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- Start by printing your name in the above box and check your section in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.
- The hourly exam itself will have space for work on each page. This space is excluded here in order to save printing resources.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications needed

1) T **F** The length of the sum of two vectors is always the sum of the length of the vectors.

Solution:
There is a triangle inequality in general. But equality only holds for parallel vectors pointing in the same direction

2) T **F** For any three vectors, $\vec{v} \times (\vec{w} + \vec{u}) = \vec{w} \times \vec{v} + \vec{u} \times \vec{v}$.

Solution:
The cross product is distributive but not commutative.

3) T **F** The set of points which satisfy $x^2 + 2x + y^2 - z^2 = 0$ is a cone.

Solution:
$x^2 + y^2 - z^2 = 0$ is a cone. Completion of the square adds an other constant and the surface is a one sheeted hyperboloid.

4) T **F** The functions $\sqrt{x + y - 1}$ and $\log(x + y - 1)$ have the same domain of definition.

Solution:
The square root is defined for 0, but the logarithm is not defined at 0.

5) **T** F If P, Q, R are 3 different points in space that don't lie in a line, then $\vec{PQ} \times \vec{RQ}$ is a vector orthogonal to the plane containing P, Q, R.

Solution:
The vectors $\vec{PQ}$ and $\vec{RQ}$ are both in the plane. The cross product is perpendicular to the plane.

6) **T** F The line $\vec{r}(t) = \langle 1 + 2t, 1 + 3t, 1 + 4t \rangle$ hits the plane $2x + 3y + 4z = 9$ at a right angle.

Solution:
The vector $\langle 2,3,4\rangle$ is in the line and perpendicular to the plane.

7) | T | F | (T highlighted)

The graph of $f(x,y)=\cos(xy)$ is a level surface of a function $g(x,y,z)$.

Solution:
Yes, it is the surface $g(x,y,z)=c$ for the function $g(x,y,z)=z-\cos(xy)$ and the constant $c=0$.

8) | T | F | (F highlighted)

For any two vectors, $\vec{v}\times\vec{w}=\vec{w}\times\vec{v}$.

Solution:
The cross product is anti commutative.

9) | T | F | (T highlighted)

If $|\vec{v}\times\vec{w}|=0$ for all vectors $\vec{w}$, then $\vec{v}=\vec{0}$.

Solution:
Assume $\vec{v}$ is not $\vec{0}$, then take $\vec{w}$ as a vector which is perpendicular to $\vec{v}$.

10) | T | F | (T highlighted)

If $\vec{u}$ and $\vec{v}$ are orthogonal vectors, then $(\vec{u}\times\vec{v})\times\vec{u}$ is parallel to $\vec{v}$.

Solution:
The vector in question is perpendicular to $\vec{u}$ and perpendicular to $\vec{u}\times\vec{v}$. Also $\vec{v}$ is perpendicular to $\vec{u}$ and $\vec{u}\times\vec{v}$.

11) | T | F | (F highlighted)

Every vector contained in the line $\vec{r}(t)=\langle 1+2t,1+3t,1+4t\rangle$ is parallel to the vector $\langle 1,1,1\rangle$.

Solution:
The line contains the point $(1,1,1)$ and a vector $\langle 2,3,4\rangle$.

12) T **F** There is a quadric $ax^2+by^2+cz^2+dx+ey+fz=e$ which is a hyperbola when intersected with the plane $z=0$, which is a hyperbola when intersected with the plane $y=0$ and which is a parabola when intersected with $x=0$.

Solution:
The answer can be found by checking through all the quadrics $ax^2+by^2+cz^2+dx+ey+fz=0$ we have seen hyperbololids, parabololids, ellipsoids, cylinders, cones. None of them has this property. P.S. If the quadrics were allowed to be alligned differently, for example if we are allowed to turn a given quadric in space, the answer changes to yes: take a cone with openeing angle larger than 90 degrees, turn it so that the xy-plane is parallel to a tangent plane, cutting the cone in a parabola. For most translations of this cone, the other two coordinate axes cut hyperbola.

