PROBLEM SESSIONS ON INTEGRATION Math 21a, Fall 2008

Knowing integrals

In a time of computer algebra systems, knowing integrals is less urgent, but it is imperative to know a few essential anti derivatives:

$\int \sin(x)\,dx = -\cos(x)$	$\int \cos(x)\,dx = \sin(x)$
$\int 1/x\,dx = \log(x)$	$\int \frac{1}{x}\,dx = \log(x)$
$\int 1\,dx = x$	$\int e^x\,dx = e^x$
$\int \sqrt{1+x}\,dx = (1+x)^{3/2}\,\frac{2}{3}$	$\int \frac{1}{1+x^2}\,dx = \arctan(x)$

Here are some integrals which are good to know but which are not so important to have available in memory:

$\int \sinh(x)\,dx = \cosh(x)$	$\int \cosh(x)\,dx = \sinh(x)$
$\int 1/\sqrt{1-x^2}\,dx = \arcsin(x)$	$\int 1/\sqrt{1+x^2}\,dx = \text{arcsinh}(x)$
$\int 1/\sin^2(x)\,dx = -\cot(x)$	$\int 1/\cos^2(x)\,dx = \tan(x)$

Substitution

Integrals of the form $\int f(g(x))g'(x)\,dx$ can be detected on the "spot" as $F(g(x))$, where F is the antiderivative of f. Examples:

$\int \tan(x)\,dx = -\log(\cos(x))$	$\int \cot(x)\,dx = \log(\sin(x))$
$\int 2x\cos(1+x^2)\,dx = \sin(1+x^2)$	$\int x^2\sqrt{1+x^3}\,dx = (1+x^3)^{3/2}\,\frac{2}{9}$
$\int \sin(x)\cos(x)\,dx = -\cos^2(x)/2$	$\int \frac{1}{x\log(x)}\,dx = \log(\log(x))$
$\int \log(x)/x\,dx = \log^2(x)/2$	$\int \sin^5(x)\cos(x)\,dx = \sin^6(x)/6$

Trigonometric forms

Knowing the trigonometric identities $\sin^2(x) + \cos^2(x) = 1, \cos(2x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x), \sin(2x) = 2\sin(x)\cos(x)$ is helpful:

$\int \tan^2(x)\ dx = \int \frac{(1-\cos^2(x))}{\cos^2(x)}\ dx$

$\int \cos^2(x)\ dx = \int \frac{1+\cos(2x)}{2}\ dx$

$\int \sin^2(x)\ dx = \int \frac{1-\cos(2x)}{2}\ dx$

$\int \sin^3(x)\ dx = \int \sin(x)(1-\cos^2(x))\ dx$

Integration by parts

Integration often works for products $x^n f(x)$. Integrating the product rule $uv' = uv - u'v$ gives

$$\int u\ dv = uv - \int duv\ .$$

Sometimes, it has to be repeated

$\int x\sin(x)\ dx = -x\cos(x) + \sin(x)$

$\int x\log(x)\ dx = x^2\log(x)/2 - x^2/2$

$\int \log(x)\ dx = \log(x)x - \int \frac{1}{x}x\ dx$

$\int xe^x\ dx = xe^x - \int e^x\ dx$

Sometimes, it has to be repeated:

$\int e^x\cos(x)\ dx = e^x\sin(x) - \int e^x\sin(x) = e^x\sin(x) - e^x\cos(x) - \int e^x\cos(x)\ dx$

$\int x^2e^x\ dx = x^2e^x - \int 2xe^x\ dx = x^2e^x - 2xe^x + \int 2e^x\ dx$

Partial fractions

Rational functions can be integrated by writing the fraction as a sum of simpler fractions:

$\int \frac{1}{1-x^2}dx = \int \frac{A}{1-x}\ dx + \int \frac{B}{1+x}\ dx$

$\int \frac{x}{1+x}dx = \int \frac{1+x}{1+x}\ dx - \int \frac{1}{1+x}\ dx$

Remarks

If you get into a mess with an integral in an exam in this course, it is an indication that you need an other approach:

other coordinate system

change the order of integration

use of an integral theorem (covered later)

PROBLEM SESSION INTEGRATION Math 21a, Fall 2008

Knowing integrals

$$\int \frac{1}{1+x} \, dx$$

$$\int x^2 + \cos(x) + \log(x) \, dx$$

Substitution

$$\int \sin(x) \cos(x) \, dx$$

$$\int \log^3(x)/x \, dx$$

Trigonometric forms

$$\int \cot^2(x) \, dx$$

$$\int \cos^3(x)\,dx$$

Integration by parts

$$\int x^2 \sin(x)\,dx$$

$$\int (x+1)\sin(x)\,dx$$

Partial fractions

$$\int \frac{1}{(x-1)(x+2)} dx$$

$$\int \frac{1}{(x^2+x)} dx$$