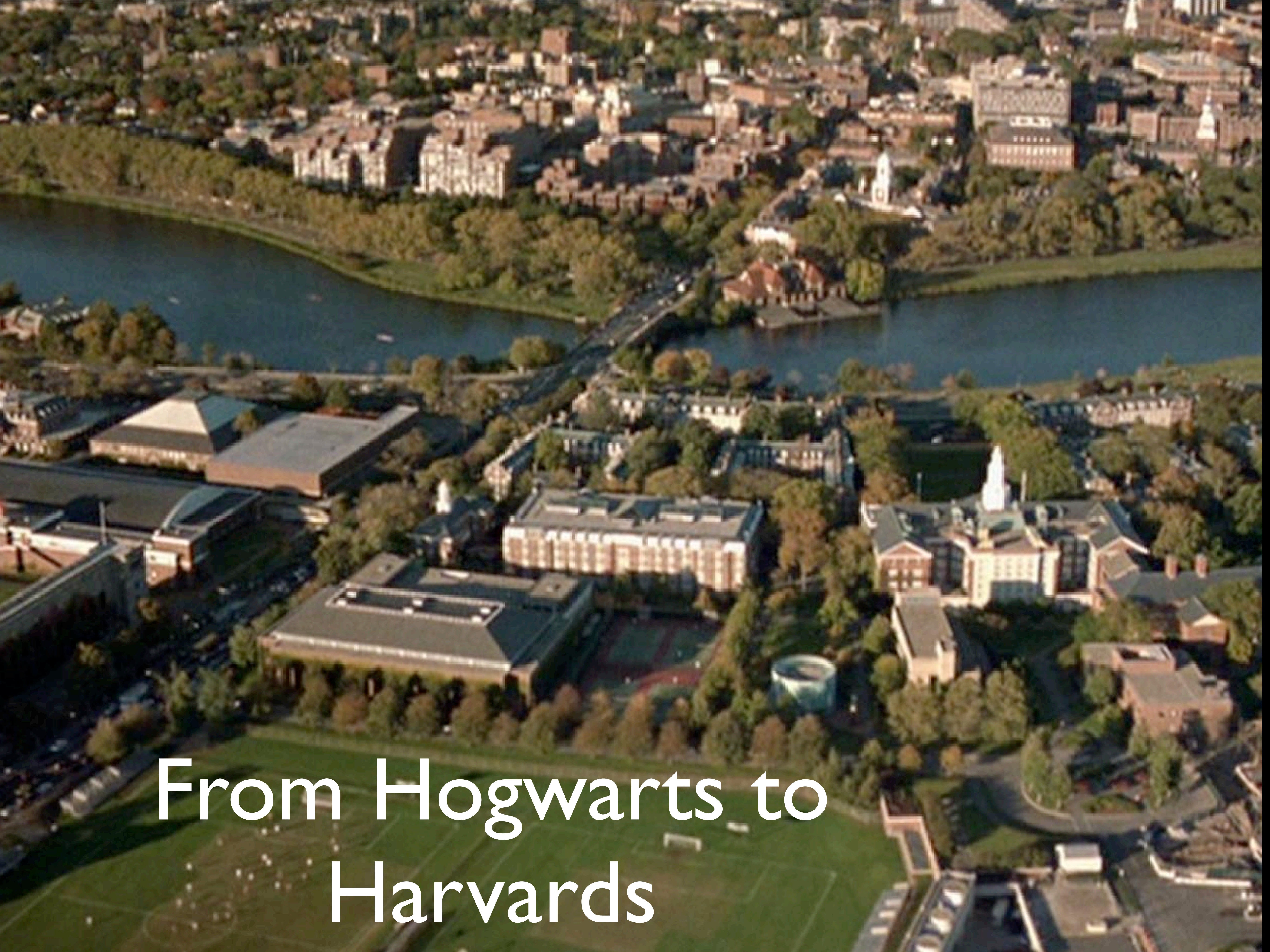
Math21a, Review Fall 2005, Part II

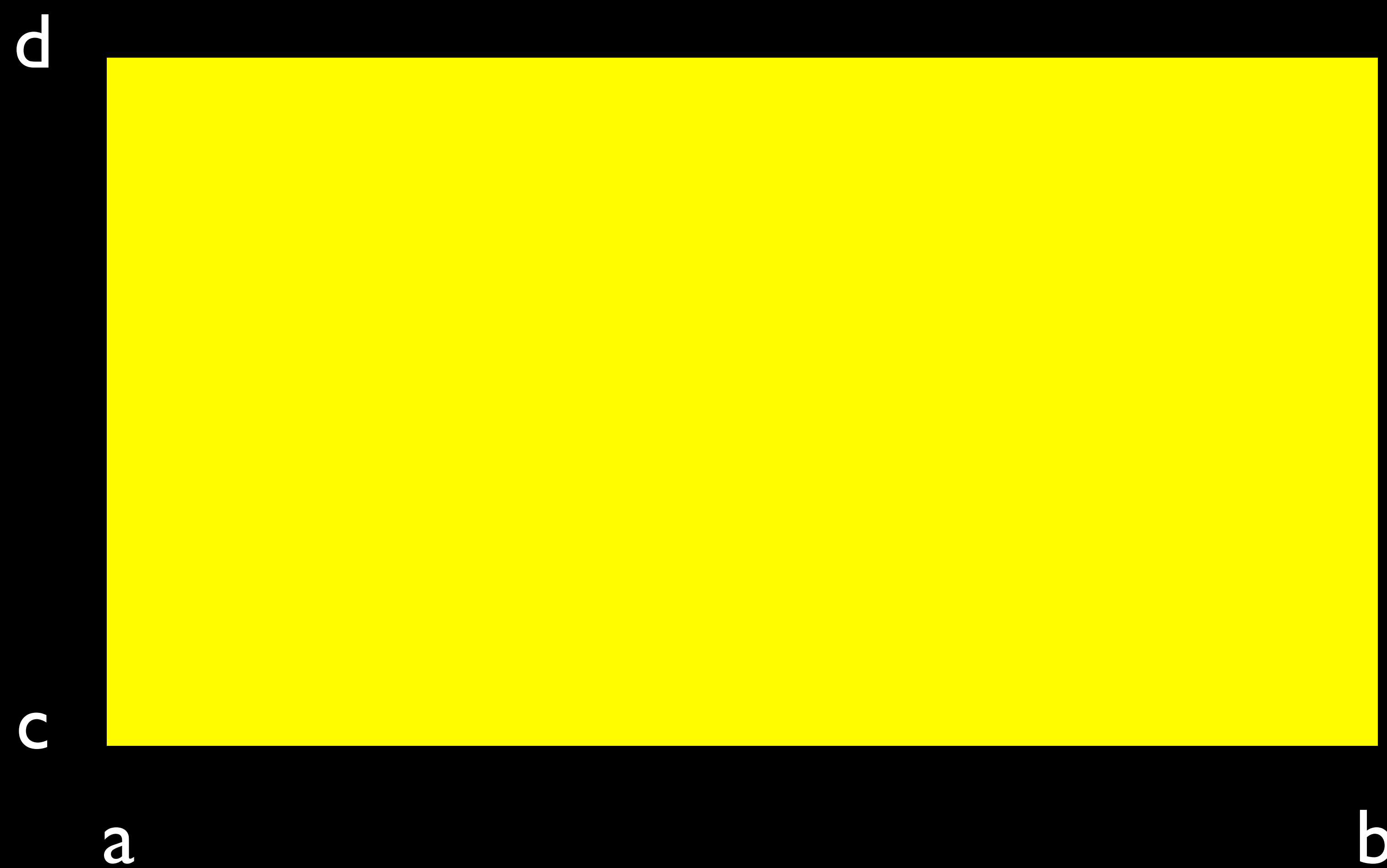
Oliver Knill, 1/10-11/2006


From Hogwarts to
Harvards


We cover first
integration

Start with Double Integrals

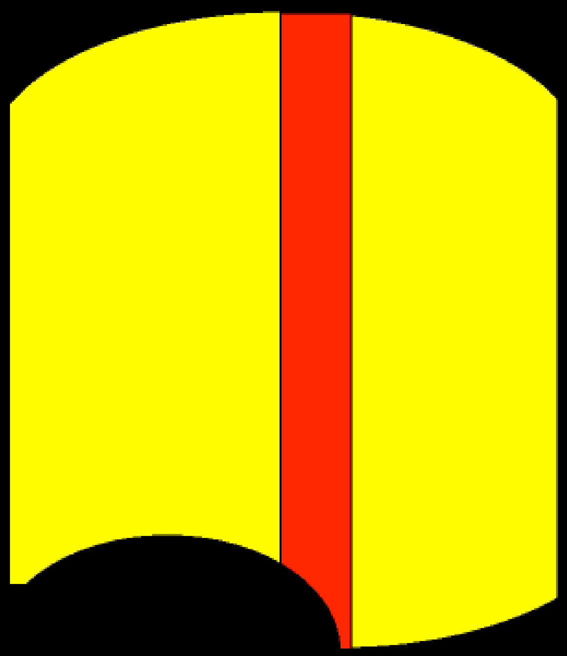
Fubini



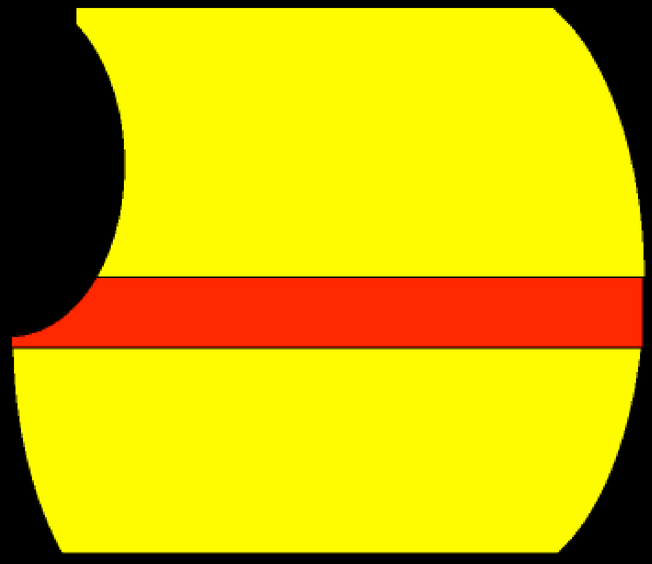
$$\int_a^b \int_c^d f(x,y)\; dydx = \int_c^d \int_a^b f(x,y)\; dxdy$$

Double Integrals

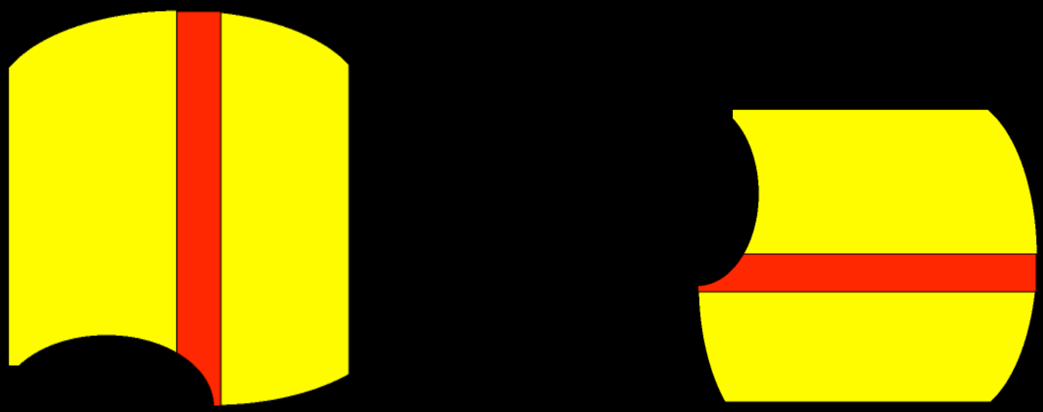
Type I



Type II

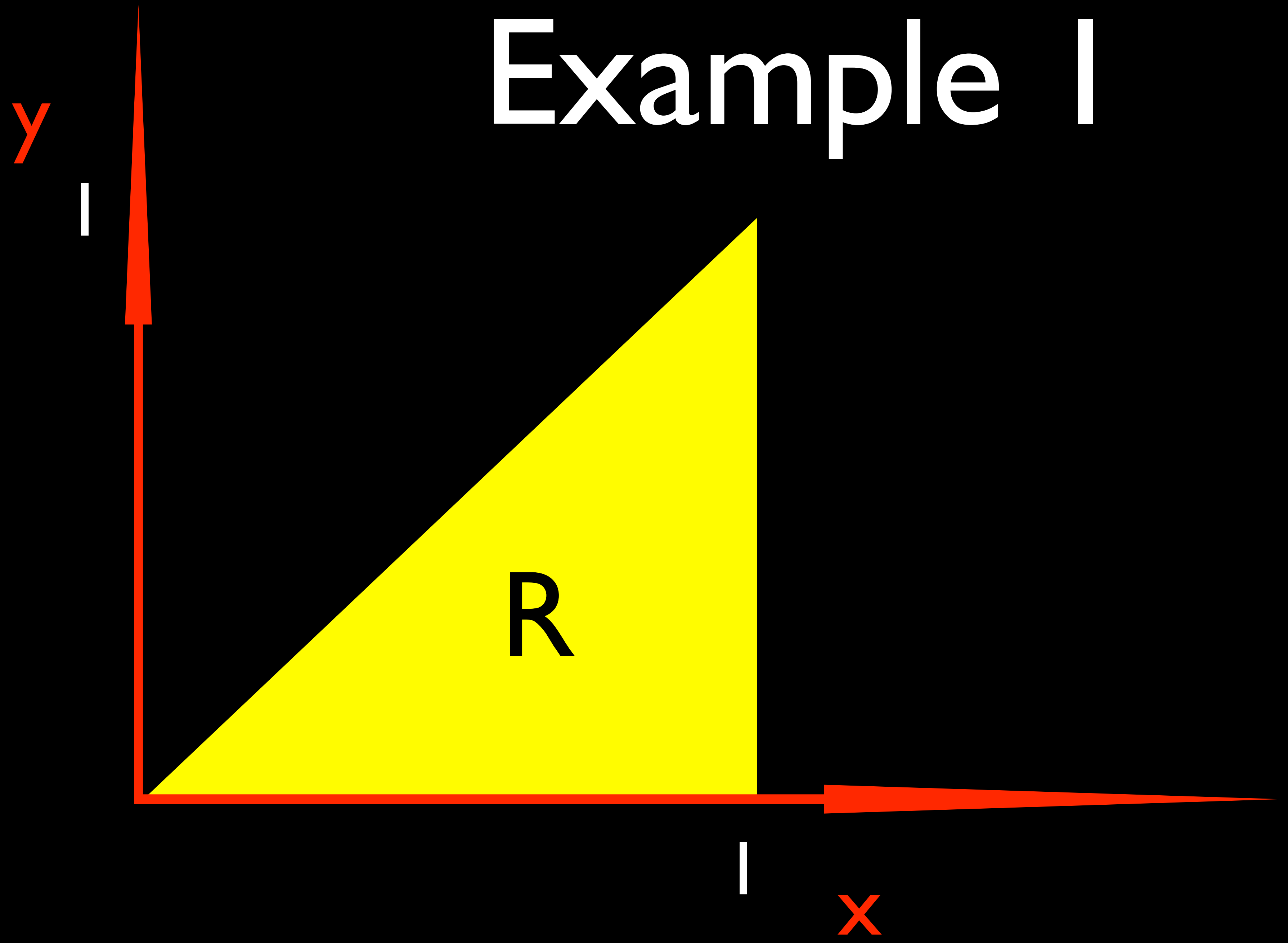


Change Type I to Type II



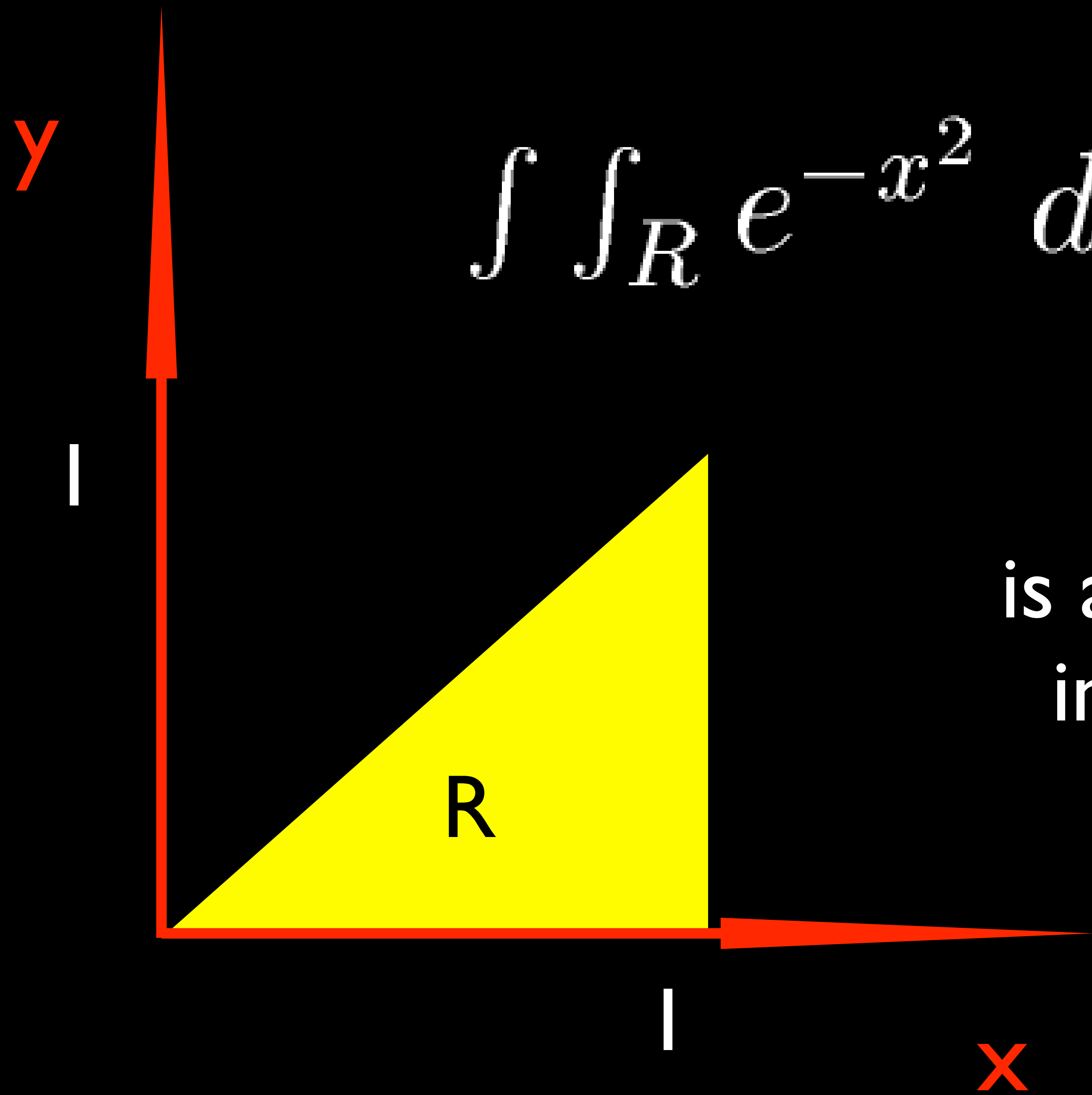
If you can not solve an integral as a type I integral, try to treat it as as type II integral

Example 1



Find: $\int\int_R e^{-x^2}\, dx\, dy$

$$\int\int_R e^{-x^2}\ dx\,dy$$

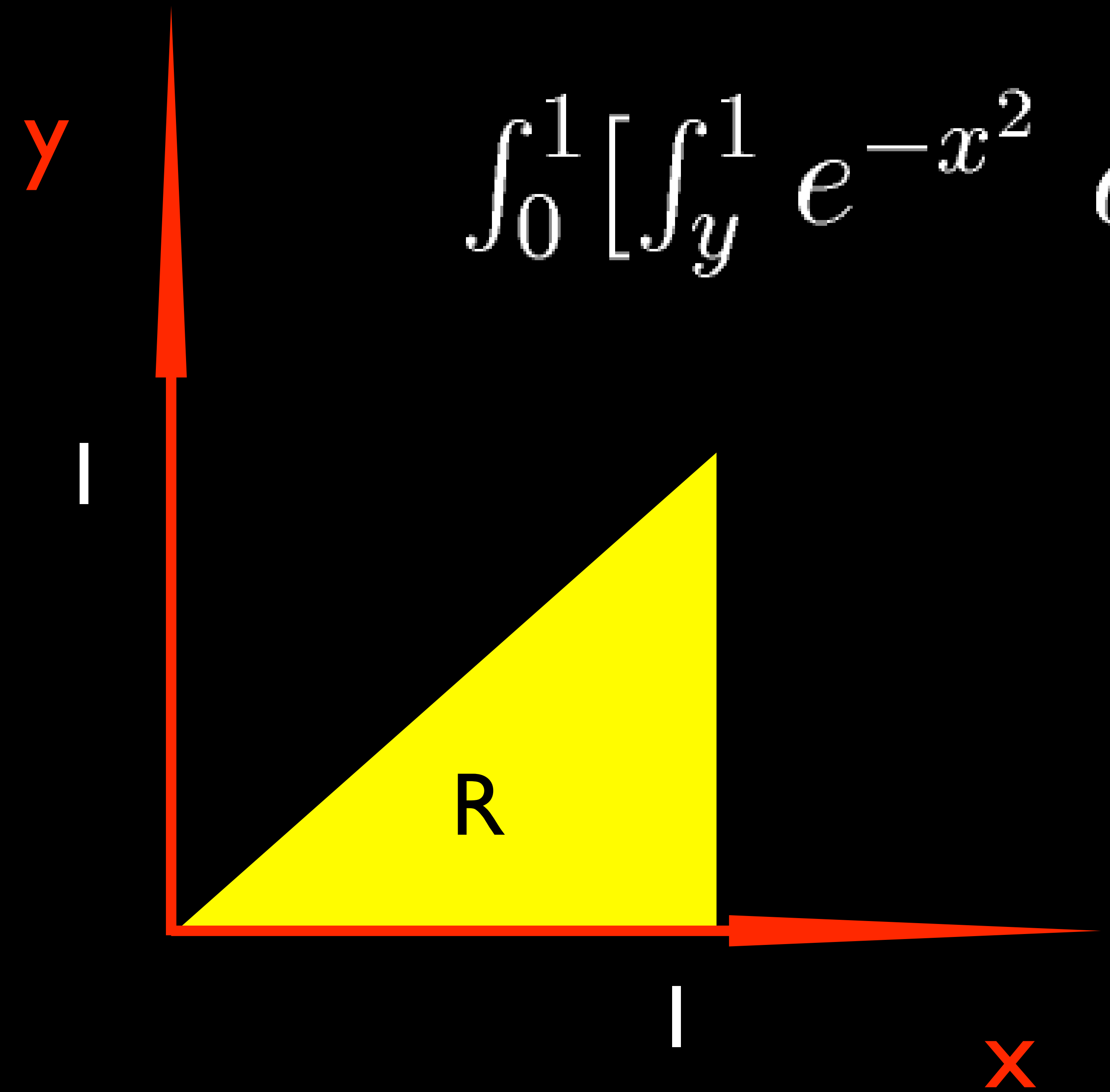


is a Type II integral

$$\int_0^1 \left[\int_y^1 e^{-x^2}\ dx\right] dy$$

Are stuck!

$$\int_0^1 \left[\int_y^1 e^{-x^2}\, dx\right] dy$$



Change order of integration:

$$\int_0^1 \left[\int_0^x e^{-x^2}\, dy\right] dx$$

$$\int_0^1 x e^{-x^2}\, dx = -\frac{e^{-x^2}}{2}\Big|_0^1 = \frac{(1-e^{-1})}{2}$$

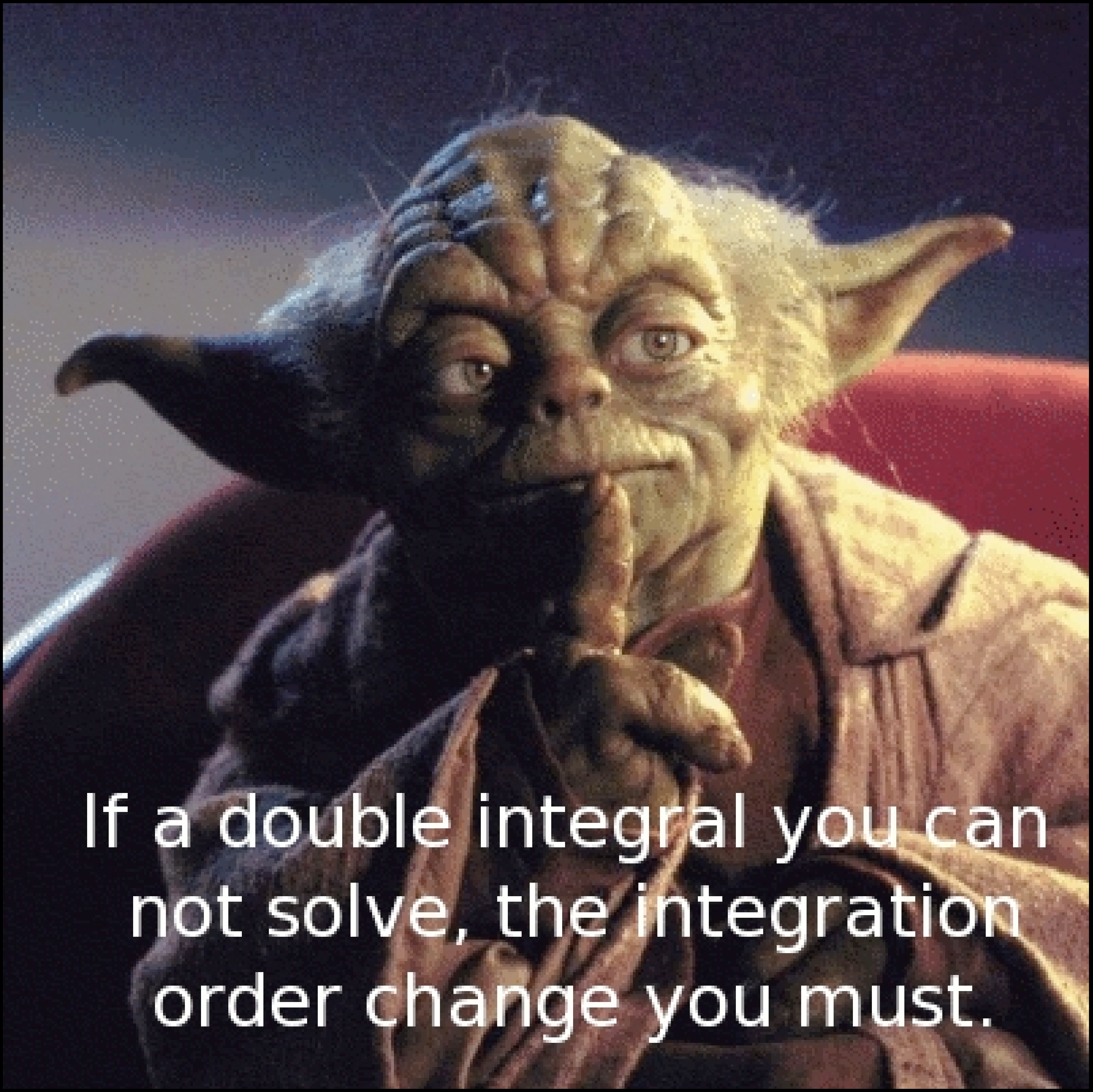
Example 2: What is

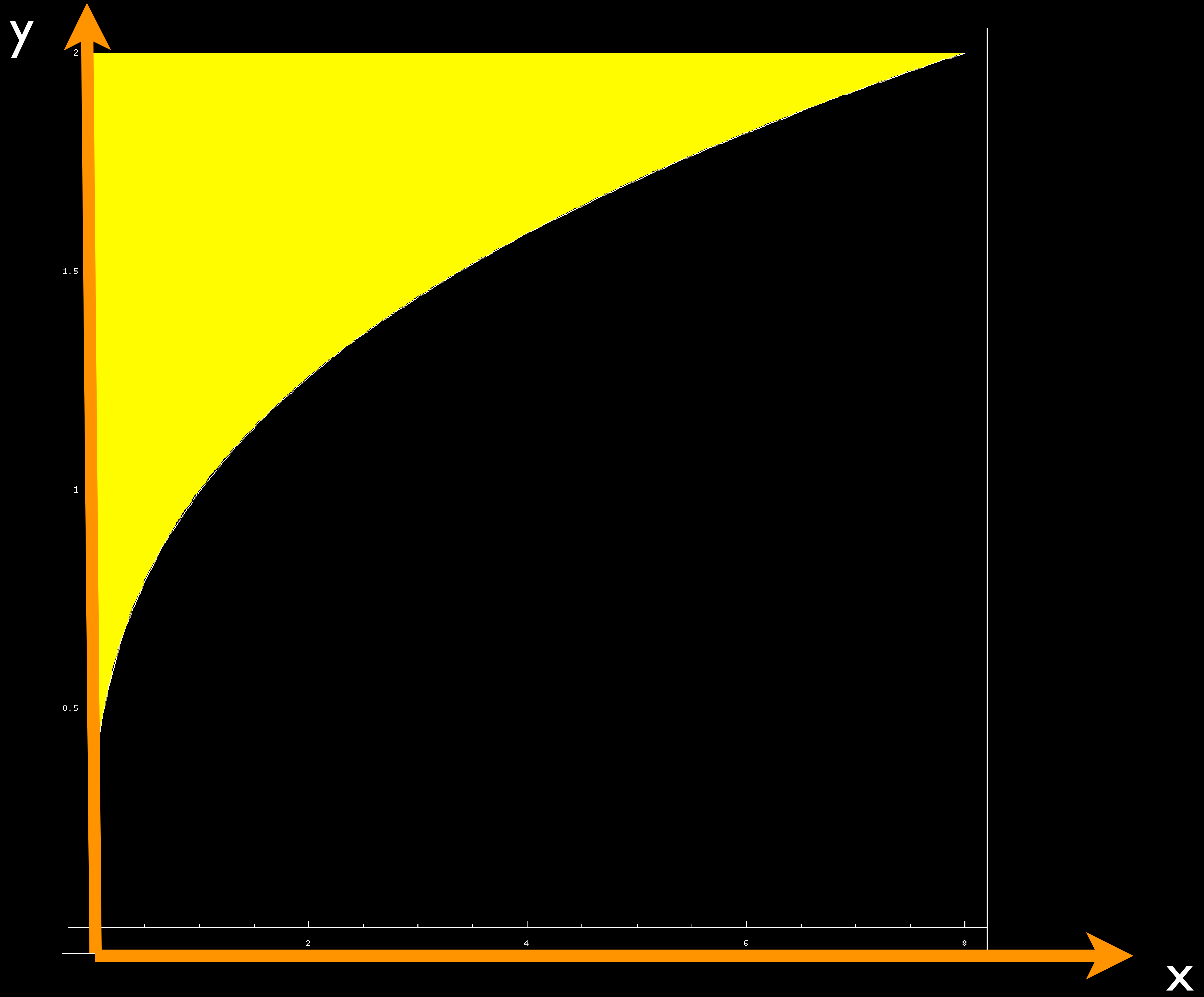
$$\int_0^8 \int_{x^{1/3}}^2 \frac{1}{y^4+1}\,dy\,dx$$

