Name:

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| MWF 12 Toby Gee |
| MWF 12 Xinwen Zhu |
| TTH 10 Jack Huizenga |
| TTH 10 Fred van der Wyck |
| TTH 11:30 Ming-Tao Chuan |
| TTH 11:30 Fred van der Wyck |

- Start by printing your name in the above box and **check your section** in the box to the left.

- Do not detach pages from this exam packet or un-staple the packet.

- Please write neatly. Answers which are illegible for the grader cannot be given credit.

- **Show your work**. Except for problems 1-3, we need to see details of your computation.

- No notes, books, calculators, computers, or other electronic aids can be allowed.

- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) [T] [F] The function $f(x, y, z) = x^2 - y^2 - z^2$ increases in the direction $\langle -3, -1, 2 \rangle / \sqrt{14}$ at the point $(1, 1, 1)$.

2) [T] [F] The unit tangent vector of the curve $\vec{r}(t) = \langle 3t, 4t, t^2 \rangle$ at time $t = 0$ is $\langle 3/5, 4/5, 0 \rangle$.

3) [T] [F] There exist two nonzero vectors $\vec{a}$ and $\vec{b}$ such that the length of the vector projection of $\vec{a}$ to $\vec{a} \times \vec{b}$ is $\frac{1}{2}|\vec{b}|$.

4) [T] [F] The arc length of the curve $\vec{r}_1(t) = \langle e^{3t^3} - 1, t^6 + 2, \sin(2t^3) \rangle$, $0 \le t \le 1$ is larger than that of $\vec{r}_2(t) = \langle e^{3t} - 1, t^2 + 2, \sin(2t) \rangle$, $0 \le t \le 1$.

5) [T] [F] The tangent plane of the graph of $f(x, y) = \sin(x) + y^3$ at $(0, 1, 1)$ is $x + 3y = 3$.

6) [T] [F] There exists a curve C on the level surface of $f(x, y, z) = x^3 + e^{yz} + \cos(y) = 2$ such that the line integral $\int_C \nabla f \cdot d\vec{r} > 0$.

7) [T] [F] If Q is the point away from the plane $3x + 5y + z = 7$ and P is the point on the plane closest to Q, then $\vec{PQ}$ is parallel to $\langle 3, 5, 1 \rangle$.

8) [T] [F] The vector field $\vec{F}(x, y, z) = \langle y^2 - xz + e^y, -yz, x^4 + y^2 - z^2 \rangle$ is the curl of a vector field $\vec{G}$.

9) [T] [F] Let $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \rangle$ and C be the unit circle oriented counterclockwise. Since $Q_x = P_y$ everywhere, Green implies $\int_C \vec{F} \cdot d\vec{r} = 0$.

10) [T] [F] By linear approximation of the function $f(x, y, z) = e^{x+y+z}$ we can estimate $f(0.1, 0.01, 0.001)$ as 1.111.

11) [T] [F] If $\vec{F}(x, y, z)$ is a vector field defined on $0 < x^2 + y^2 + z^2 < 4$ and $\text{curl}(\vec{F}) = 0$ everywhere on this solid, then $\vec{F} = \nabla f$ for some function f.

12) [T] [F] The tangent plane of the surface $x^2 + y^4 + z^6 = 6$ at $(2, 1, 1)$ is perpendicular to the line $\vec{r}(t) = \langle 1 + 2t, 3 + 2t, -4 + 3t \rangle$.

13) [T] [F] Given two curves $C_1 : \vec{r}_1(t) = \langle t, t^3 \rangle$, $0 \le t \le 1$ and $C_2 : \vec{r}_2(s) = \langle s, s^5 \rangle$, $0 \le s \le 1$ $f(x, y) = \sin(x^2 y)$. Then $\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$.

14) [T] [F] If $f(x, y)$ has a global maximum, then the discriminant function $D(x, y) = f_{xx} f_{yy} - f_{xy}^2$ has a global maximum.

15) [T] [F] Let $\vec{F}(x, y, z) = \langle x, y, z \rangle$ and S the surface boundary of the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$ oriented by outward normal vectors. Then $\int \int_S \vec{F} \cdot d\vec{S} = 0$.

16) [T] [F] Let $\vec{F}(x, y, z) = \langle x/3, y/3, z/3 \rangle$ and S the unit sphere oriented by the outward normal vectors. Then $\int \int_S \text{curl}(\vec{F}) \cdot d\vec{S}$ is the volume of the unit ball.

