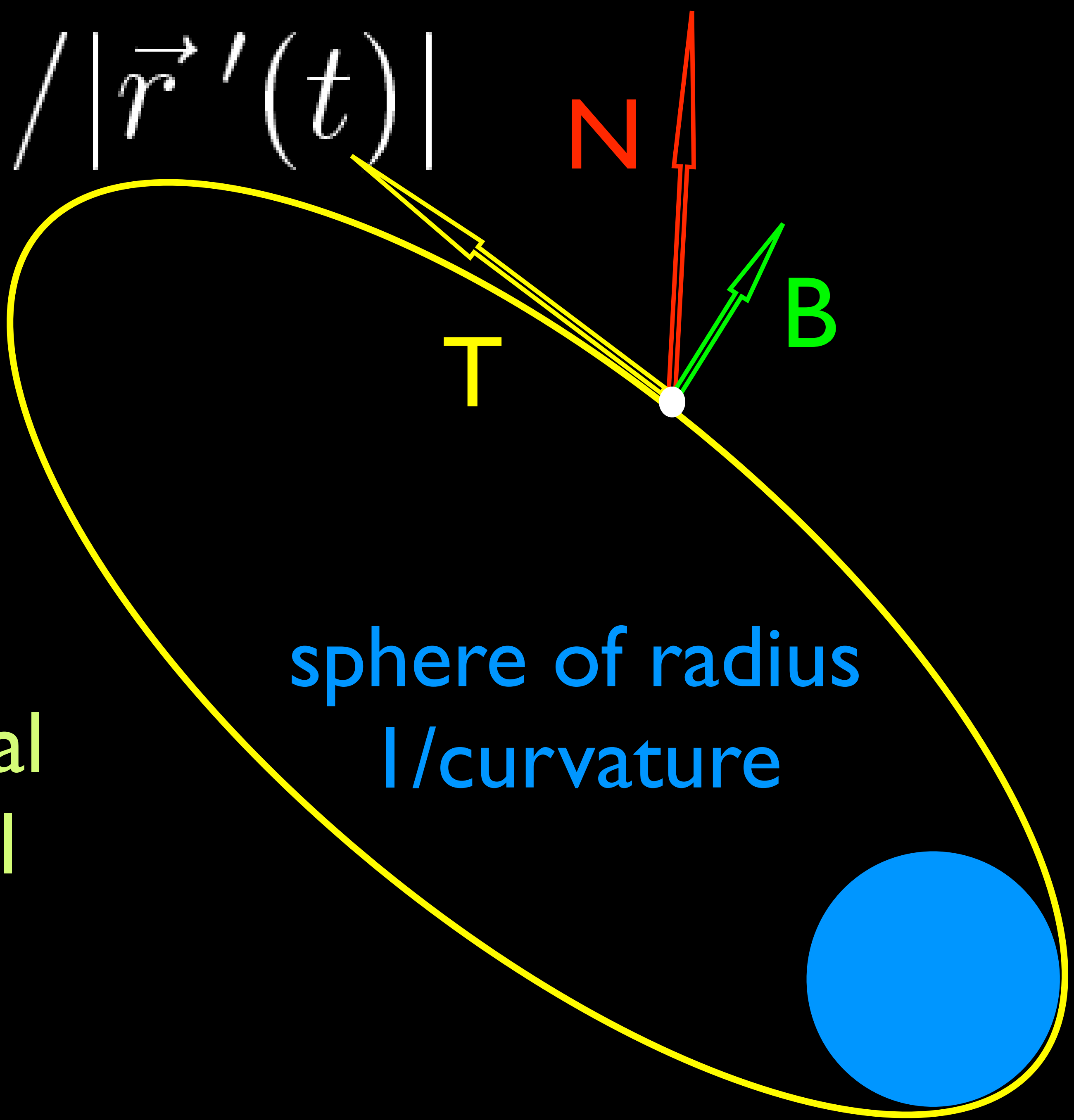
$$\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)|$$

$$\vec{N}(t) = \vec{T}'(t)/|\vec{T}'(t)|$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

$$\kappa(t) = |\vec{T}'(t)|/|\vec{r}\,'(t)|$$

$$= \frac{|r\,'(t) \times r\,''(t)|}{|r\,'(t)|^3}$$

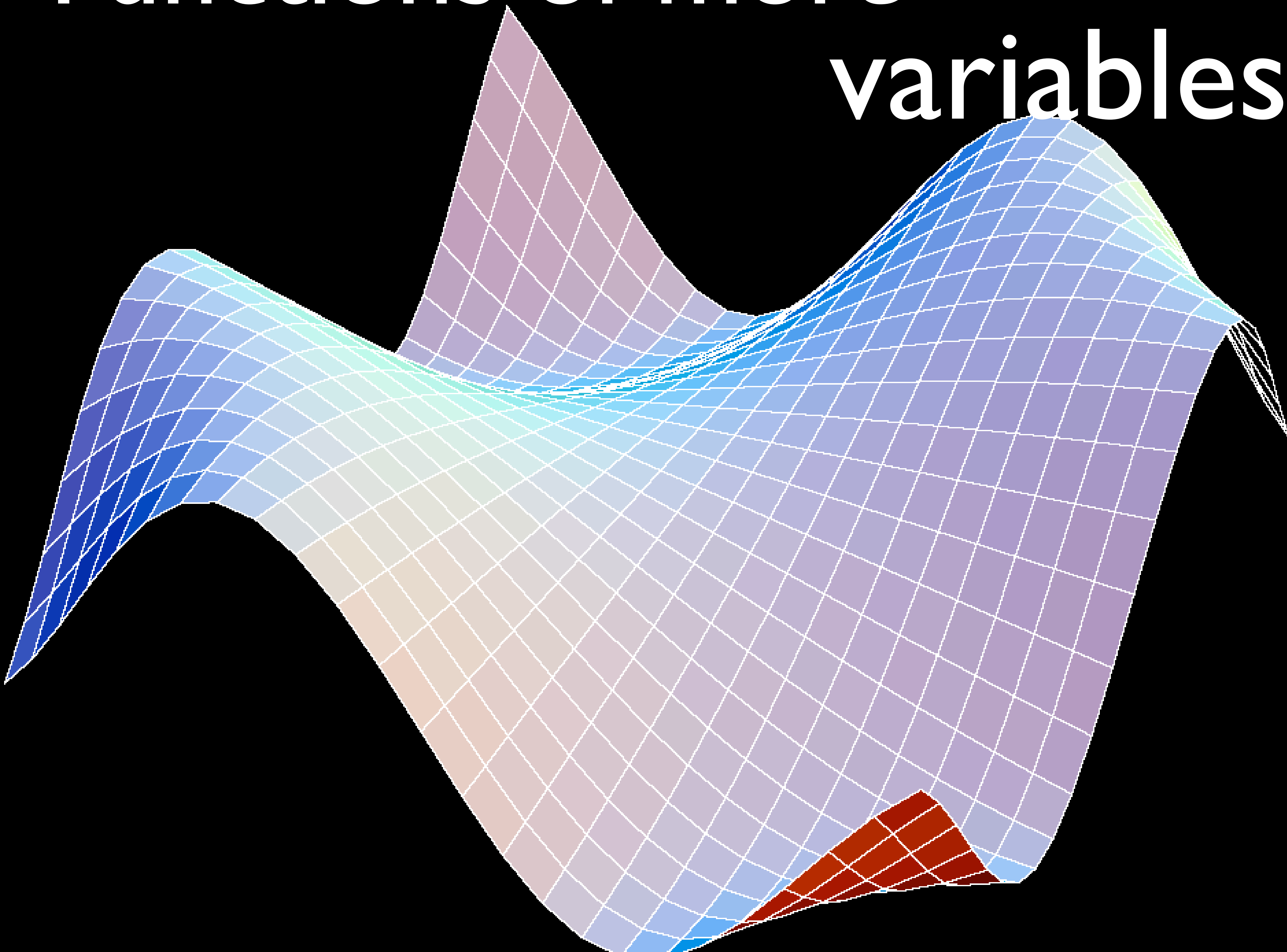


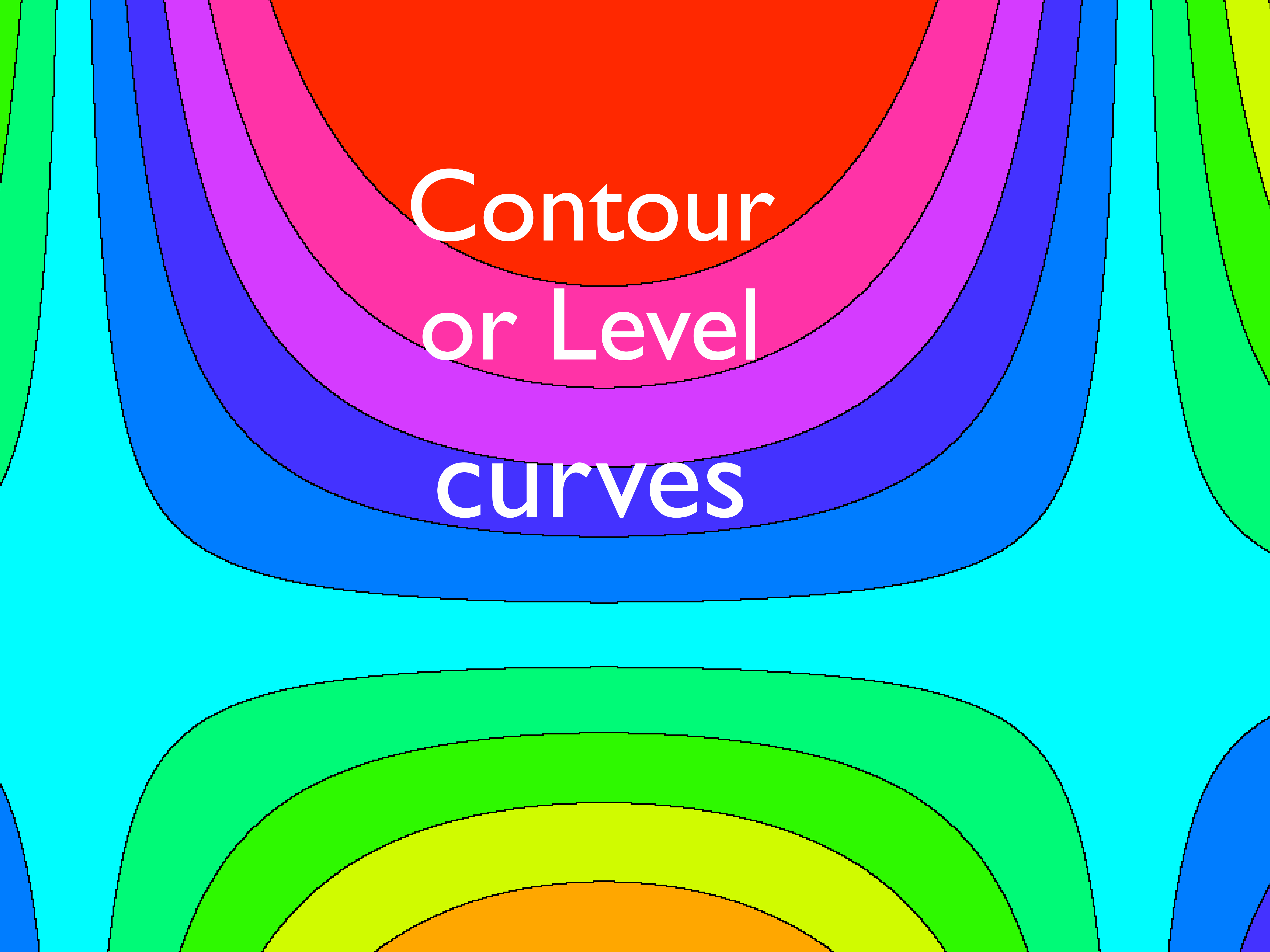
Unit tangent, unit normal and binormal vectors ore normal to each other