13) T **F** The curvature of the curve $2\vec{r}(4t)$ at $t=0$ is twice the curvature of the curve $\vec{r}(t)$ at $t=0$.

Solution:
The curvature of the first curve is 1/2 of the curvature of the second curve.

14) T **F** The set of points which satisfy $x^2-2y^2-3z^2=0$ form an ellipsoid.

Solution:
The surface is an elliptical cone.

15) T **F** If $\vec{v}\times\vec{w}=(0,0,0)$, then $\vec{v}=\vec{w}$.

Solution:
The two vectors can be parallel and nonzero.

16) T **F** Every vector contained in the line $\vec{r}(t)=\langle 1+2t,1+3t,1+4t\rangle$ is parallel to the vector $\langle 1,1,1\rangle$.

Solution:
It is parallel to $\langle 2,3,4\rangle$

17) **T** F Two nonzero vectors are parallel if and only if their cross product is $\vec{0}$.

Solution:
You can use the formula $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\alpha)$. If this is zero, then either one of the vectors is the zero vector or $\sin(\alpha) = 0$. In all cases, this can be considered parallel.

18) | T | F | The vector $\vec{u} \times (\vec{v} \times \vec{w})$ is always in the same plane together with $\vec{v}$ and $\vec{w}$.

Solution:
Let $\vec{n} = (\vec{v} \times \vec{w})$ be the vector perpendicular to the plane spanned by $\vec{v}$ and $\vec{w}$. Then $\vec{u} \times (\vec{v} \times \vec{w}) = \vec{u} \times \vec{n}$ is perpendicular to $\vec{n}$. It is therefore parallel to the plane.

19) | T | F | The line $\vec{r}(t) = \langle 1 + 2t, 1 + 2t, 1 - 4t \rangle$ hits the plane $x + y + z = 9$ at a right angle.

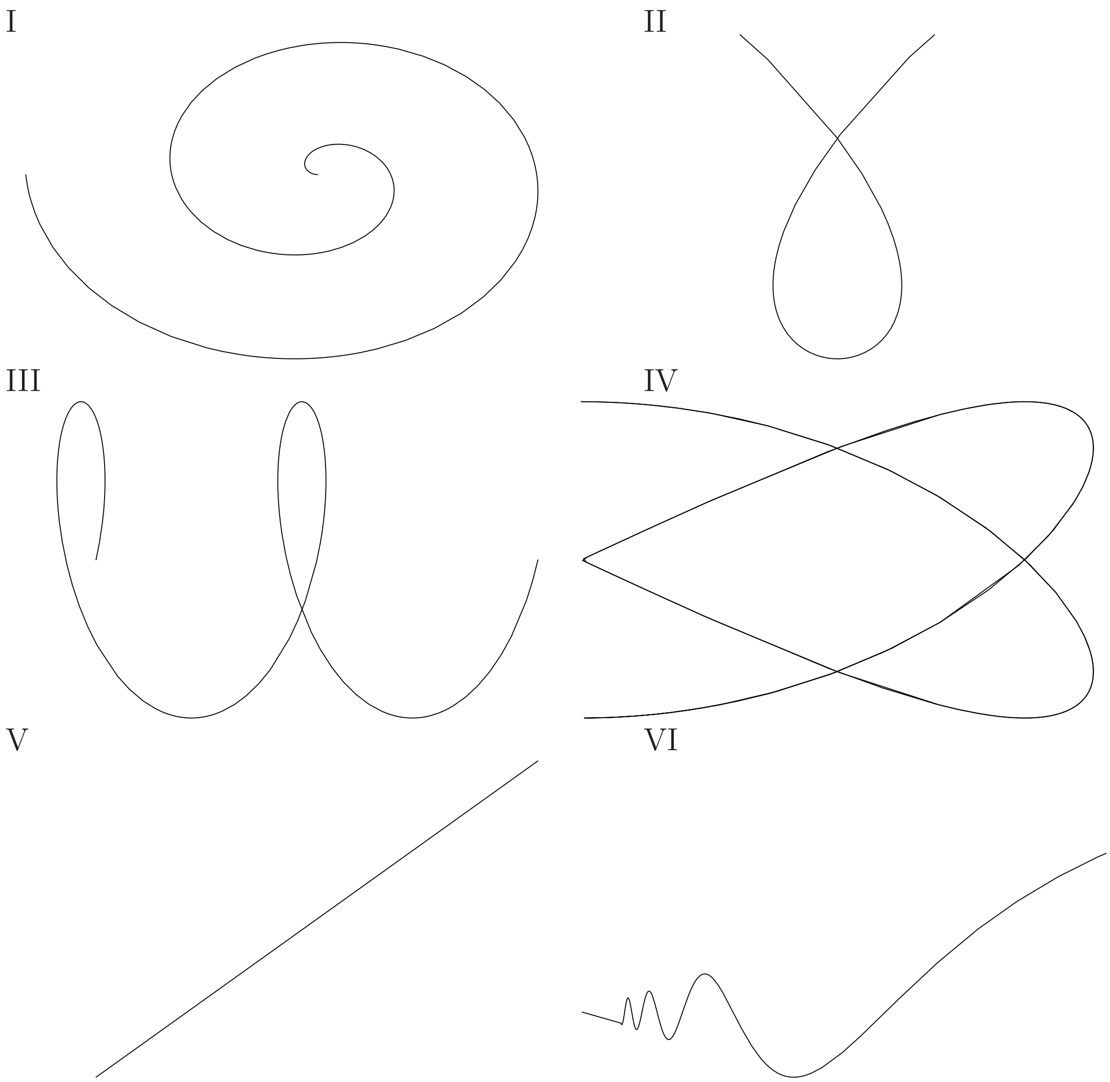
Solution:
In order to be perpendicular, the velocity vector would have to be parallel to $\langle 1, 1, 1 \rangle$.

