

**Solution to Problem D in week 6 Problem D:**

a) Using the chain rule, verify that in full generality, the energy of a Hamiltonian system is preserved: for every path  $\vec{r}(t) = \langle x(t), y(t) \rangle$  solving the system, we have

$$H(x(t), y(t)) = \text{const} .$$

b) What is the relation between the level curves of the function  $H(x, y)$  and the solution curves  $r(t) = \langle x(t), y(t) \rangle$  of the system?

c) Determine whether the Hamiltonian system with energy  $H(x, y) = x^4 + y^4$  can have paths which go to infinity.

d) Determine whether the Hamiltonian system with energy  $H(x, y) = x^4 - y^4$  has solution paths for which the position goes to infinity.

**Solution:**

a) We differentiate  $H(\vec{r}(t))$  with respect to  $t$  and use the chain rule:

$$\frac{d}{dt}H(x(t), y(t)) = H_x(x(t), y(t))x'(t) + H_y(x(t), y(t))y'(t) .$$

Now we substitute  $x'(t) = H_y$  and  $y'(t) = -H_x$  from the Hamiltonian system:

$$\frac{d}{dt}H(x(t), y(t)) = H_x(x(t), y(t))H_y(x(t), y(t)) + H_y(x(t), y(t))(-H_x(x(t), y(t))) = 0 .$$

So,  $H(x(t), y(t))$  is constant.

b) The level curves are the solution curves of the system because the solutions stay on the level curves at all times.

c) No, because the level curves of  $H$  are bounded curves also the solution curves have remain in a bounded region. Especially, if  $x^4 + y^4 = 1$ , then  $|x| \leq 1$  and  $|y| \leq 1$  so that the level curve is bounded. The solution curves according to b) are therefore bounded too.

d) Yes, there are solution paths which go to infinity because the level curves of  $H$  are unbounded curves. These curves look like hyperboloids, the level curves of  $x^2 - y^2$ . There is no way that a solution curve can reverse direction on a level curve of  $H$ : that would lead to a point, where  $x'(t) = y'(t) = 0$  and so  $H_y(x, y) = 0$  and  $H_x(x, y) = 0$ . But then  $H_x = 4x^3$  and  $H_y = 4y^3$  which can only happen if  $(x, y) = (0, 0)$ .