Look at the inner integral:

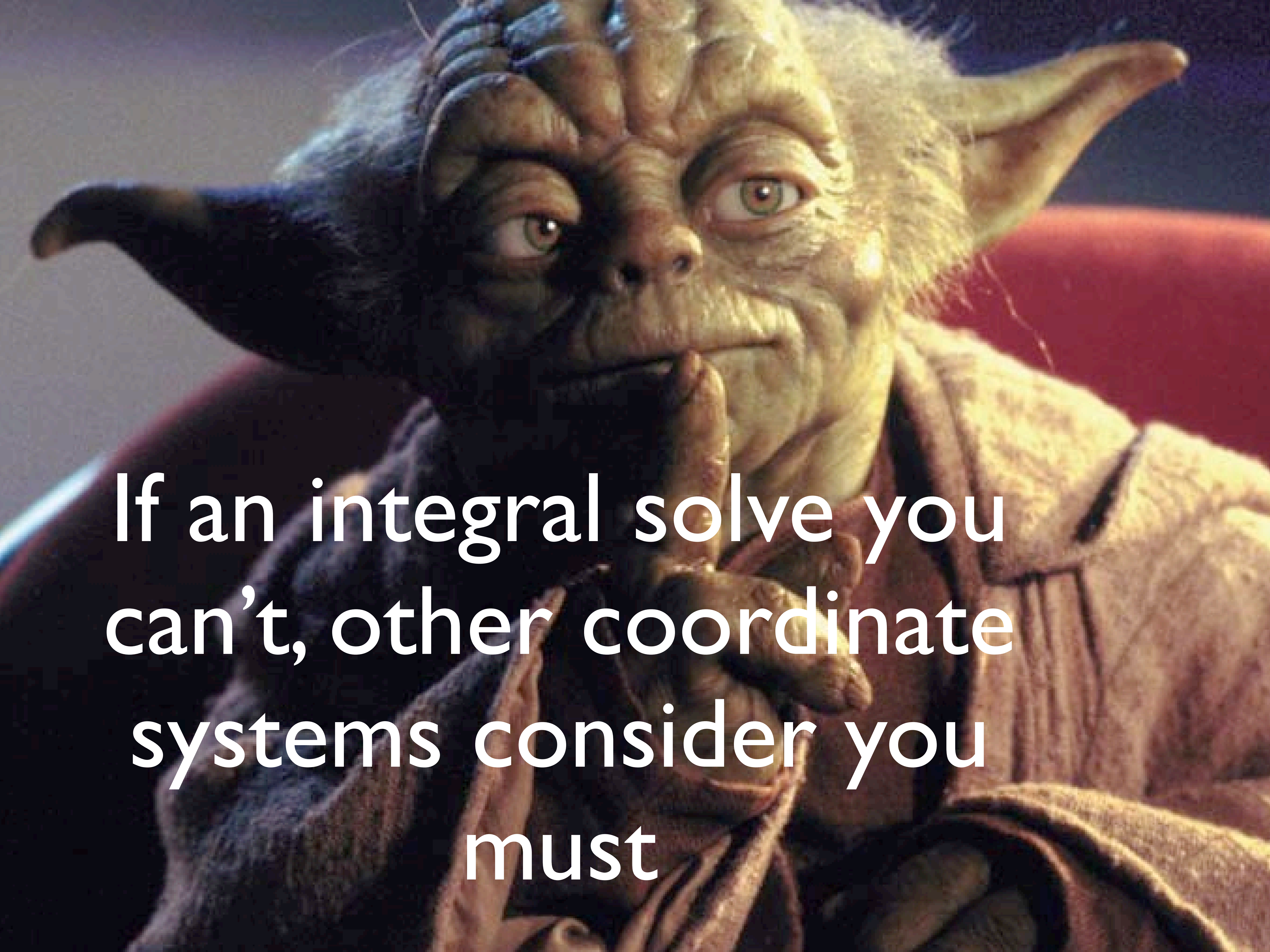
$$\int_0^8 \int_{x^{1/3}}^2 \frac{1}{y^4+1}\, dy\, dx$$

It does not look good.


If a double integral you can
not solve, the integration
order change you must.



y
2
1.5
1
0.5
2
4
6
8
x


If an integral solve you
can’t, other coordinate
systems consider you
must

Polar coordinates

$$\int_0^{2\pi} \int_0^{r(\theta)} r \, dr \, d\theta$$

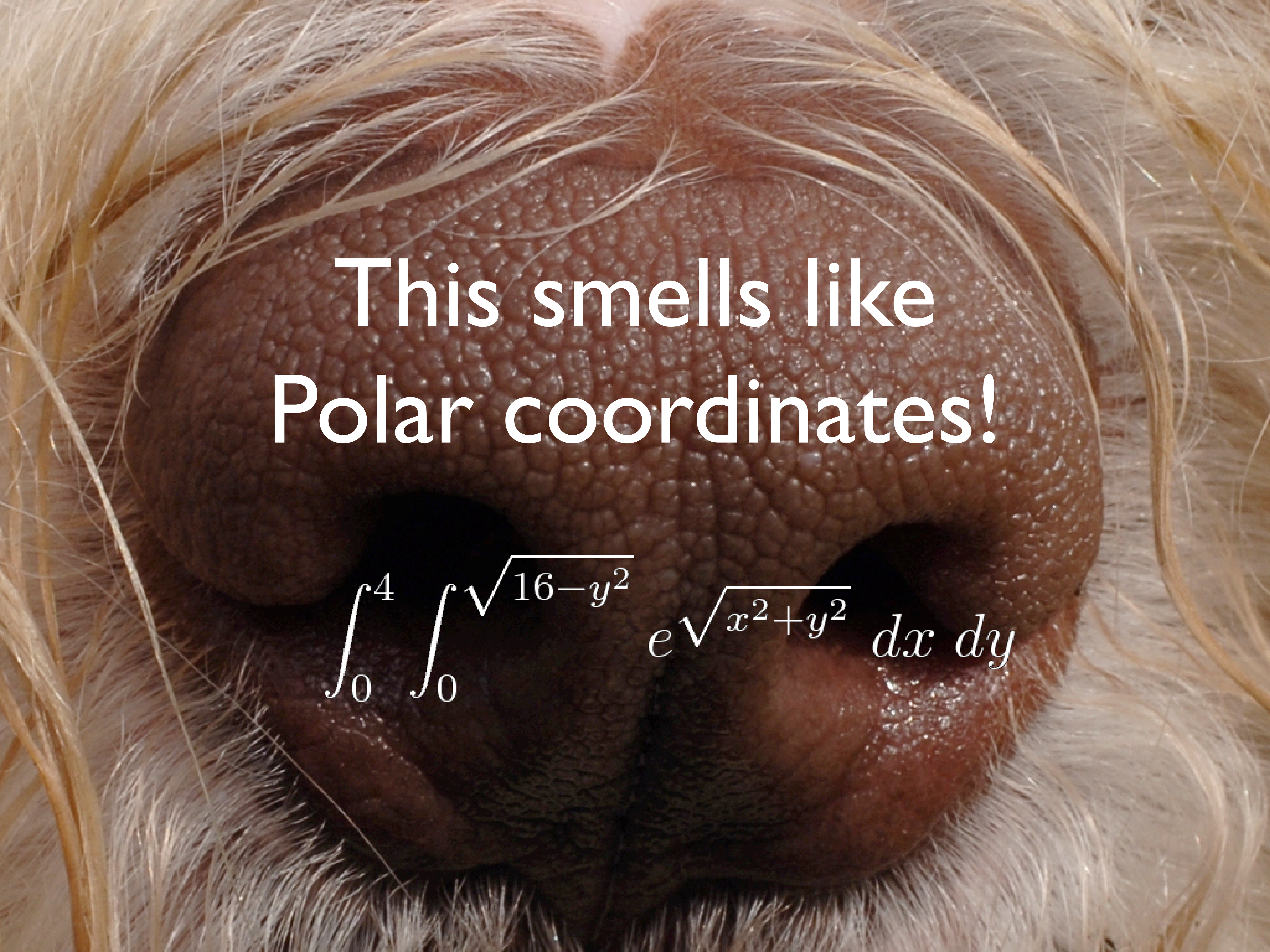
Example

$$r(\theta) = |\cos(n\theta)|$$

$$\int_0^{2\pi} \frac{\cos^2(n\theta)}{2} \, d\theta = \frac{\pi}{2}$$

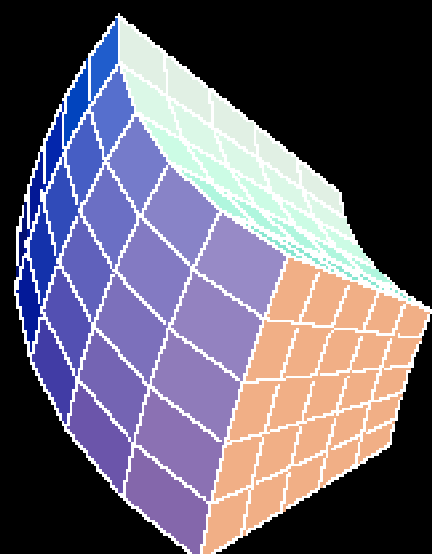
Problem 3: What is

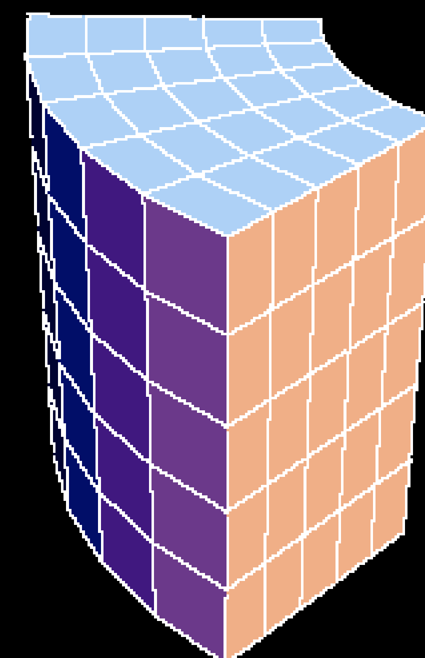
$$\int_0^4 \int_0^{\sqrt{16-y^2}} e^{\sqrt{x^2+y^2}} \, dx \, dy$$


This smells like
Polar coordinates!
$\int_0^4 \int_0^{\sqrt{16-y^2}} e^{\sqrt{x^2+y^2}}\, dx\, dy$

Triple Integrals

Spherical/Cylindrical Coordinates

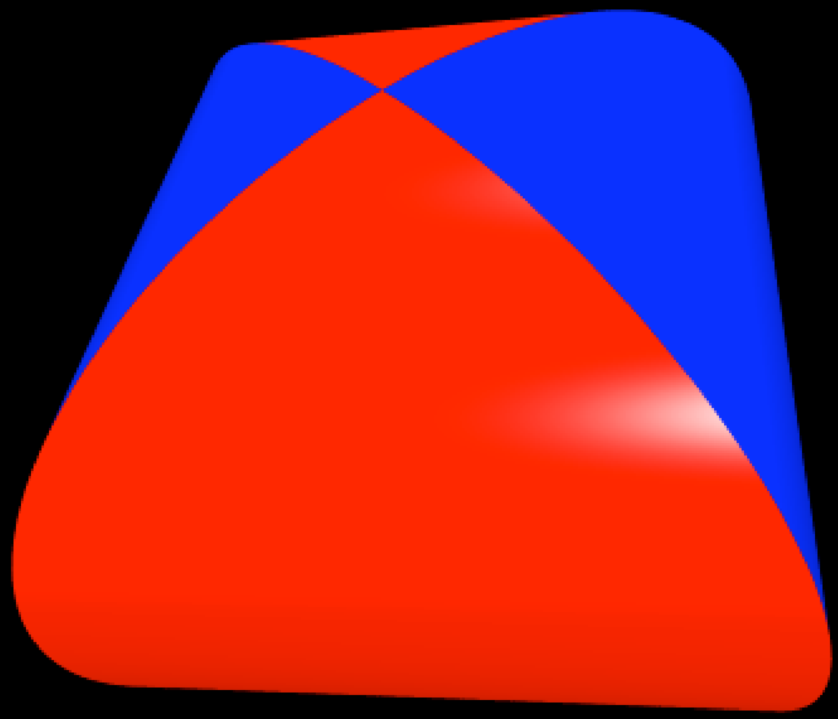


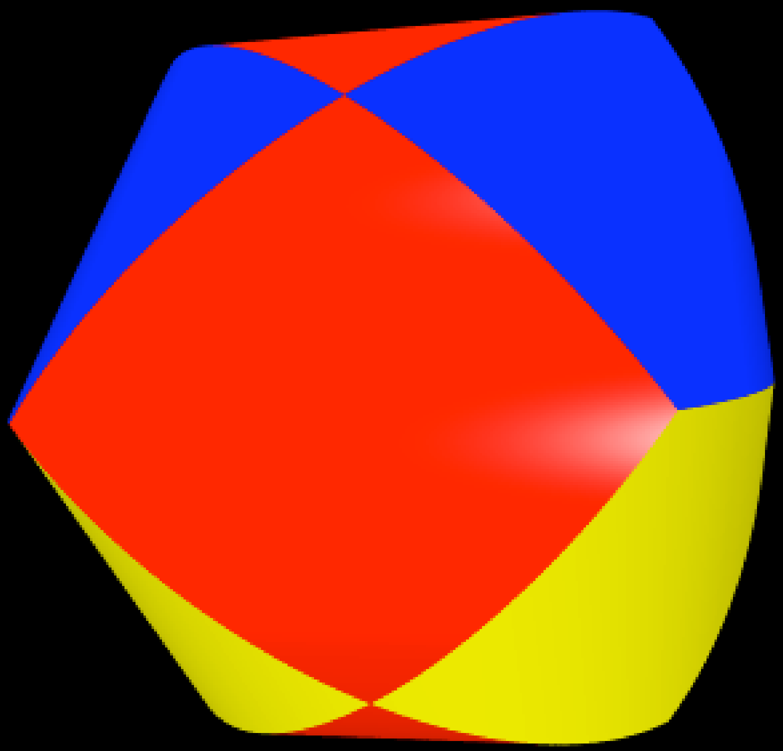


Remember the integration factors? You should know them by heart.

Triple Integrals

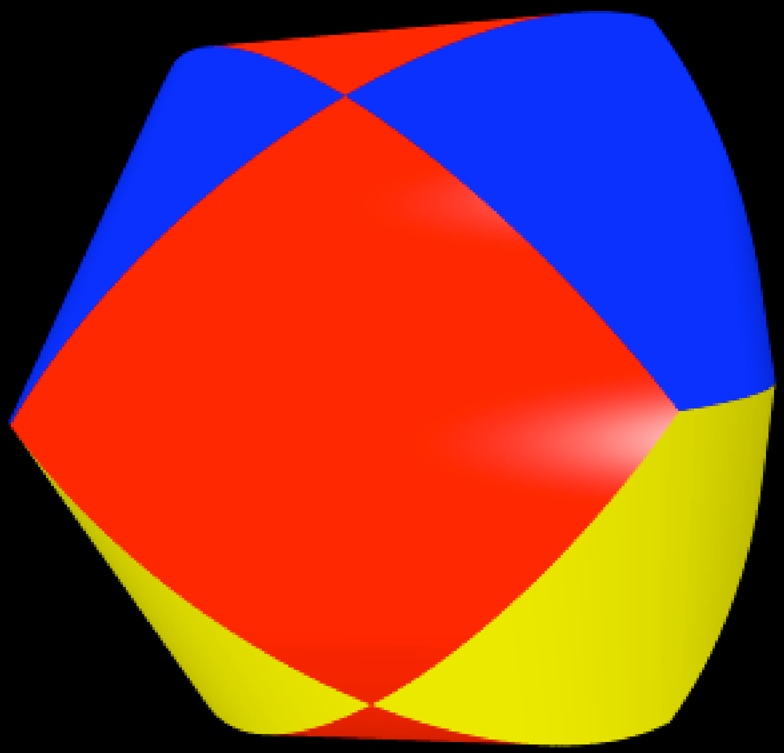
Example: Volume computation

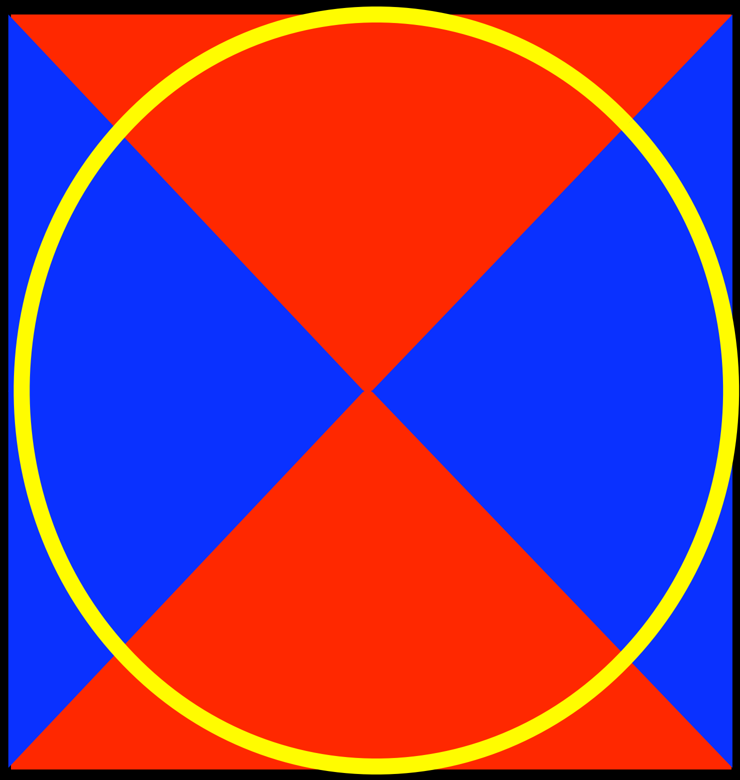




$$\int\int\int_E 1\ dV = \mathrm{Vol(E)}$$

Make good pictures!





Example: Volume of the intersection of the three cylinders:

$$8\int_{-\pi/4}^{\pi/4}\int_0^1 \sqrt{1-r^2\sin^2(t)}r\ drdt = -\frac{16}{3}+8\sqrt{2}$$

Always draw a figure!

The key for setting up and solving a 3D integral
is a good figure and the right coordinate system.

What if I can't?

Order of integration,

Coordinates

Good picture


Lighten up, dude, its just an exam!

Problem:

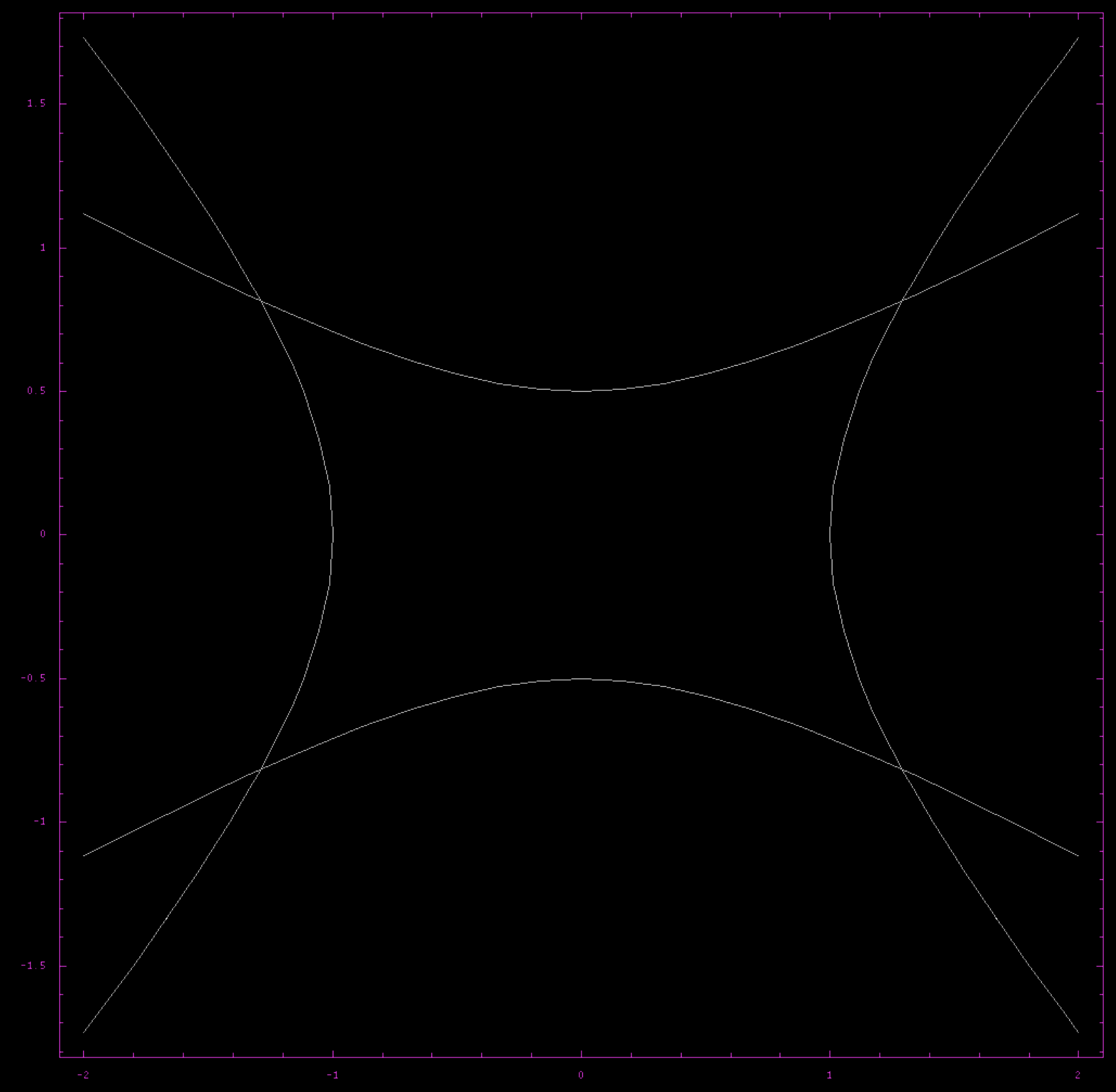
Find the volume of the solid bound by the one sheeted hyperboloid