Thats all I have to say about that.

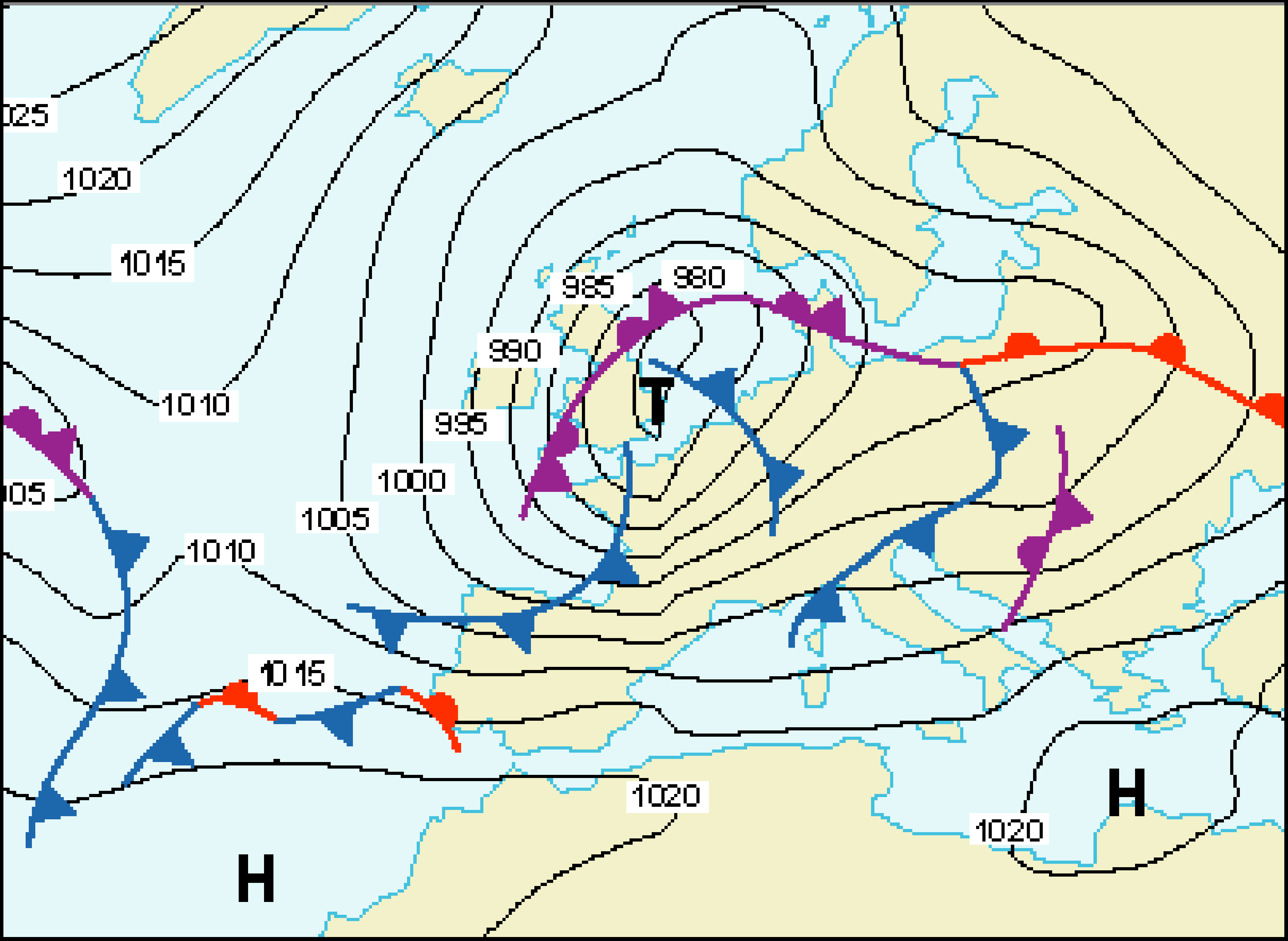
3) Functions etc

Functions of more variables

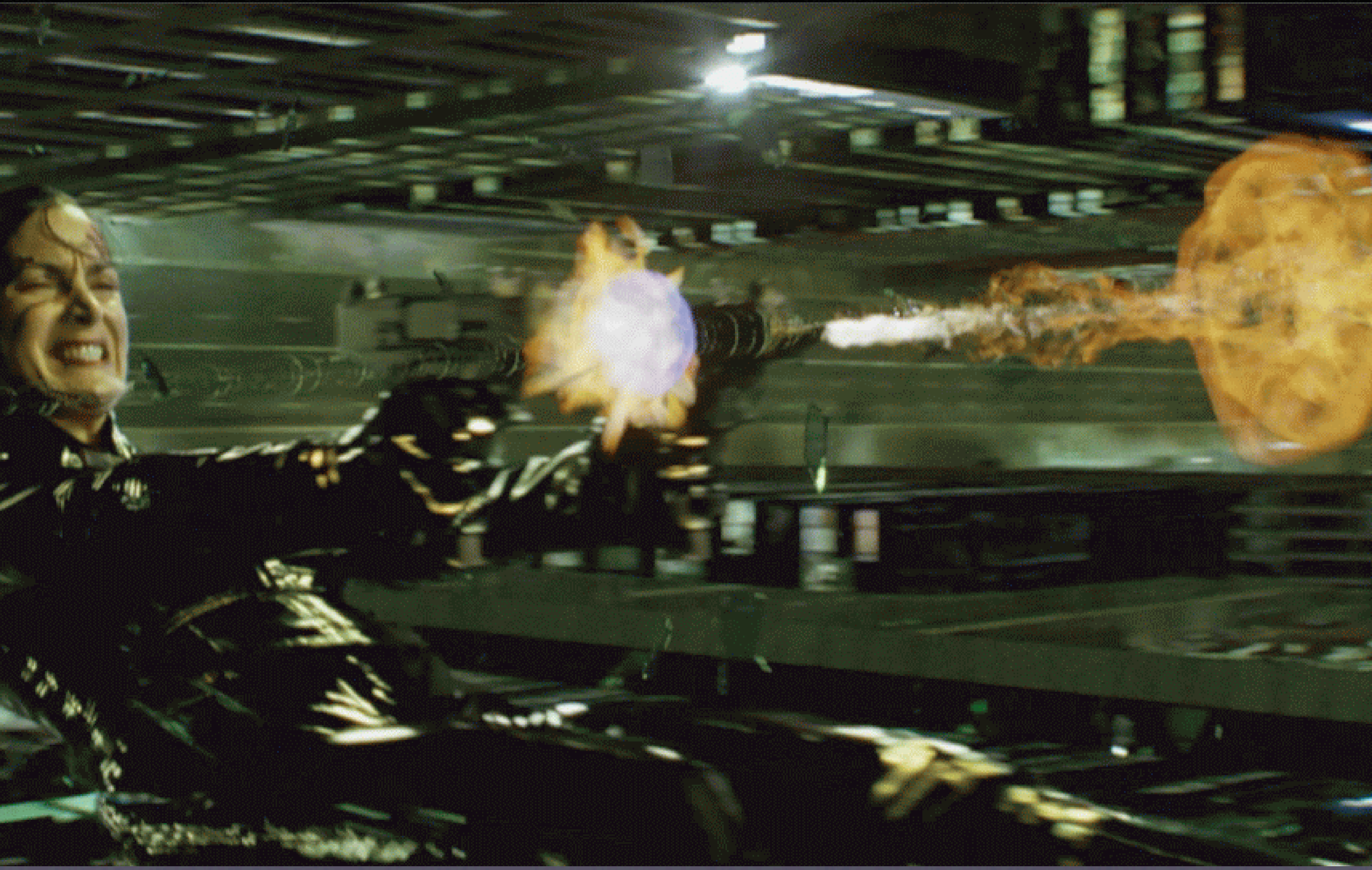



Contour
or Level
curves

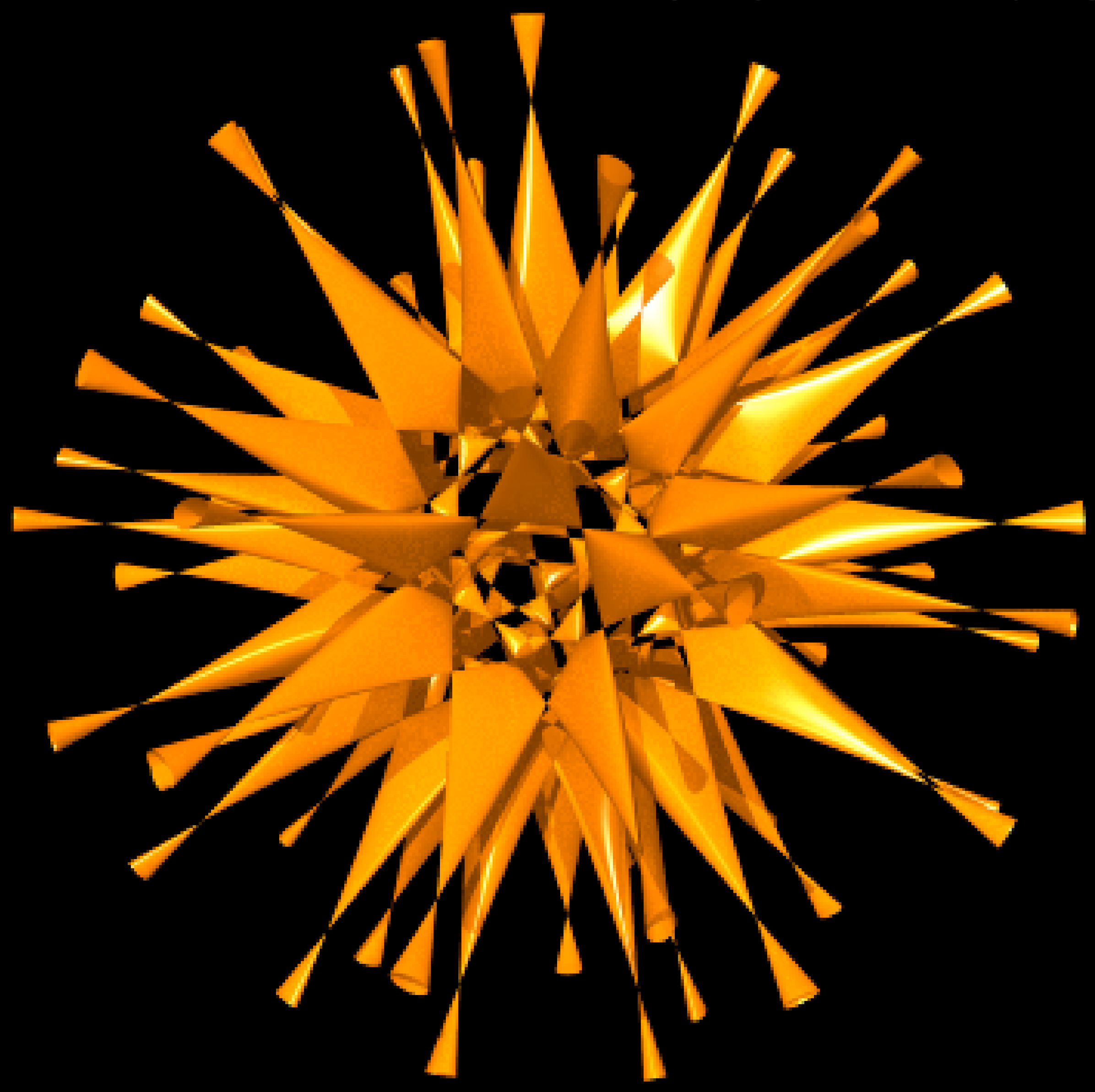
and their Gradients

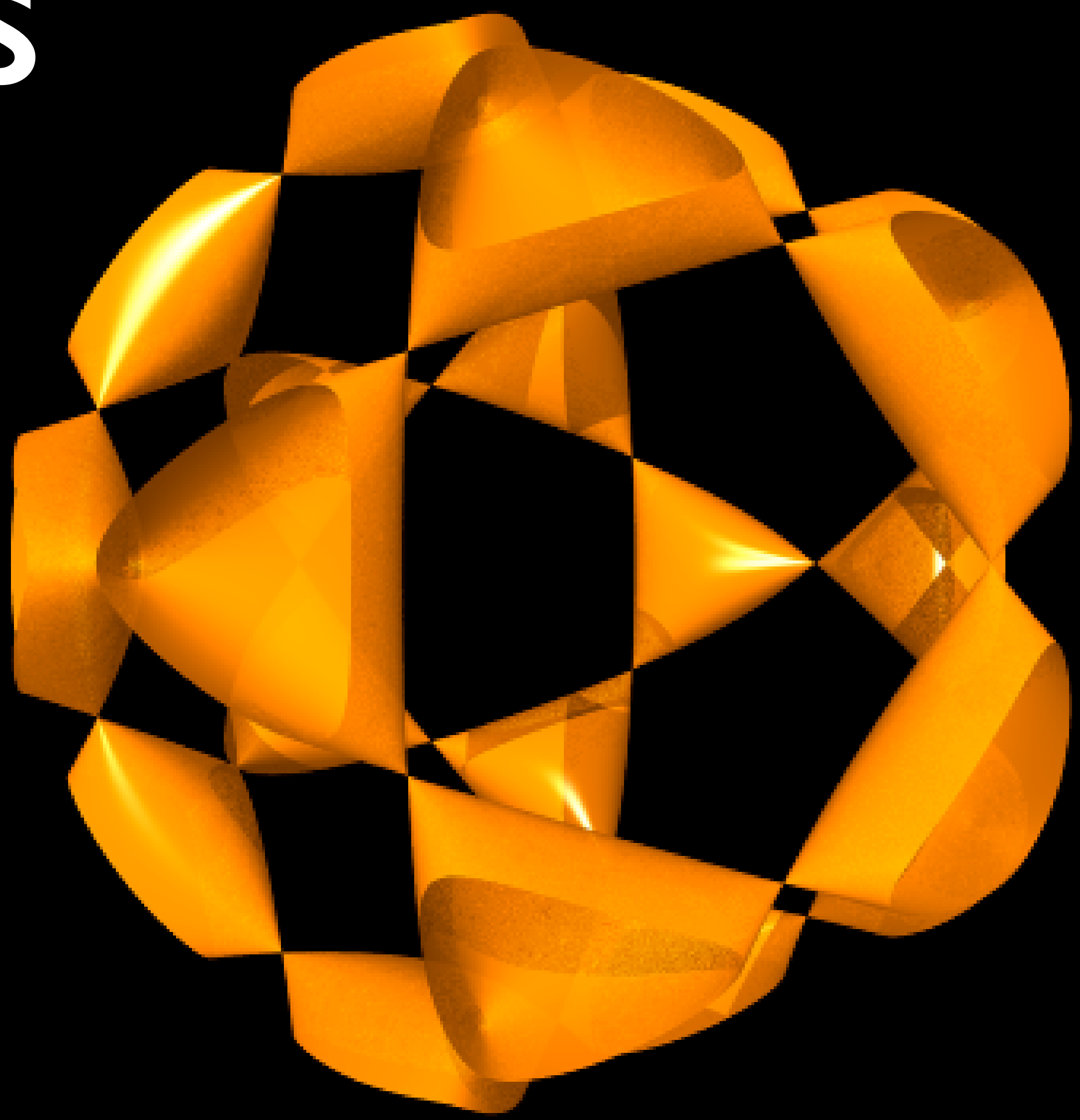


Surfaces



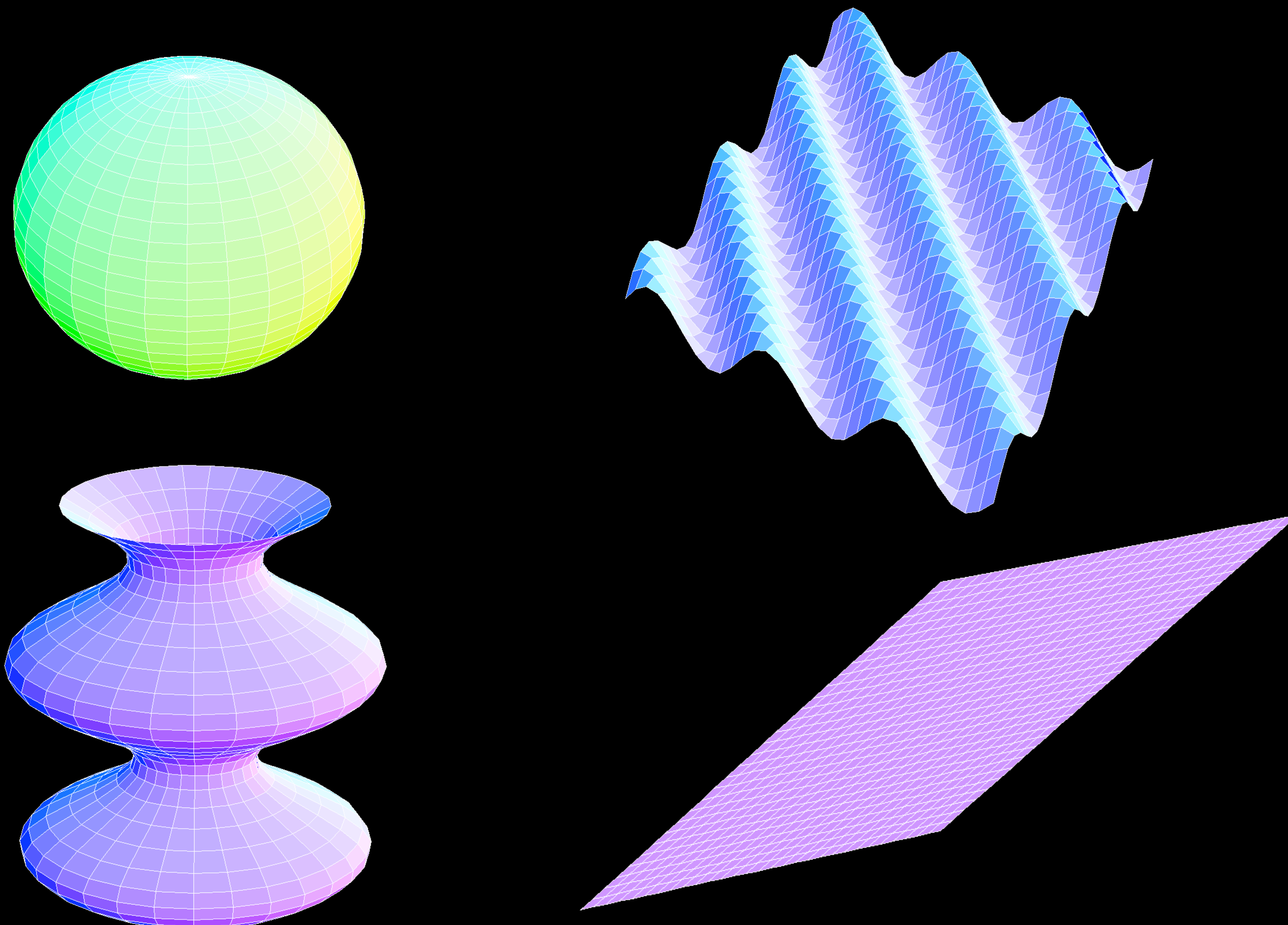
