Name:

MWF 9 Jameel Al-Aidroos
MWF 10 Andrew Cotton-Clay
MWF 10 Oliver Knill
MWF 10 HT Yau
MWF 11 Ana Caraiani
MWF 11 Chris Phillips
MWF 11 Ethan Street
MWF 12 Toby Gee
MWF 12 Xinwen Zhu
TTH 10 Jack Huizenga
TTH 10 Fred van der Wyck
TTH 11:30 Ming-Tao Chuan
TTH 11:30 Fred van der Wyck

- Start by printing your name in the above box and **check your section** in the box to the left.

- Do not detach pages from this exam packet or unstaple the packet.

- Please write neatly. Answers which are illegible for the grader cannot be given credit.

- **Show your work**. Except for problems 1-3, we need to see details of your computation.

- No notes, books, calculators, computers, or other electronic aids can be allowed.

- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points)

1) **T** **F** For any two vectors $\vec{v}$ and $\vec{w}$ one has $\text{proj}_{\vec{v}}(\vec{v} \times \vec{w}) = \vec{0}$.

2) **T** **F** Any parameterized surface S is either the graph of a function $f(x, y)$ or a surface of revolution.

3) **T** **F** If the directional derivative $D_{\vec{v}}(f)$ of f into the direction of a unit vector $\vec{v}$ is zero, then $\vec{v}$ is perpendicular to the level curve of f.

4) **T** **F** The linearization $L(x, y)$ of $f(x, y) = 5x - 100y$ at $(0, 0)$ satisfies $L(x, y) = 5x - 100y$.

5) **T** **F** If a parameterized curve $\vec{r}(t)$ intersects a surface $\{f = c\}$ at a right angle, then at the point of intersection we have $\nabla f(\vec{r}(t)) \times \vec{r}\,'(t) = 0$.

6) **T** **F** The curvature of the curve $\vec{r}(t) = \langle \cos(3t^2), \sin(6t^2) \rangle$ at the point $\vec{r}(1)$ is larger than the curvature of the curve $\vec{r}(t) = \langle 2\cos(3t), 2\sin(6t) \rangle$ at the point $\vec{r}(1)$.

7) **T** **F** At every point (x, y, z) on the hyperboloid $4x^2 - y^2 + z^2 = 10$, the vector $\langle 4x, -y, z \rangle$ is parallel to the hyperboloid.

8) **T** **F** The set $\{\phi = \pi/2, \theta = \pi/2\}$ in spherical coordinates is the positive y-axis.

9) **T** **F** The integral $\int_0^1 \int_0^{2\pi} r^2 \sin(\theta) \; d\theta \, dr$ is equal to the area of the unit disk.

10) **T** **F** If three vectors $\vec{u}, \vec{v}$ and $\vec{w}$ attached at the origin are in a common plane, then $\vec{u} \cdot ((\vec{v} + \vec{u}) \times \vec{w}) = 0$.

11) **T** **F** If a function $f(x, y)$ has a local minimum at $(0, 0)$, then the discriminant D must be positive.

12) **T** **F** The integral $\int_0^1 \int_y^1 f(x, y) \, dy \, dx$ represents a double integral over a bounded region in the plane.

13) **T** **F** The following identity is true: $\int_0^3 \int_0^x x^2 \, dy dx = \int_0^3 \int_y^3 x^2 \, dx \, dy$

14) **T** **F** There is a quadric of the form $ax^2 + by^2 + cz^2 = d$, for which all three traces are hyperbola.

15) **T** **F** The curvature of a space curve $\vec{r}(t)$ is a vector perpendicular to the acceleration vector $\vec{r}\,''(t)$.

16) **T** **F** Assume S is the unit sphere oriented so that the normal vector points outside. Let S^+ be the upper hemisphere and S^- the lower hemisphere. If a vector field $\vec{F}$ has divergence zero, then the flux of $\vec{F}$ through S^+ is equal to the flux of $\vec{F}$ through S^-.

17) **T** **F** If a vector field $\vec{F}$ is defined at all points in three-space except at the origin and $\text{curl}(\tilde{F}) = \tilde{0}$ everywhere, then the line integral of $\vec{F}$ around any closed path not passing through the origin is zero.