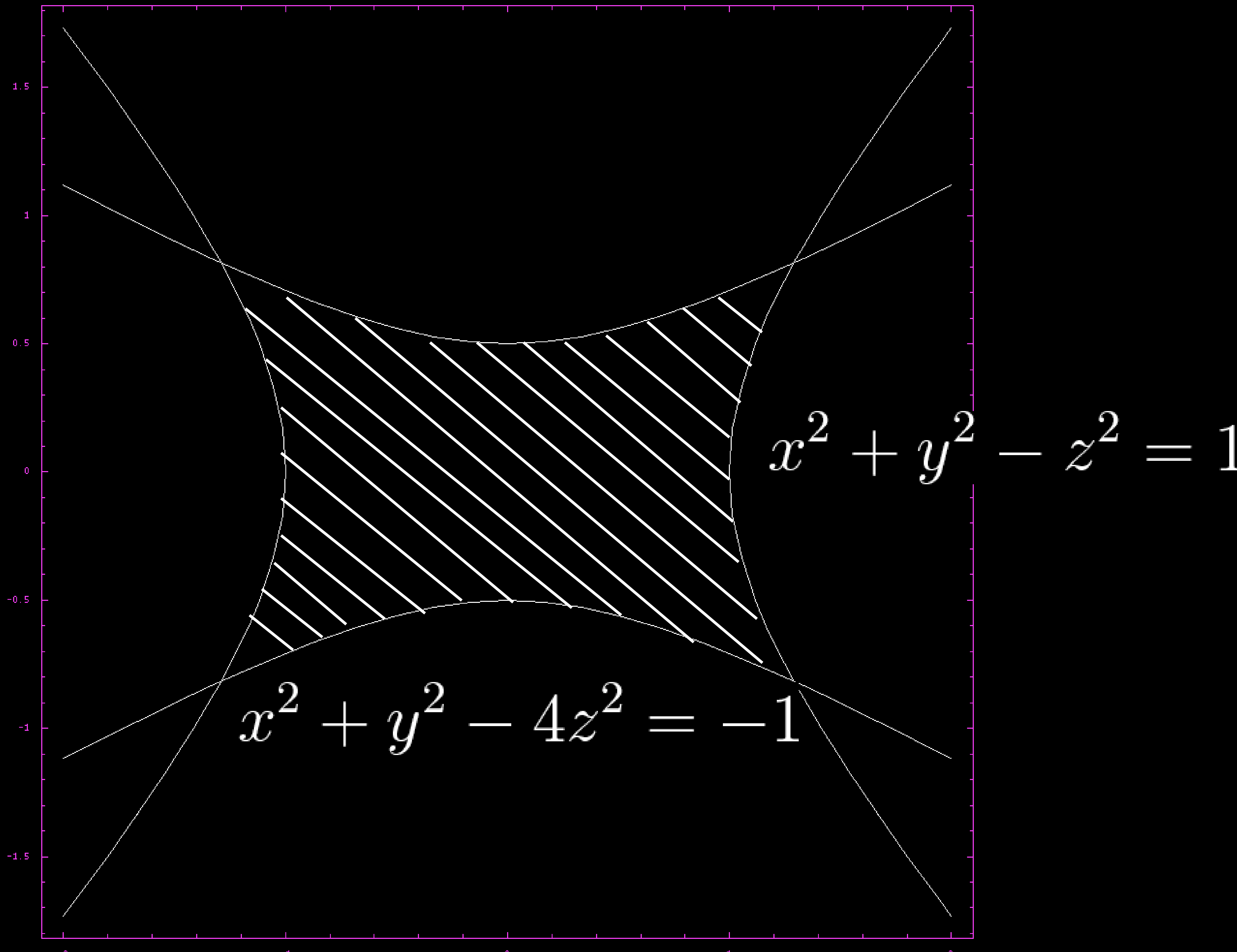
$$x^2 + y^2 - z^2 = 1$$

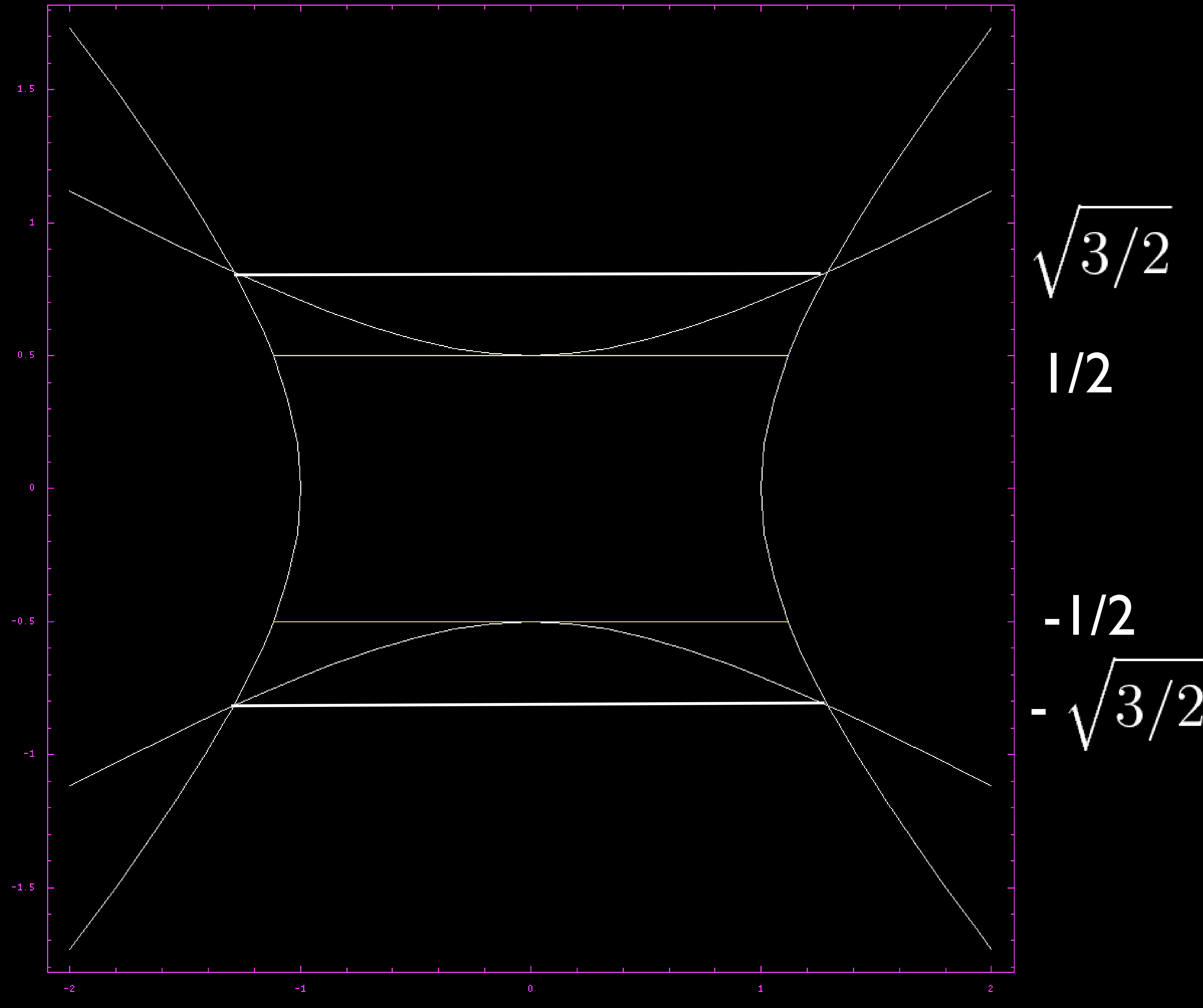
and the two sheeted hyperboloid

$$x^2 + y^2 - 4z^2 = -1$$

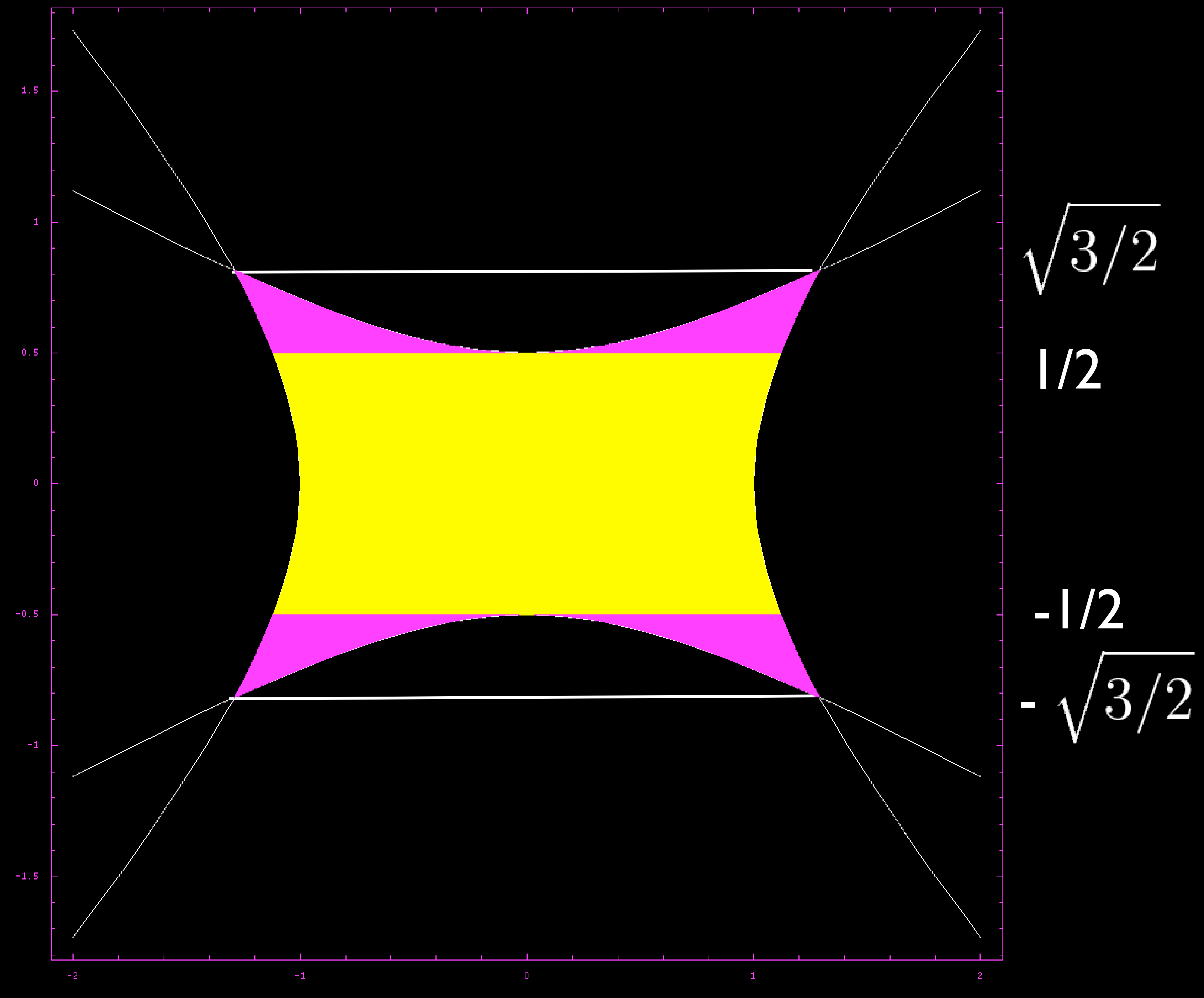
It is crucial to have a good picture




$x^2 + y^2 - z^2 = 1$
$x^2 + y^2 - 4z^2 = -1$
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1
0
1
2



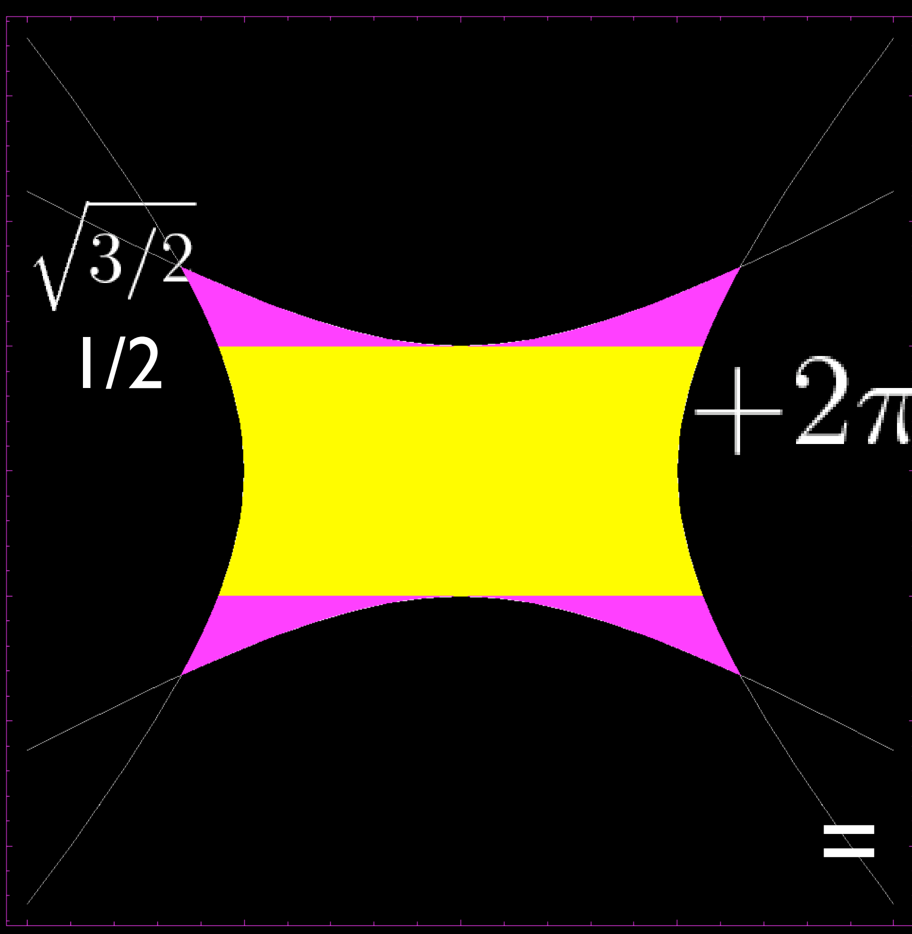
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1
0
1
2
$\sqrt{3/2}$
1/2
-1/2
- $\sqrt{3/2}$



1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1
0
1
2
$\sqrt{3/2}$
1/2
-1/2
$-\sqrt{3/2}$

$$2\int_0^{1/2}\int_0^{\sqrt{1+z^2}}\int_0^{2\pi} r\,d\theta\,dr\,dz$$

$$+2\int_{1/2}^{\sqrt{2/3}}\int_{\sqrt{-1+4z^2}}^{\sqrt{1-z^2}}\int_0^{2\pi} r\,d\theta\,dr\,dz$$



=

$$2\int_0^{1/2}\pi(1+z^2)\,dz$$

$$+2\pi\int_{1/2}^{\sqrt{2/3}}(1+z^2)-(-1+4z^2)\,dz$$

=

Quiz coming up

Win some Swiss Chocolate

Problem: take the standard double cone

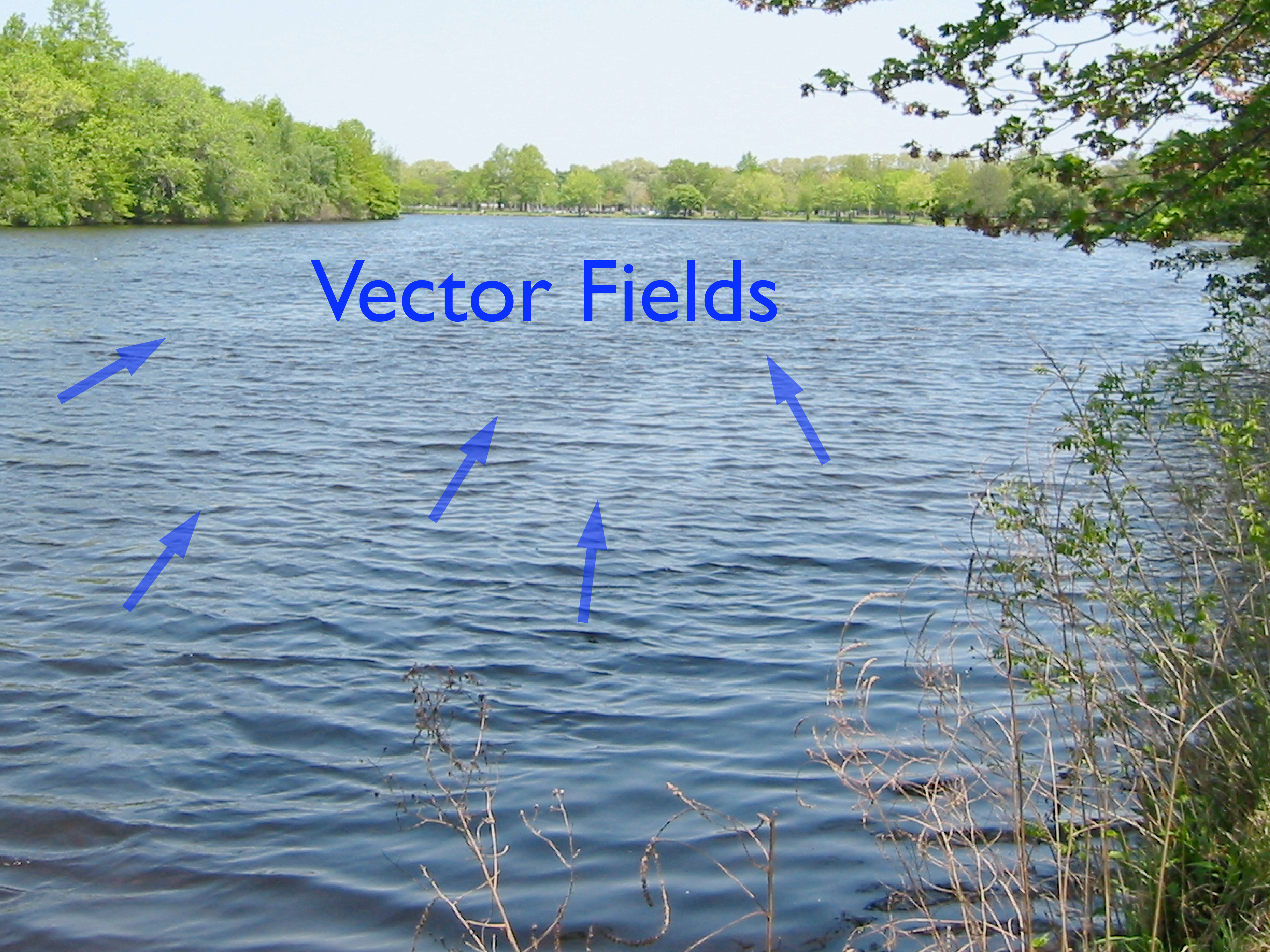
$x^2 + y^2 - z^2 = 1$

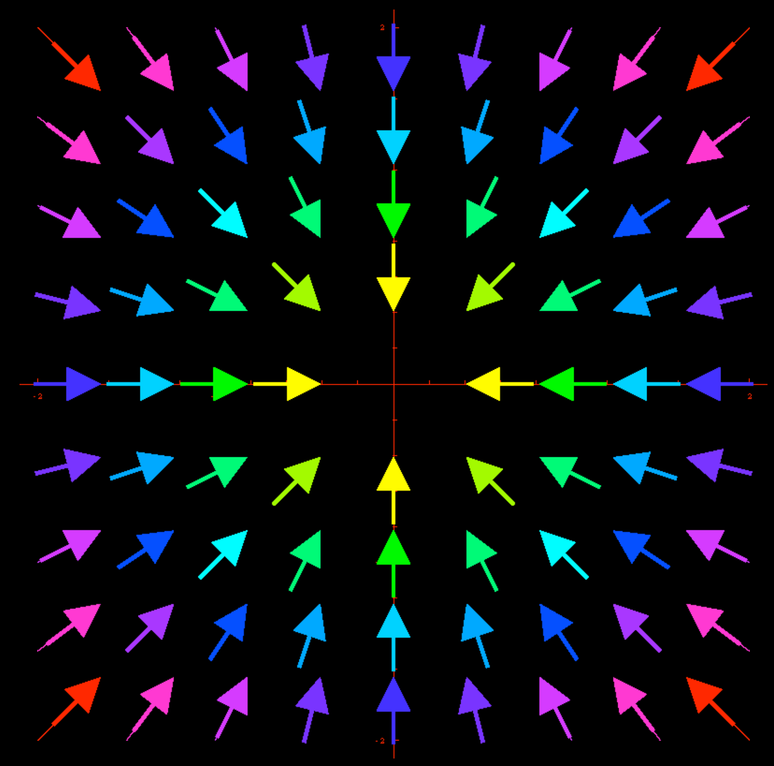
Move the upper cone by 1/2 down and the lower cone by 1/2 up. Find the volume of the intersection these two half cones.

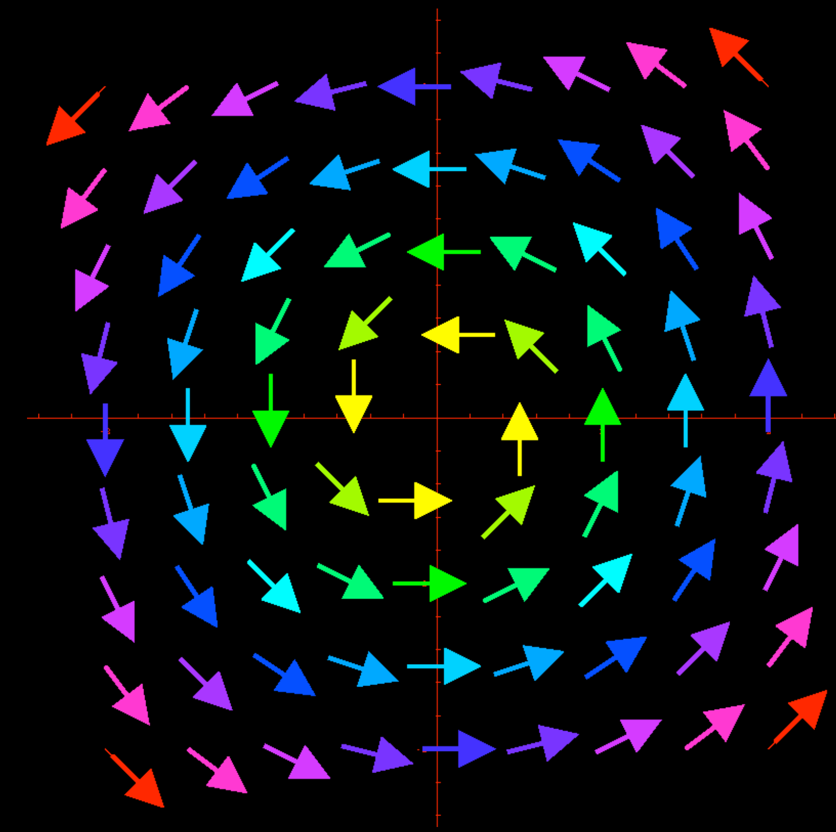
Three bits of wisdom:

- Make a good picture
- Consider other coordinate systems
- Switch order of integration if necessary