2 ways to represent surfaces





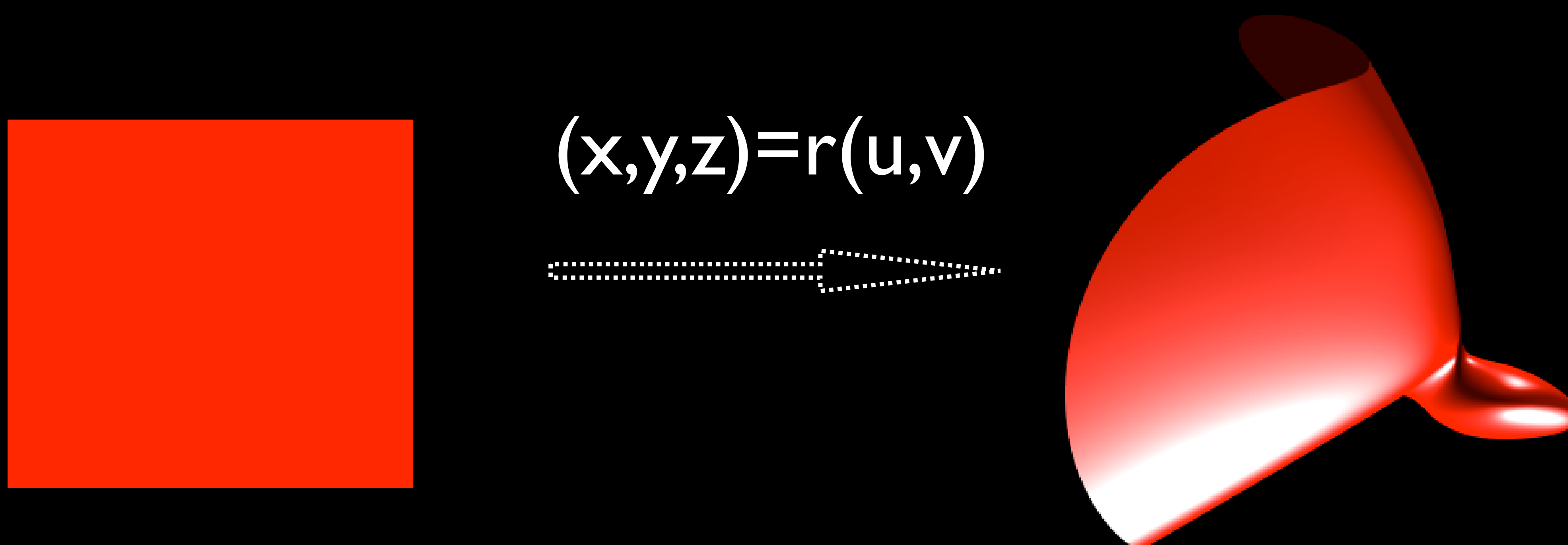
- Implicit surface $g(x,y,z)=c$
- Parametric surface $r(u,v)$

Surfaces you should know:



Surface Area

$$r(u,v) = (x(u,v), y(u,v), z(u,v))$$
$$r(R) = S$$
$$\text{Area}(S) = \int\int_R |r_u \times r_v| \, du dv$$



Thats all I have to say about that.

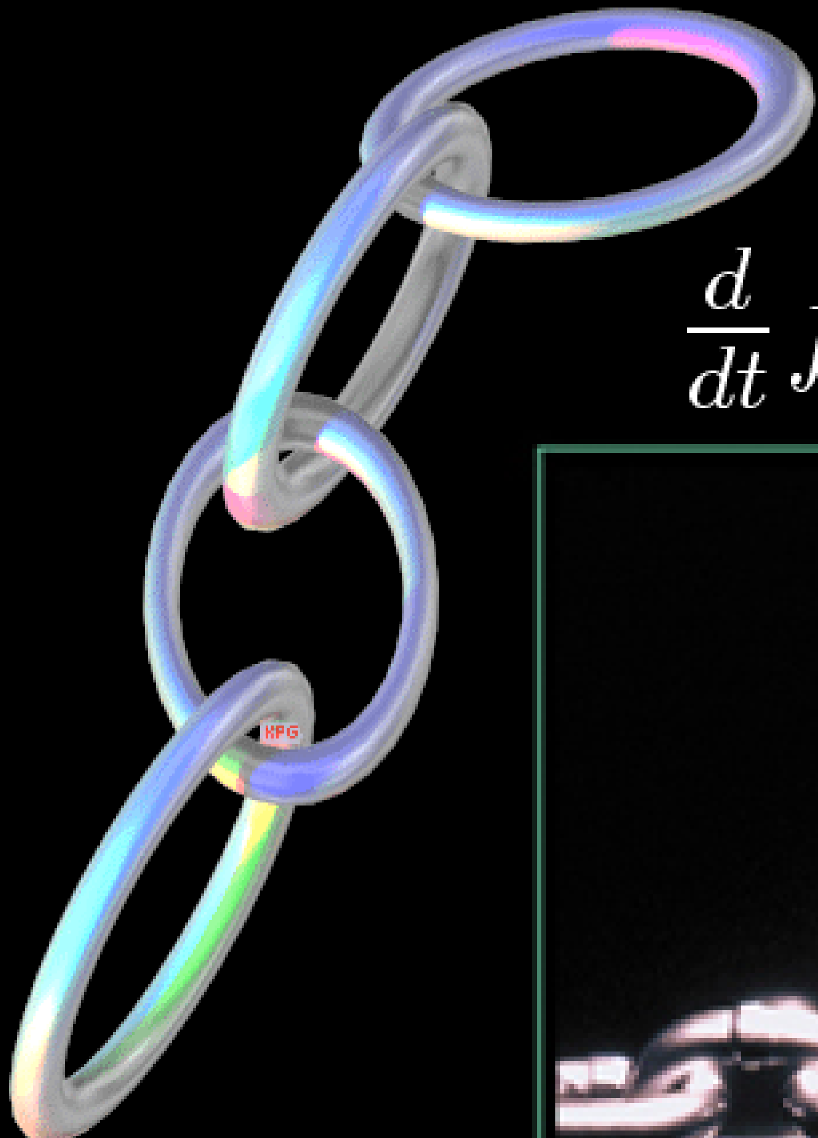