20) | T | F | The intersection of the ellipsoid $x^2/3 + y^2/4 + z^2/3 = 1$ with the plane $y = 1$ is a circle.

Solution:
Just set $y = 1$ in that equation.

Problem 2a) (10 points)

Match the cruves with their parametric definitions.



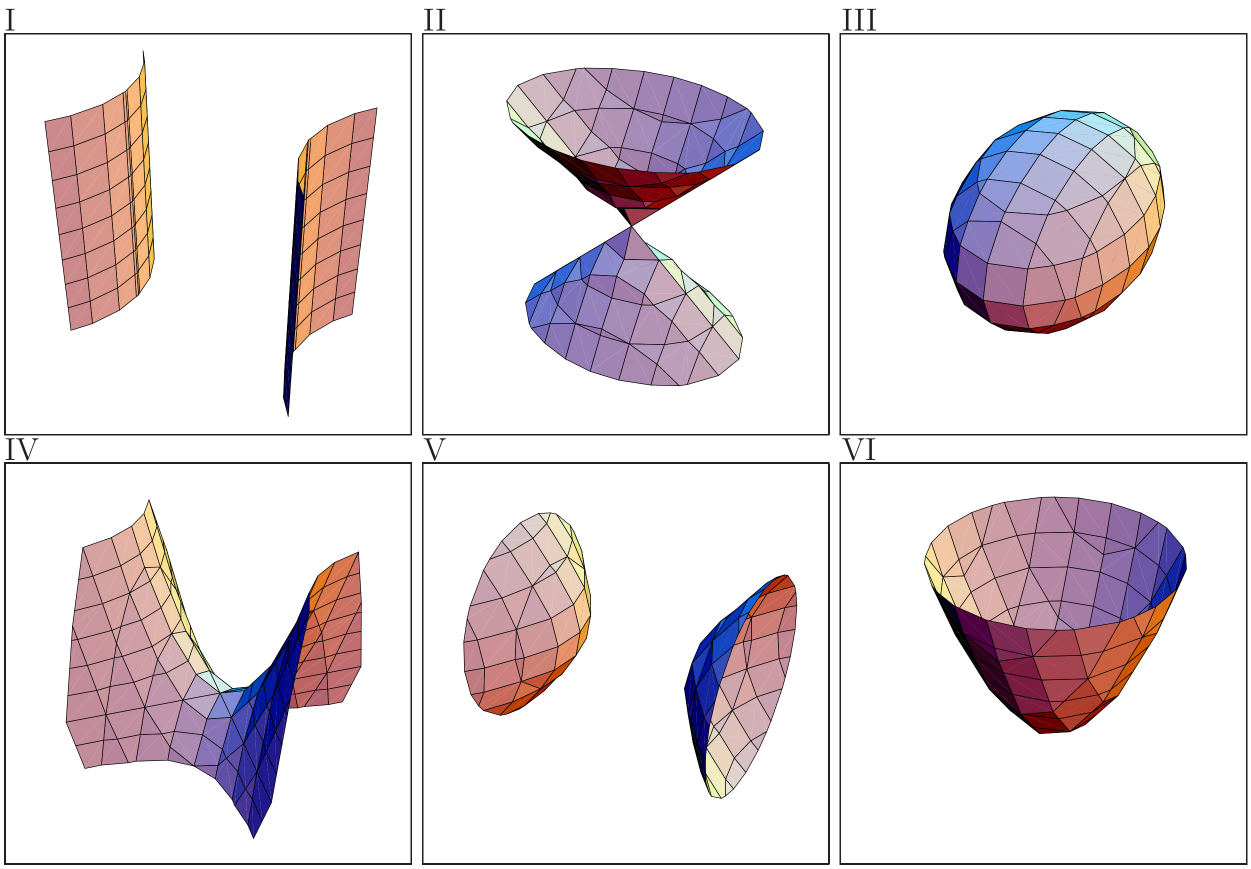
Enter I,II,III,IV,V or VI here	Parametric equation for the curve
	$\vec{r}(t) = \langle t, \sin(1/t)t \rangle$
	$\vec{r}(t) = \langle t^3 - t, t^2 \rangle$
	$\vec{r}(t) = \langle t + \cos(2t), \sin(2t) \rangle$
	$\vec{r}(t) = \langle \lvert\sin(2t)\rvert, \cos(3t) \rangle$
	$\vec{r}(t) = \langle 1 + t, 5 + 3t \rangle$
	$\vec{r}(t) = \langle -t\cos(t), 2t\sin(t) \rangle$

Solution:

Enter I,II,III,IV,V or VI here	Parametric equation for the curve
VI	$\vec{r}(t) = \langle t, \sin(1/t)t \rangle$
II	$\vec{r}(t) = \langle t^3 - t, t^2 \rangle$
III	$\vec{r}(t) = \langle t + \cos(2t), \sin(2t) \rangle$