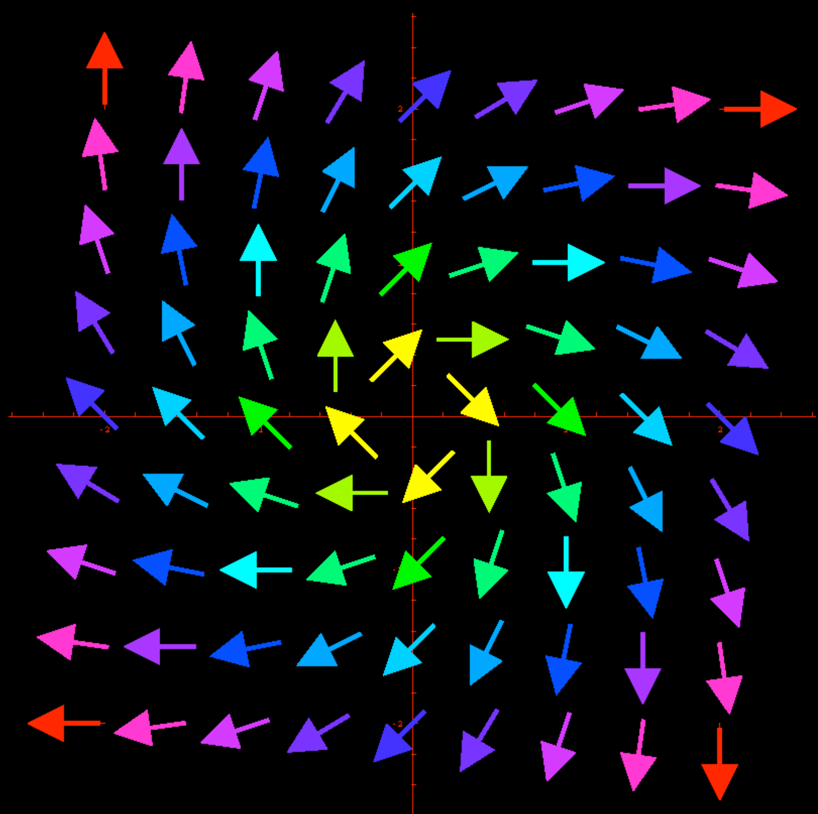
Vector Fields

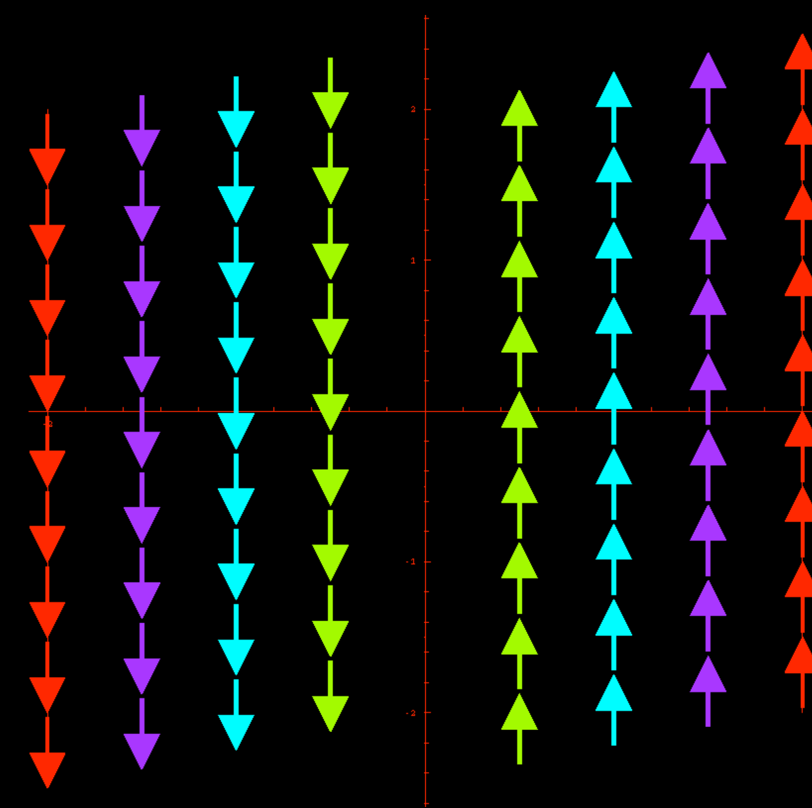

Vector Fields

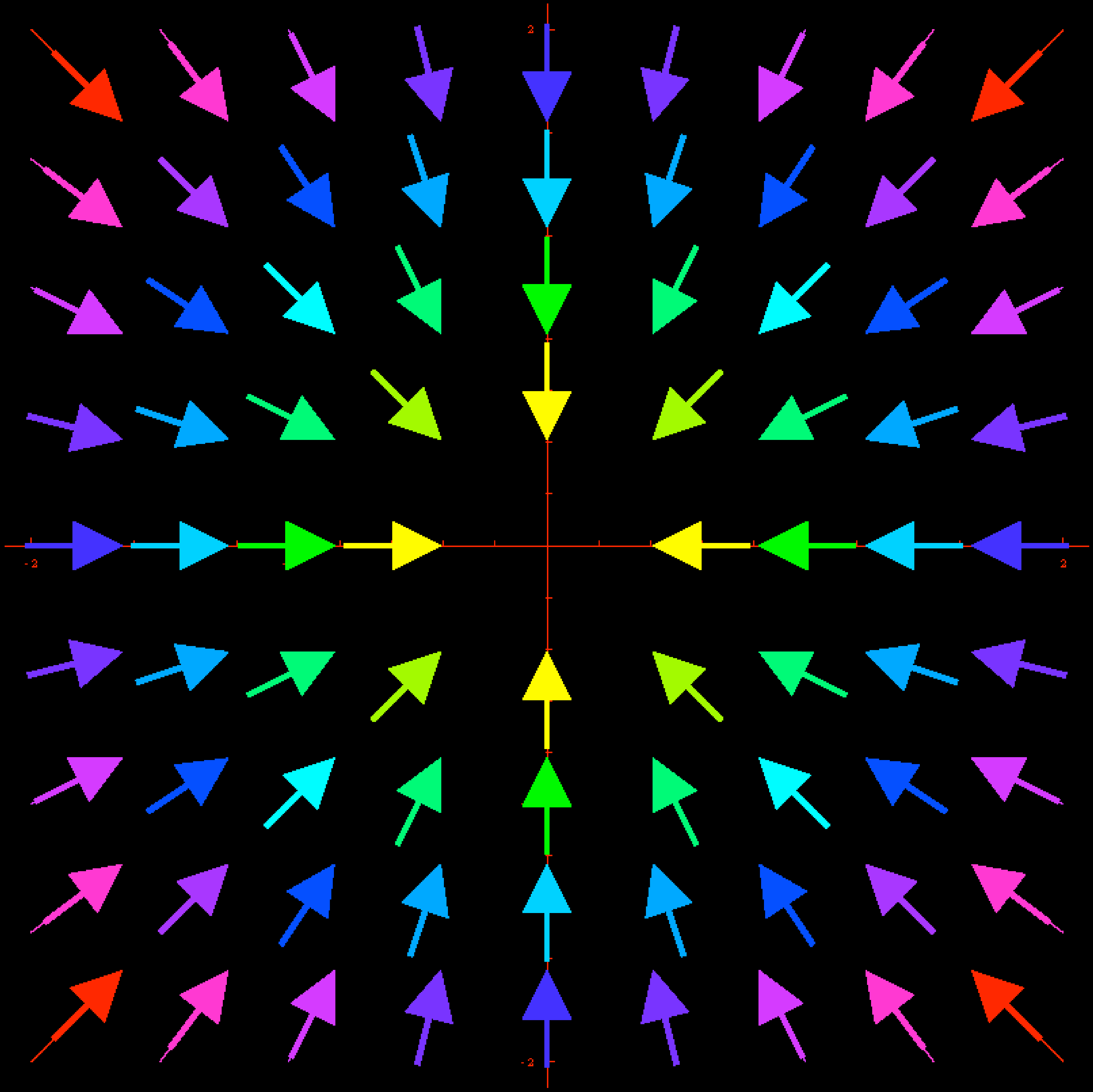


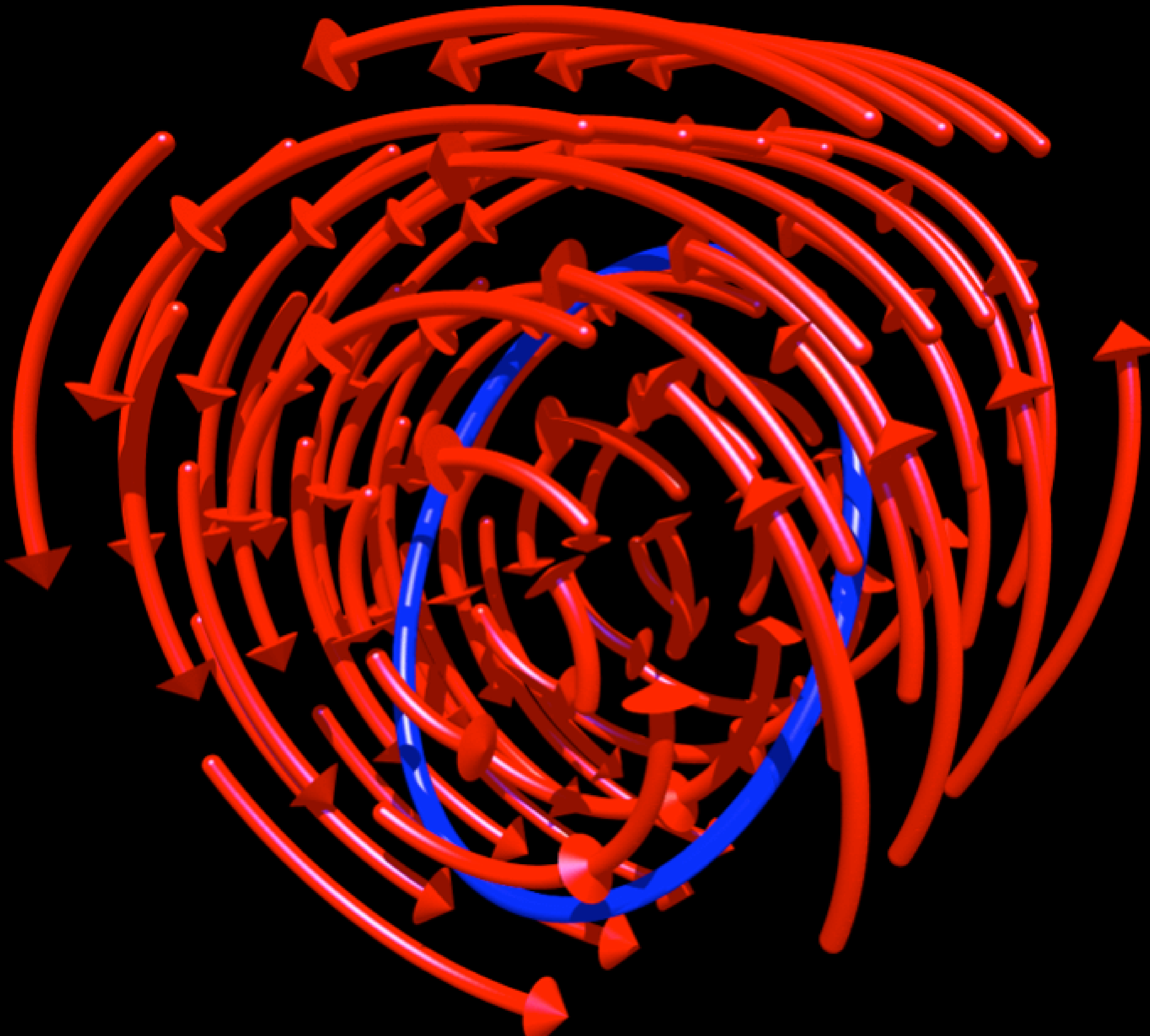


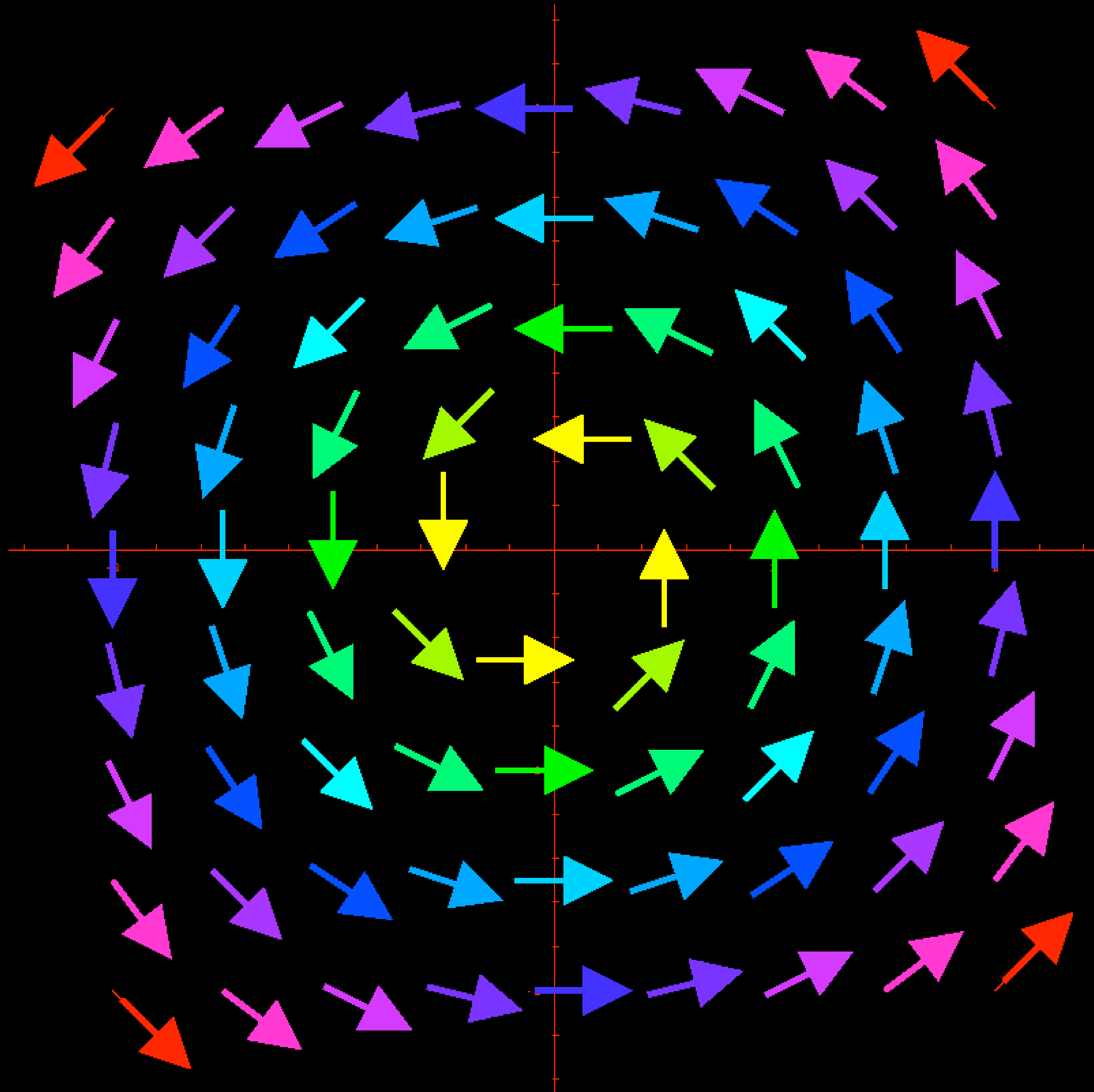
Vector fields

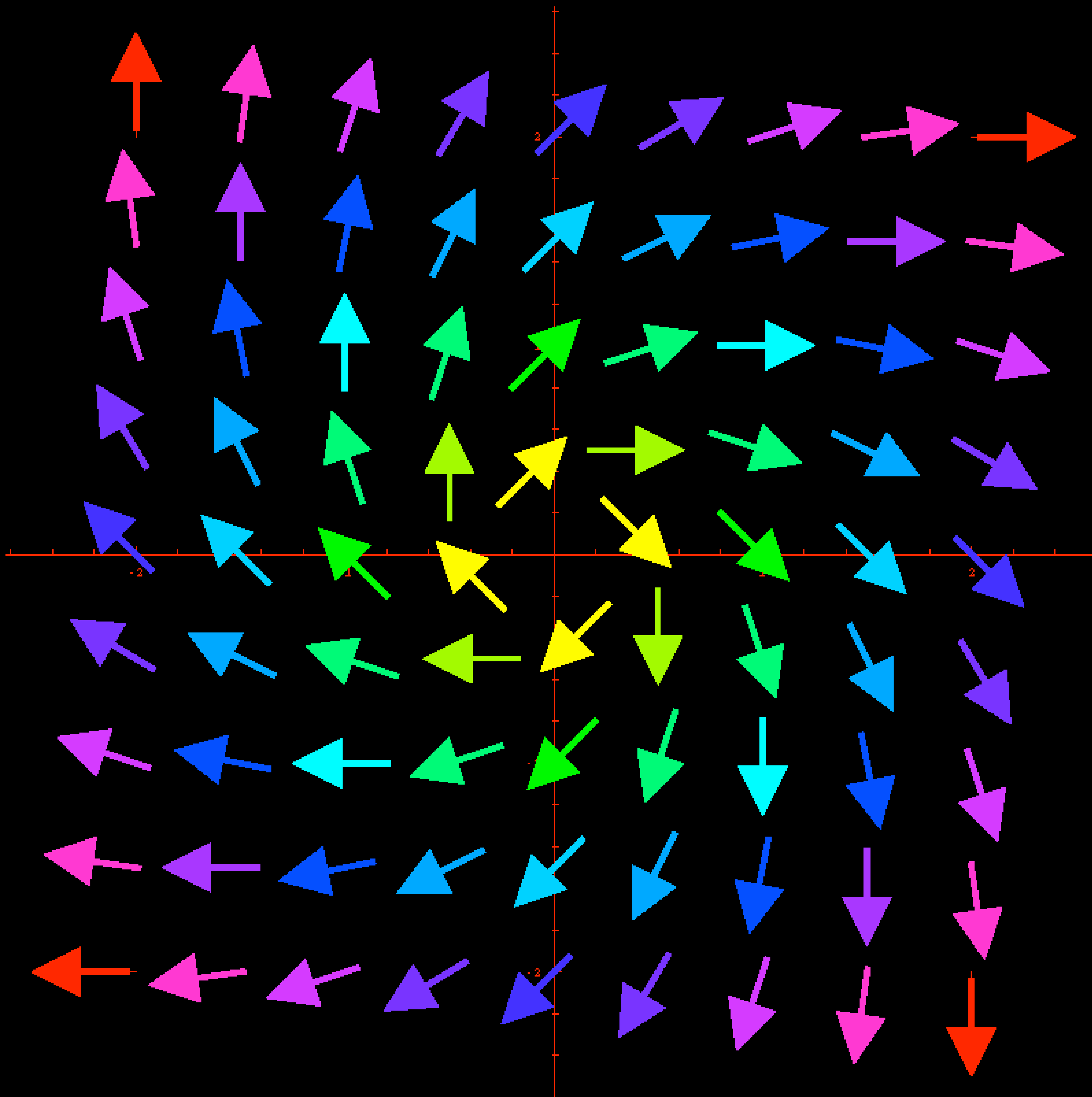


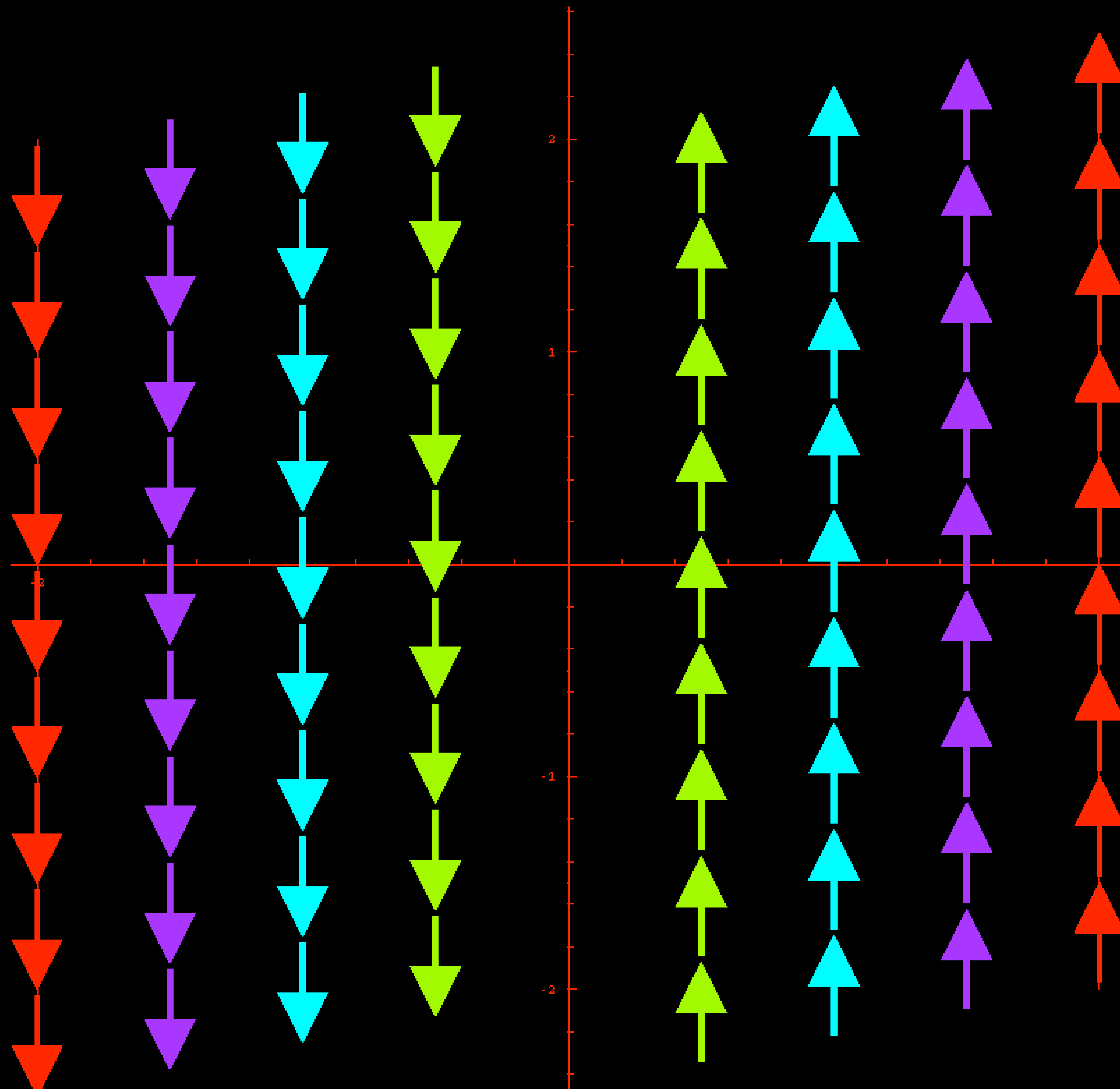



2
-2
2
-2



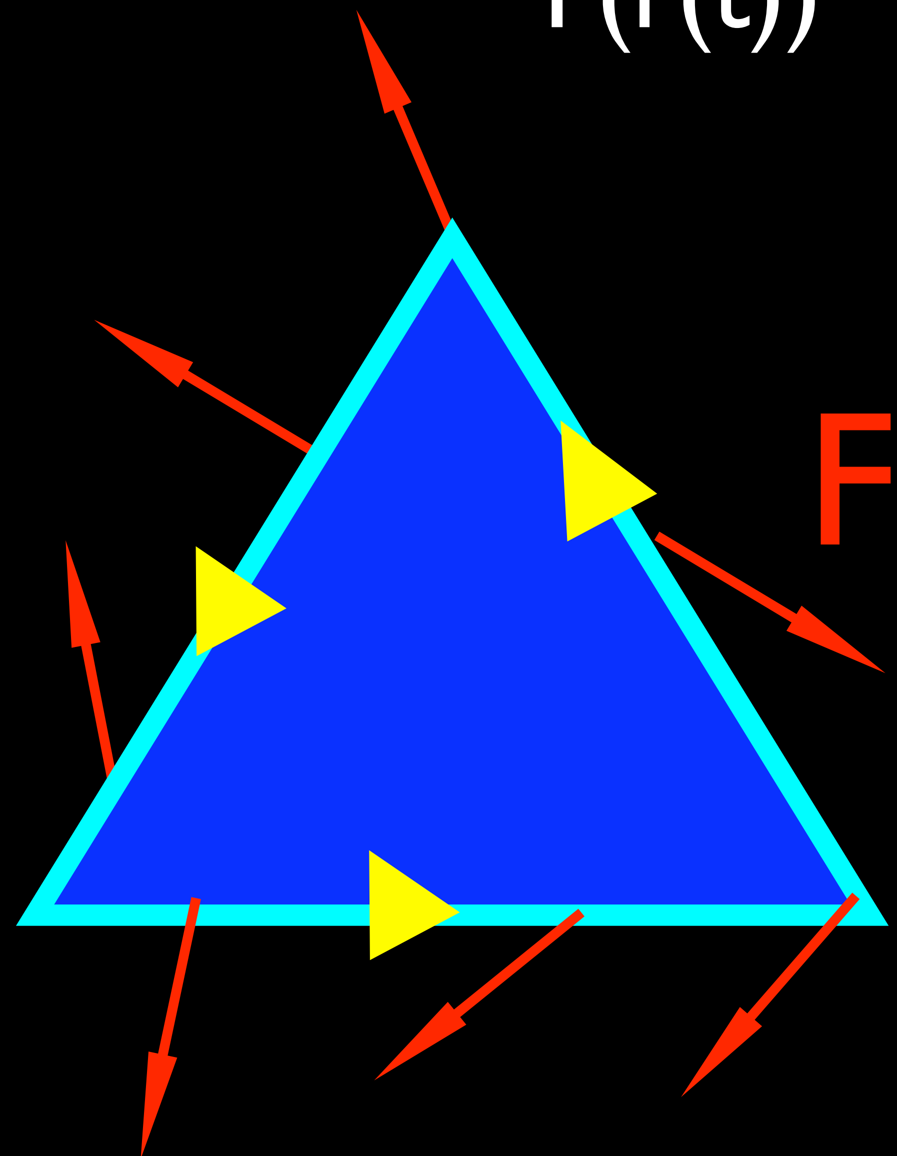





2
1
-1
-2

Lineintegrals

F(r(t)) “Force at point r(t)”



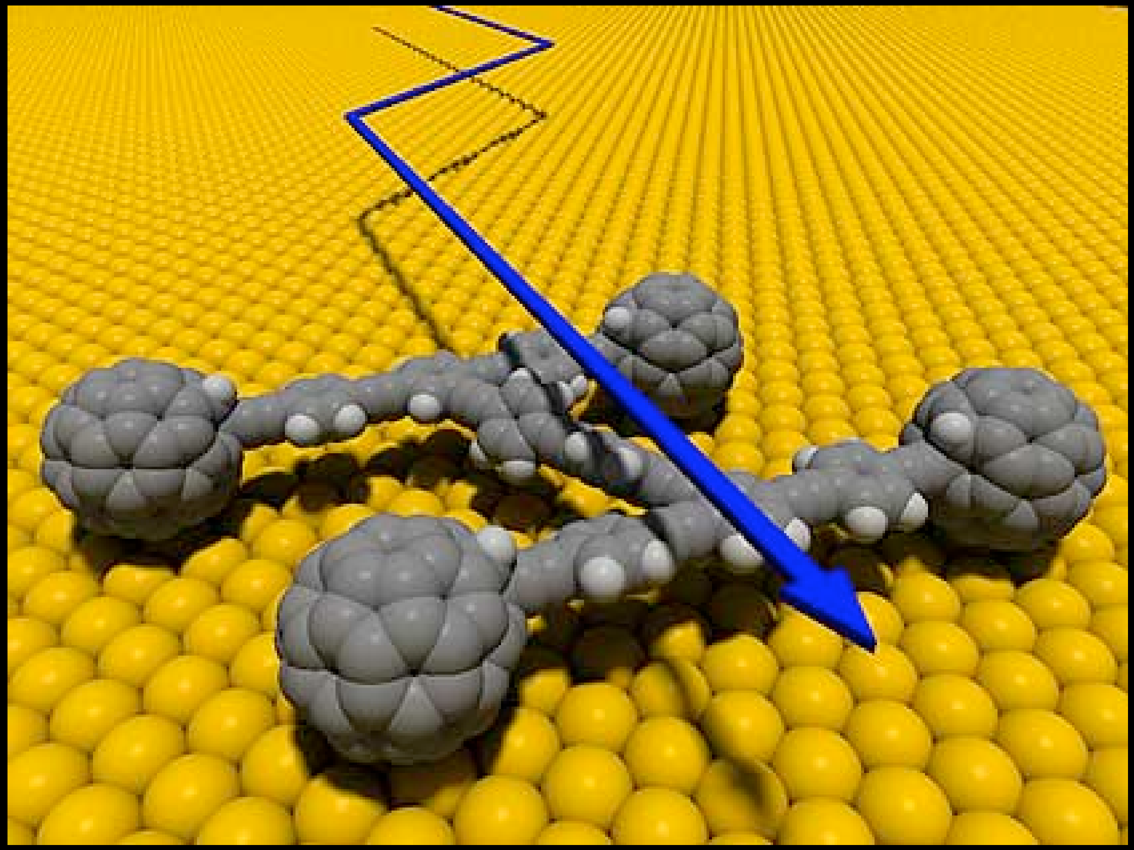
F (r(t)) . r’(t)

= “Power”

Integrating this over the time interval gives an energy.

Is this line integral positive or negative?

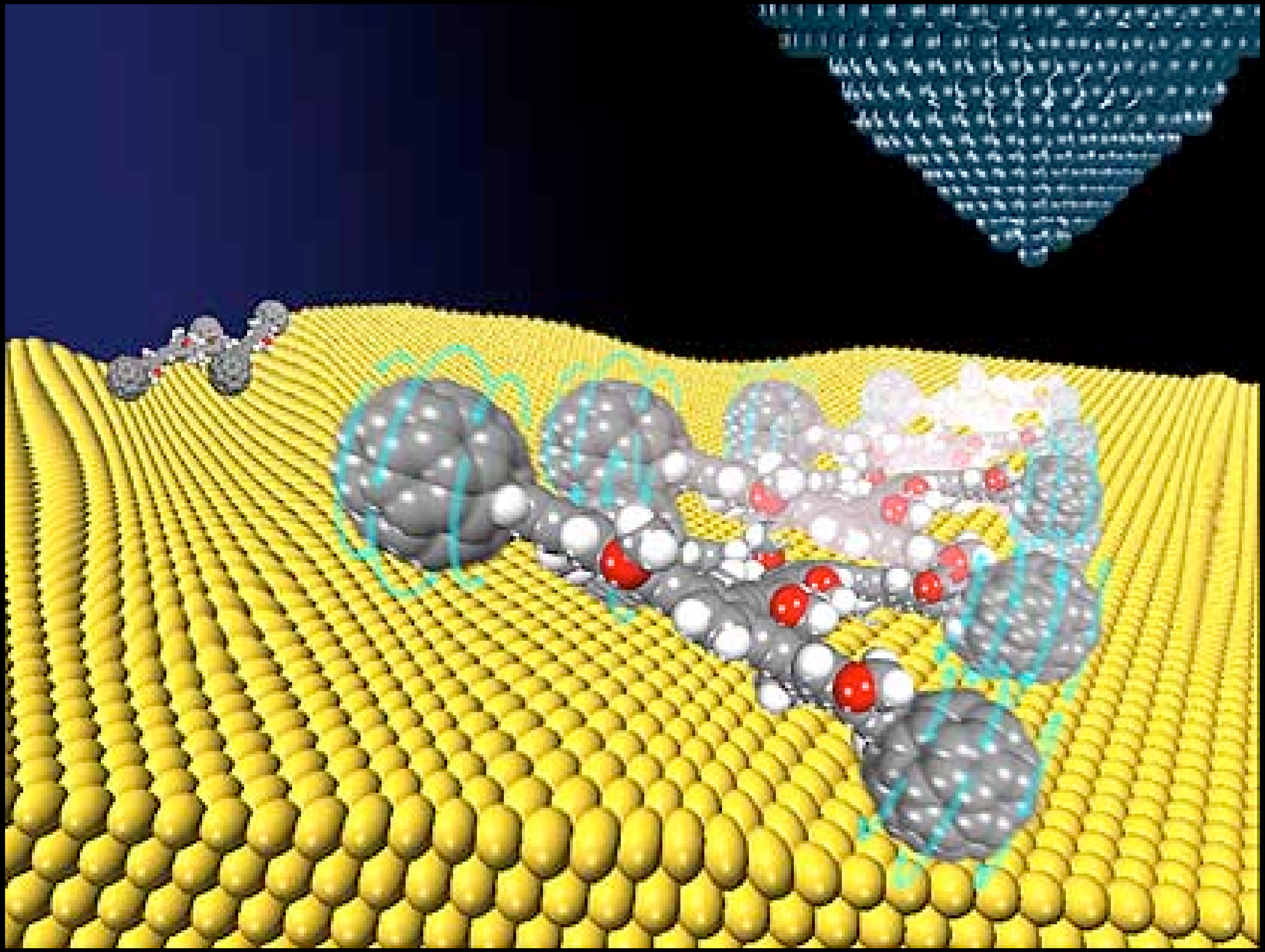
Nano car problem



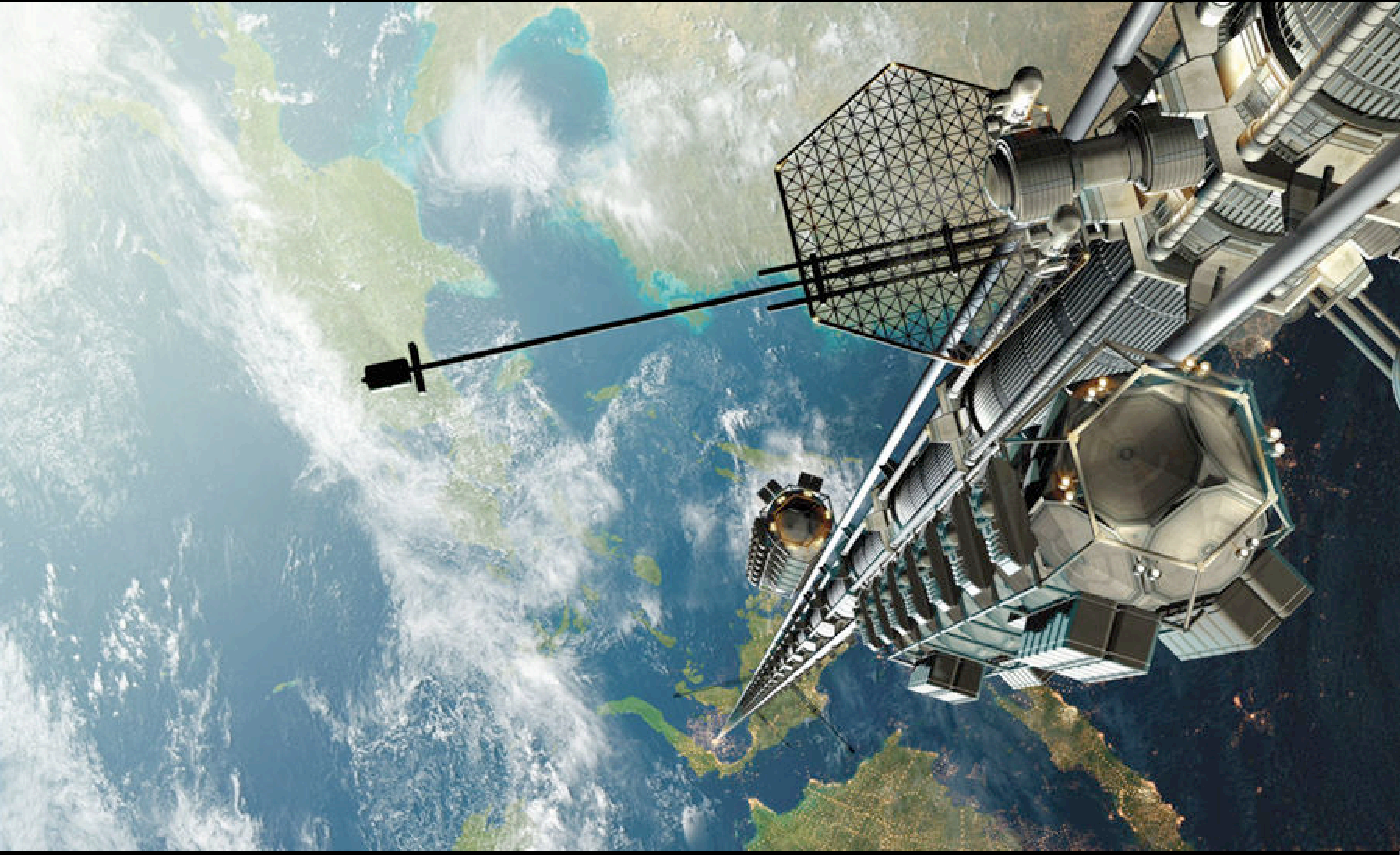
At Rice university, one has build the first nanocar, it is 20’000 times smaller than the thickness of a hair and made of one molecule. The “street” is a gold plate which is heated up.