Derivatives

$$\partial_x f = f_x = \frac{\partial f}{\partial x}$$

$$\nabla f(x, y, z) = (f_x, f_y, f_z) = \mathrm{grad}(f)$$

$$D_v f(x, y, z) = \nabla f(x, y, z) \cdot v$$

Chain Rule



$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

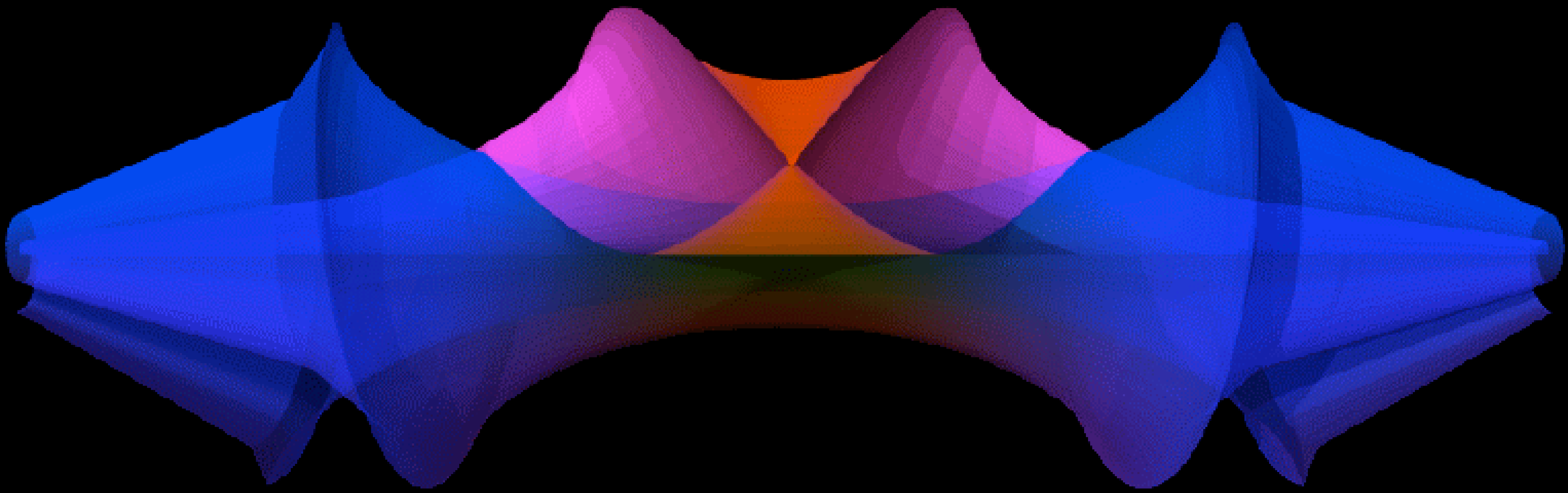
Implicit Differentiation

g(x,y,z) = 0 defines z = f(x,y)