IV	$\vec{r}(t) = \langle \lvert\sin(2t)\rvert, \cos(3t) \rangle$
V	$\vec{r}(t) = \langle 1 + t, 5 + 3t \rangle$
I	$\vec{r}(t) = \langle -t\cos(t), 2t\sin(t) \rangle$

Problem 2b) (4 points)

Match the equations with the surfaces.



Enter I,II,III,IV,V,VI here	Equation
	$x^2 - y^2 - z^2 = 1$
	$x^2 + 2y^2 = z^2$
	$2x^2 + y^2 + 2z^2 = 1$
	$x^2 - y^2 = 5$
	$x^2 - y^2 - z = 1$
	$x^2 + y^2 - z = 1$

Solution:

Enter I,II,III,IV,V,VI here	Equation
V	$x^2 - y^2 - z^2 = 1$
II	$x^2 + 2y^2 = z^2$
III	$2x^2 + y^2 + 2z^2 = 1$
I	$x^2 - y^2 = 5$
IV	$x^2 - y^2 - z = 1$
VI	$x^2 + y^2 - z = 1$

Problem 3) (10 points)

a) (6 points) Find a parameterization of the line of intersection of the planes $3x - 2y + z = 7$ and $x + 2y + 3z = -3$.

b) (4 points) Find the symmetric equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

representing that line.