Nano car problem

The car is exposed to a force field $F(x,y,z) = (y\,z, x, z)$ from the surface and pushed along a path $r(t) = (t, \cos(t), \sin(t))$, where t goes from 0 to pi. What work is done on the car?

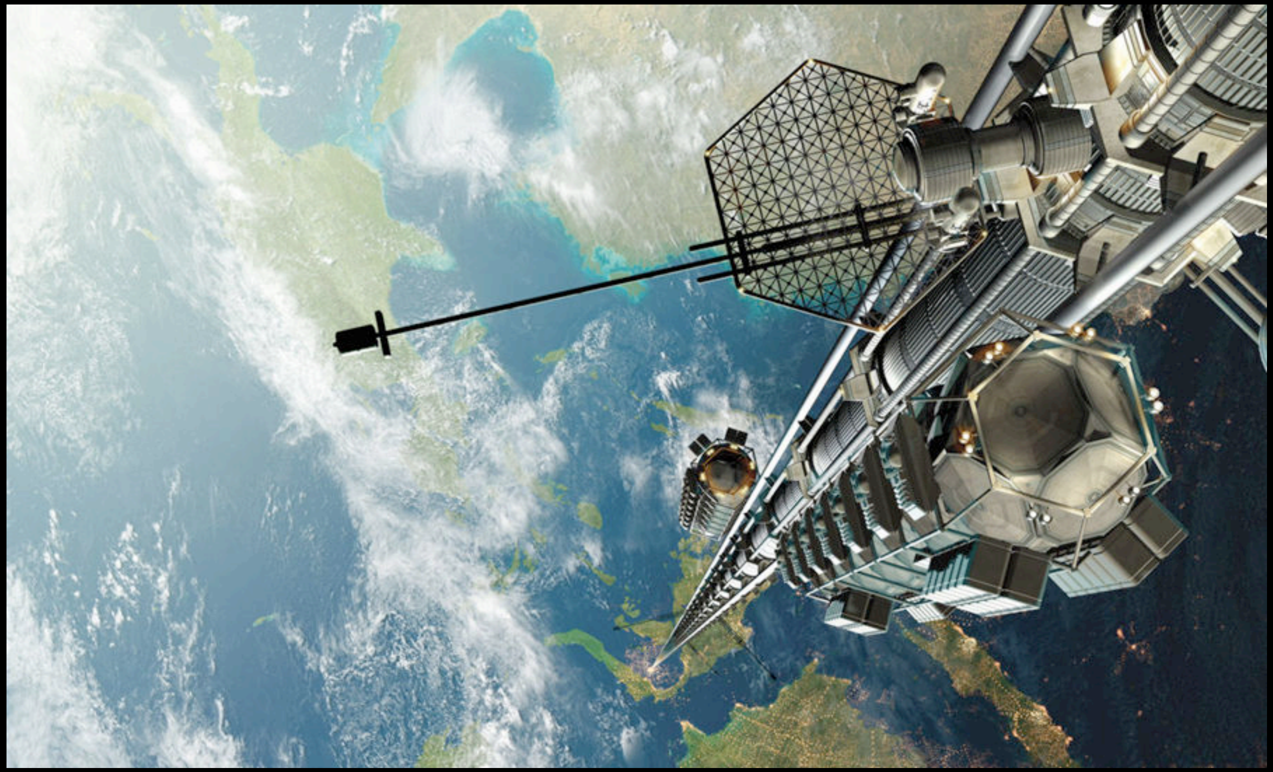


The space elevator problem



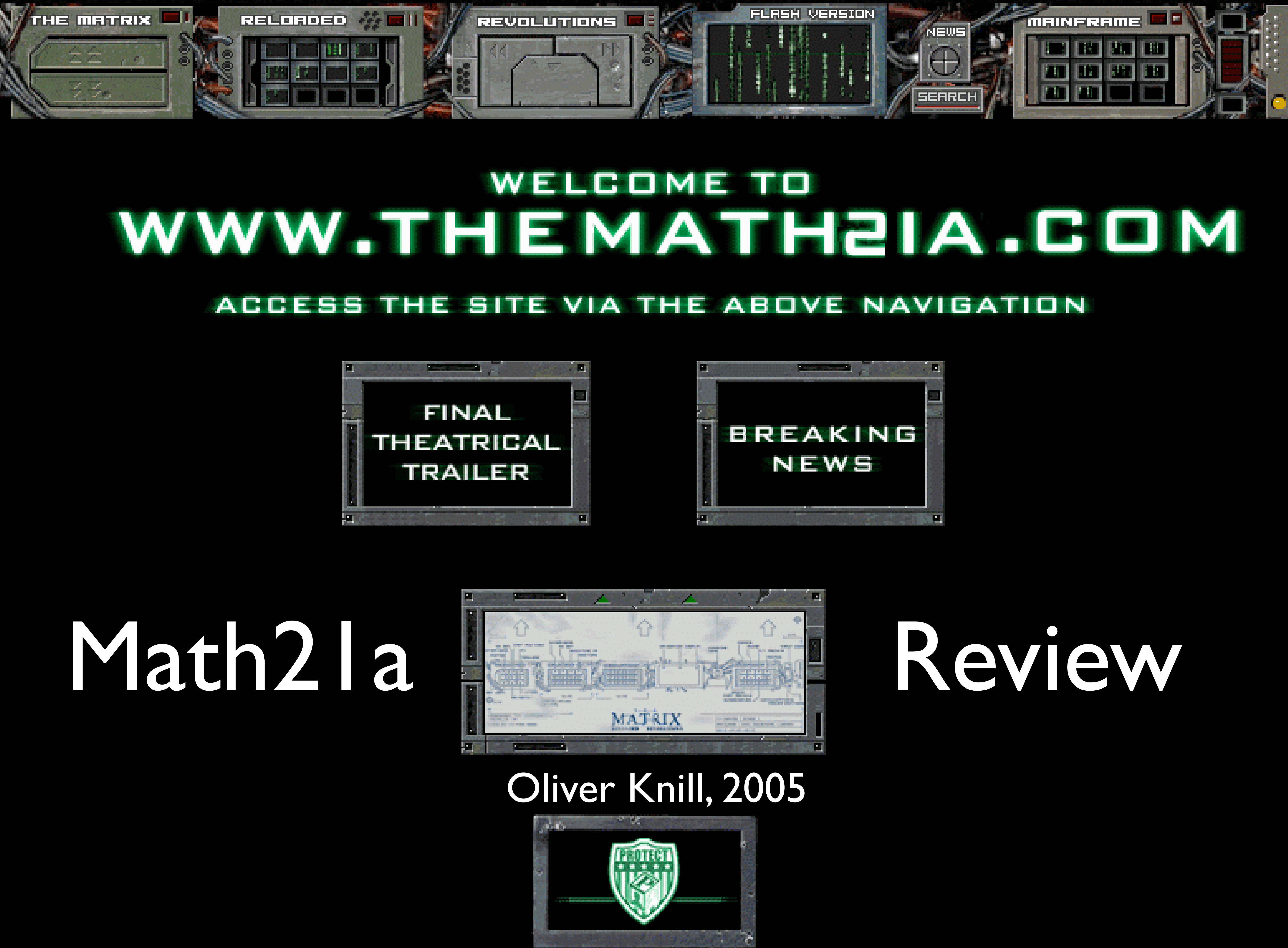
Problem: We are in the year 2094. Many space elevators have been built. The gravitational field of the earth is the gradient of the function $f(x,y,z) = x^2 + y^2 + 0.999 z^2$.

What work is necessary to go from the earth ground (0,0,6000) to the point (0,0,66000)?



Good luck! After the break we continue with integral theorems.


Math21a, Review
Fall 2005, Part III
After a break
Oliver Knill, 1/10-11/2006



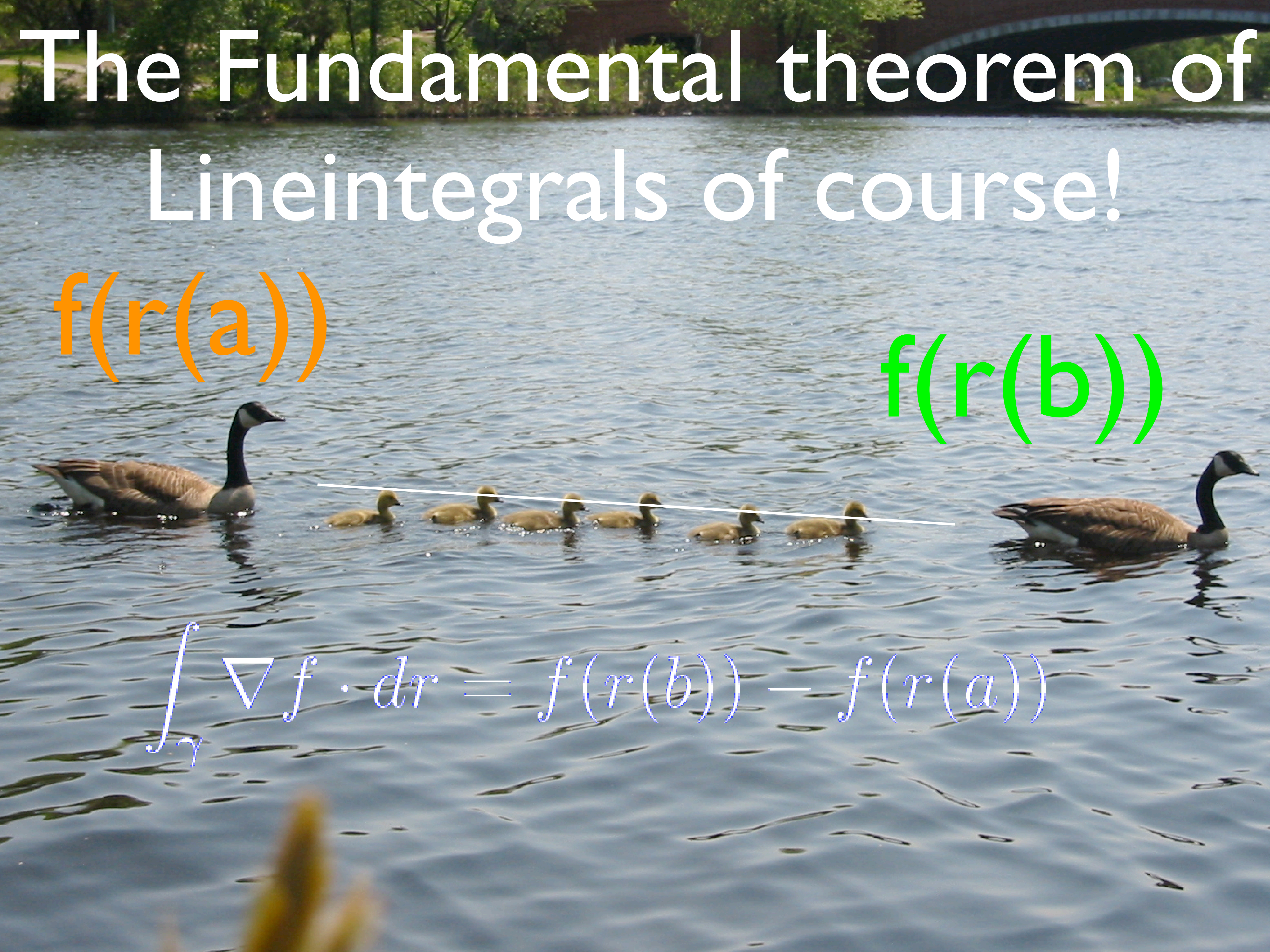
Math21a Review

Oliver Knill, 2005

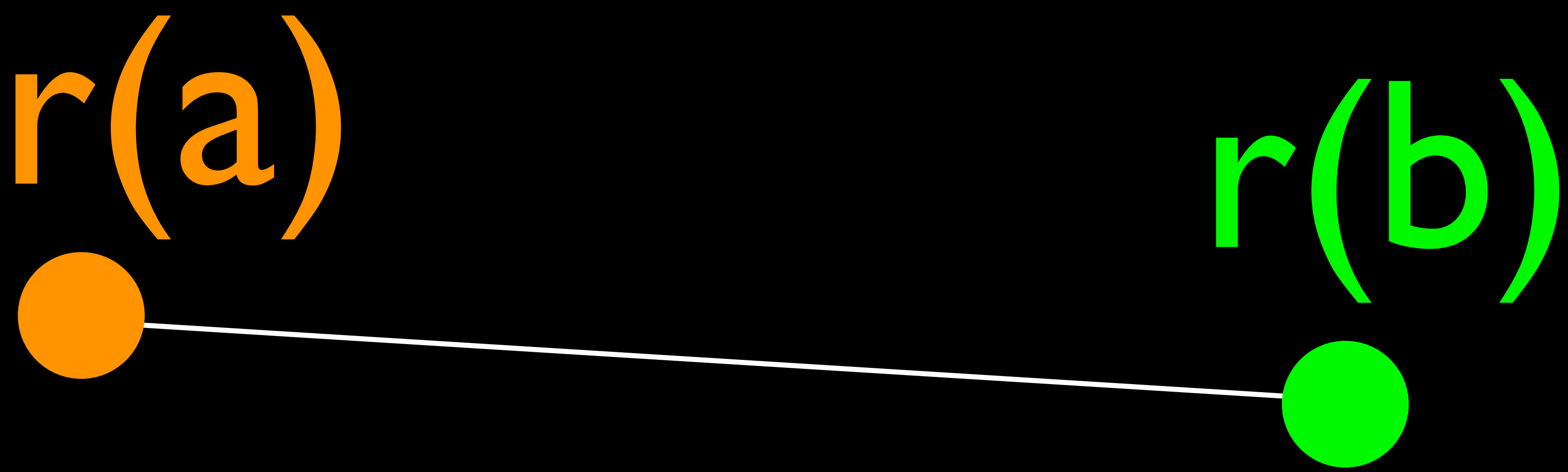
Table of content

- We review the theoretical material using about 60 slides.
- There are 4 problems solved on the blackboard.

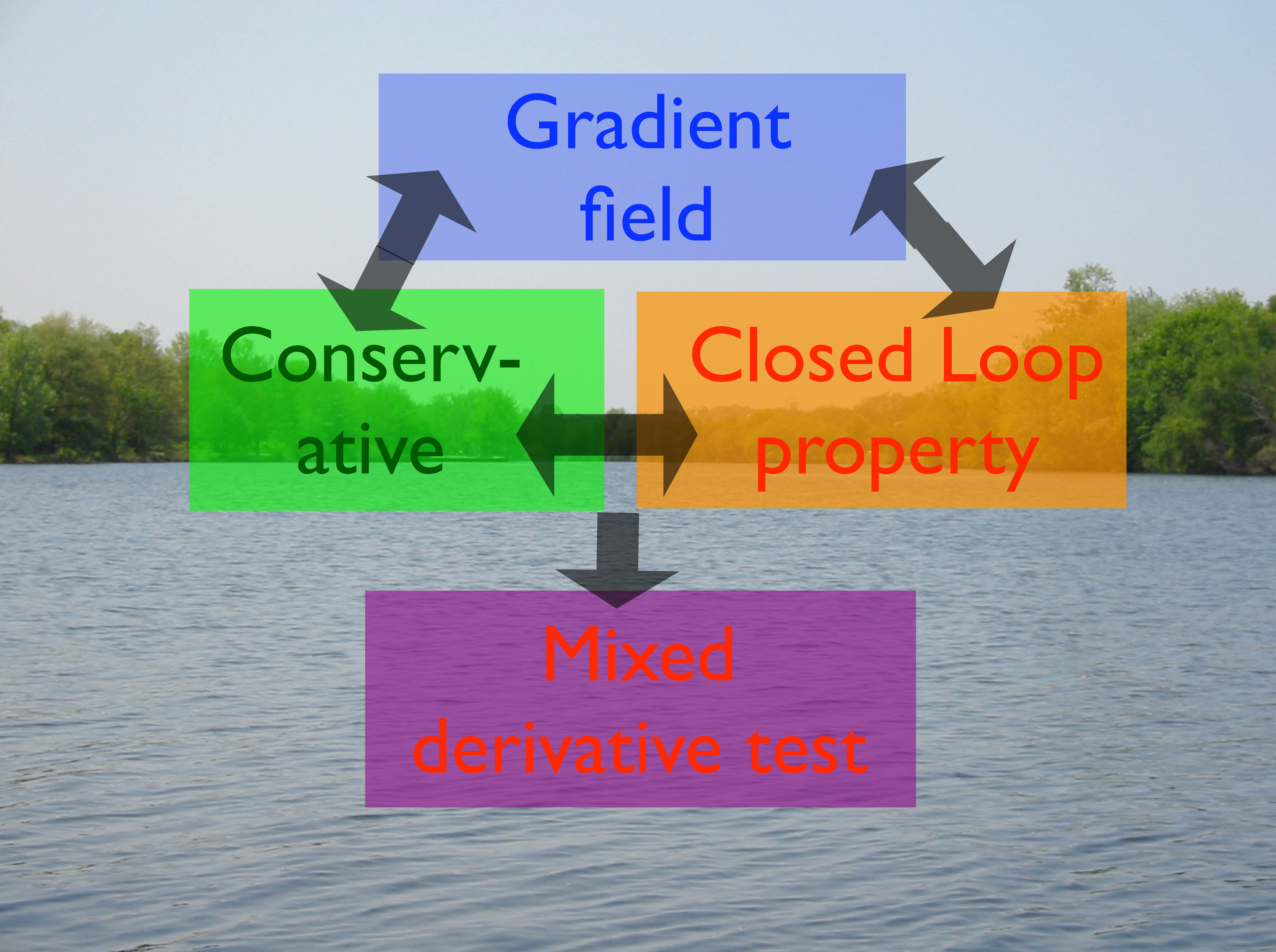

What do you think,
when you see this picture?


The Fundamental theorem of Lineintegrals of course!
f(r(a))
f(r(b))
$\int_\gamma \nabla f \cdot dr = f(r(b)) - f(r(a))$

Fundamental theorem of Lineintegrals



$$\int_\gamma \nabla f \cdot dr = f(r(b)) - f(r(a))$$

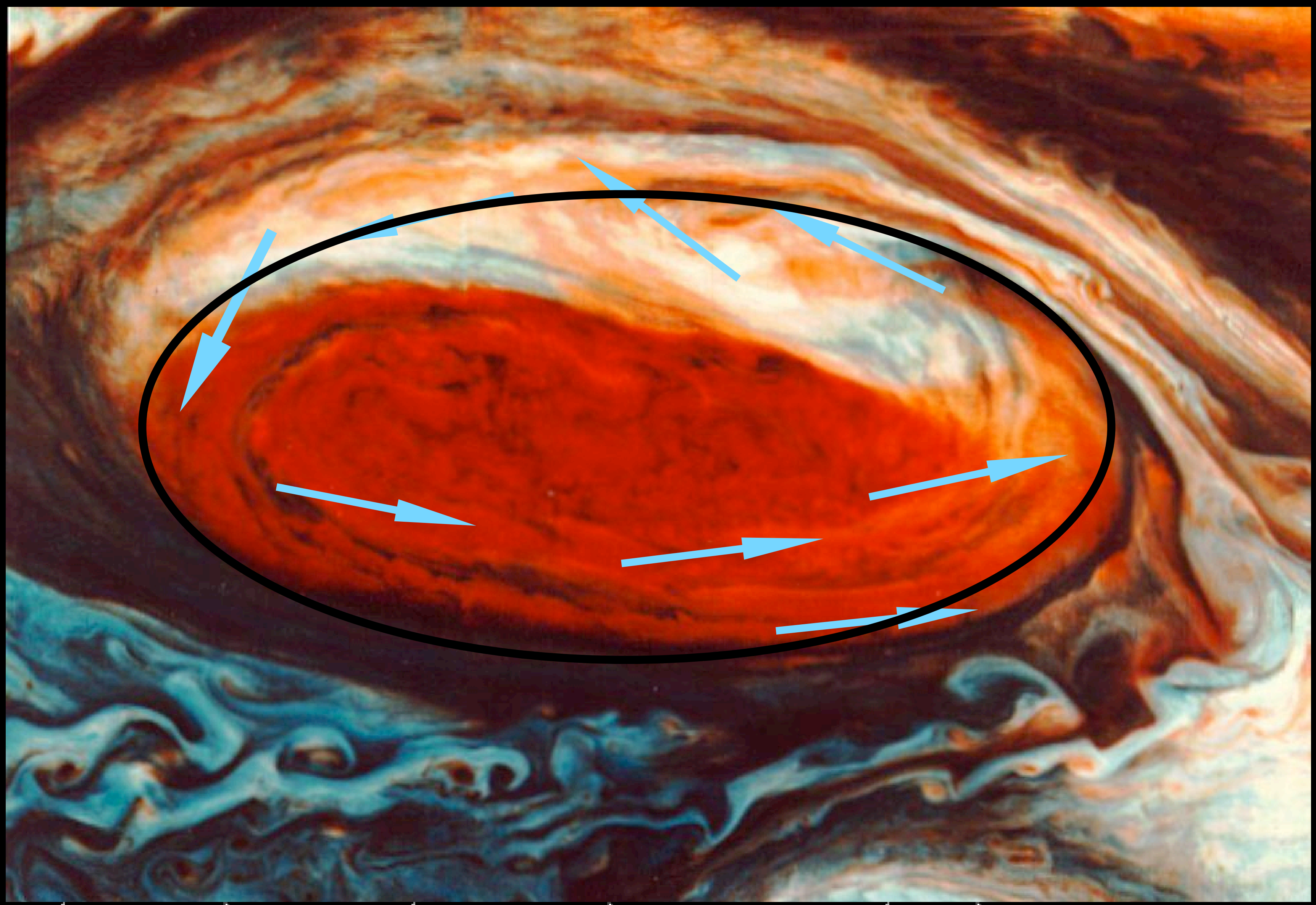

Gradient field
Conserv-ative
Closed Loop property
Mixed derivative test


What do you think, if you see this?


Greens theorem
of course!

Greens Theorem

$$\int_C F \cdot dr = \int\int_R \mathrm{curl}(F)\, dxdy$$



$$F(x,y) = (P,Q),\ \mathrm{curl}(F) = Q_x - P_y.$$


A line integral problem

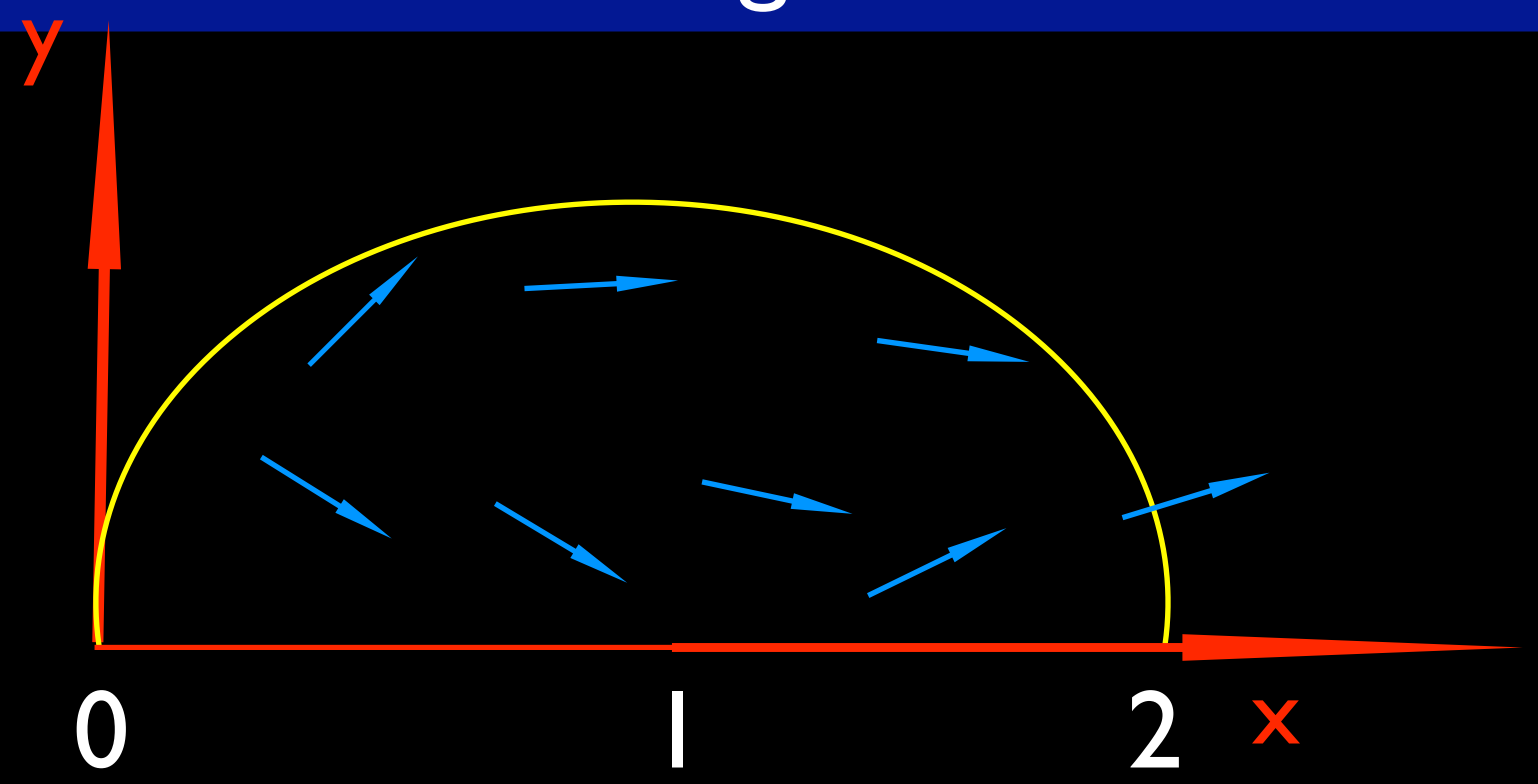
Problem 1

Find the line integral of the vector field

$F(x,y)=(y^2\cos(x) +3, x+2y \sin(x))$

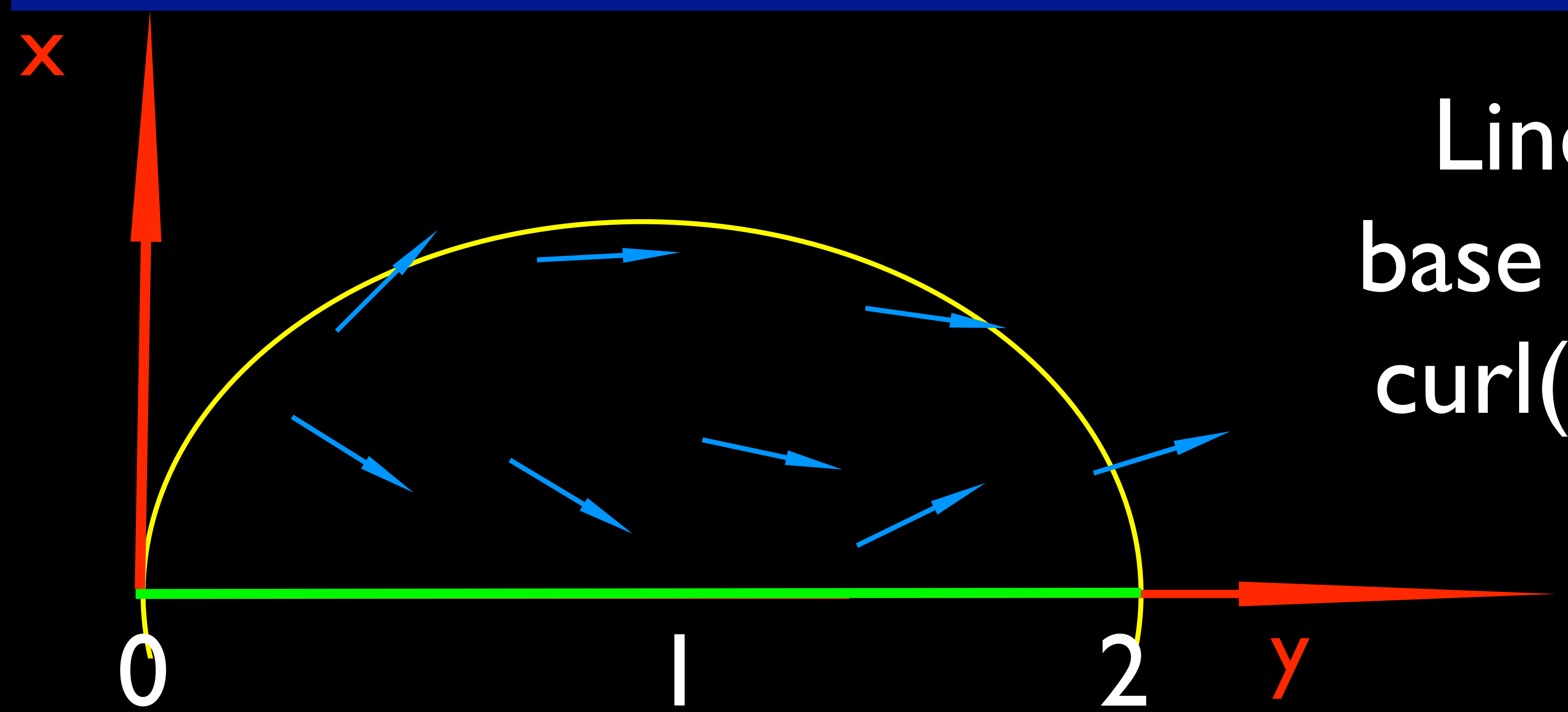
along the upper semi circle (counterclockwise)

given below.



