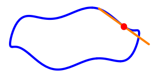
LINEAR APPROXIMATION — Math21a, O. Knill

HOMEWORK: Section 11.4: 2,4,26,30,32 Due Mon Oct 31

LINEAR APPROXIMATION.

1D: The **linear approximation** of a function $f(x)$ at a point x_0 is the linear function $\boxed{L(x) = f(x_0) + f'(x_0)(x - x_0)}$. The graph of L is tangent to the graph of f at x_0.



2D: The **linear approximation** of a function $f(x,y)$ at (x_0, y_0) is

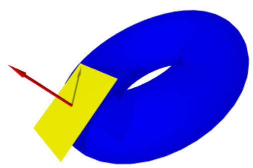
$$\boxed{L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)}$$

The level curve of g is tangent to the level curve of f at (x_0, y_0). The graph of L is tangent to the graph of f.

3D: The **linear approximation** of a function $f(x,y,z)$ at (x_0, y_0, z_0) by

$$\boxed{L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x - x_0) + f_y(x_0,y_0,z_0)(y - y_0) + f_z(x_0,y_0,z_0)(z - z_0)}.$$

The level surface of L is tangent to the level surface of f at (x_0, y_0, z_0).



We will write next week the linearization using $\nabla f = (f_x, f_y)$ as

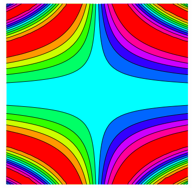
$$\boxed{L(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0)}$$

JUSTIFYING THE LINEAR APPROXIMATION.

If the second variable $y = y_0$ is fixed, then we have a one dimensional situation where the only variable is x. Now $f(x, y_0) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0)$ is the linear approximation. Similarly, if $x = x_0$ is fixed y is the single variable, then $f(x_0, y) = f(x_0, y_0) + f_y(x_0, y_0)(y - y_0)$. Knowing the linear approximations in both the x and y varibles, we can get the general linear approximation by $f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$.

PREVIEW: An other justification will be shown next week. We will see that the vector $(a,b) = (f_x, f_y)$ is perpendicular to the level curve at (x_0, y_0). Because the line $ax + by = ax_0 + by_0$ has also the vector (a,b) perpendicular to the curve and the curve and line pass through the same point (x_0, y_0), they are tangent. The line is the best among all lines passing through (x_0, y_0).

EXAMPLE (2D) Find the linear approximation of the function $f(x,y) = \sin(\pi x y^2)$ at the point $(1,1)$. We have $(f_x(x,y), y_f(x,y) = (\pi y^2 \cos(\pi x y^2), 2y\pi \cos(\pi x y^2))$ which is at the point $(1,1)$ equal to $\nabla f(1,1) = (\pi \cos(\pi), 2\pi \cos(\pi)) = (-\pi, 2\pi)$. The linear function approximating f is $L(x,y) = f(1,1) + (f_x(1,1), f_y(1,1)) \cdot (x - 1, y - 1) = 0 - \pi(x-1) - 2\pi(y-1) = -\pi x - 2\pi y + 3\pi$. The level curves of G are the lines $x + 2y = const$. The line which passes through $(1,1)$ satisfies $x + 2y = 3$.



Application: $-0.00943407 = f(1+0.01, 1+0.01) \sim g(1+0.01, 1+0.01) = -\pi 0.01 - 2\pi 0.01 + 3\pi = -0.00942478$.

EXAMPLE (3D) Find the linear approximation to $f(x,y,z) = xy + yz + zx$ at the point $(1,1,1)$.

We have $f(1,1,1) = 3$, $\nabla f(x,y,z) = (y+z, x+z, y+x), \nabla f(1,1,1) = (2,2,2)$. Therefore $L(x,y,z) = f(1,1,1) + (2,2,2) \cdot (x-1, y-1, z-1) = 3 + 2(x-1) + 2(y-1) + 2(z-1) = 2x + 2y + 2z - 3$.