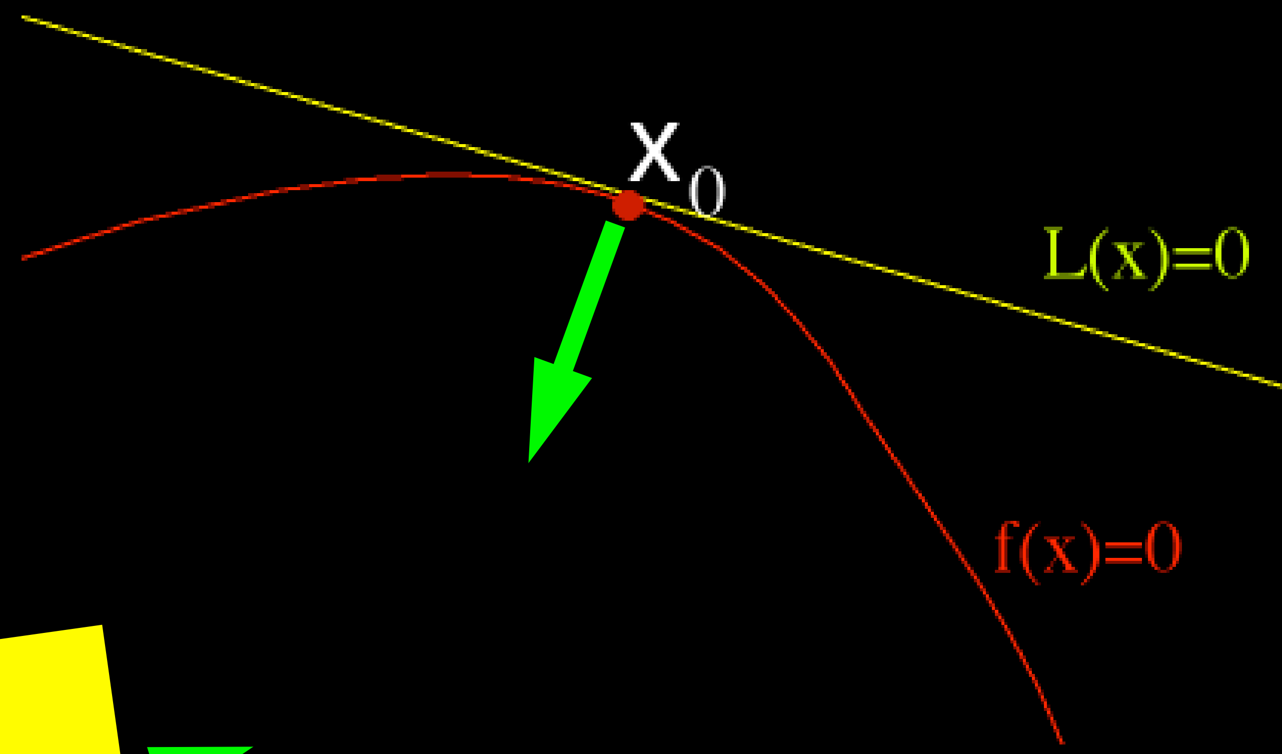
$$z_x = -g_x(x,y,z)/g_z(x,y,z)$$

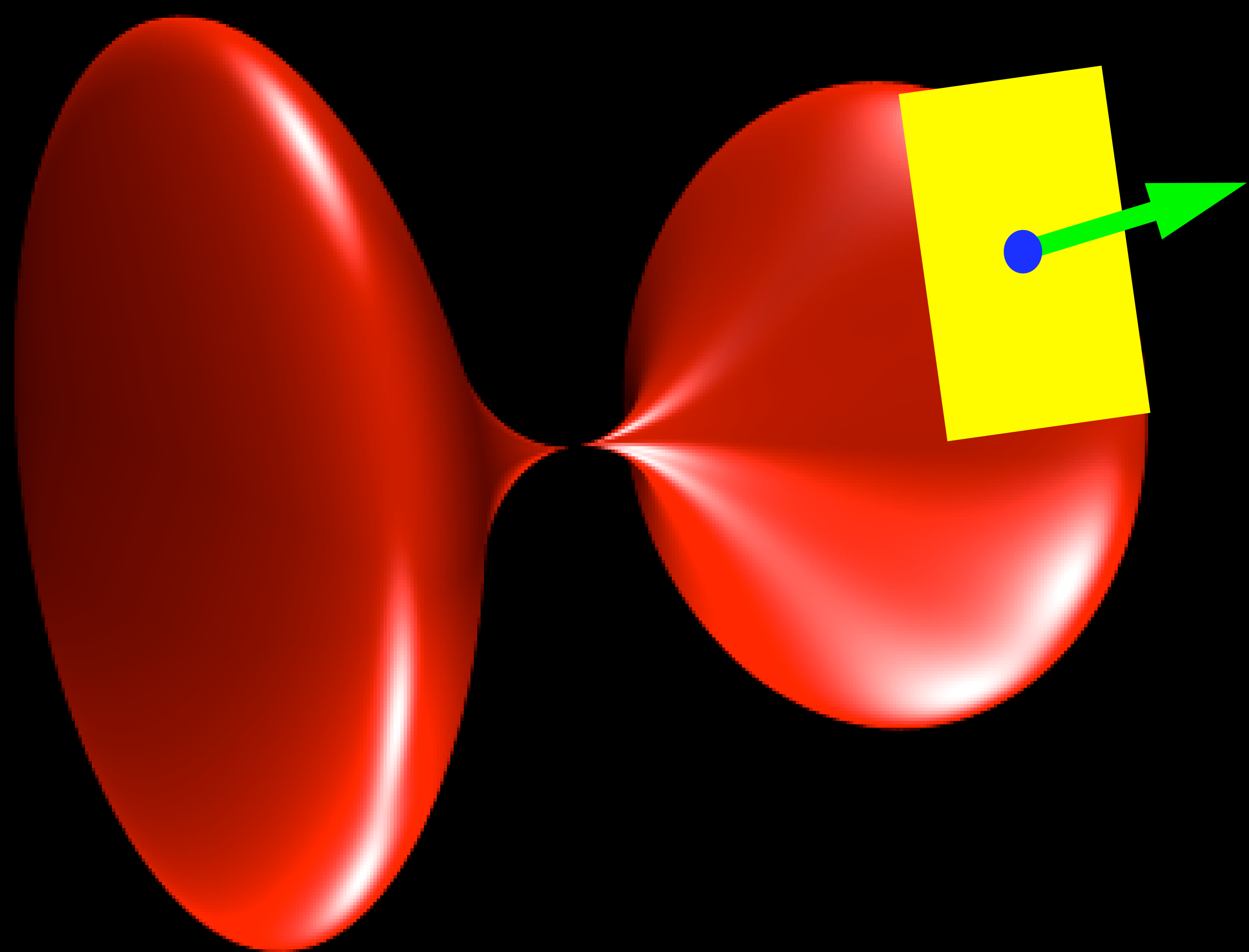
Example:

$$x^5 + y^5 - z - z^7 - 1 = 0$$
$$z = g(x, y)$$
$$g_x(x, y) = -f_x(x, y, z)/f_z(x, y, z) = -5x^4/(1 + 7z^6)$$
$$g_x(1, 0) = -5$$



Gradients and Tangents





Crucial: the normal vector (a,b,c) is the gradient. The equation of the plane is

ax+by+cz=d

where d is found at the end.

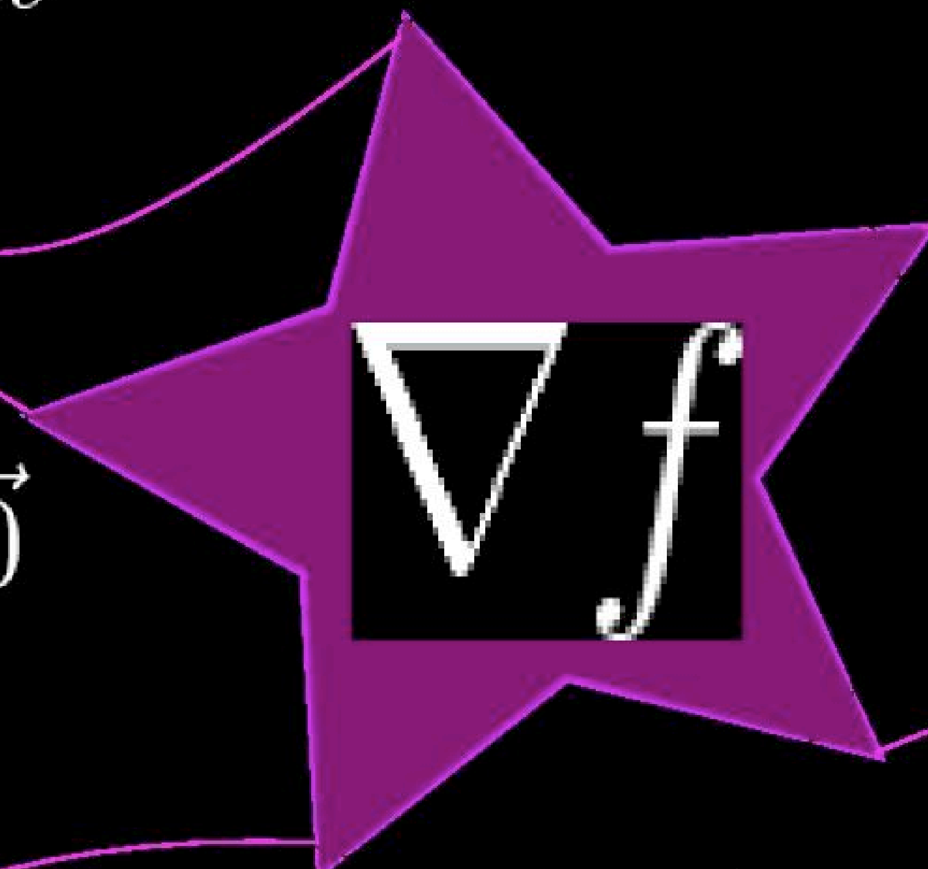
Linear Approximation

Estimation

Tangent plane

$$L(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{x}_0)(\vec{x} - \vec{x}_0)$$