Use Green!

The vector field
$F(x,y)=(y^2 \cos(x) +3, x+2y \sin(x))$
has curl(F)=1

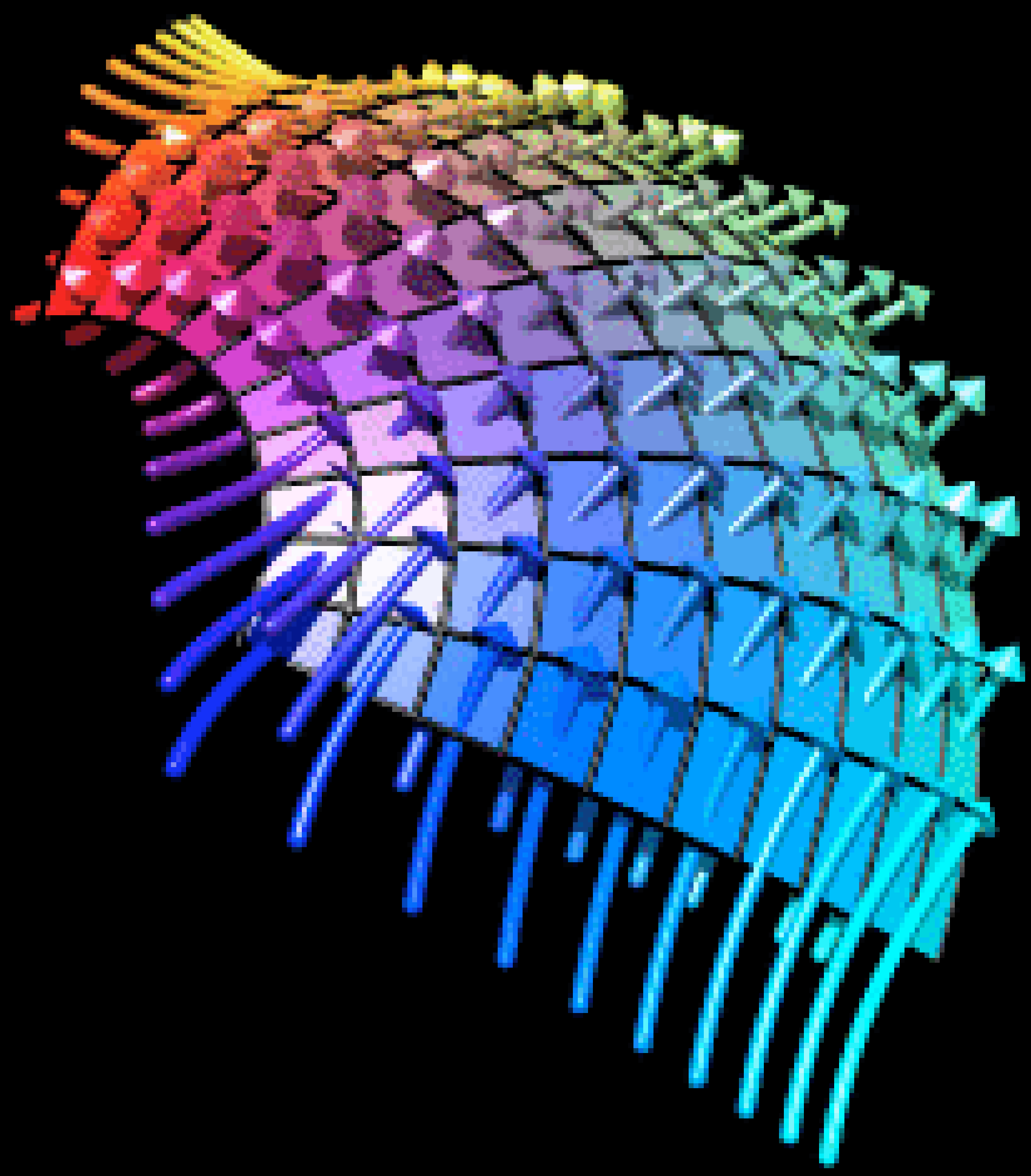


Line integral along base is 6. Integral of curl(F) over region is pi/2.

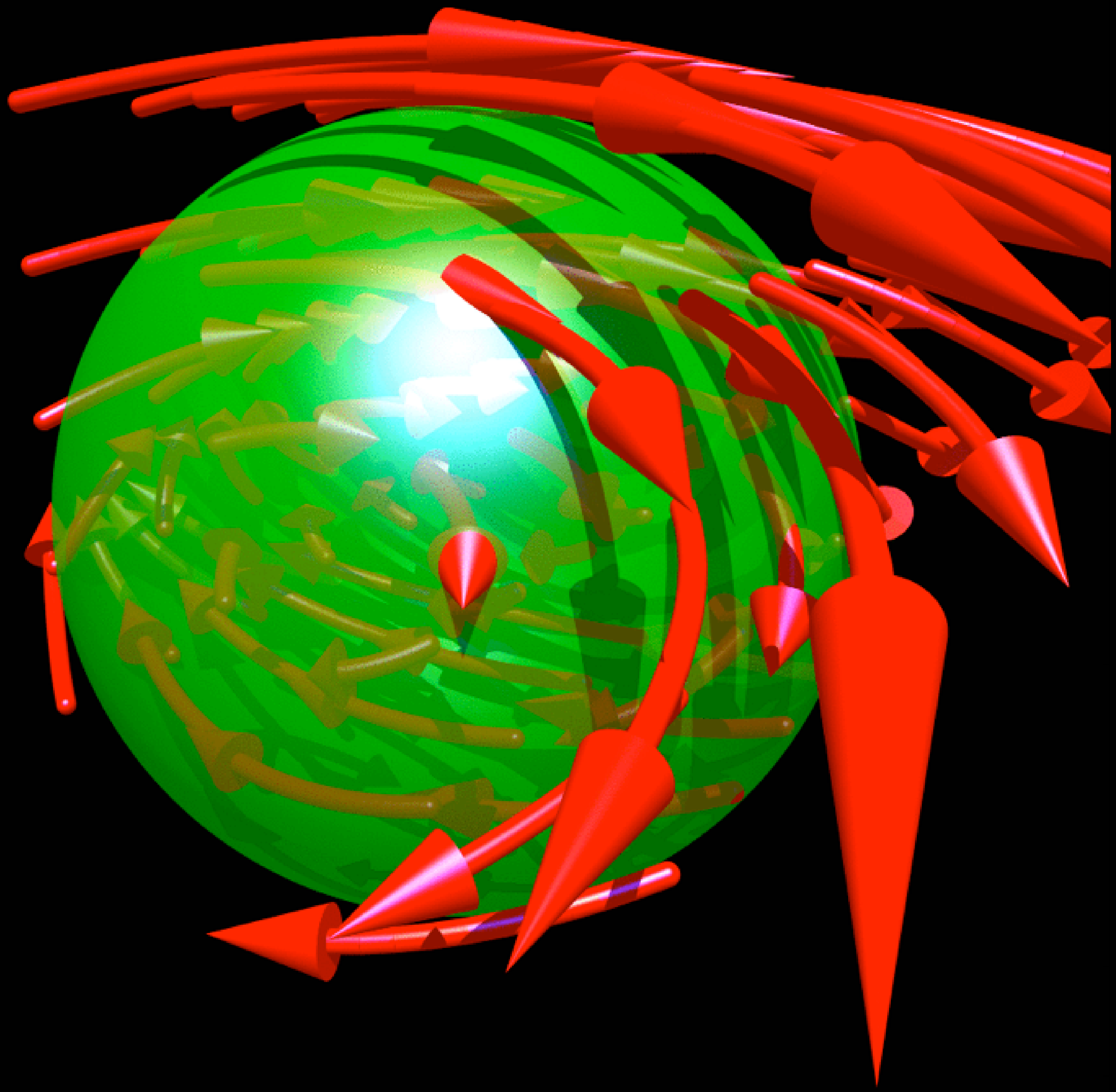
Answer: X=pi/2-6


Flux

Flux



Flux

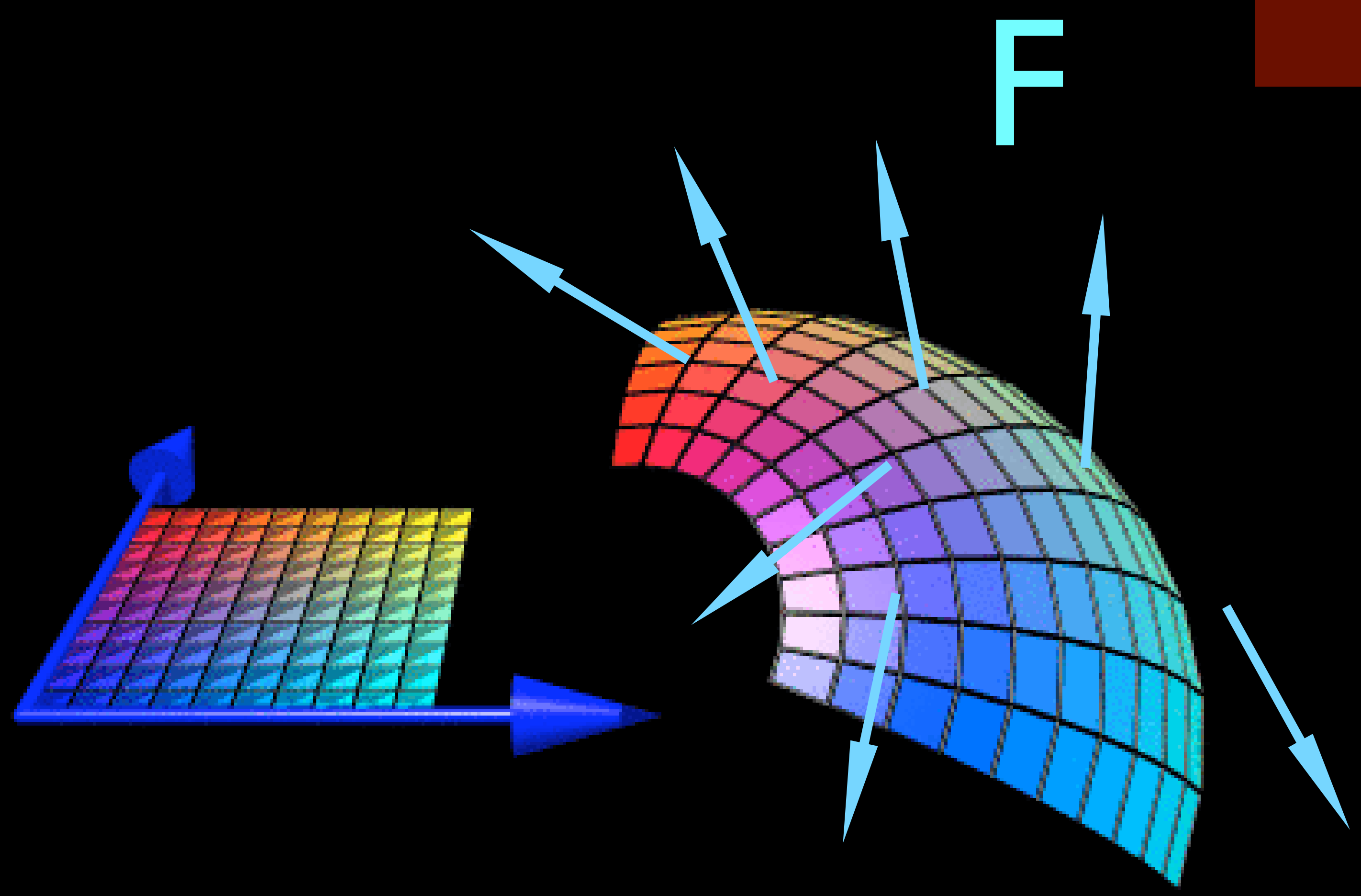


Flux integrals

F(r(u,v)) "Fluid velocity"

$F(r(u,v)) \cdot r_u \times r_v =$

"Flux component"

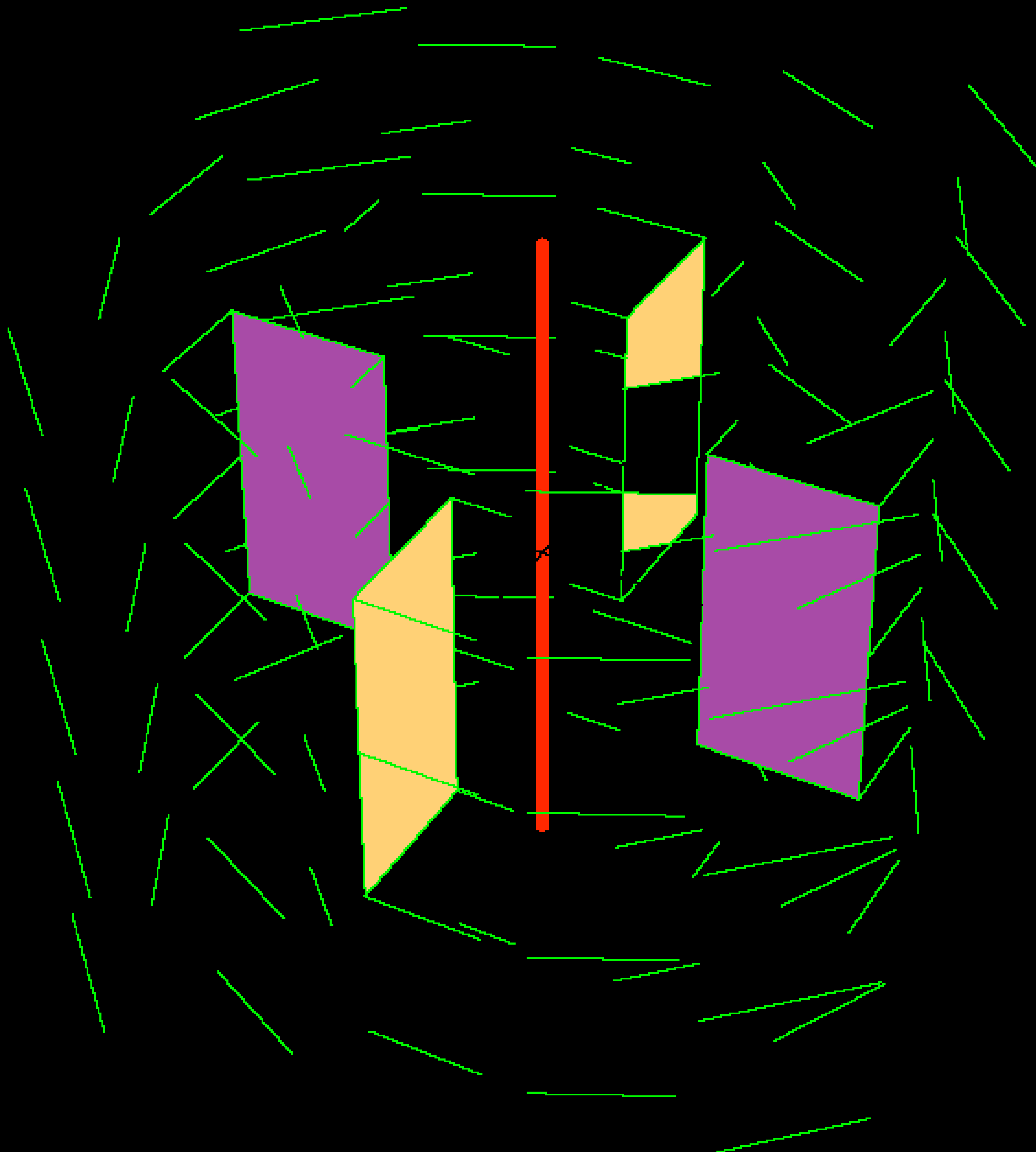


Integrating this over the surface gives the flux.

3D Curl

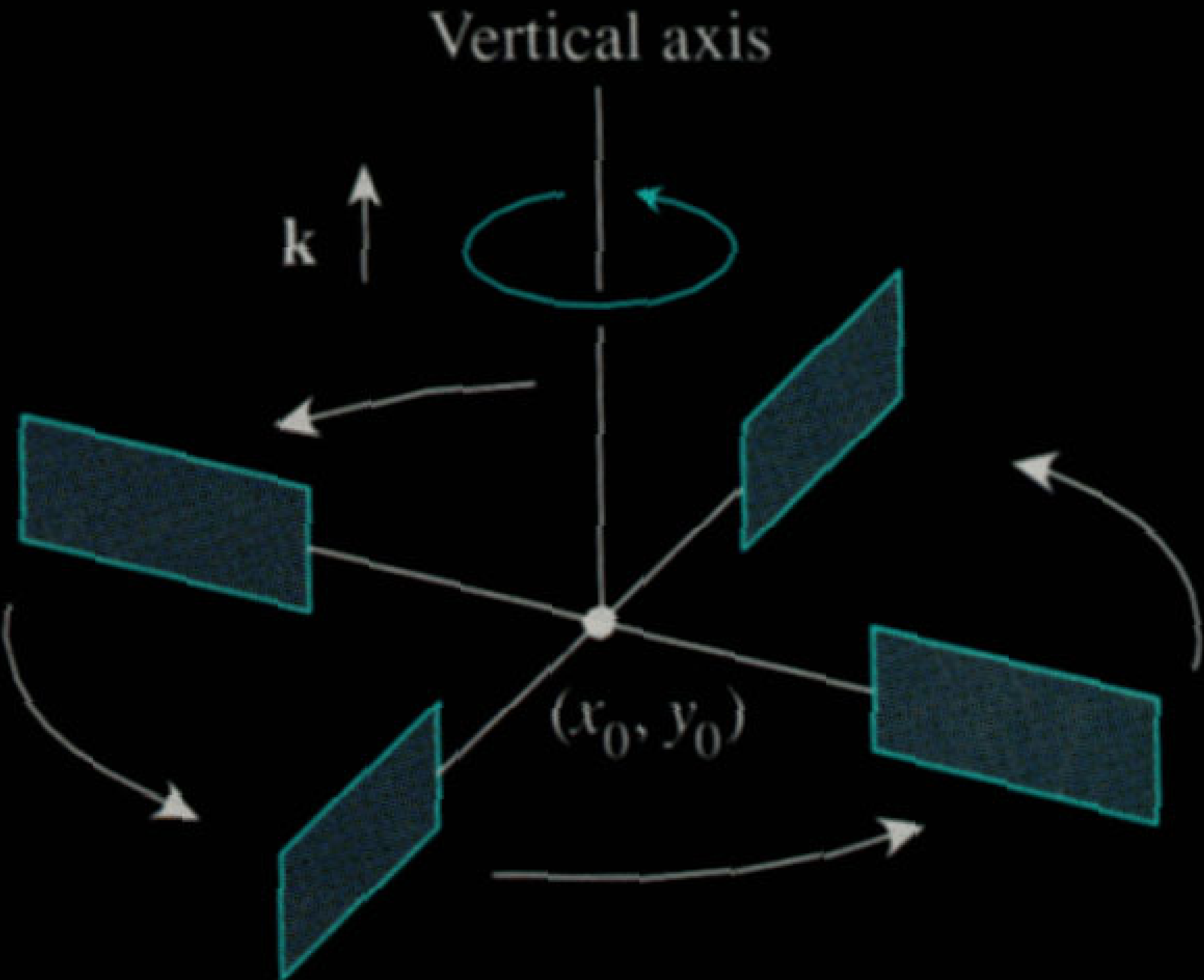
3D Curl

$$\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ \text{P} & \text{Q} & \text{R} \end{bmatrix}$$



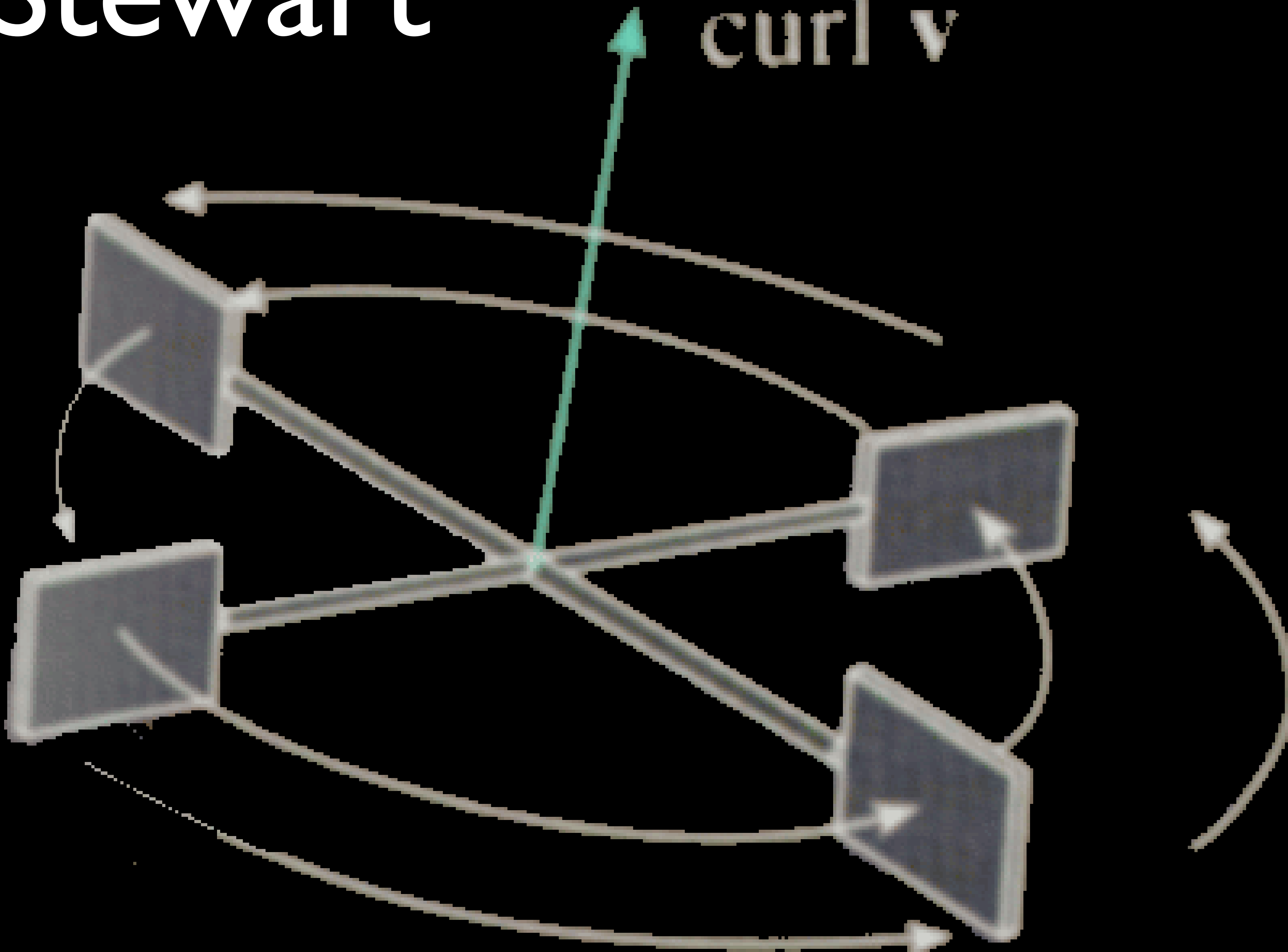
In essentially all multivariable textbooks, the paddlewheel appears.

Thomas



Curl $\mathbf{F}(x_0, y_0) \cdot \mathbf{k} > 0$
Counterclockwise circulation

Stewart



Edwards-Penny

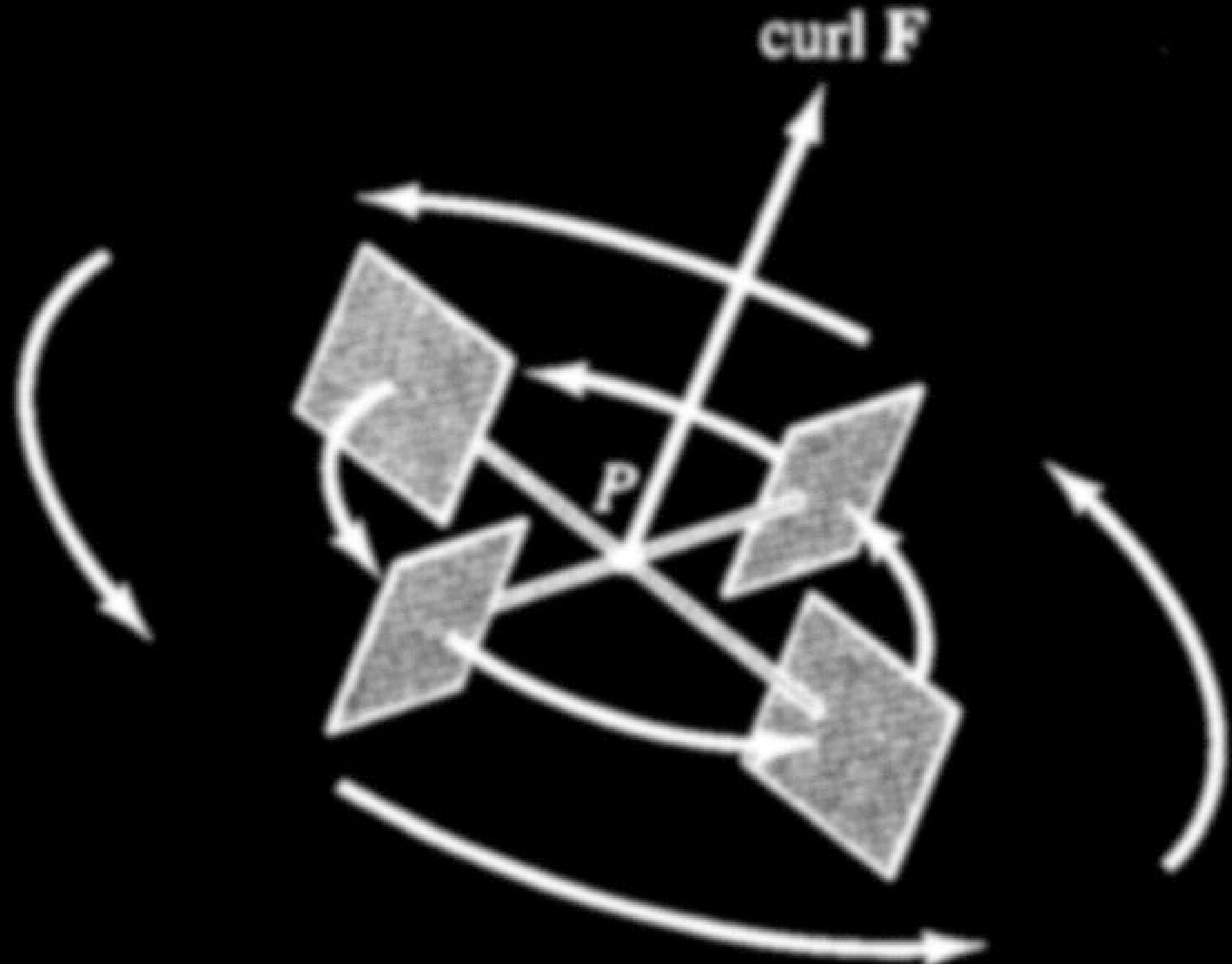


FIGURE 15.7.6 The paddle-wheel interpretation of curl **F**.