Solution:
a) The line of intersection has the direction $(3, -2, 1) \times (1, 2, 3) = 8(-1, -1, 1)$. The parameterization is $\boxed{\vec{r}(t) = (1, -2, 0) + t(-1, -1, 1)}$.
b) If a line contains the point (x_0, y_0, z_0) and a vector $\langle a, b, c\rangle$, then the symmetric equation is

$$(x - x_0)/a = (y - y_0)/b = (z - z_0)/c \ .$$

In our case, where $(x_0, y_0, z_0) = (1, -2, 0)$ and $(a, b, c) = (-1, -1, 1)$, the symmetric equations are $\boxed{x - 1 = y + 2 = -z}$.

Problem 4) (10 points)

a) (4 points) Find the area of the parallelogram with vertices $P = (1, 0, 0)$ $Q = (0, 2, 0)$, $R = (0, 0, 3)$ and $S = (-1, 2, 3)$.

b) (3 points) Verify that the triple scalar product has the property $[\vec{u}+\vec{v}, \vec{v}+\vec{w}, \vec{w}+\vec{u}] = 2[\vec{u}, \vec{v}, \vec{w}]$.

c) (3 points) Verify that the triple scalar product $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ has the property

$$|[\vec{u}, \vec{v}, \vec{w}]| \leq ||\vec{u}|| \cdot ||\vec{v}|| \cdot ||\vec{w}||$$

Solution:
a) One has to realize which vectors form the sides of the parallelogram. The solution is $|\vec{PQ} \times \vec{PR}| = 7$.
b) $[u+v, v+w, w+u] = [u,v,w] + [u,v,u] + [u,w,w] + [u,w,u] + [v,v,w] + [v,v,u] + [v,w,w] + [v,w,u]$. Any term, where two parallel vectors appear is zero. So, only $2[u,v,w]$ remains on the right hand side.
c) Build the parallelepiped spanned by u, v, w and note that one can shear it in such a way that it is contained in the box of size $||\vec{u}||$ and $||\vec{v}||$ and $||\vec{w}||$. You can also see the identity by using angle formulas for the dot product $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos(\alpha)$ and the the length of the cross produc $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\beta)$

$$|[\vec{u}, \vec{v}, \vec{w}]| \leq |\vec{u}||\vec{v}||\vec{w}||\cos(\alpha)||\sin(\beta)|$$

where β is the angle between $\vec{v}$ and $\vec{w}$ and where α is the angle $\vec{v} \times \vec{w}$ and $\vec{u}$.

Problem 5) (10 points)

Find the distance between the two lines

$$\vec{r}_1(t) = \langle t, 2t, -t \rangle$$

and

$$\vec{r}_2(t) = \langle 1+t, t, t \rangle \ .$$

Solution:
The point $P = (0,0,0)$ is on the first line. The point $Q = (1,0,0)$ on the second line. The vector $\vec{v} = \langle 1, 2, -1 \rangle$ in the first line and $\vec{w} = \langle 1, 1, 1 \rangle$ in the second line. We have $\vec{n} = \langle 3, -2, -1 \rangle$. Now, the distance is $3/\sqrt{14}$. $(Q-P) \cdot \vec{n}/|\vec{n}| = \langle 1, 0, 0 \rangle \cdot \langle 3, -2, -1 \rangle / |n| = 3/\sqrt{14}$.

Problem 6) (10 points)

Find an equation for the plane that passes through the origin and whose normal vector is parallel to the line of intersection of the planes $2x + y + z = 4$ and $x + 3y + z = 2$.

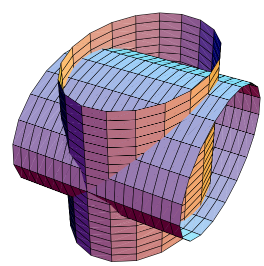
Solution:
The line of intersection is parallel to the cross product of $\vec{v} = \langle 2, 1, 1 \rangle$ and $\vec{w} = \langle 1, 3, 1 \rangle$ which is $\langle -2, -1, 5 \rangle$. This vector is perpendicular to the plane we are looking for. The equation of the plane is $-2x - y + 5z = 0$.