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$



$$\nabla f = \vec{0}$$

$$\nabla f(\vec{x}_0) = (a, b, c)$$

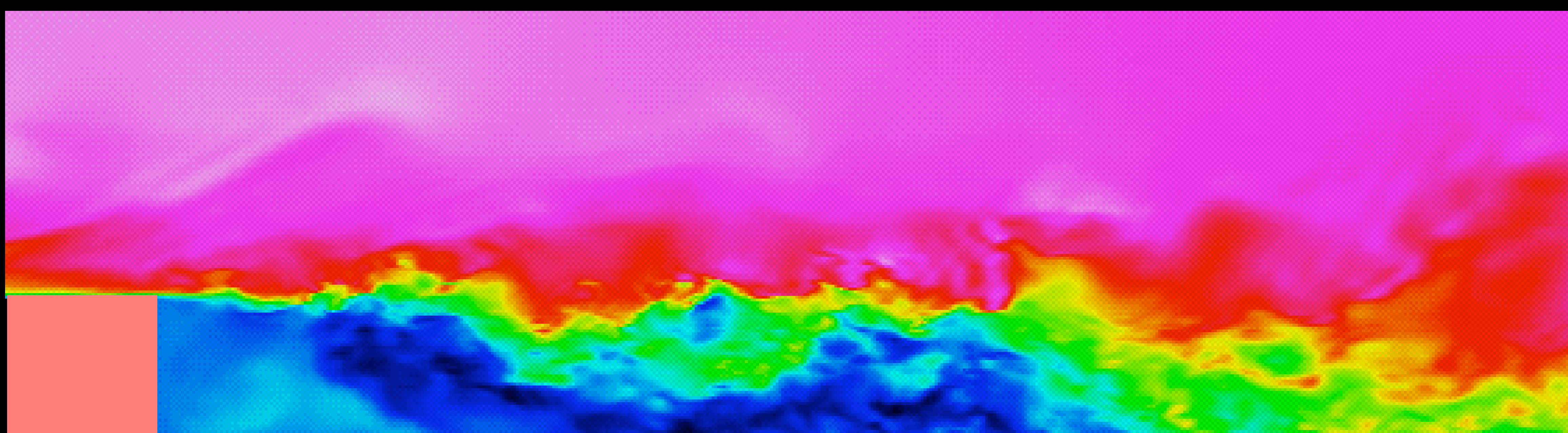
$ax + by + cz = d$ tangent plane

$f(x, y) \sim L(x, y)$ near (x_0, y_0)

PDE's

Equations in which partial derivatives appear: PDE's

$$\frac{d}{dt}u + u \cdot \nabla u = \nu \Delta u - \nabla p + f$$
$$\mathrm{div} u = 0$$



PDE's

PDE examples appearing in this course are Clairot's theorem:

$$f_{xy} - f_{yx} = 0.$$

or identities like curl (grad(f))=0
or div(curl(F))=0 which are
based on Clairot.

PDE's

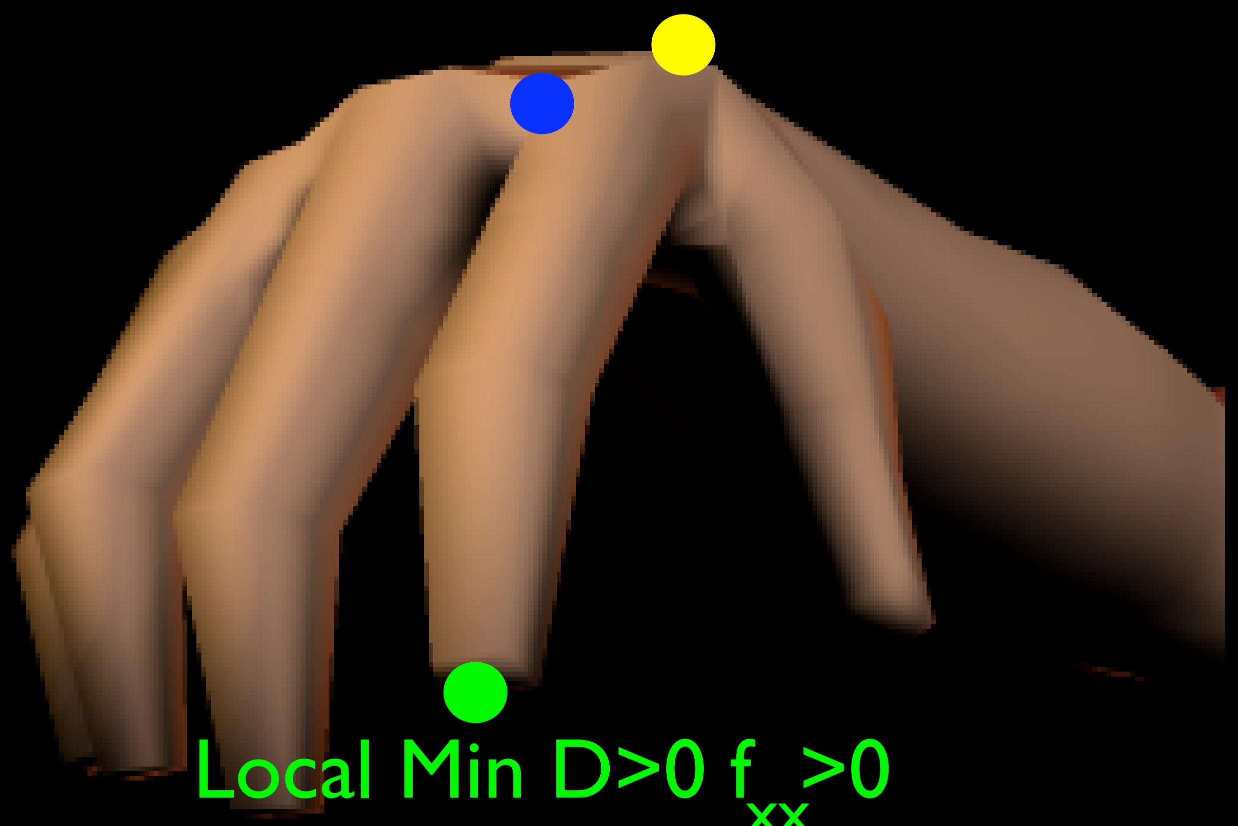
You should be able to verify that certain functions satisfy a PDE like

$u_{xx} + u_{yy} = c^2 u_{tt}$	wave
$\mu(u_{xx} + u_{yy}) = u_t$	heat
$u_t = c u_x$	transport

Thats all I have to say about that.