18) **T** **F** Every vector field which satisfies $\text{curl}(\vec{F}) = \vec{0}$ everywhere in the entire three dimensional space can be written as $\vec{F} = \text{grad}(f)$ for some scalar function f.

19) **T** **F** Let $\vec{F}$ be a vector field and let S be an oriented surface $\vec{r}(u, v)$. Then $20 \int \int_S \vec{F} \cdot dS = \int \int_S \vec{G} \cdot dS$, where $\vec{G} = 20\vec{F}$.
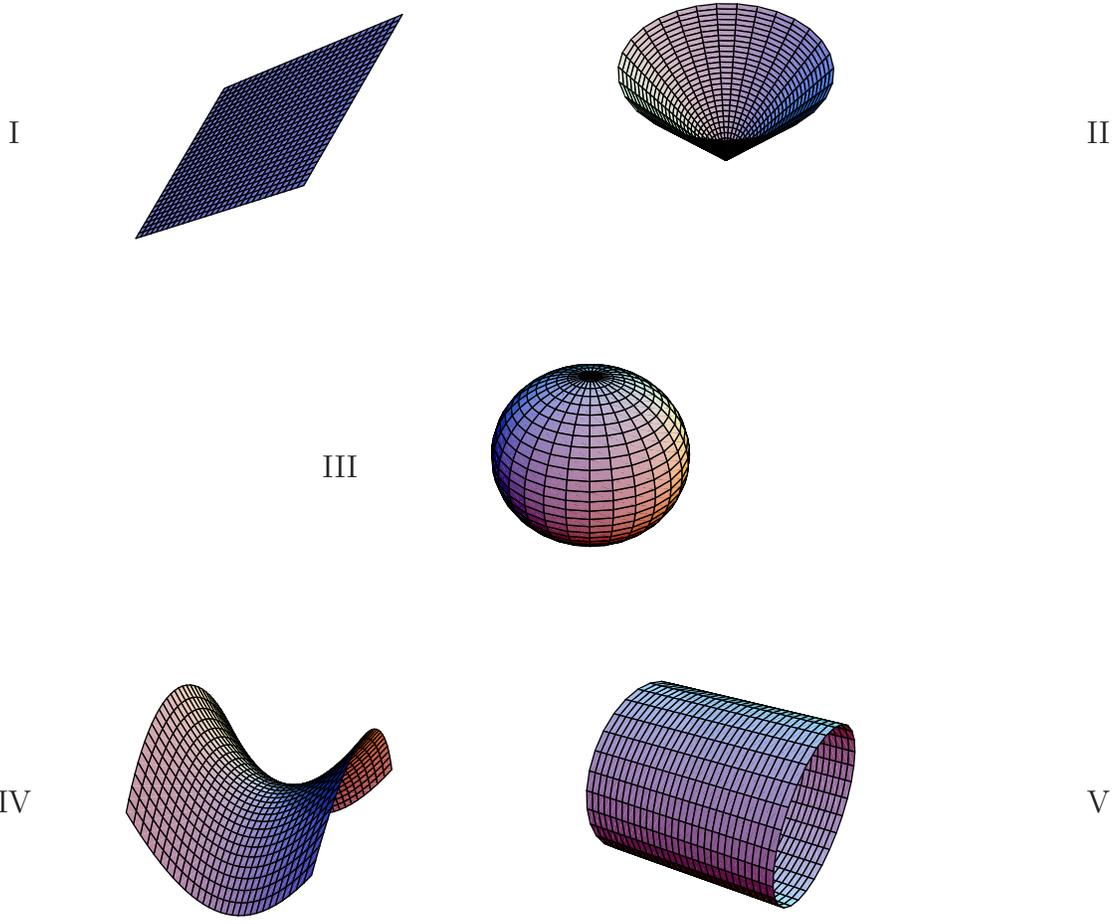
20) T F   Consider the surface S given by the equation $z^2 = f(x, y)$. If $(x, y, z) = (x, y, \sqrt{f(x, y)})$ is a point on the surface with maximal distance from the origin, it is a local maximum of $g(x, y) = x^2 + y^2 + f(x, y)$.