Marsden Tromba

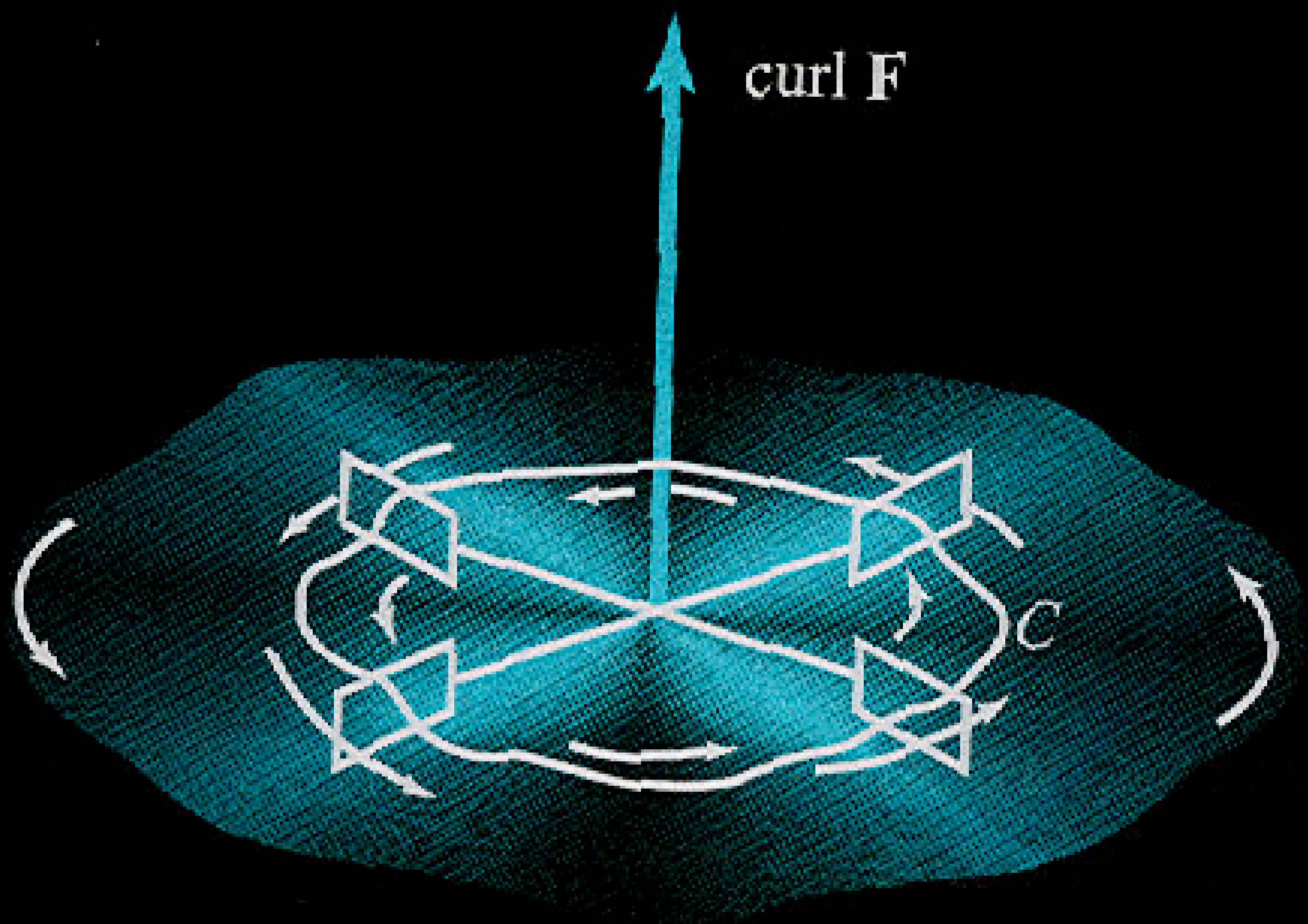
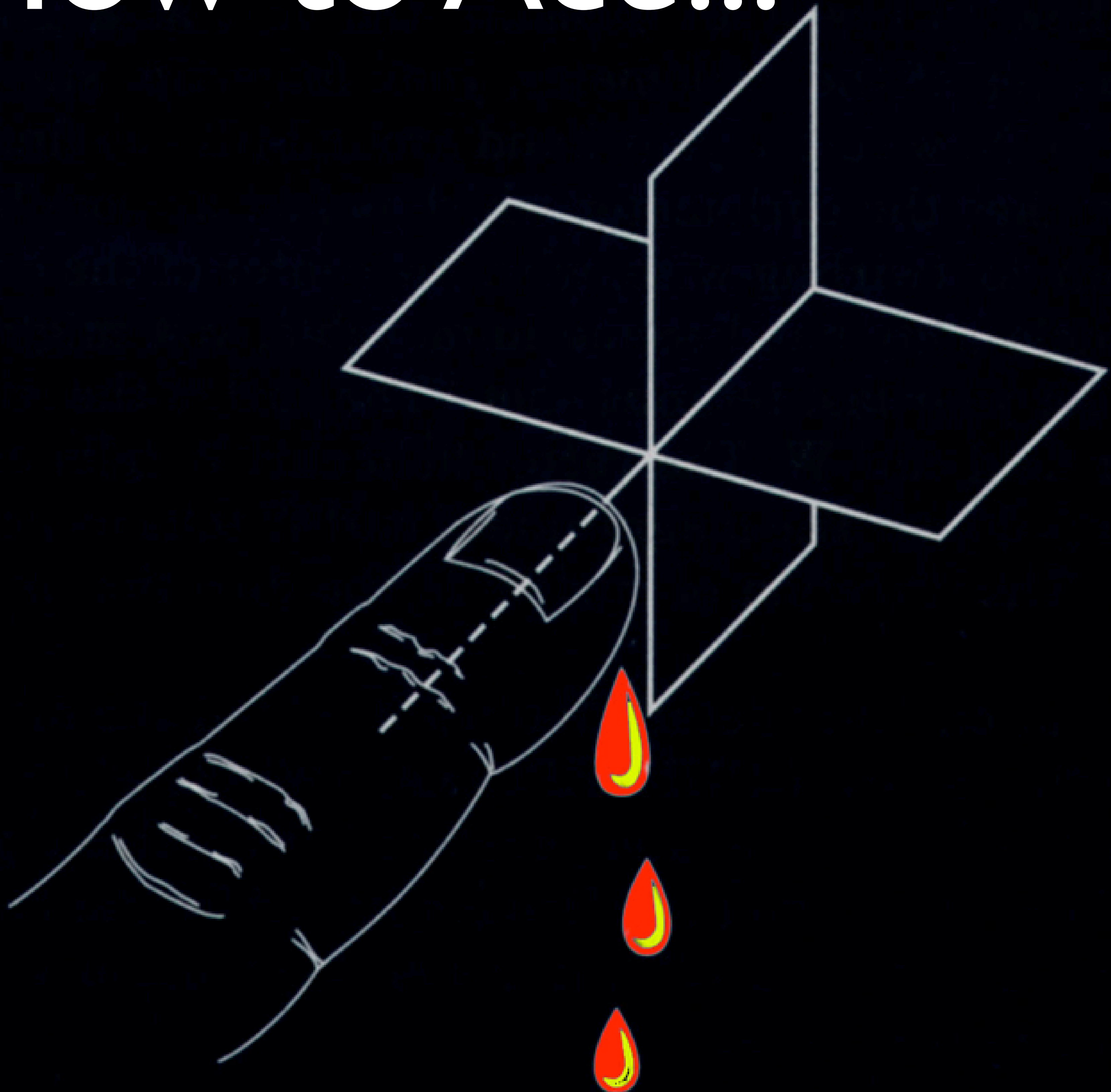


Figure 8.3.4 $\int_C \mathbf{F} \cdot$
paddle wheel in a flu
rotate around its axis.

How to Ace...



Curl

The wind velocity around Harvard yard is $F(x,y,z)=(2x,\sin(x),yz)$

Is this a conservative field?

How fast rotates the wheel at the point (1,1,3)

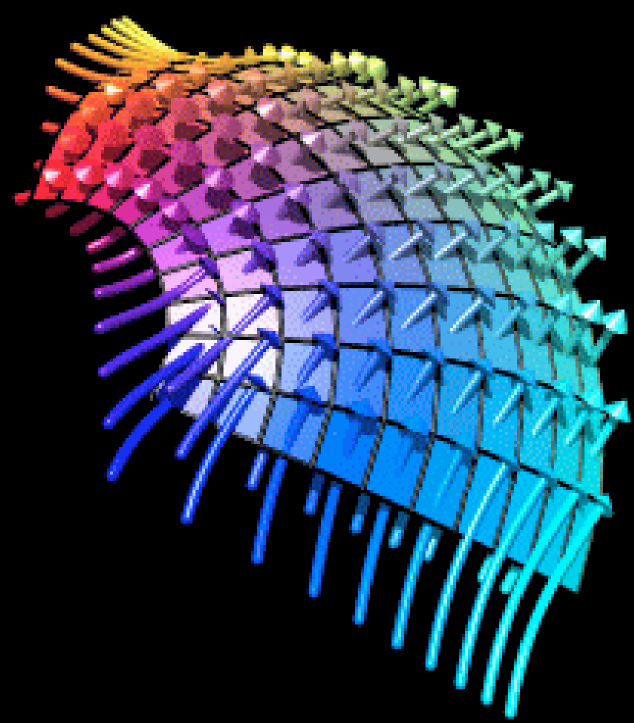

Stokes

Stokes Theorem

Stokes Theorem

$$\mathrm{curl}(P, Q, R) = (R_y - Q_z, P_z - R_x, Q_x - P_y)$$

$$\int_C F \cdot dr = \int\int_S \mathrm{curl}(F) \cdot dS$$



Fluid Dynamics

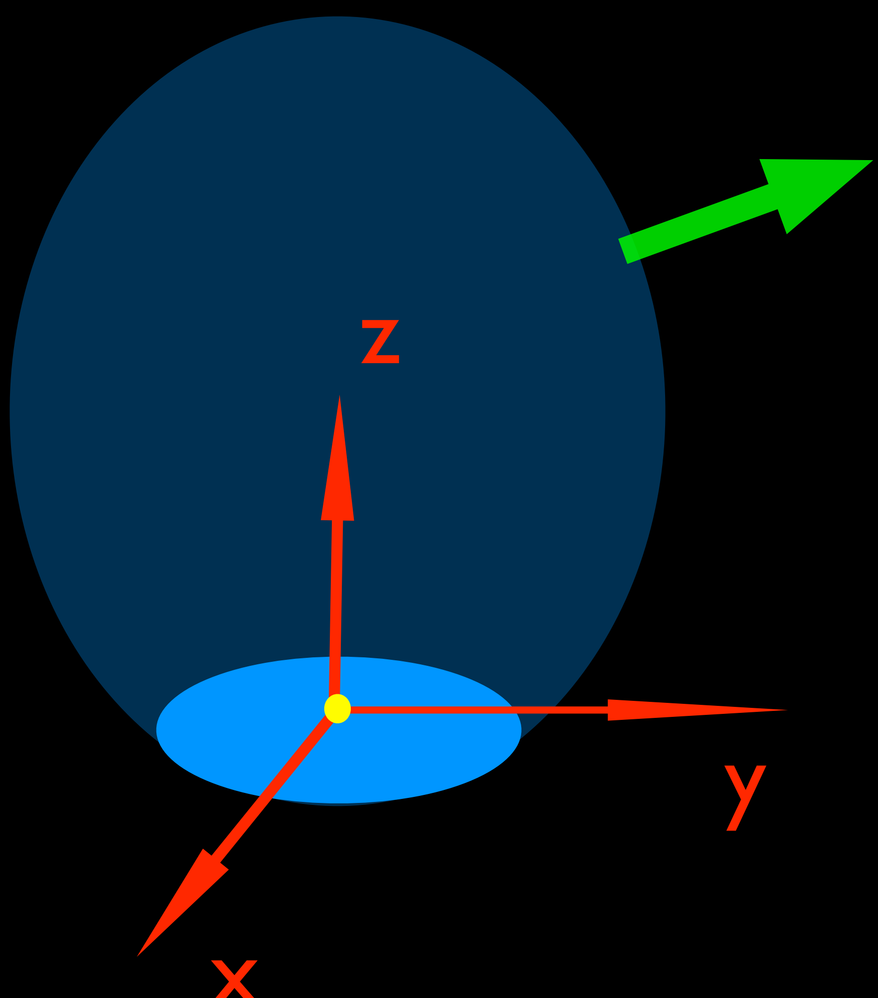
PHOTO COURTESY OF SCALED COMPOSITES, LLC

Problem 2

Blimp Comeback

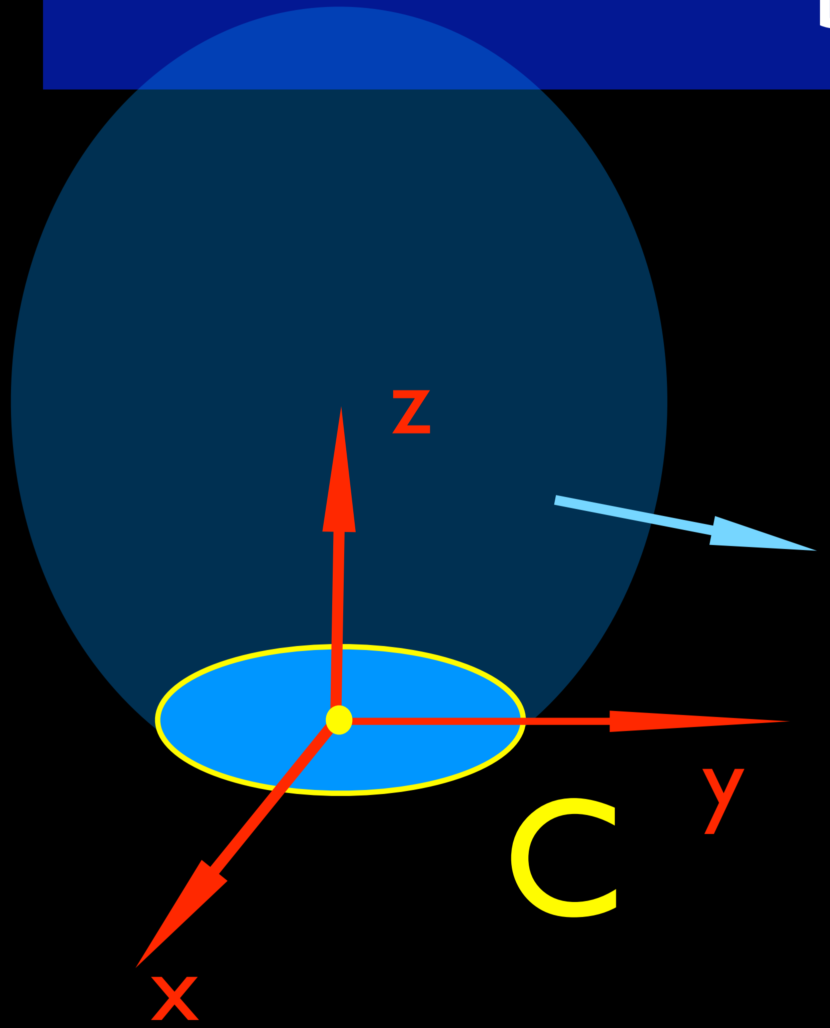
Problem

The velocity field in a hot air balloon is F=curl(G), where G=(-y,x,0). What is the flux of F through the surface of a hot air balloon for which the opening is the unit circle in the xy plane



Use Stokes Theorem

The flux of curl(F) through the balloon is equal to the lineintegral of F=(-y,x,0) along the boundary curve C.



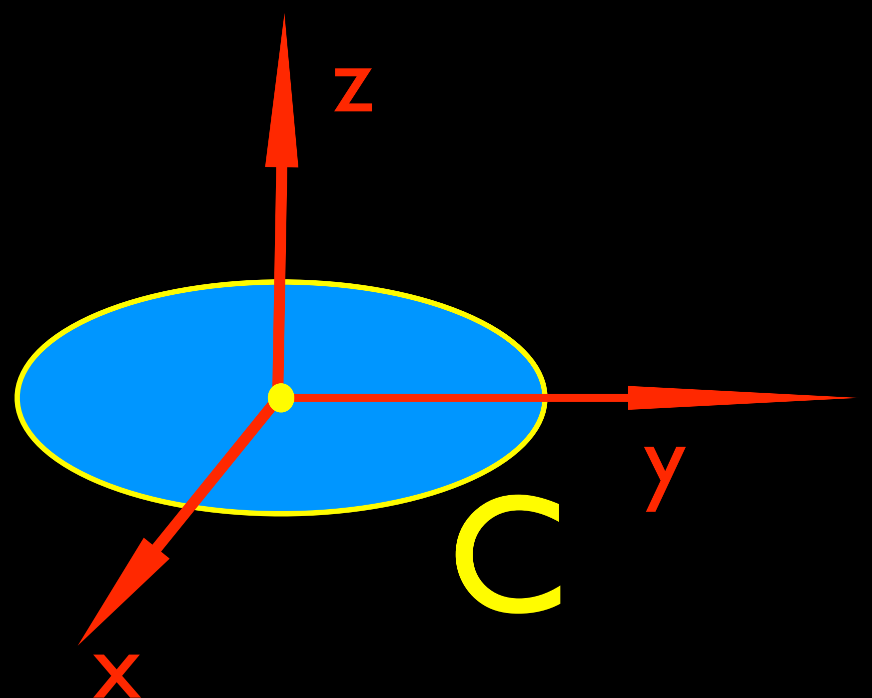
Use Stokes Theorem

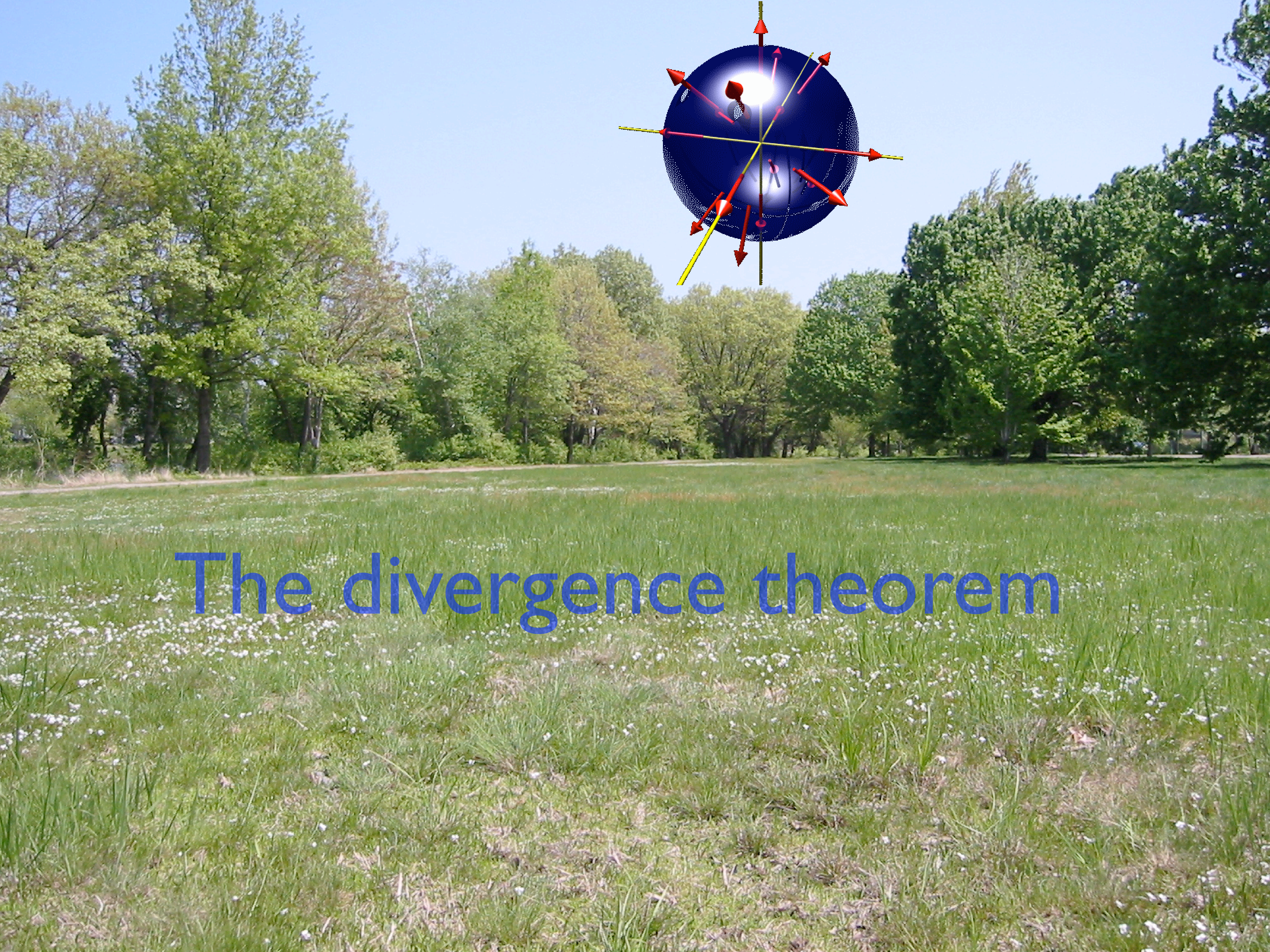
C: $r(t) = (\cos(t),\sin(t),0)$. $r'(t)=(-\sin(t),\cos(t),0)$

$F(r(t))=(-\sin(t),\cos(t),0)$

$F(r(t)) \cdot r'(t)= 1$

The line integral is 2pi, and so is the flux.



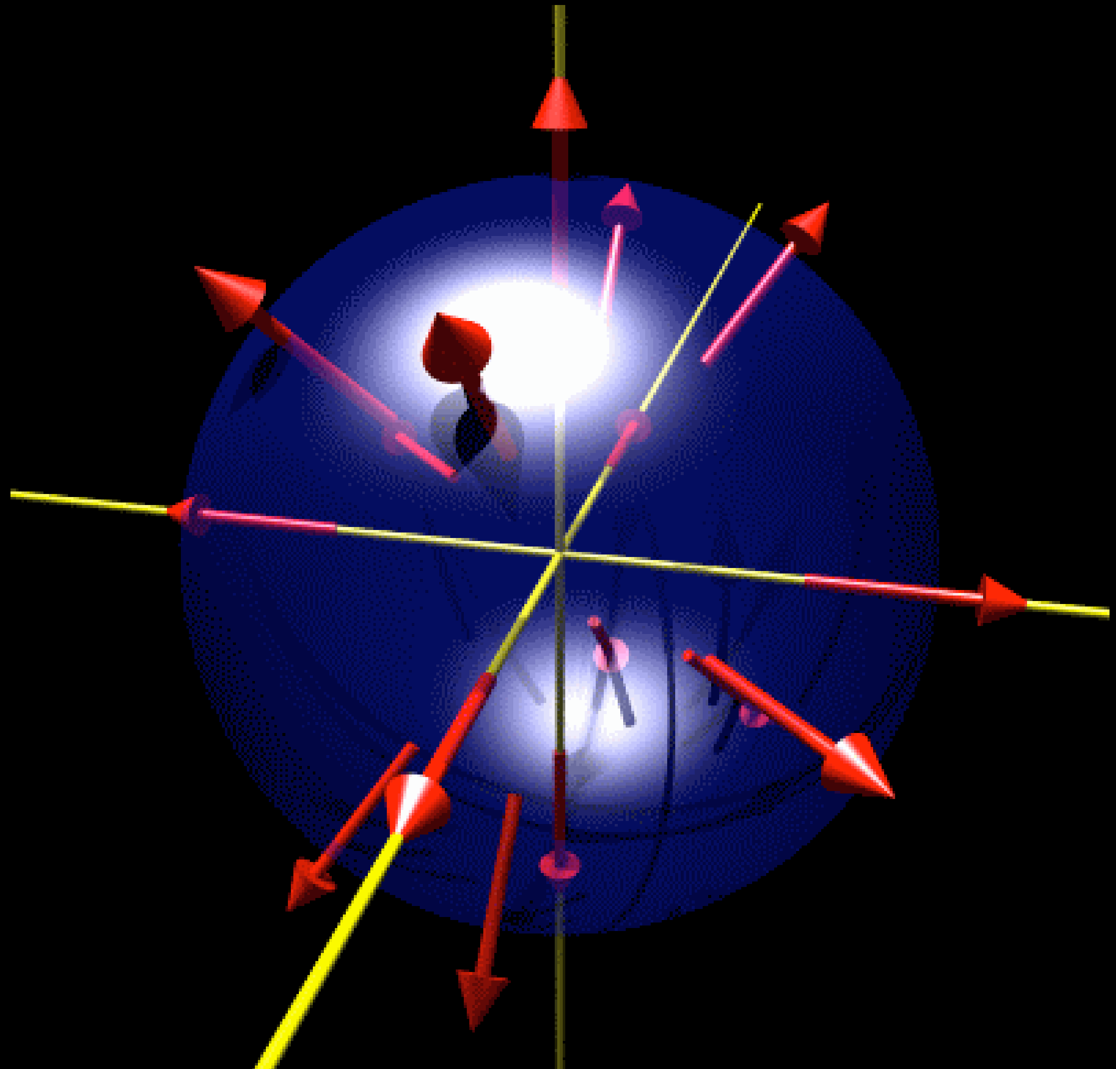

The divergence theorem

Divergence Theorem

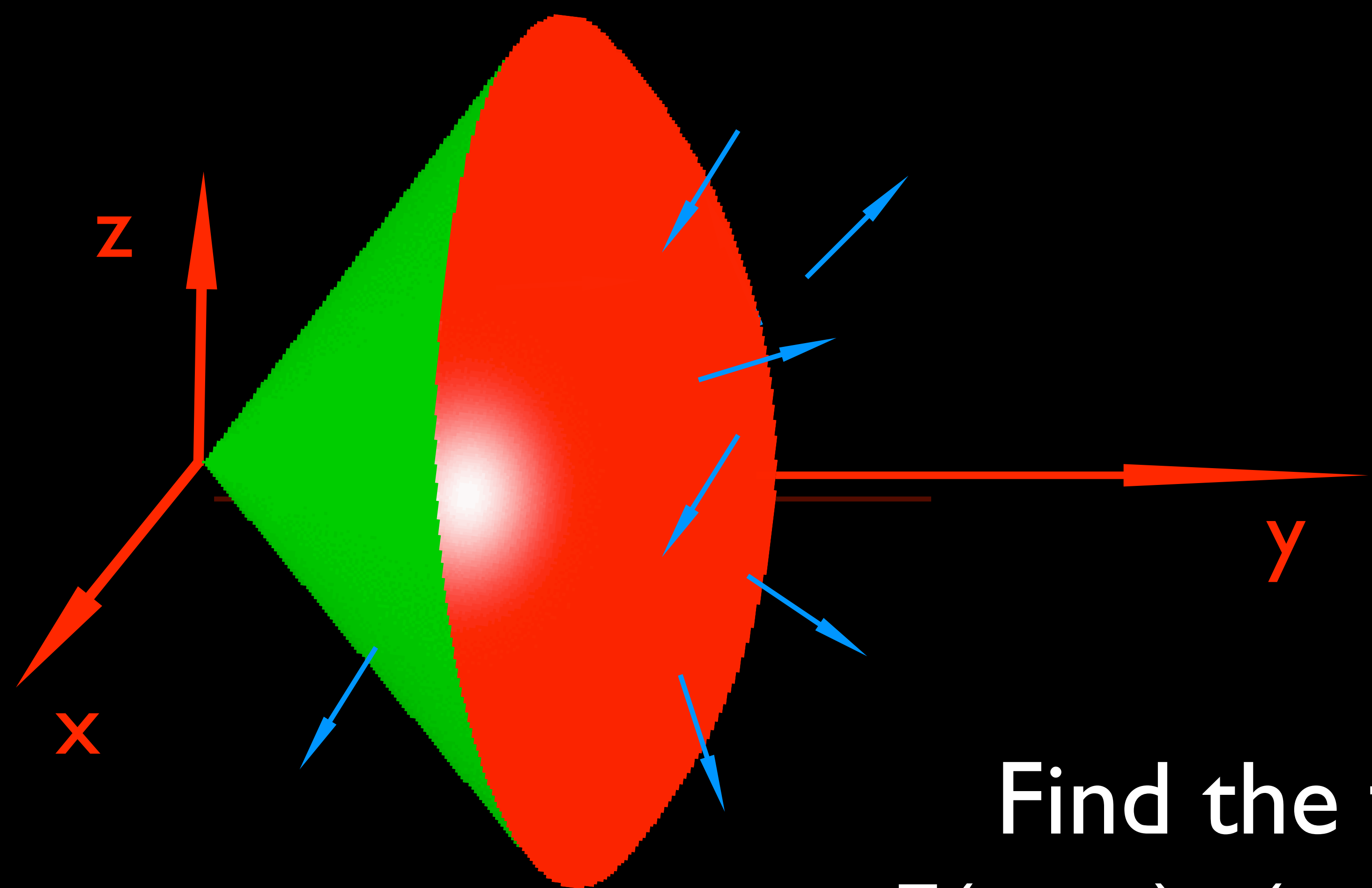
Divergence Theorem

$$\mathrm{div}(F) = P_x + Q_y + R_z$$

$$\int\int_S F \cdot dS = \int\int\int_E \mathrm{div}(F)\ dV$$



Problem



Find the flux of the vector field
F(x,y,z)=(x-sin(y),y+1,log(3+y sin(x))
through the cone |x|=|z| intersected
with
y>0 and 0<|(x,y,z)|<2