Extrema

Second Derivative Test



Problem 6: Find all critical points of

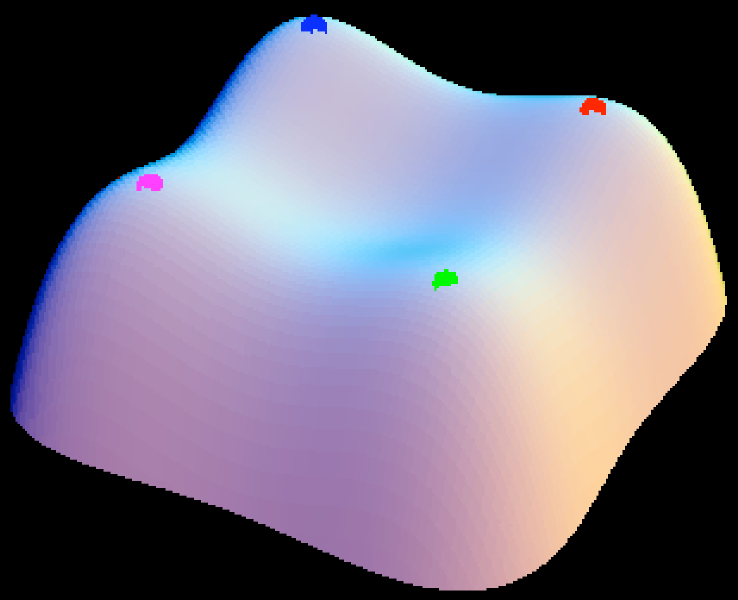
$$f(x, y) = 2x^2 + 2y^2 - x^4 - y^4$$

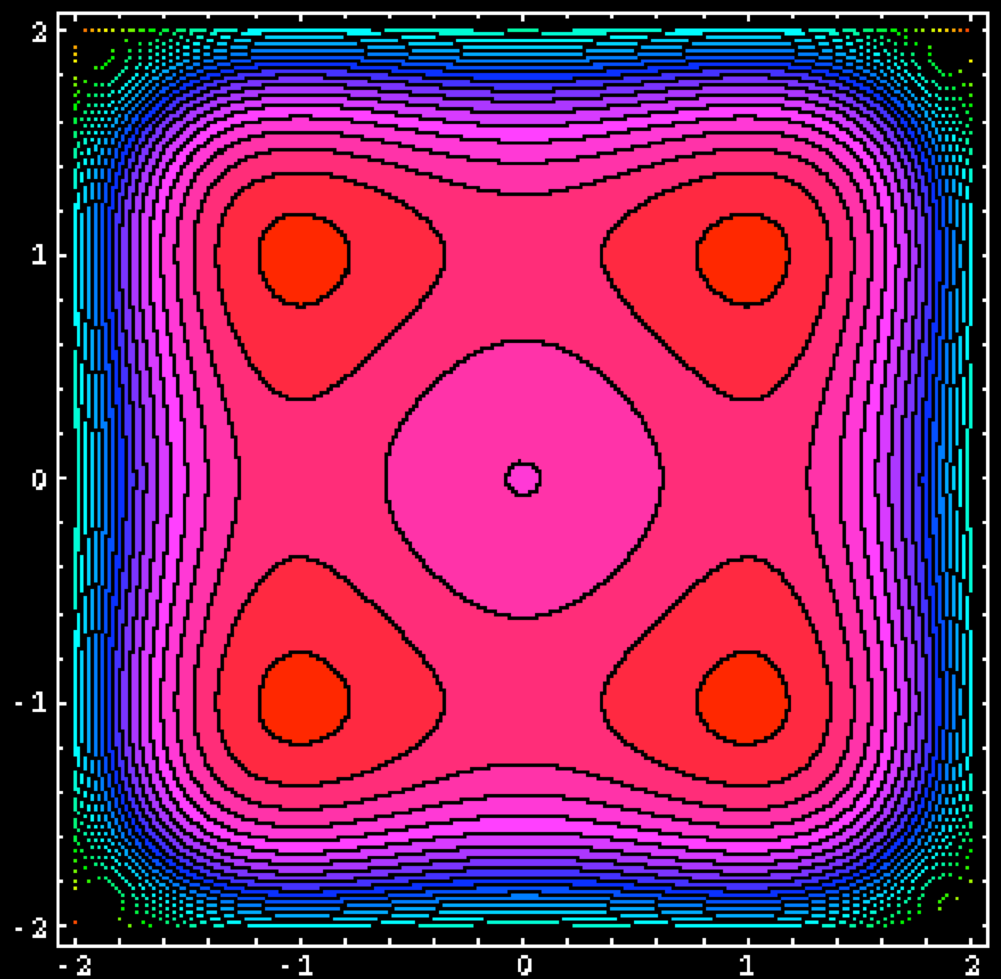
and classify them.

Problem 5: extremize

$$f(x,y) = 2x^2 + 2y^2 - x^4 - y^4$$

$$\nabla f(x,y) = (4x - 4x^3, 4y - 4y^3)$$





Quiz coming up

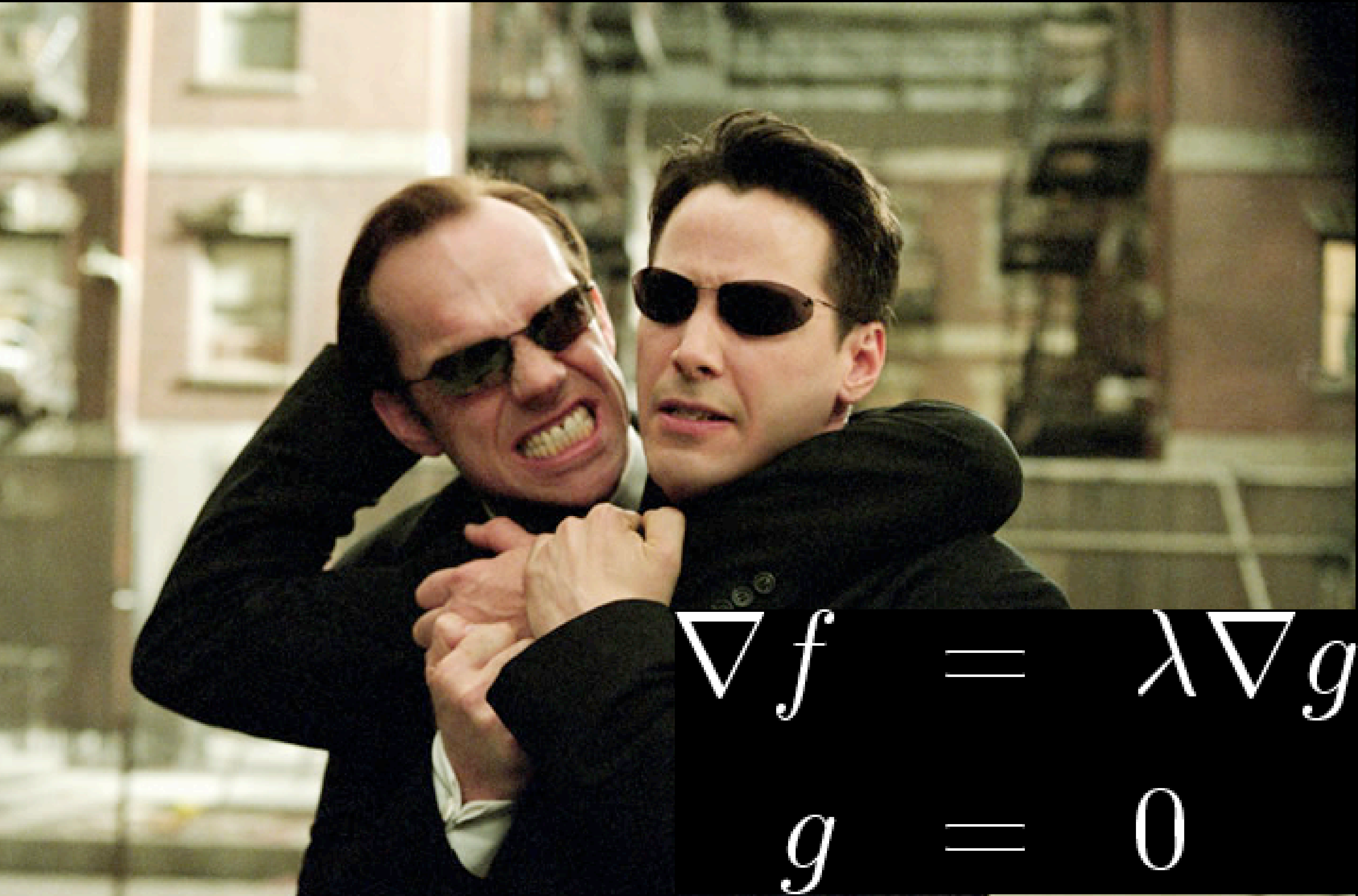
Win some Swiss Chocolate

An exam is like a box of chocolates. You never know what you are gonna get.

Sweet Problem:

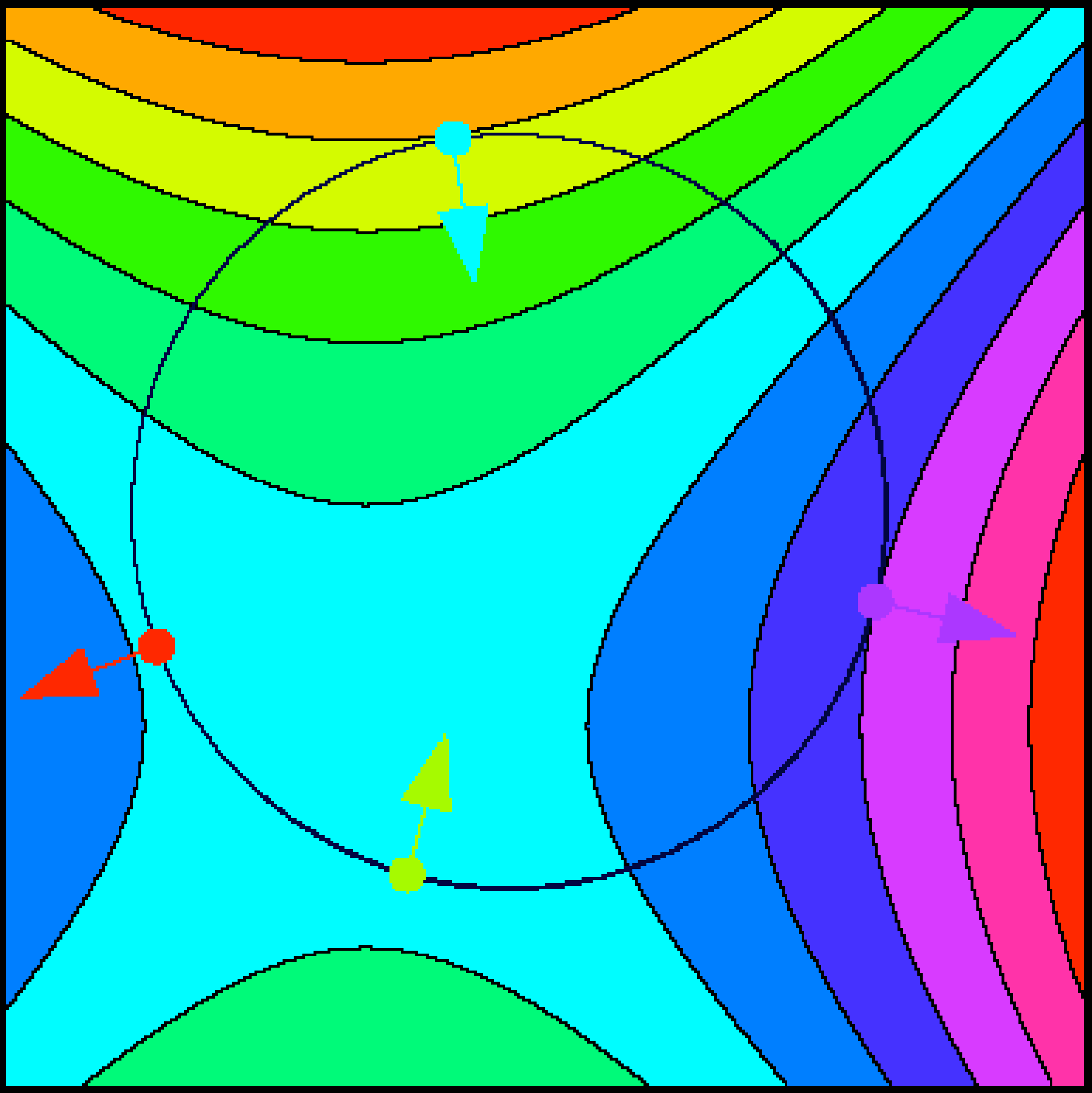
You know that the Laplacian of f (x,y)is zero everywhere. What can you say about the critical points of f?