17) [T] [F] In three dimensional space there exist two nonzero vector fields $\vec{F}$ and $\vec{G}$ such that $\text{curl}(\vec{F}) = \text{div}(\vec{G})$.

18) [T] [F] The vector field $\vec{F}(x, y, z) = \langle \cos(y), \cos(z), \cos(x) \rangle$ has the property that $\vec{F} = \text{curl}(\text{curl}(\vec{F}))$.

19) [T] [F] There exists a vector field $\vec{F}(x, y, z)$ defined on $\mathbf{R}^3$ such that every line integral $\int_C \vec{F} \cdot d\vec{r}$ of $\vec{F}$ over a closed curve C is equal to 0, but not every surface integral $\int \int_S \vec{F} \cdot dS$ over a closed surface S is equal to 0.

20) [T] [F] Whenever $\vec{F} = \nabla f$, for some function $f(x, y)$ defined on the annulus $\frac{1}{2} \le x^2 + y^2 \le 2$, then $\int_C \vec{F} \cdot d\vec{r} = 0$, where C is the circle $x^2 + y^2 = 1$.

Problem 2) (10 points)

a) (5 points) We match in this problems vector fields with properties of vector fields and formulas for vector fields. A field $\vec{F}$ is **divergence free** if $\operatorname{div}(\vec{F}) = 0$ everywhere in the plane. A field $\vec{F}$ is **irrotational**, if $\operatorname{curl}(\vec{F}) = \vec{0}$ everywhere in the plane. In the last two columns of the following table, check the boxes which apply.

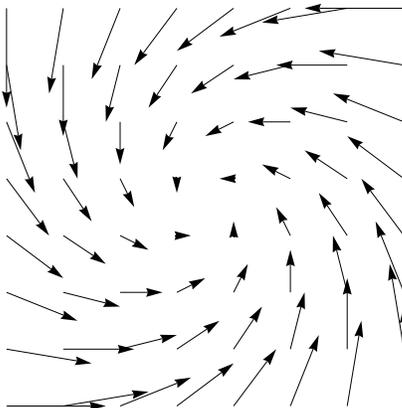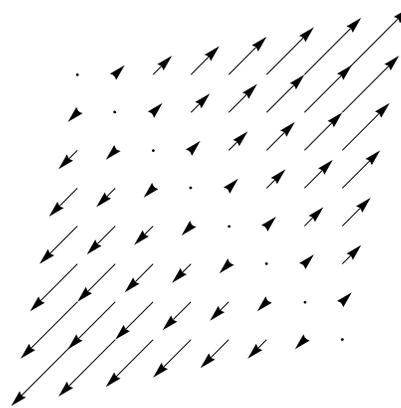| field	enter I-IV	divergence free	irrotational
$\vec{F}(x, y) = \langle -y, x \rangle$			
$\vec{F}(x, y) = \langle y, x \rangle$			
$\vec{F}(x, y) = \langle -x - y, x - y \rangle$			
$\vec{F}(x, y) = \langle x + y, x + y \rangle$			

I
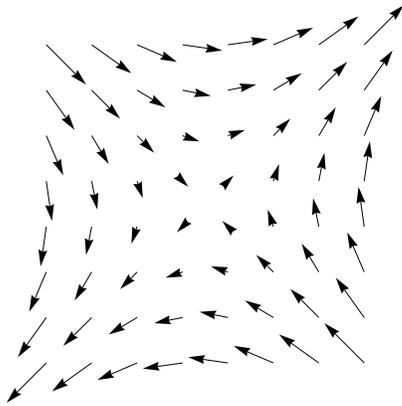
II
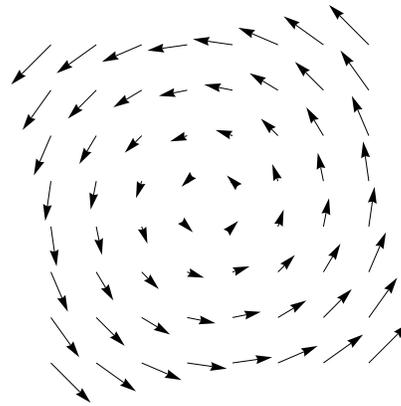
III

IV

b) (5 points) Match the following names of partial differential equations with functions $u(t, x)$ which satisfy the differential equation and with formulas defining these equations.

equation	A-D	1-4
wave		
heat		
transport		
Laplace		

A	$u(t, x) = t^2 + x^2$
B	$u(t, x) = t^2 - x^2$
C	$u(t, x) = \sin(x + t)$
D	$u(t, x) = x^2 + 2t$