Match the parameterized surface formulas and pictures with the formulas for the implicit surfaces. No justifications are needed.

A)	$\vec{r}(u,v) = \langle 1 + u, v, u + v \rangle$
B)	$\vec{r}(u,v) = \langle v\cos(u), v\sin(u), v \rangle$
C)	$\vec{r}(u,v) = \langle \cos(u)\sin(v), \sin(u)\sin(v), \cos(v) \rangle$
D)	$\vec{r}(u,v) = \langle u, v, u^2 - v^2 \rangle$
E)	$\vec{r}(u,v) = \langle v, \sin(u), \cos(u) \rangle$

I

II

III

IV

V

Enter A),B),C),D),E) here	Enter I),II),III),IV),V) here	Equation
		$y^2 + z^2 = 1$
		$x + y - z = 1$
		$x^2 + y^2 + z^2 = 1$
		$x^2 + y^2 - z^2 = 0$
		$x^2 - y^2 - z = 0$

Problem 3) (10 points)

Match the formulas and theorems with their names. No justifications are needed.

N) $\int_a^b \int_c^d f(x,y)\, dy dx = \int_c^d \int_a^b f(x,y)\, dx dy$

U) $f_{xy}(x,y) = f_{yx}(y,x)$

M) $\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w}$

E) $\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}$

R) $\frac{d}{dt}f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$

A) $\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos(\alpha)$

L) $(\vec{u}\times\vec{v})\cdot\vec{w}$

F) $|\vec{v}\cdot\vec{w}| \leq |\vec{v}|\cdot|\vec{w}|$

I) $|\vec{v}|^2 + |\vec{w}|^2 = |\vec{v}-\vec{w}|^2$ if $\vec{v}\cdot\vec{w} = 0$

G) $|\vec{v}+\vec{w}| \leq |\vec{v}| + |\vec{w}|$

Enter letters here	Object or theorem
	Fubini theorem
	Clairaut theorem
	vector projection
	scalar projection
	chain rule
	dot product formula
	scalar triple product
	Cauchy-Schwarz inequality
	Pythagorean theorem
	Triangle inequality

Problem 4) (10 points)

Let L be the line $\vec{r}(t) = \langle t, 0, 0\rangle$. We are also given a point $Q = (3,3,0)$ in space.

a) (2 points) What is the distance $d((x,y,z), Q)$ between a general point (x,y,z) and Q?

b) (3 points) What is the distance $d((x,y,z), L)$ between the point (x,y,z) and the line L?

c) (3 points) Find the equation for the set C of all points (x,y,z) satisfying
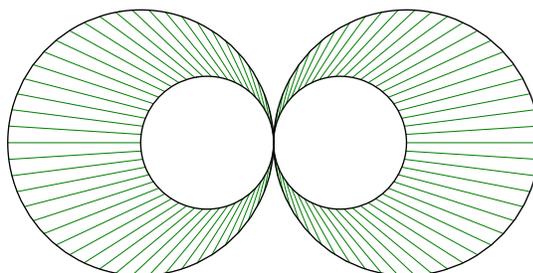
$$d((x,y,z), Q) = d((x,y,z), L) .$$

d) (2 points) Identify the surface.

Problem 5) (10 points)

Find the area of the region in the plane given in polar coordinates by

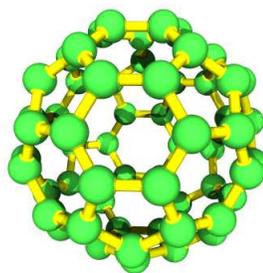$$\{(r,\theta) \mid |\cos(\theta)| \leq r \leq 2|\cos(\theta)|, 0 \leq \theta < 2\pi \} .$$

The region is the shaded part in the figure.

A microscopic bucky ball C60 is located on a gold surface. The surface produces the electric potential $f(x, y) = x^4 + y^4 - 2x^2 - 8y^2 + 5$.

a) (7 points) Find all critical points of f and classify them.

b) (3 points) The fullerene will settle at a global minimum of $f(x, y)$. Find the global minima of the function $f(x, y)$.

