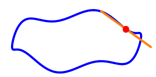
LINEAR APPROXIMATION — Math21a, O. Knill

LINEAR APPROXIMATION.

1D: The **linear approximation** of a function $f(x)$ at a point x_0 is the linear function

$$L(x) = f(x_0) + f'(x_0)(x - x_0).$$



The graph of L is tangent to the graph of f at x_0.

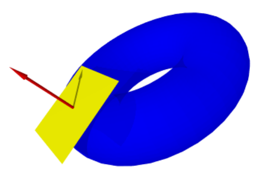
2D: The **linear approximation** of a function $f(x, y)$ at (x_0, y_0) is

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The level curve of g is tangent to the level curve of f at (x_0, y_0). The graph of L is tangent to the graph of f.

3D: The **linear approximation** of a function $f(x, y, z)$ at (x_0, y_0, z_0) by

$$L(x, y, z) = f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0).$$



The level surface of L is tangent to the level surface of f at (x_0, y_0, z_0).

Using $\nabla f = \langle f_x, f_y \rangle$, the linearization can be written as

$$L(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0)$$

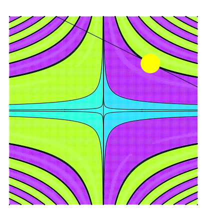
HOW CAN IT BE USED? Linearization is important because linear functions are easier to deal with. Using linearization, one can estimate function values near known points.

JUSTIFYING THE LINEAR APPROXIMATION.

If the second variable $y = y_0$ is fixed, then we have a one-dimensional situation where the only variable is x. Now $f(x, y_0) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0)$ is the linear approximation. Similarly, if $x = x_0$ is fixed y is the single variable, then $f(x_0, y) = f(x_0, y_0) + f_y(x_0, y_0)(y - y_0)$. Knowing the linear approximations in both the x and y variables, we can get the general linear approximation by $f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

An other justification uses the chain rule: the vector $\langle a, b \rangle = \langle f_x, f_y \rangle$ is perpendicular to the level curve at (x_0, y_0). Because the line $ax + by = ax_0 + by_0$ has also the vector (a, b) perpendicular to the curve and the curve and line pass through the same point (x_0, y_0), they are tangent. The line is the best among all lines passing through (x_0, y_0).

EXAMPLE (2D) Find the linear approximation of the function $f(x, y) = \sin(\pi x y^2)$ at the point $(1, 1)$. We have $(f_x(x, y), y_f(x, y) = (\pi y^2 \cos(\pi x y^2), 2y\pi \cos(\pi x y^2))$ which is at the point $(1, 1)$ equal to $\nabla f(1, 1) = (\pi \cos(\pi), 2\pi \cos(\pi)) = (-\pi, 2\pi)$. The linear function approximating f is $L(x, y) = f(1, 1) + (f_x(1, 1), f_y(1, 1)) \cdot (x - 1, y - 1) = 0 - \pi(x - 1) - 2\pi(y - 1) = -\pi x - 2\pi y + 3\pi$. The level curves of G are the lines $x + 2y = const$. The line which passes through $(1, 1)$ satisfies $x + 2y = 3$.



ESTIMATION. We continue the example and compare the value of f with the value of the linear approximation. $-0.00943407 = f(1 + 0.01, 1 + 0.01) \sim L(1 + 0.01, 1 + 0.01) = -\pi 0.01 - 2\pi 0.01 + 3\pi = -0.00942478$.

EXAMPLE (3D) Find the linear approximation to $f(x, y, z) = xy + yz + zx$ at the point $(1, 1, 1)$.

We have $f(1, 1, 1) = 3$, $\nabla f(x, y, z) = (y + z, x + z, y + x), \nabla f(1, 1, 1) = (2, 2, 2)$. Therefore $L(x, y, z) = f(1, 1, 1) + (2, 2, 2) \cdot (x - 1, y - 1, z - 1) = 3 + 2(x - 1) + 2(y - 1) + 2(z - 1) = 2x + 2y + 2z - 3$.

EXAMPLE (3D). Use the best linear approximation to $f(x, y, z) = e^x \sqrt{y} z$ to estimate the value of f at the point $(0.01, 24.8, 1.02)$.