1	$u_t(t, x) = u_x(t, x)$
2	$u_{tt}(t, x) = u_{xx}(t, x)$
3	$u_{tt}(t, x) = -u_{xx}(t, x)$
4	$u_t(t, x) = u_{xx}(t, x)$

Problem 3) (10 points)

a) (6 points) Select 6 of the integrals $A - H$ in the lower tables and match them with their names in the following table:

name	label A-H
line integral	
flux integral	
surface area	
arc length	
volume	
area	

$\int \int_R x^2 - y^2 \, dxdy$	A
$\int \int_R 1 \, dxdy$	B
$\int \int \int_R 1 \, dxdydz$	C
$\int \int \int_R x^2 + z^2 \, dxdydz$	D

$\int \int_R \vec{F}(\vec{r}(u, v)) \cdot \vec{r}_u \times \vec{r}_v \, dudv$	E
$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$	F
$\int_a^b |\vec{r}'(t)| \, dt$	G
$\int \int_R |\vec{r}_u \times \vec{r}_v| \, dudv$	H

b) (4 points)

derivative	enter A-D
divergence	
curl	
gradient	
directional derivative	

The middle column of the following table is obtained by applying a derivative operation to the object in the left column. Fill in the correct label (A-D) of that operation into the above table.

object	derivative	label
$\vec{F}(x, y, z) = \langle -y, x, x \rangle$	$\langle 0, -1, 2 \rangle$	A
$\vec{F}(x, y, z) = \langle x^2, y, x \rangle$	$2x + 1$	B
$f(x, y, z) = x^2 + y^2 + z$	$\langle 2x, 2y, 1 \rangle$	C
$f(x, y, z) = x^3 + 5y^2$	$10y$	D

Problem 4) (10 points)

Consider the tetrahedron with vertices

$$A = (0, 1, -1), B = (4, 0, -1), C = (2, 1, 3), \text{ and } D = (2, 2, 0).$$

a) (3 points) What is the area of the parallelogram spanned by $\vec{AB}$ and $\vec{AD}$?

b) (3 points) Find the volume of the parallelepiped spanned by $\vec{AC}, \vec{AB}$ and $\vec{AD}$.

c) (4 points) Determine the distance between the two skew lines AB and CD.
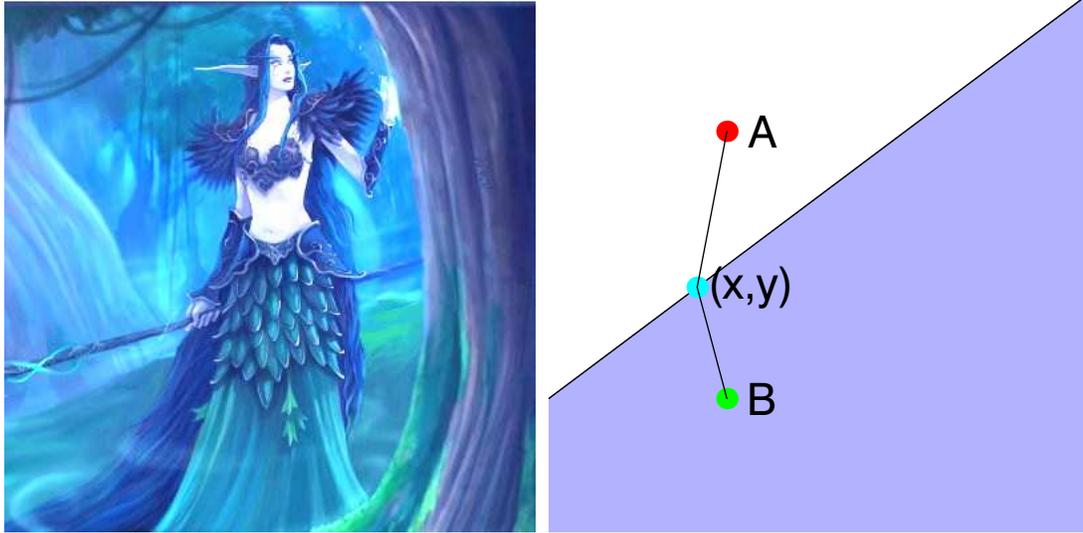
Problem 5) (10 points)

a) (5 points) The curl of $\vec{F}(x, y) = \langle -e^{xy}, y \rangle$ is equal to a scalar function $f(x, y)$. Estimate $f(1.1, 0.001)$ by linear approximation.

b) (5 points) Using the same function as in a), the equation $f(x, y) = \text{curl}(\vec{F})(x, y) = 1$ defines y as a function $g(x)$ of x near $x = 1$. Find $g'(1)$.