A circular wheel with boundary $g(x, y) = x^2 + y^2 = 1$ has the boundary point (x, y) connected to two points $A = (-2, 0)$ and $B = (3, 1)$ by rubber bands. The potential energy

at position (x, y) is by Hooks law equal to $f(x, y) = (x + 2)^2 + y^2 + (x - 3)^2 + (y - 1)^2$, the sum of the squares of the distances to A and B. Our goal is to find the position (x, y) for which the energy is minimal. To find this position for which the wheel is at rest, minimize $f(x, y)$ under the constraint $g(x, y) = 1$.
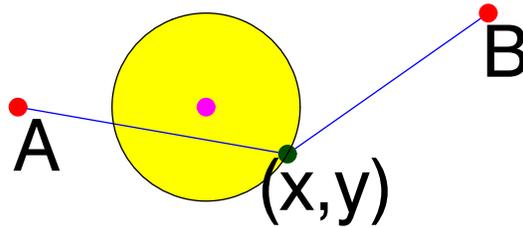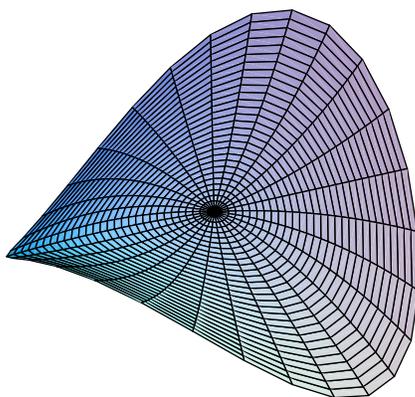

Problem 8) (10 points)

---

a) (5 points) Find the surface area of the parameterized surface

$$\vec{r}(u, v) = \langle u - v, u + v, uv \rangle$$

with $u^2 + v^2 \leq 1$.

b) (3 points) Find an implicit equation $g(x, y, z) = 0$ for this surface.

c) (2 points) What is the name of the surface?

Problem 9) (10 points)

a) (4 points) Find the tangent plane to the surface $f(x, y, z) = zx^5 + y^5 - z^5 = 1$ at the point $(1, 1, 1)$.

b) (3 points) Find the linearization $L(x, y, z)$ of $f(x, y, z)$ at at the point $(1, 1, 1)$.

c) (3 points) Near the point $(1, 1, 1)$, the surface can be written as a graph $z = g(x, y)$. Find the partial derivative $g_x(1, 1)$.

Problem 10) (10 points)

A tower E with base $0 \leq x \leq 1, 0 \leq y \leq x$ has a roof $f(x, y) = \sin(1 - y)/(1 - y)$. Find the volume of this solid. The solid is given in formulas by

$$E = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq x, 0 \leq z \leq f(x, y) \}.$$
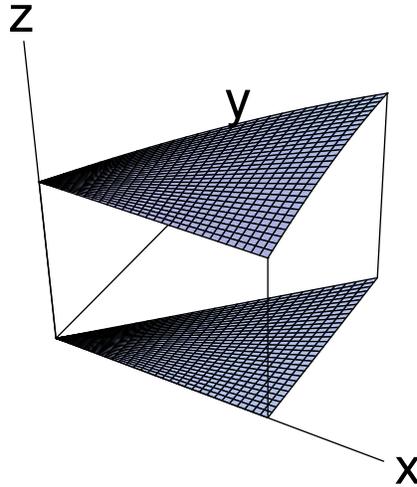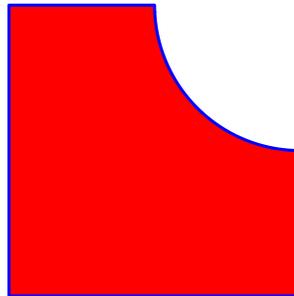
Let $\vec{F} = \langle y, 2x + \tan(\tan(y)) \rangle$ be a vector field in the plane and let C be the boundary of the region