Problem 7) (10 points)

The intersection of the two surfaces $x^2 + \frac{y^2}{2} = 1$ and $z^2 + \frac{y^2}{2} = 1$ consists of two curves.

a) (4 points) Parameterize each curve in the form $\vec{r}(t) = (x(t), y(t), z(t))$.
b) (3 points) Set up the integral for the arc length of one of the curves.
c) (3 points) What is the arc length of this curve?

Solution:
a) Fix first $x(t), y(t)$ to satisfy the first equation then get $z(t)^2 = \cos^2(t)$ and $z = \pm\cos(t)$by solving the second equation for z. $\vec{r}(t) = (\cos(t), \sqrt{2}\sin(t), \pm\cos(t))$.
b) We find the velocity $\vec{r}'(t) = (-\sin(t), \sqrt{2}\cos(t), -\sin(t))$ and then the speed $|\vec{r}'(t) = \sqrt{\sin^2(t) + 2\cos^2(t) + \sin^2(t)} = \sqrt{2}$. The length is $\int_0^{2\pi} |\vec{r}'(t)| \; dt = \int_0^{2\pi} \sqrt{2} \; dt$. Also an expression like $\boxed{\int_0^{2\pi} \sqrt{\sin^2(t) + 2\cos^2(t) + \sin^2(t)} \; dt}$ is here correct at this stage.
c) Evaluate the integral $\boxed{2\sqrt{2}\pi}$.



Problem 8) (10 points)

a) (6 points) Find the curvature $\kappa(t)$ of the space curve $\vec{r}(t) = \langle -\cos(t), \sin(t), -2t \rangle$ at the point $\vec{r}(0)$.

b) (4 points) Find the curvature $\kappa(t)$ of the space curve $\vec{r}(t) = \langle -\cos(5t), \sin(5t), -10t \rangle$ at the point $\vec{r}(0)$.

Hint. Use one of the two formulas for the curvature

$$\kappa(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} \; ,$$

where $\vec{T}(t) = \vec{r}\;'(t)/|\vec{r}\;'(t)|$. The curvatures in b) can be derived from the curvature in a).

There is no need to redo the calculation, but we need a justification.

Solution:
a) We use the second formula for the curvature: $\vec{r}\,'(t) = \langle \sin(t), \cos(t), -2\rangle$. $\vec{r}\,''(t) = \langle \cos(t), -\sin(t), 0\rangle$. The speed of the curve satisfies $|\vec{r}\,'(t)| = \sqrt{5}$. The vector $\vec{r}\,'(t) \times \vec{r}\,''(t)$ is $(-2\sin(t), -2\cos(t), -1)$ which has length $\sqrt{5}$. therefore, the curvature is constant $\boxed{\kappa(t) = 1/5}$.
b) Because the curvature is independent of the parametrization, the curvature is agin 1/5.

Problem 9) (10 points)

For each of the following, fill in the blank with < (less than), > (greater than), or = (equal). Justify your answer completely.

1. The arc length of the curve parameterized by $\vec{f}(t) = \langle \cos 2t, 0, \sin 2t\rangle$, $0 \le t \le \pi$. ______ The arc length of the curve parameterized by $\vec{g}(u) = \langle 3, 2\cos u^2, 2\sin u^2\rangle$, $0 \le u \le \sqrt{\pi}$.

2. The arc length of the curve parameterized by $\vec{f}(t) = \langle t^2, 2\cos t, 2\sin t\rangle$, $0 \le t \le 2\pi$. ______ The arc length of the curve parameterized by $\vec{g}(u) = \langle u^4, 2\cos u^2, 2\sin u^2\rangle$, $0 \le u \le 2\pi$.

