

## Problem E: Corner detection in Computer Vision

In this exercise we derive and discuss a formula for the curvature of a level curve  $f(x, y) = c$  at a point  $(x_0, y_0)$ . It is used in computer vision.

a) Verify that the curvature of the level curve is the absolute value of the directional derivative of

$$g(x, y) = \arctan\left(\frac{f_y}{f_x}\right)$$

in the direction

$$\vec{v} = \langle -f_y, f_x \rangle / \sqrt{f_x^2 + f_y^2}.$$

### Solution:

As given,  $g(x, y)$  is the angle of the gradient vector. The curvature of a curve is  $\kappa = |\vec{T}'|$  if  $\vec{T}$  is the unit tangent vector of a curve  $\vec{r}(t)$  with speed 1. Because  $\vec{T}$  is a unit vector, we have  $\vec{T}(t) = \langle \cos(\alpha(t)), \sin(\alpha(t)) \rangle$ . The curvature is the rate of change of  $\vec{T}$  which is

$$|\vec{T}'(t)| = |\langle -\sin(\alpha(t)), \cos(\alpha(t)) \rangle| |\alpha'(t)| = |\alpha'(t)|.$$

Because the gradient vector to the level curve is perpendicular to the velocity vector, this is also the rate of change of the angle of the gradient vector:

$$\kappa = |\vec{T}'(t)| = |\alpha'(t)| = \left| \frac{d}{dt} g(\vec{r}(t)) \right| = |\nabla g(\vec{r}(t)) \cdot \vec{r}'(t)| = |D_{\vec{r}'(t)} g| = |D_{\vec{v}} g|.$$

For the last step, we have used that both  $\vec{v}$  and  $\vec{r}'(t)$  are unit vectors perpendicular to the gradient so that  $\vec{v} = \pm \vec{r}'$ .

b) Find an expression for

$$D_{\vec{v}} g(x, y)$$

whose absolute value you have identified with the curvature  $\kappa$  in a).

### Solution:

First, we compute the gradient vector of  $g(x, y) = \arctan\left(\frac{f_y}{f_x}\right)$ . For that we have to remember first  $\arctan'(x) = 1/(1+x^2)$ . Because

$$\begin{aligned} \frac{\partial}{\partial x} \frac{f_y}{f_x} &= \frac{f_x f_{xy} - f_y f_{xx}}{f_x^2} \\ \frac{\partial}{\partial y} \frac{f_y}{f_x} &= \frac{f_x f_{yy} - f_y f_{xy}}{f_x^2}, \end{aligned}$$

we have

$$g_x(x, y) = \left( \frac{\partial}{\partial x} \frac{f_y}{f_x} \right) \left( 1 + \frac{f_y^2}{f_x^2} \right)^{-1} = \left( \frac{f_x f_{xy} - f_y f_{xx}}{f_x^2} \right) \left( 1 + \frac{f_y^2}{f_x^2} \right)^{-1} = \frac{f_x f_{xy} - f_y f_{xx}}{f_x^2 + f_y^2}$$

and

$$g_y(x, y) = \left( \frac{\partial}{\partial y} \frac{f_y}{f_x} \right) \left( 1 + \frac{f_y^2}{f_x^2} \right)^{-1} = \left( \frac{f_x f_{yy} - f_y f_{xy}}{f_x^2} \right) \left( 1 + \frac{f_y^2}{f_x^2} \right)^{-1} = \frac{f_x f_{yy} - f_y f_{xy}}{f_x^2 + f_y^2}.$$

Therefore

$$\vec{v} \cdot \nabla g(x, y) = \frac{\langle -f_y, f_x \rangle}{\sqrt{f_x^2 + f_y^2}} \cdot \frac{\langle f_x f_{xy} - f_y f_{xx}, f_x f_{yy} - f_y f_{xy} \rangle}{f_x^2 + f_y^2}$$

which is

$$= \frac{f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2}{(f_x^2 + f_y^2)^{3/2}}.$$