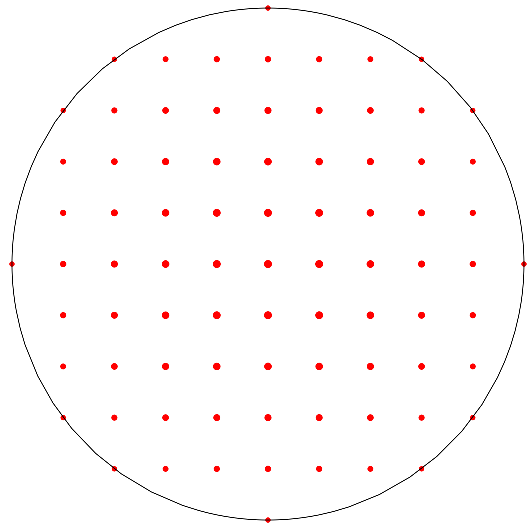
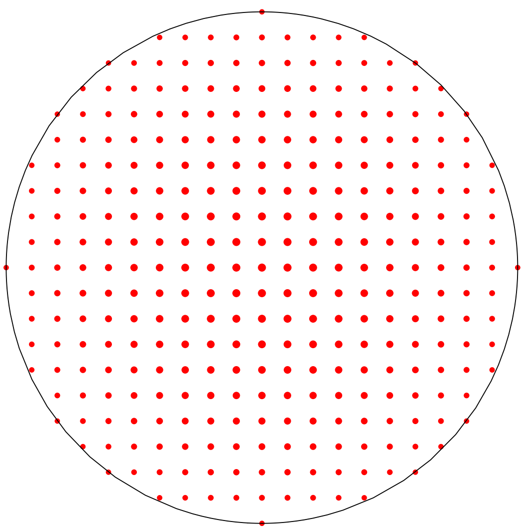
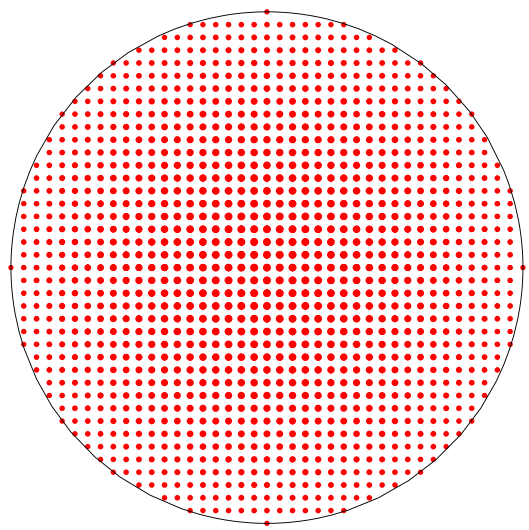
The double integral of a function f over a region R is defined as the limit of $\frac{1}{n^2}\sum_{x_{ij}\in R} f(x_{ij})$, when $n \to \infty$ if $x_{ij} = (i/n, j/n)$. Let us look at the case of a **disc** of radius r and $n = 5$ and the case $f(x,y) = 1$, where we compute the area. What is the approximation area, which we get by counting the number of lattice points inside R?



The **Gauss circle problem** asks to estimate the number of lattice points $g(r) = \pi r^2 + E(r)$ enclosed by the circle r. By counting, one gets $g(10) = 317, g(100) = 31417, g(1000) = 3141549, g(10000) = 314159053$.





Estimating the error $g(n)/n^2 - \pi$ is called the **"Gauss circle problem"**. If one writes $E(n) = g(n) - \pi n^2$, one believes that for every $\theta > 1/2$, there is a constant C such that $E(n) \le Cn^\theta$. Gauss knew that this is true for $\theta = 1$. It is now known to be true up to $\theta = 46/73$. The picture shows $E(n)/\sqrt{n}$ for n up to 1000.