3. The arc length of the curve parameterized by $\vec{f}(t) = \langle 1 + 3t^2, 2 - t^2, 5 + 2t^2\rangle$, $0 \le t \le 1$. ______ The arc length of the curve parameterized by $\vec{g}(u) = \left\langle \frac{1}{2}u^2, u, \frac{2\sqrt{2}}{3}u^{3/2}\right\rangle$, $0 \le u \le 2$.

4. The arc length of the curve parameterized by $\vec{f}(t) = \langle \sin t, \cos t, t\rangle$, $1 \le t \le 5$. ______ The arc length of the curve parameterized by $\vec{g}(u) = \langle u\sin u, u\cos u, u\rangle$, $1 \le u \le 5$.

Solution:

1. The left curve is a circle of radius 1 so has arc length 2π. The right curve is half of a circle of radius 2, so it also has arc length 2π.

2. The left curve is a subset of the right curve; it is the portion of the right curve with $0 \leq u \leq \sqrt{2\pi}$. Therefore, the right curve has greater arc length.

3. The left curve is a line segment from $(1, 2, 5)$ to $(4, 1, 7)$, which has length $\sqrt{14}$. To find the length of the right curve, we use the arc length formula, which says the length is $\int_0^2 \|\vec{g}'(u)\| \, du$. We calculate $\vec{g}'(u) = \langle u, 1, \sqrt{2u} \rangle$, so $\|\vec{g}'(u)\| = \sqrt{u^2 + 2u + 1} = u+1$, and the arc length is $\int_0^2 (u + 1) \, du = 4$, which is greater than $\sqrt{14}$.

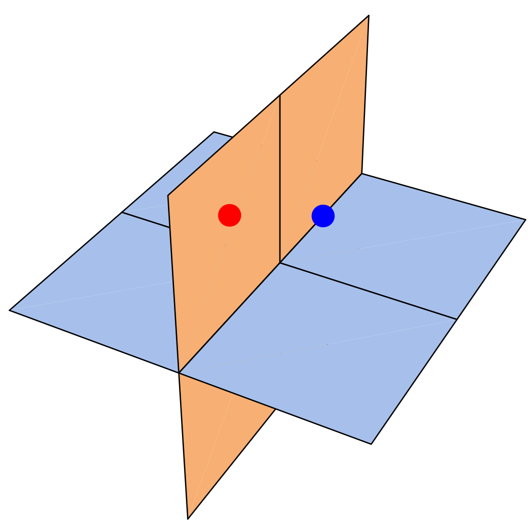
4. Both curves spiral upward the same amount, but the coils of the right curve are always wider, so the right curve has greater arc length. Alternatively, it's easy to see that $\|f'(t)\| < \|g'(t)\|$ whenever $t > 1$, so $\int_1^5 \|f'(t)\| \, dt$ must be smaller than $\int_1^5 \|g'(t)\| \, dt$.

Problem 10) (10 points)

Given the plane $x + y + z = 6$ containing the point $P = (2, 2, 2)$. Given is also a second point $Q = (3, -2, 2)$.

a) (5 points) Find the equation $ax + by + cz = d$ for the plane through P and Q which is perpendicular to the plane $x + y + z = 6$.

b) (5 points) Find the symmetric equation for the intersection of these two planes.



Solution:

a) The vector $\vec{v} = \langle 1, 1, 1 \rangle$ is perpendicular to the first plane and so parallel to the second plane. The vector $\vec{w} = \vec{QP} = \langle 1, -4, 0 \rangle$ is also in the second plane. Therefore, $\vec{n} = \vec{v} \times \vec{w} = \langle 4, 1, -5 \rangle$ is perpendicular to the second plane. The plane has the equation $\boxed{4x + y - 5z = d = 0}$. The constant $d = 0$ was obtained here by plugging in a point like $P = (2, 2, 2)$.

b) To get the intersection line, construct the vector $\vec{n} \times \vec{v} = \langle 6, -9, 3 \rangle$ which is parallel to the line. The symmetric equation is

$\boxed{\frac{x-2}{6} = \frac{y-2}{-9} = \frac{z-2}{3}}$.