Integral Theorems

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$

$$\int\int_R \mathrm{curl}(F)\ dA = \int_C F \cdot dr$$

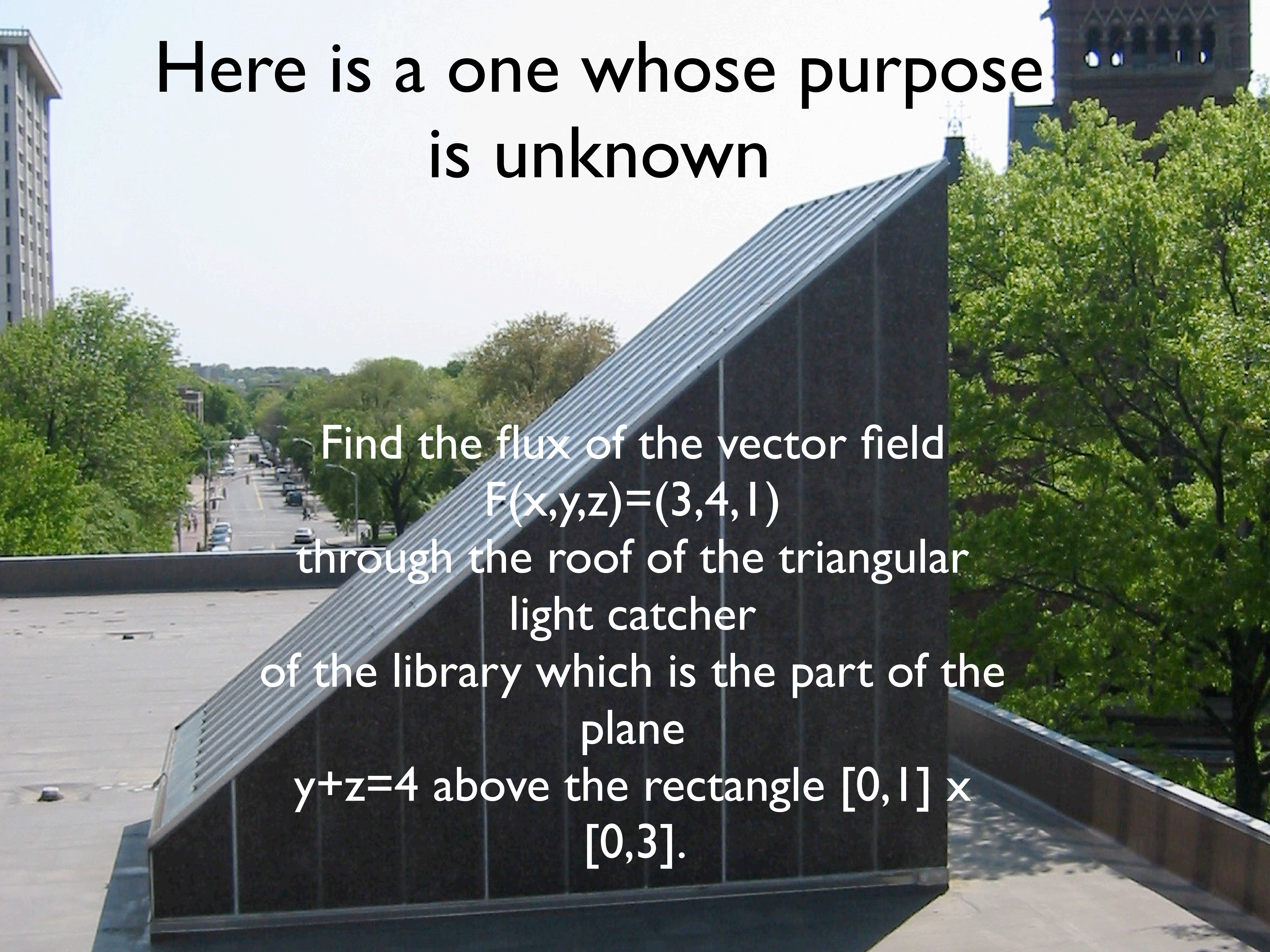
$$\int\int_S \mathrm{curl}(F) \cdot dS = \int_C F \cdot dr$$

$$\int\int\int_B \mathrm{div}(F)\ dV = \int\int_S F \cdot dS$$


"That's all Folks!"
P
Q

There would be tons of other problems:

Here is a one whose purpose is unknown



Find the flux of the vector field
$F(x,y,z)=(3,4,1)$
through the roof of the triangular light catcher
of the library which is the part of the plane
$y+z=4$ above the rectangle $[0,1] \times [0,3]$.

Integrals Overview

line and flux integrals:

$$\int_C F \cdot dr = \int_C F(r(t)) \cdot r'(t)\ dt$$

$$\int \int_S F \cdot dS = \int \int_R F(r(u,v)) \cdot (r_u \times r_v)\ dudv$$

length and area:

$$\int_C 1 ds = \int_a^b |r'(t)|\ dt$$

$$\int \int_S 1 dS = \int \int_R |r_u \times r_v|\ dudv$$

double and triple integrals:

Derivatives

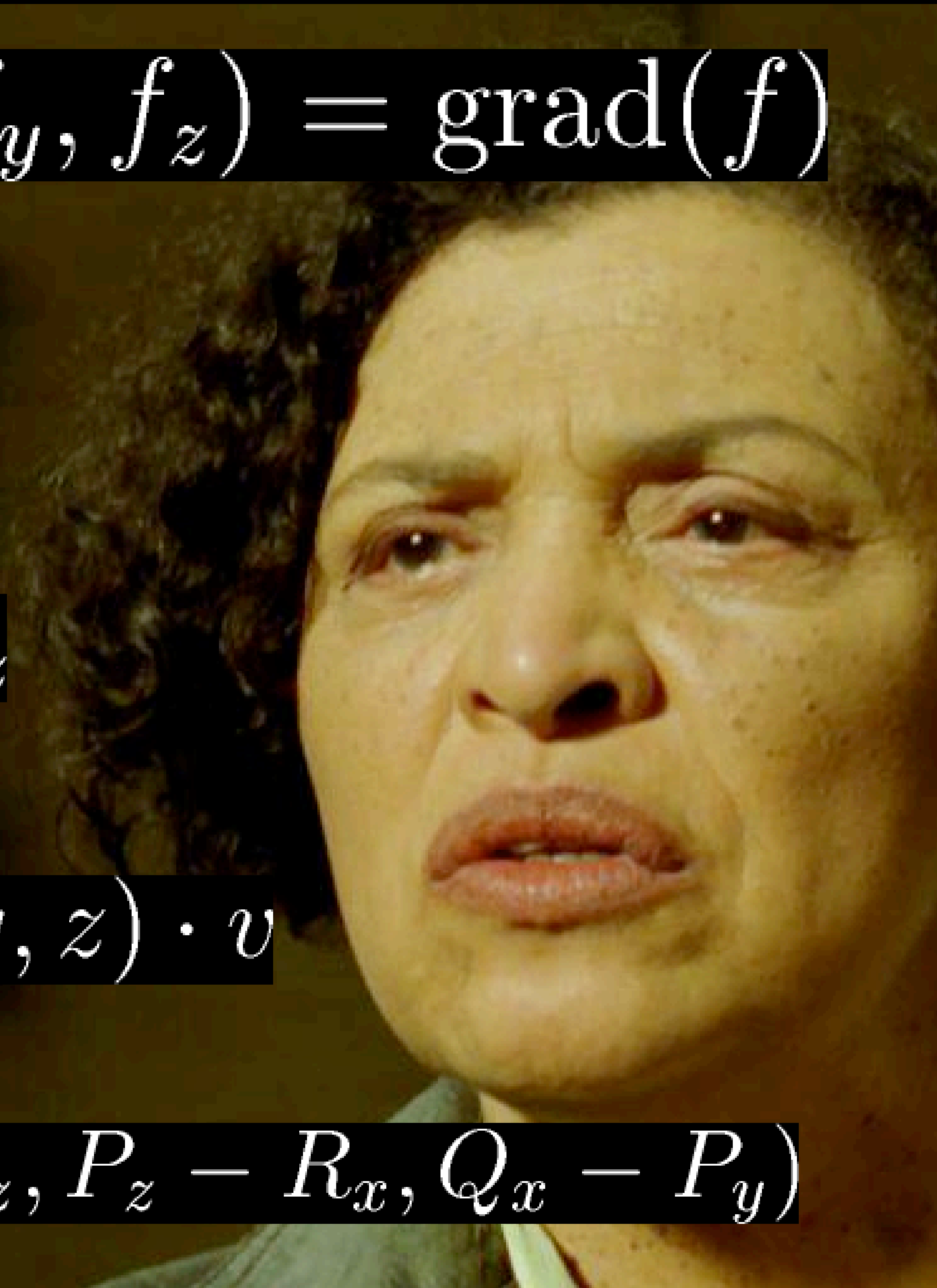
$$\nabla f(x, y, z) = (f_x, f_y, f_z) = \mathrm{grad}(f)$$

$$\partial_x f = f_x = \frac{\partial f}{\partial x}$$

$$\mathrm{div}(F) = P_x + Q_y + R_z$$

$$D_v f(x, y, z) = \nabla f(x, y, z) \cdot v$$

$$\mathrm{curl}(P, Q, R) = (R_y - Q_z, P_z - R_x, Q_x - P_y)$$



Derivatives Overview

$$\mathrm{grad}(f) = (f_x, f_y, f_z)$$

$$\mathrm{curl}(F) = N_x - M_y$$

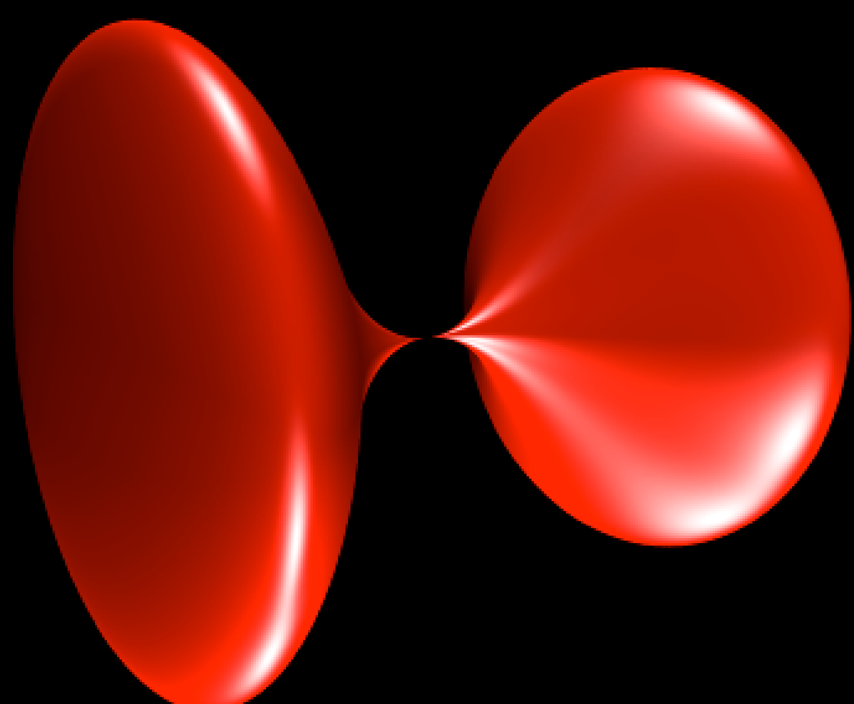
$$\mathrm{curl}(F) = (P_y - N_z, M_z - P_x, N_x - M_y)$$

$$\mathrm{div}(F) = M_x + N_y + P_z$$

Identities

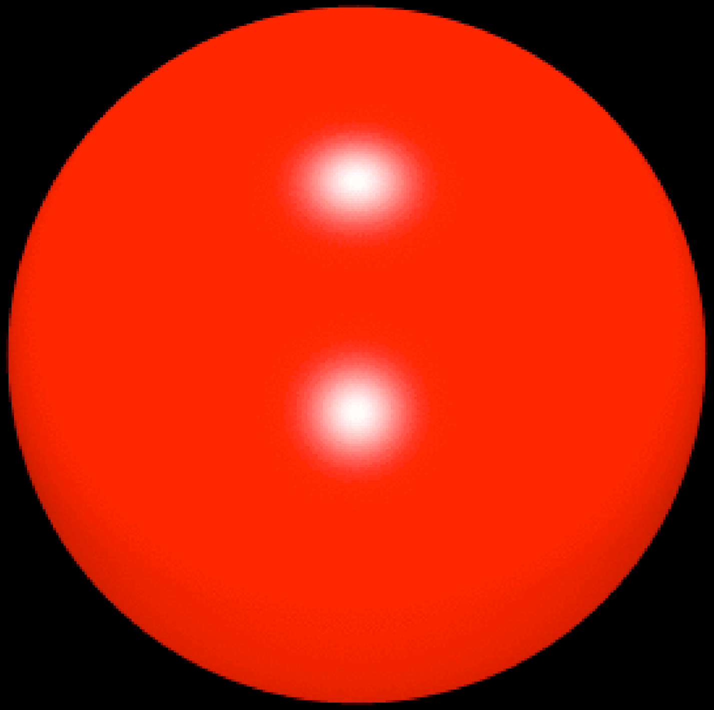
- div(curl(F))=0
- curl(grad(f))=0

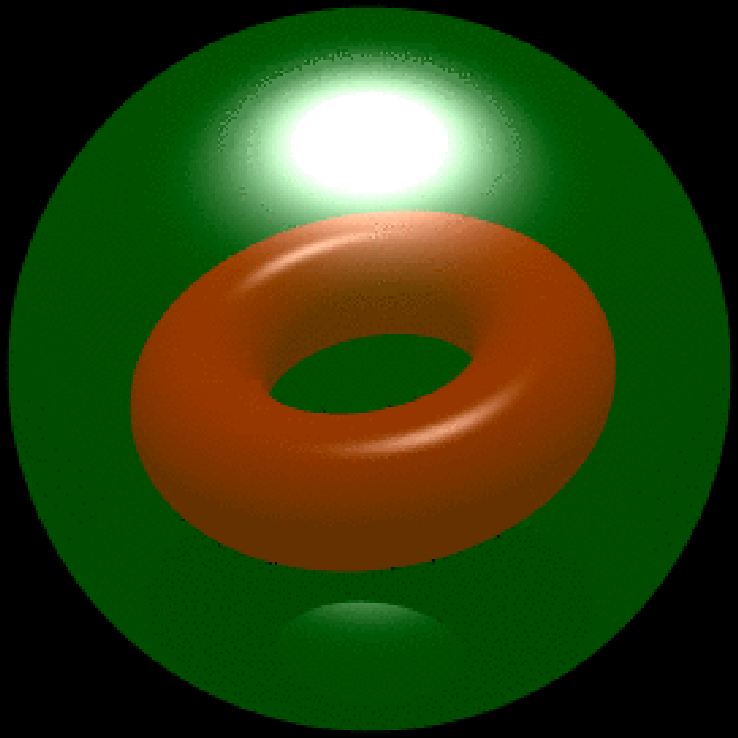
The line integral of F=grad(f) along a closed curve is zero. (Either Stokes of fundamental Theorem of Line integrals)

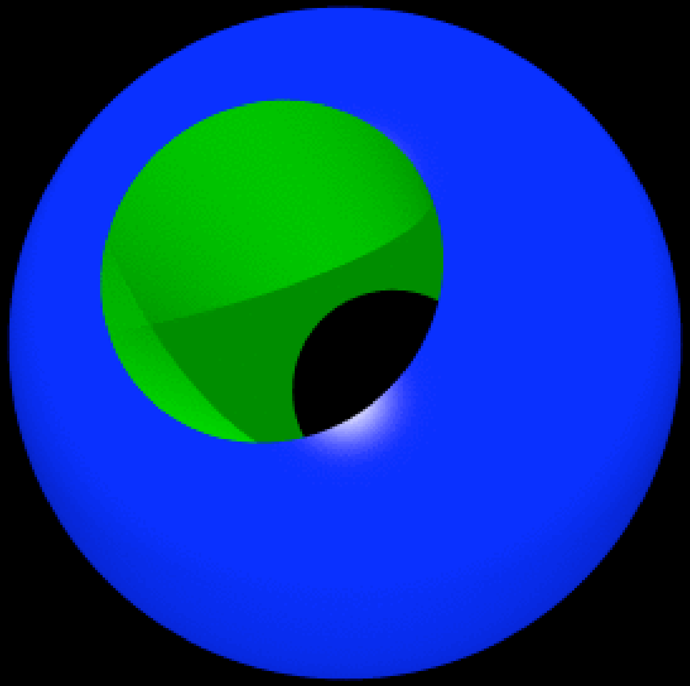


The flux of F=curl(G) through a closed surface is zero. (Either Divergence theorem of Stokes Theorem)

Simply connected

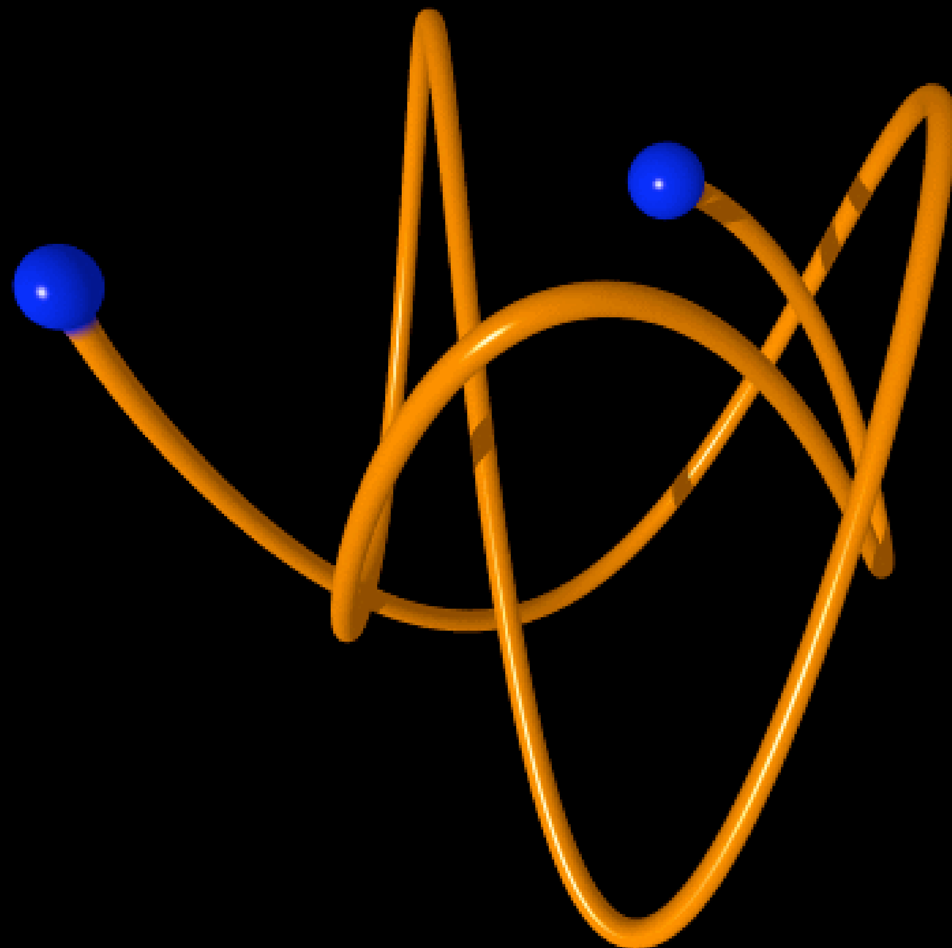


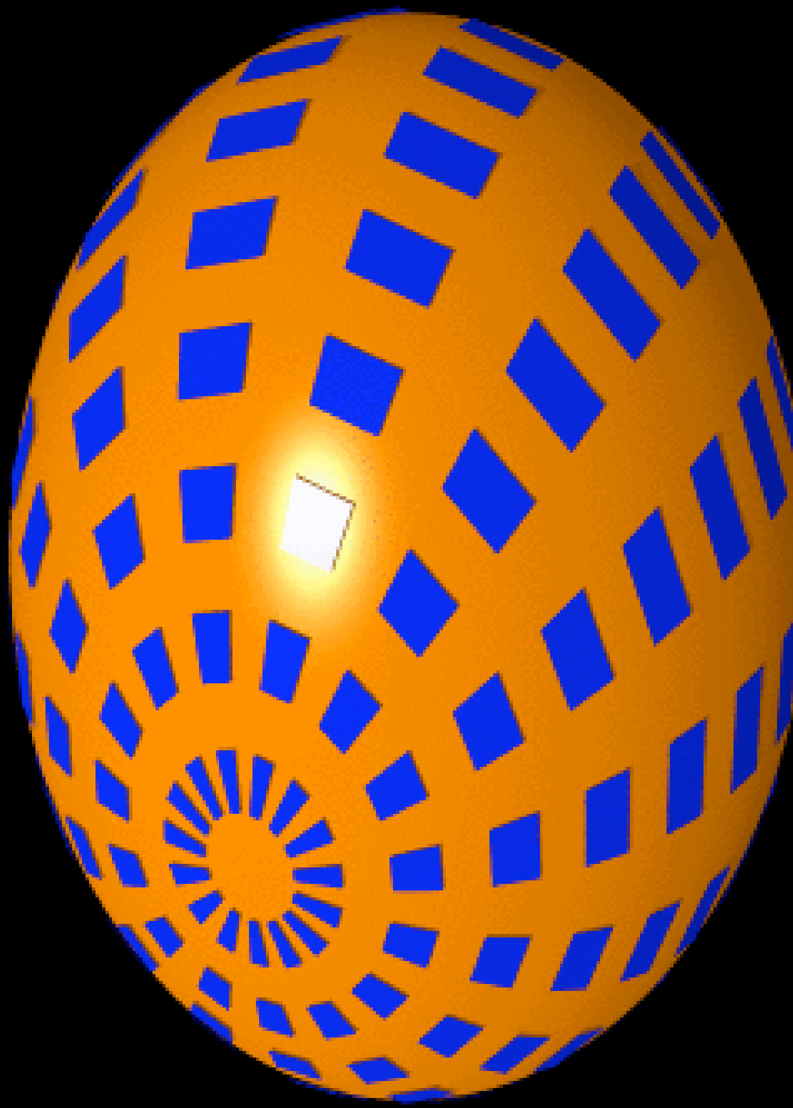


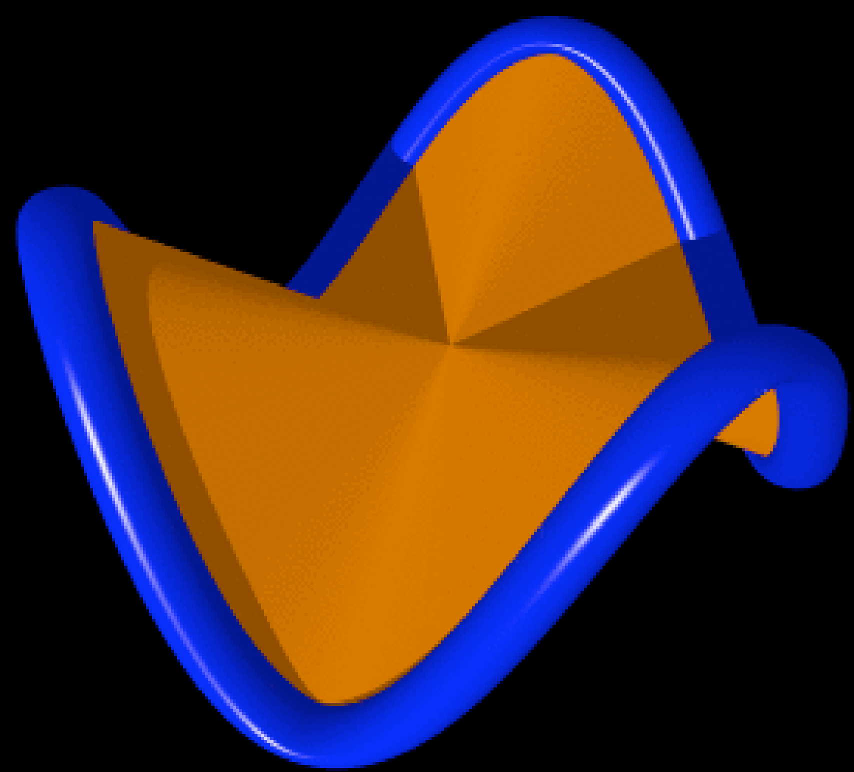


Simply connected ?

Boundary







Derivatives Overview

$$\mathrm{grad}(f) = (f_x, f_y)$$

$$\mathrm{curl}(F) = Q_x - P_y$$

$$\mathrm{grad}(f) = (f_x, f_y, f_z)$$

$$\mathrm{curl}(F) = (R_y - Q_z, P_z - R_x, Q_x - P_y)$$

$$\mathrm{div}(F) = P_x + Q_y + R_z$$

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Only regular and physics sections

length and area:

$$\int_C 1 ds = \int_a^b |r'(t)|\ dt$$

$$\int\int_S 1 dS = \int\int_R |r_u \times r_v|\ dudv$$

Double integrals

triple integrals