LINEAR APPROXIMATION APPLICATIONS — Math21a, O. Knill

COMPUTATION TRICKS.

What is the fifth root of 244? Because $3^5 = 243 = x_0$, we can linearize the function $f(x) = x^{1/5}$ near the point $x_0 = 243$. Because $f(x)$ is close to $L(x) = f(x_0) + f'(x_0)(x - x_0) = 3 + x_0^{-4/5}/5(x - 243)$ and $L(244) = 3 + 243^{-4/5}/5 \cdot 1 = 3 + (1/81)/5 = 3 + 1/405 = 1216/405 = 3.00246914$. The actual result is 3.002465081. The difference is 4 over one million. We have found a rational number, which is an up to the 5'th digit acurate approximation to the fifth root. An other example: we can estimate $\sqrt{10} = 3+1/(2\sqrt{9}) = 3+1/6$. The actual result is 3.16228 which is pretty close. Within the squareroots of $1 - 100$ we can get like this an approximation up to 2 percent. Except for $n = 7, 20, 30$ where the error is less than 3 percent.

PHYSICAL LAWS IN GENERAL.

Many physical laws are in fact linear approximations to more complicated laws. One could say that a large fraction of physics consists of understand nature with linear laws.

LINEAR STABILITY ANALYSIS.

In physics, complicated situations can occur. Usually, many unknown parameters are present and the only way to analyze the situation theoretically is to assume that things depend linearly on these parameters. The analysis of the linear situation allows then to predict for example the stability of the system with respect to perturbations. Sometimes, the stability of the linearized system will imply the stability of the perturbation.

ERROR ANALYSIS.

Error analysis is based on linear approximation. Assume, you make a measurement of a function $f(a, b, c)$, where a, b, c are parameters. Assume, you know the numbers a, b, c up to accuracy ϵ. How precise do you know the values $f(a, b, c)$? Because $f(a_0 + \epsilon_a, b_0 + \epsilon_b, c_0 + \epsilon_c)$ is about $f(a_0, b_0, c_0) + \nabla f(a_0, b_0, c_0) \cdot (\epsilon_a, \epsilon_b, \epsilon_c)$, the answer is that we know F up to accuracy $|\nabla f(a_0, b_0, c_0)|\epsilon$.

POWER LAWS.

Some laws in physics are given by functions of the form $g(x, y) = x^\alpha y^\beta$. An example is the Cobb-Douglas formula in economics. Such dependence on x or y is called **power law behavior**. If we consider $f = \log(g)$, and introduce $a = \log(x), b = \log(y)$, then this becomes $f(a, b) = \log(g(x, y)) = a\alpha + b\beta$. Power laws become linear laws in a logarithmic scale. But they usually are linear approximations to more complicated nonlinear relations.

ELECTRONICS.

If we apply a voltage difference U at the ends of a resistor R, then a current I flows. The relation $U = RI$ is called **Ohms law**. In logarithmic coordinates $\log(U) = \log(R) + \log(I)$, this is a linear law. In reality, the relation between current, voltage and resistance is more complicated. For example, if the resistor heats up, then its characteristics begin to change. Nonlinear resistors are used for example in synthesizers or in radars. While Ohm's law works **extremely well**, the nonlinear behavior can have important consequences for example to stabilize systems or to protect equipment against over-voltages.

THERMODYNAMICS.

If $l(T)$ is the length of an object with temperature T, then $l(T) = l(T_0) + c(T - T_0)$, where the expansion coefficient c depends on the material. (Trick question: What happens if you heat a ring, does the inner ring become smaller or bigger?). The volume of a hot air balloon and therefore its lift capacity grows like $c(T - T_0)^3$. The law of expansion is only an approximation.

OCEANOGRAPHY.

For oceanographers, it is important to know the water density $\rho(T, S)$ in dependence on the **temperature** T (Kelvin) and the **salinity** S (psu). If we would include the pressure P (Bar), then we had a function $\rho(T, S, P)$ of three variables. Near a specific point (T, S, P) the density can be approximated by a linear function giving a law which is precise enough.

ENGENEERING.

Hooks law tells that the force of a spring is proportional to the length with which it is pulled: $F(l) = c(l - l_0)$, where l_0 is the length when the spring is relaxed. This allows to measure weights or to cushion shocks. However, this law is only good in a certain range. If the spring is pulled too strongly, then more force is needed. Such a nonlinear behavior is needed for example in shock absorbers.

MECHANICS

For small amplitudes, the pendulum motion $\ddot{x} = -g\sin(x)$ can be approximated by $\ddot{x} = -gx$, the harmonic oscillator. Nonlinear (partial) differential equations like $u_{xx} + u_{yy} + u_{zz} = F(x, y, z)$ are often approximated by linear differential equations.

CARTOGRAPHY.

It was well known already to the Greeks that we live on a sphere. On a sphere a triangle however the sum of its angles adds up to more than 180 degrees and every straight line (great circles) crosses every other line at least twice. Despite this, a city map can perfectly assume that the coordinate system is Cartesian. When drawing a plan of a house, an architect can assume that the house stands on a plane (the level curve of the linearization $G(x, y)$ of $F(x, y)$ defining the surface of the earth.)

RELATIVITY

Newton's law tells that $r''(t)$, the acceleration of a particle is proportional to the force F which acts on the mass point: $r''(t) = F/m$. For a constant force and zero initial velocity this implies $r'(t) = tF/m$. This law can not apply for all times, because we can not reach the speed of light with a massive body. In special relativity, the Newton axiom is replaced with $d/dt(r'(t)m(t)) = F$, where the mass $m(t)$ depends on the velocity. This gives $v(t) = (tF/m_0)\frac{1}{\sqrt{1+F^2t^2/(c^2m_0^2)}}$. Linearization at $t = 0$ produces the classical law $v(t) = tF/m_0$.

ECONOMICS.

The mathematician Charles W. Cobb and the economist Paul H. Douglas found in 1928 empirically a formula $F(L, K) = bL^\alpha K^\beta$ giving the total production F of an economic system as a function of the amount of labor L and the capital investment K. This is a linear law in logarithmic coordinates. The formula actually had been found by linear fit of empirical data. In general, the production depends in a more complicated way on labor and capital investment. For example, with increase of labor and investment, logistic constraints will become relevant.

CHEMISTRY.

The ideal gas law $PV = RT$ relates the pressure, the volume and the temperature of an ideal gas using a constant R called the Avogadro number. This law $T = f(P, V)$ is linear in logarithmic scales. This law is only an approximation and has to be replaced by the van der Waals law, which takes into account the molecular interactions as well as the volume of the molecules.