$$G = \{0 \le x \le 2, 0 \le y \le 2, (x-2)^2 + (y-2)^2 \ge 1\}$$

oriented counter clock-wise. Compute the line integral

$$\int_C \vec{F} \cdot \vec{dr} \, .$$

Let S be the surface of a turbine blade parameterized by $\vec{r}(s,t) = \langle s\cos(t), s\sin(t), t \rangle$ for $t \in [0, 6\pi]$ and $s \in [0, 1]$. Let $\vec{F} = \mathrm{curl}(\vec{G})$ denote the velocity field of the water velocity, where $\vec{G}(x, y, z) = \langle -y + (x^2 + y^2 - 1), x + (x^2 + y^2 - 1), (x^2 + y^2 - 1) \rangle$. Compute the

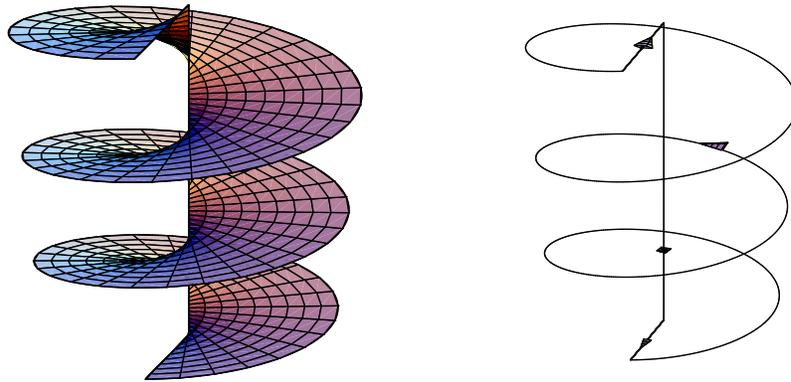power of the turbine which is given by the flux of $\vec{F} = \text{curl}(\vec{G})$ through S.

Hint. The boundary C of the surface S consists of 4 paths:
$\vec{r_1}(t) = \langle \cos(t), \sin(t), t \rangle$, $t \in [0, 6\pi]$.
$\vec{r_2}(s) = \langle 1 - s, 0, 6\pi \rangle$, $s \in [0, 1]$.
$\vec{r_3}(t) = \langle 0, 0, 6\pi - t \rangle$, $t \in [0, 6\pi]$.
$\vec{r_4}(s) = \langle s, 0, 0 \rangle$, $s \in [0, 1]$.

Problem 13) (10 points)

Let $\vec{F}(x, y, z) = \langle z^2, -z^5 + z \sin(e^{\sin(x)}), (x^2 + y^2) \rangle$. Let S denote the part of the graph $z = 9 - x^2 - y^2$ lying above the xy-plane oriented so that the normal vector points upwards. Find the flux of $\vec{F}$ through the surface S.

Hint. You might also want to look at the surface $D = \{x^2 + y^2 \le 9, z = 0\}$ lying in the xy-plane.

We are given two vector fields $\vec{F}(x, y, z) = \langle x, y, z \rangle$ and $\vec{G}(x, y, z) = \langle y, z, x \rangle$. We are also given a brezel surface S bounding a solid of volume 3 and a surface U which is part of S and has been left after cutting away a piece of the brezel. The surface U together with two discs D_1 and D_2 form a closed surface bounding a "cut brezel". The first surface S does not have a boundary and is oriented so that the normal vector points outwards. The second surface U has the same orientation than S and as a boundary the union C of two curves C_1 and C_2 which are oriented so that U is "to the left", when you look into the direction of the velocity vector and if the normal vector to U points "up".

a) (2 points) Compute $\int \int_S \vec{F} \, d\vec{S}$.

b) (2 points) Find $\int \int_S \vec{G} \, d\vec{S}$.

c) (2 points) Verify that the vector field $\vec{A} = \langle -xy, -yz, -zx \rangle$ is a **vector potential** of $\vec{G}$ meaning that $\vec{G} = \text{curl}(\vec{A})$.

d) (2 points) Express $\int_C \vec{A} \cdot d\vec{r}$ as a flux integral through U.

e) (2 points) Compute $\int_C \vec{F} \cdot d\vec{r}$.