THE GRADIENT   Math21a

Problem A: $\vec{r}(u,v) = (u, v^2, u^2 + v^2)$ is a parametrized surface S.

a) Find an implicit equation $g(x,y,z) = 0$ for this surface.
b) Use a) to find the tangent plane at the point $(1,1,2)$.
Problem B:

a) Why are the vectors $r_u(u,v)$ and $r_v(u,v)$ tangent to the S?
b) Use a) to find the tangent plane at the point $(1,1,2)$ again.
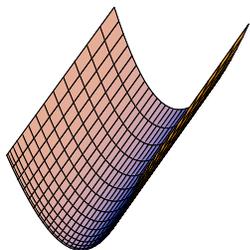
GRADIENT. Define the **gradient** of a function $f(x,y)$ as $\nabla f(x,y) = (f_x(x,y), f_y(x,y))$ or $\nabla f(x,y,z) = (f_x(x,y,z), f_y(x,y,z), f_z(x,y,z))$ if the function has three variables.

ETYMOLOGY. The name "Nabla" means "Egyption harp". Its origin is believed to be related to the Hebrew word "nevel"=harp which seems to have the same aramaic origin.

REMINDER CHAIN RULE. The chain rule in multivariable calculus can be written more compactly as

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$$

It looks like the 1D chain rule, but the derivative f' is replaced with the gradient and the derivative of r is the velocity.

GRADIENTS AND LEVEL CURVES/SURFACES.

**Gradients are orthogonal to level curves and level surfaces.**

PROOF. Every curve $\vec{r}(t)$ on the level curve or level surface satisfies $\frac{d}{dt} f(\vec{r}(t)) = 0$. By the chain rule, $\nabla f(\vec{r}(t))$ is perpendicular to the tangent vector $\vec{r}'(t)$.
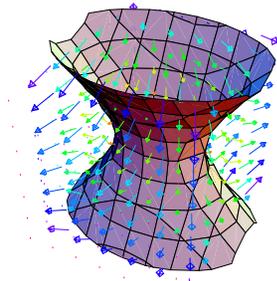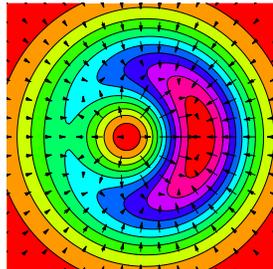

REMINDER TANGENT LINE AND TANGENT PLANE.
Because $\vec{n} = \nabla f(x_0, y_0) = \langle a, b \rangle$ is perpendicular to the level curve $f(x,y) = c$ through (x_0, y_0), the equation for the tangent line is $ax + by = d$, $a = f_x(x_0, y_0)$, $b = f_y(x_0, y_0)$, $d = ax_0 + by_0$. This can be written ccompactly as

$$\nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = 0$$

This equation means that the gradient of f is perpendicular to any vector $(\vec{x} - \vec{x}_0)$ in the plane.
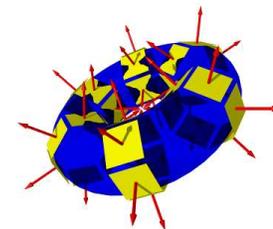
PROBLEM. Compute the tangent plane to the surface $3x^2 y + z^2 - 4 = 0$ at the point $(1,1,1)$.
SOLUTION $\nabla f(x,y,z) = \langle 6xy, 3x^2, 2z \rangle$. And $\nabla f(1,1,1) = \langle 6, 3, 2 \rangle$. The plane is $6x + 3y + 2z = 11$.
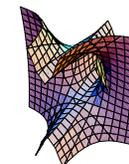
REMINDER: LINEARIZATION. Using the gradient, we can write the linearization of $f(x,y)$

$$L(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0)$$

This is the analogue of $L(x) = f(x_0) + f'(x_0)(x - x_0)$ in one dimensions.

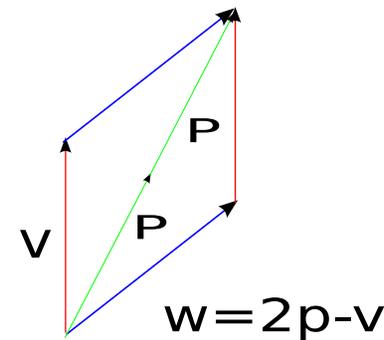PROBLEM. Reflect the ray $\vec{r}(t) = \langle 1 - t, -t, 1 \rangle$ at the surface
$$x^4 + z^2 + z^6 = 6 .$$

Solution. $\vec{r}(t)$ hits the surface at the time $t = 2$ in the point $(-1, -2, 1)$. The velocity vector in that ray is $\vec{v} = \langle -1, -1, 0 \rangle$ The normal vector at this point is $\nabla f(-1, -2, 1) = \langle -4, 4, 6 \rangle = \vec{n}$.

Remember the formula for the reflection derived in the midterm? It was

$$R(\vec{v}) = 2\text{Proj}_{\vec{n}}(\vec{v}) - \vec{v} .$$

We have $\text{Proj}_{\vec{n}}(\vec{v}) = 8/68 \langle -4, -4, 6 \rangle$. Therefore, the reflected ray is $\vec{w} = (4/17)\langle -4, -4, 6 \rangle - \langle -1, -1, 0 \rangle$.

PROBLEM. Given the curve $x^4 + y^4 = 1$ in the plane. Let $f(x,y)$ denote the distance to that curve. Show that f satisfies the PDE $f_x^2 + f_y^2 = 1$.

PROOF. The partial differential equation tells that $||\nabla f|| = 1$ everywhere in the plane. This means that the level curves of f have equal distance. In other words, they are also the level curves of the distance function to the curve.