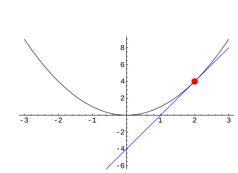
Solution. Take $(x_0, y_0, z_0) = (0, 25, 1)$, where $f(x_0, y_0, z_0) = 5$. The gradient is $\nabla f(x, y, z) = (e^x \sqrt{y} z, e^x z/(2\sqrt{y}), e^x \sqrt{y})$. At the point $(x_0, y_0, z_0) = (0, 25, 1)$ the gradient is the vector $(5, 1/10, 5)$. The linear approximation is $L(x, y, z) = f(x_0, y_0, z_0) + \nabla f(x_0, y_0, z_0)(x - x_0, y - y_0, z - z_0) = 5 + (5, 1/10, 5)(x - 0, y - 25, z - 1) = 5x + y/10 + 5z - 2.5$. We can approximate $f(0.01, 24.8, 1.02)$ by $5 + (5, 1/10, 5) \cdot (0.01, -0.2, 0.02) = 5 + 0.05 - 0.02 + 0.10 = 5.13$. The actual value is $f(0.01, 24.8, 1.02) = 5.1306$, very close to the estimate.

ABOUT DIMENSIONS. Do not mix up dimensions! For functions $f(x, y)$ of two variables, the linear approximation is a function $L(x, y)$ of two variables. We have tangency in two different dimensions: the level curves of f are tangent to the level curves of L at (x_0, y_0). But we also know that the graph of L is tangent to the graph of f.

TANGENT LINES. Because $\vec{n} = \nabla f(x_0, y_0) = \langle a, b \rangle$ is perpendicular to the level curve $f(x, y) = c$ through (x_0, y_0), the equation for the tangent line is

$$ax + by = d,\ a = f_x(x_0, y_0),\ b = f_y(x_0, y_0),\ d = ax_0 + by_0$$

The tangent line is a level curve of $L(x, y)$.
Example: Find the tangent to the graph of the function $g(x) = x^2$ at the point $(2, 4)$. Solution: the level curve $f(x, y) = y - x^2 = 0$ is the graph of a function $g(x) = x^2$ and the tangent at a point $(2, g(2)) = (2, 4)$ is obtained by computing the gradient $\langle a, b \rangle = \nabla f(2, 4) = \langle -g'(2), 1 \rangle = \langle -4, 1 \rangle$ and forming $-4x + y = d$, where $d = -4 \cdot 2 + 1 \cdot 4 = -4$. The answer is $-4x + y = -4$ which is the line $y = 4x - 4$ of slope 4. Graphs of 1D functions are curves in the plane, you have computed tangents in single variable calculus.



TANGENT PLANES. The tangent plane to the surface $g(x, y, z) = z - f(x, y) = 0$ at $(x_0, y_0, z_0 = f(x_0, z_0)$ is $-f_x x - f_y y + z = -f_x x_0 - f_y y_0 + z_0$. This can be read as $z = z_0 + f_x(x - x_0) + f_y(y - y_0)$. Calling the right hand side $L(x, y)$ shows that the **graph** of L is tangent to the graph of f at (x_0, y_0).

AVOID THE TERM TOTAL DIFFERENTIAL. Aiming to estimate the change $\Delta f = f(x, y) - f(x_0, y_0)$ of f for points $(x, y) = (x_0, y_0) + (\Delta x, \Delta y)$ near (x_0, y_0), we can estimate it with the linear approximation which is $L(\Delta x, \Delta y) = f_x(x_0, y_0)\Delta x + f_y \Delta y$. In an old-fashioned notation, one writes $df = f_x dx + f_y dy$ and calls df the **total differential**. One can **totally avoid** the notation of the **total differential**, it is confusing and out of date.

IN CLASS PROBLEM: Find the linear approximation $L(x, y)$ to

$$f(x, y) = \sin(x + 2y) + 3y$$

at the point $(0, \pi/2)$ and Estimate $f(1.01, \pi/2 - 0.03)$.