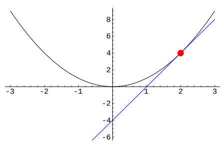
EXAMPLE (3D). Use the best linear approximation to $f(x,y,z) = e^x \sqrt{y} z$ to estimate the value of f at the point $(0.01, 24.8, 1.02)$.

Solution. Take $(x_0, y_0, z_0) = (0, 25, 1)$, where $f(x_0, y_0, z_0) = 5$. The gradient is $\nabla f(x,y,z) = (e^x \sqrt{y} z, e^x z/(2\sqrt{y}), e^x \sqrt{y})$. At the point $(x_0, y_0, z_0) = (0, 25, 1)$ the gradient is the vector $(5, 1/10, 5)$. The linear approximation is $L(x,y,z) = f(x_0,y_0,z_0) + \nabla f(x_0,y_0,z_0)(x - x_0, y - y_0, z - z_0) = 5 + (5, 1/10, 5)(x - 0, y - 25, z - 1) = 5x + y/10 + 5z - 2.5$. We can approximate $f(0.01, 24.8, 1.02)$ by $5 + (5, 1/10, 5) \cdot (0.01, -0.2, 0.02) = 5 + 0.05 - 0.02 + 0.10 = 5.13$. The actual value is $f(0.01, 24.8, 1.02) = 5.1306$, very close to the estimate.

TANGENT LINE FOR 1D FUNCTIONS? Because $\vec{n} = \nabla f(x_0, y_0) = \langle a, b \rangle$ is perpendicular to the level curve $f(x,y) = c$ through (x_0, y_0), the equation for the tangent line is

$$\boxed{ax + by = d,\ a = f_x(x_0,y_0),\ b = f_y(x_0,y_0),\ d = ax_0 + by_0}$$

Example: Find the tangent to the graph of the function $g(x) = x^2$ at the point $(2,4)$. Solution: the level curve $f(x,y) = y - x^2 = 0$ is the graph of a function $g(x) = x^2$ and the tangent at a point $(2, g(2)) = (2,4)$ is obtained by computing the gradient $\langle a, b \rangle = \nabla f(2,4) = \langle -g'(2), 1 \rangle = \langle -4, 1 \rangle$ and forming $-4x + y = d$, where $d = -4 \cdot 2 + 1 \cdot 4 = -4$. The answer is $\boxed{-4x + y = -4}$ which is the line $y = 4x - 4$ of slope 4. Graphs of 1D functions are curves in the plane, you have computed tangents in single variable calculus.



TANGENT PLANES TO 3D FUNCTIONS. The tangent plane to the surface $g(x,y,z) = z - f(x,y) = 0$ at $(x_0, y_0, z_0 = f(x_0, z_0)$ is $-f_x x - f_y y + z = -f_x x_0 - f_y y_0 + z_0$. This can be read as $z = z_0 + f_x(x - x_0) + f_y(y - y_0)$. Calling the right hand side $L(x,y)$ shows that the graph of L is tangent to the graph of f at (x_0, y_0).

MIXING UP DIMENIONS. Do not mix up dimensions. For functions of two variables $f(x,y)$, the linear approximation is a function $L(x,y)$ of two variables. We have tangency in two different dimensions: the level curves of f are tangent to the level curves of L at (x_0, y_0). But we also know that the graph of L is tangent to the graph of f.

TOTAL DIFFERENTIAL. Aiming to estimate the change $\Delta f = f(x,y) - f(x_0, y_0)$ of f for points $(x,y) = (x_0, y_0) + (\Delta x, \Delta y)$ near (x_0, y_0), we can estimate it with the linear approximation which is $L(\Delta x, \Delta y) = f_x(x_0, y_0)\Delta x + f_y \Delta y$. In an old-fashioned notation, one writes also $df = f_x dx + f_y dy$ and calls df the **total differential**. One can **totally avoid** the notation of the **total differential**.