Extrema with constraints



$$\nabla f = \lambda \nabla g$$
$$g = 0$$

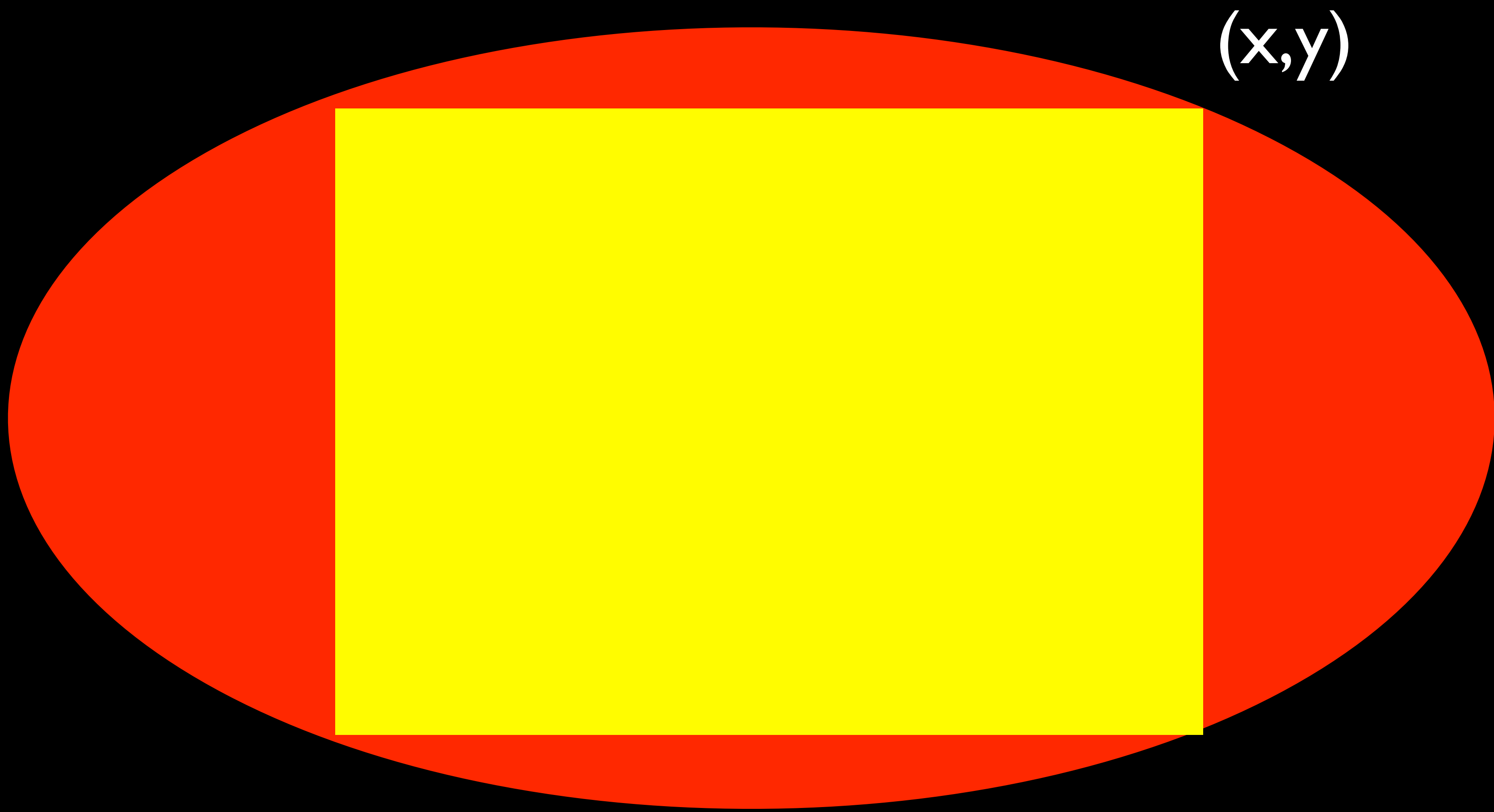
Lagrange



$$\nabla f = \lambda \nabla g$$
$$g = 0$$

Problem

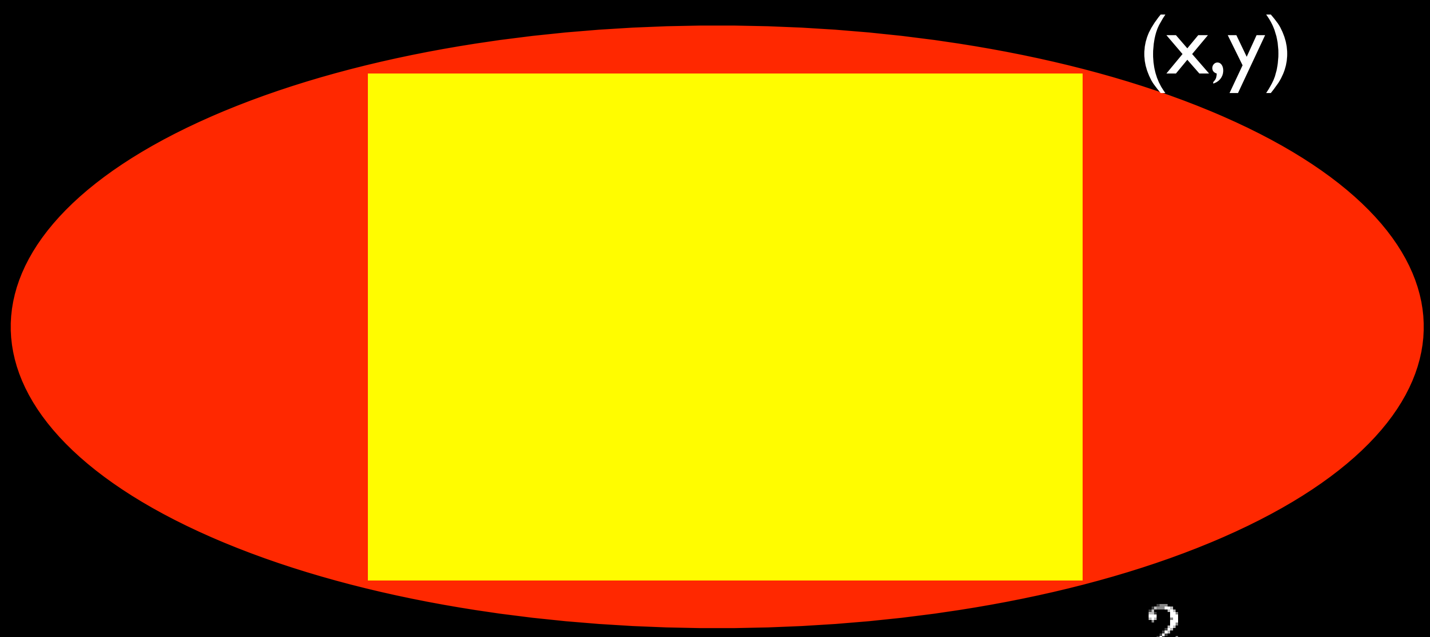
Find the largest rectangle inscribed in the ellipse E



Ellipse E: $$\frac{x^2}{4} + y^2 = 1$$

Lagrange Problem

$$f(x, y) = 4xy$$
$$g(x, y) = x^2/4 + y^2 - 1 = 0$$



Ellipse E: $\frac{x^2}{4} + y^2 = 1$

Solution

$$f(x, y) = 4xy$$
$$g(x, y) = x^2/4 + y^2 - 1 = 0$$

Lagrange equations

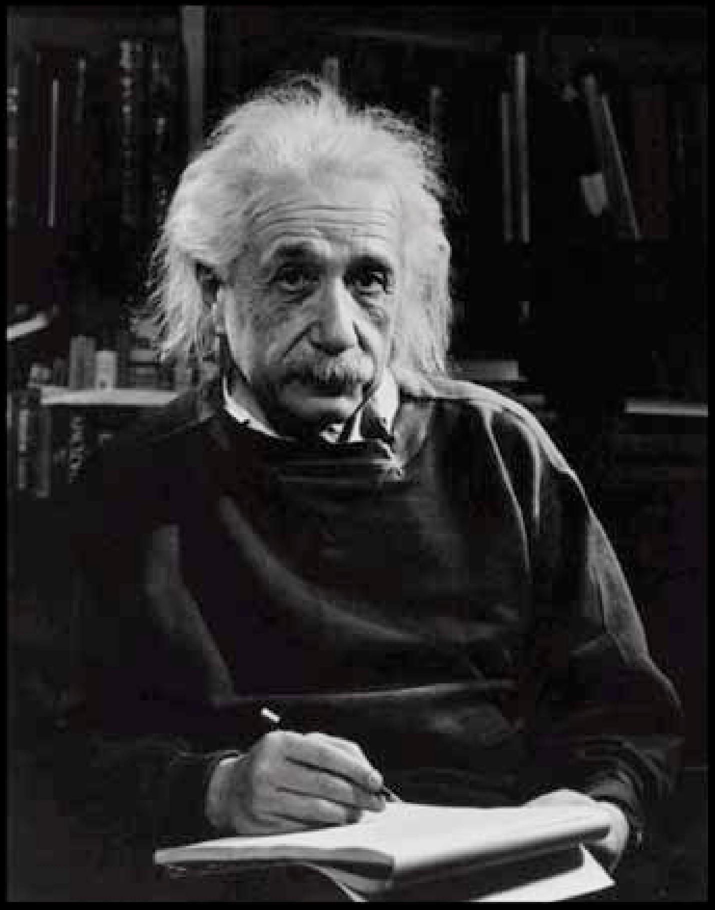
$$4y = \lambda x/2$$
$$4x = \lambda 2y$$
$$g(x, y) = 0$$

$$x = 2y, \frac{x^2}{4} + y^2 = 1$$
$$\Rightarrow y = 1/\sqrt{2}, x = \sqrt{2}$$

An other problem if time permits:

The earths surface is the unit sphere. An alian species shines a mental enhancement ray in our direction with intensity $f(x,y,z) = xy + z$. Where is the maximal intensity on earth (we become Einsteins) where is the minimal intensity (we become slugs)

Source: How to ace calculus, p. 165



Thats all I have to say about that.


Math21a, Review
Fall 2005, Part II
Tomorrow at the
same time.
Oliver Knill, 1/10-11/2006