

INTEGRATION

Math 21a, Fall 2009

Knowing integrals

In a time of computer algebra systems, knowing integrals is less urgent, but it is imperative to know a few essential anti derivatives:

$$\int \sin(x) dx = -\cos(x)$$

$$\int \cos(x) dx = \sin(x)$$

$$\int x^n dx = x^{n+1}/(n+1)$$

$$\int \frac{1}{x} dx = \log(x)$$

$$\int 1 dx = x$$

$$\int e^x dx = e^x$$

$$\int \sqrt{1+x} dx = (1+x)^{3/2} \frac{2}{3}$$

$$\int \frac{1}{1+x^2} dx = \arctan(x)$$

Here are some integrals which are good to know but which are not so important to have available in memory:

$$\int \sinh(x) dx = \cosh(x)$$

$$\int \cosh(x) dx = \sinh(x)$$

$$\int 1/\sqrt{1-x^2} dx = \arcsin(x)$$

$$\int 1/\sqrt{1+x^2} dx = \operatorname{arcsinh}(x)$$

$$\int 1/\sin^2(x) dx = -\cot(x)$$

$$\int 1/\cos^2(x) dx = \tan(x)$$

Substitution

Integrals of the form $\int f(g(x))g'(x) dx$ can be detected on the "spot" as $F(g(x))$, where F is the antiderivative of f . Examples:

$$\int \tan(x) dx = -\log(\cos(x))$$

$$\int \cot(x) dx = \log(\sin(x))$$

$$\int 2x \cos(1+x^2) dx = \sin(1+x^2)$$

$$\int x^2 \sqrt{1+x^3} dx = (1+x^3)^{3/2} \frac{2}{9}$$

$$\int \sin(x) \cos(x) dx = -\cos^2(x)/2$$

$$\int \frac{1}{x \log(x)} dx = \log(\log(x))$$

$$\int \log(x)/x dx = \log^2(x)/2$$

$$\int \sin^5(x) \cos(x) dx = \sin^6(x)/6$$

Trigonometric forms

Knowing the trigonometric identities $\sin^2(x) + \cos^2(x) = 1$, $\cos(2x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$, $\sin(2x) = 2\sin(x)\cos(x)$ is helpful:

$$\int \tan^2(x) dx = \int \frac{(1-\cos^2(x))}{\cos^2(x)} dx$$

$$\int \cos^2(x) dx = \int \frac{1+\cos(2x)}{2} dx$$

$$\int \sin^2(x) dx = \int \frac{1-\cos(2x)}{2} dx$$

$$\int \sin^3(x) dx = \int \sin(x)(1-\cos^2(x)) dx$$

Integration by parts

Integration often works for products $x^n f(x)$. Integrating the product rule $uv' = uv - u'v$ gives

$$\int u dv = uv - \int duv.$$

Sometimes, it has to be repeated

$$\int x \sin(x) dx = -x \cos(x) + \sin(x)$$

$$\int x \log(x) dx = x^2 \log(x)/2 - x^2/2$$

$$\int \log(x) dx = \log(x)x - \int \frac{1}{x} x dx$$

$$\int x e^x dx = x e^x - \int e^x dx$$

Sometimes, it has to be repeated:

$$\int e^x \cos(x) dx = e^x \sin(x) - \int e^x \sin(x) dx = e^x \sin(x) - e^x \cos(x) - \int e^x \cos(x) dx$$

$$\int x^2 e^x dx = x^2 e^x - \int 2x e^x dx = x^2 e^x - 2x e^x + \int 2e^x dx$$

Partial fractions

Rational functions can be integrated by writing the fraction as a sum of simpler fractions:

$$\int \frac{1}{1-x^2} dx = \int \frac{A}{1-x} dx + \int \frac{B}{1+x} dx$$

$$\int \frac{x}{1+x} dx = \int \frac{1+x}{1+x} dx - \int \frac{1}{1+x} dx$$

Remarks

If you get into a mess with an integral in an exam in this course, it is an indication that you need another approach:

other coordinate system

change the order of integration

use of an integral theorem (covered later)

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$$\int \frac{1}{1+x} dx$$

$$\int x^2 + \cos(x) + \log(x) dx$$

Substitution

$$\int \sin(x) \cos(x) dx$$

$$\int \log^3(x)/x dx$$

Trigonometric forms

$$\int \cot^2(x) dx$$

$$\int \cos^3(x) dx$$

Integration by parts

$$\int x^2 \sin(x) dx$$

$$\int (x+1) \sin(x) dx$$

Partial fractions

$$\int \frac{1}{(x-1)(x+2)} dx$$

$$\int \frac{1}{(x^2+x)} dx$$