Problem 6) (10 points)

Find all the critical points of the function $f(x, y) = y^3 - 3y^2 + 4x + x^2 - 3$ and classify them by telling whether they are local maxima, local minima or saddle points.

Problem 7) (10 points)

A nightelf in the game World of Warcraft runs from $A = (0, 2)$ to $B = (0, 0)$ along a straight line segment from A to (x, y) and swims through the lake $x - y \geq -1$ from (x, y) to a gold chest located at $B = (0, 0)$ again on a straight line segment. The effort from A to (x, y) is the square of the distance from A to (x, y). Her effort from (x, y) to B is 2 times the squared distance from (x, y) to B. Using the Lagrange method, find the choice of a drop point (x, y) on the lake shore that minimizes her effort.

Problem 8) (10 points)

Evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r} \,,$$

where C is the curve given by

$$\vec{r}(t) = \langle \frac{t\pi}{2}, 1 - t, t^3 \rangle, 0 \le t \le 1$$

and

$$\vec{F}(x, y, z) = \langle e^{y^2} + z\cos(xz), 2xye^{y^2}, x\cos(xz) \rangle \,.$$
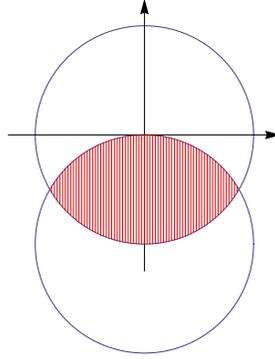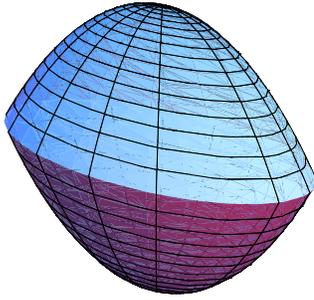
Problem (9) (10 points)

The picture shows an unidentified flying object (UFO). Although it is unidentified, we know its shape. One part of the surface

$$x^2 + y^2 + z^2 = 4$$

and the other part of the surface is

$$x^2 + y^2 + (z + 2)^2 = 4 \,.$$

Find the surface area of the UFO.

Problem 10) (10 points)

Evaluate the following integral

$$\int_0^2 \int_1^3 \int_{z^2}^4 xz \cos(y^2) \, dy\,dx\,dz \ .$$

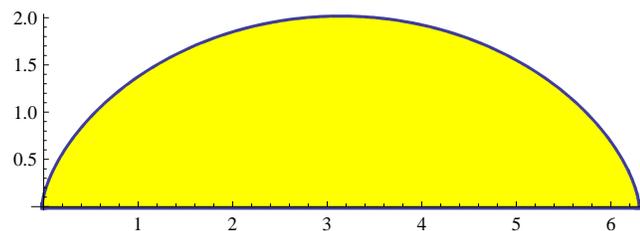**Problem 11) (10 points)**

Let $\vec{F}(x, y, z) = \langle x + yz, xye^{-xz}, e^{-xz} \rangle$. Find

$$\int\int_S \vec{F} \cdot d\vec{S} \ ,$$

where S is the surface $z = 1 - x^2 - y^2, z \geq 0$ oriented so that the normal vector points upwards.

Problem 12) (10 points)

Find the area of the region on the plane enclosed by the curve $\vec{r}(t) = \langle t - \sin(t), 1 - \cos(t) \rangle$ with $0 \leq t \leq 2\pi$ and the x-axes.

Problem 13) (10 points)

Evaluate the integral

$$\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S} \ ,$$

where $\vec{F}(x, y, z) = \langle xe^{y^2}z^3 + 2xyze^{x^2+z}, x + z^2e^{x^2+z}, ye^{x^2+z} + ze^x \rangle$ and where S is the part of the ellipsoid $x^2 + y^2/4 + (z + 1)^2 = 2, z > 0$ oriented so that the normal vector points upwards.

Problem 14) (10 points)

Let E be the rectangular solid $0 \le x \le a, 0 \le y \le b, 0 \le z \le 1$ and let S be the boundary of E. The surface S consists of 6 planar pieces where each is oriented so that the normal vector points outwards. Given the vector field

$$\vec{F} = \langle -x^2 - 4xy, -yz, 12z \rangle \ ,$$

for which parameters a, b is the flux integral

$$\int\int_S \vec{F} \cdot d\vec{S}$$

a global maximum?