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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) | T | F | The length of the curve $\vec{r}(t) = \langle \sin(t), t^4 + t, \cos(t) \rangle$ on $t \in [0, 1]$ is the same as the length of the curve $\vec{r}(t) = \langle \sin(t^2), t^8 + t^2, \cos(t^2) \rangle$ on $[0, 1]$.

Solution:
This is a consequence of the chain rule: $\int_a^b |r(s(t))'| \; dt = \int_a^b |r'(s(t))||s'(t)|dt = \int_{s(a)}^{s(b)} |r'(s)|ds$.

2) | T | F | The parametric surface $\vec{r}(u, v) = (5u - 3v, u - v - 1, 5u - v - 7)$ is a plane.

Solution:
Yes, the coordinate functions $r(u, v) = (x(u, v), y(u, v), z(u, v))$ are all linear.

3) | T | F | Any function $u(x, y)$ that obeys the differential equation $u_{xx} + u_x - u_y = 1$ has no local maxima.

Solution:
If $\nabla u = (u_x, u_y) = (0, 0)$, then $u_{xx} = 1$ which is incompatible with a local maximum, where $u_{xx} > 0$ by the second derivative test.

4) | T | F | The length of the vector projection of $\vec{b}$ onto a vector $\vec{a}$ is smaller or equal than the length of the vector $\vec{b}$.

Solution:
The length is $\vec{b}$ times the absolute value of the cos of the angle between the vectors.

5) | T | F | If $f(x, y)$ is a function such that $f_x - f_y = 0$ then f is called conservative.

Solution:
The notion of conservative applies to vector fields and not to functions.

6) | T | F | $(\vec{u} \times \vec{v}) \cdot \vec{w} = (\vec{u} \times \vec{w}) \cdot \vec{v}$ for all vectors $\vec{u}, \vec{v}, \vec{w}$ in space.

Solution:
While $|(\vec{u} \times \vec{w}) \cdot \vec{v}|$ and $|(\vec{u} \times \vec{w}) \cdot \vec{v}|$ are both equal to the volume of the parallelepiped determined by $\vec{u}, \vec{v}$ and $\vec{w}$, the sign is different. An example: for $\vec{u} = \langle 1,0,0 \rangle, \vec{v} = \langle 0,1,0 \rangle, \vec{w} = \langle 0,0,1 \rangle$, we have $(\vec{u} \times \vec{v}) \cdot \vec{w} = 1$ and $(\vec{u} \times \vec{w}) \cdot \vec{v} = -1$.

7) | T | **F** | The equation $\rho = \phi/4$ in spherical coordinates is half a cone.

Solution:
The equation $\rho = \phi/4$ defines a heart shaped rotational symmetric surface. The surface $\phi = c = const$ would define half a cone for $c \in [0, \pi]$.

8) | **T** | F | If $f(x,y) = \frac{x^3}{x^2+y^2}$ and $(x,y) \to (0,0)$ then $f(x,y) \to 0$.

Solution:
Use polar coordinates.

9) | **T** | F | $\int_0^1 \int_0^x 1 \, dydx = 1/2$.

Solution:
This is the area of half of the unit square.

10) | **T** | F | Let $\vec{a}$ and $\vec{b}$ be two vectors which are perpendicular to a given plane Σ. Then $\vec{a} + \vec{b}$ is also perpendicular to Σ.

Solution:
If $\vec{v}$ is a vector in the plane, then $\vec{a} \cdot \vec{v} = 0$ and $\vec{b} \cdot \vec{v} = 0$ then also $(\vec{a} + \vec{b}) \cdot \vec{v} = 0$.

11) | **T** | F | If $g(x,t) = f(x - vt)$ for some function f of one variable $f(z)$ then g satisfies the differential equation $g_{tt} - v^2 g_{xx} = 0$.

Solution:
Actually one could show that $g(x,t) = f(x - vt) + h(x + vt)$ is the general solution of the wave equation $g_{tt} - v^2 g_{xx} = 0$.

12) T F

If $f(x,y)$ is a continuous function on $\mathbf{R}^2$ such that $\int\int_D f\ dA \geq 0$ for any region D then $f(x,y) \geq 0$ for all (x,y).

Solution:
Assume $f(a,b) < 0$ at some point (a,b), then $f(x,y) < 0$ in a small neighborhood D of (a,b) and also $\int\int_D f\ dA < 0$ contradicting the assumption.

13) T F

Assume the two functions $f(x,y)$ and $g(x,y)$ have both the critical point $(0,0)$ which are saddle points, then $f+g$ has a saddle point at $(0,0)$.

Solution:
Example $f(x,y) = x^2 - y^2/2, g(x,y) = -x^2/2 + y^2$ have both a saddle point at $(0,0)$ but $f+g = x^2/2 + y^2/2$ has a minimum at $(0,0)$.

14) T F

If $f(x,y)$ is a function of two variables and if $h(x,y) = f(g(y), g(x))$, then $h_x(x,y) = f_y(g(y), g(x))g'(y)$.

Solution:
The correct identity would be $h_x(x,y) = f_y(g(y), g(x))g'(x)$ according to the chain rule. More explanation: y is a constant in this problem. If we use $x = t$ we can use the familiar formula for the chain rule: with $r(t) = \langle g(y), g(t)\rangle$ and $d/dt f(r(t)) = \nabla f(r(t)) \cdot r'(t)$ $= \langle f_x(g(y), g(t)), f_y(g(y), g(t))\rangle \cdot \langle 0, g'(t)\rangle = f_y(g(y), g(t))g'(t)$.

15) T F

If we rotate a line around the z axis, we obtain a cylinder.

Solution:
The surface could also be a one-sheeted hyperboloid or a cone.

16) T F

If $u(x,y)$ satisfies the transport equation $u_x = u_y$ everywhere in the plane, then the vector field $\vec{F}(x,y) = \langle u(x,y), u(x,y)\rangle$ is a gradient field.

Solution:
$\vec{F} = \langle P, Q\rangle = \langle u, u\rangle$. From $u_x = u_y$ we get $Q_x - P_y = 0$ which implies that $\vec{F}$ is a gradient field.

17) T F

$3\ \text{grad}(f) = \frac{d}{dt} f(x+t, y+t, z+t)$.

Solution:
The left hand side is a vector field, the right hand side a function.

18) | T | **F** | If $\vec{F}$ is a vector field in space and f is equal to the line integral of $\vec{F}$ along the straight line C from $(0,0,0)$ to (x,y,z), then $\nabla f = \vec{F}$.

Solution:
This would be true if $\vec{F}$ were a conservative vector field. In that case, f would be a potential. In general this is false: for example if $\vec{F}(x,y,z) = \langle 0, x, 0 \rangle$, then $\int_C \vec{F} \cdot \vec{dr} = x^2/2$ and $\nabla f(x,y,z) = \langle x, 0, 0 \rangle$ which is different from $\vec{F}$.

19) | **T** | F | The line integral of $\vec{F}(x,y) = (x,y)$ along an ellipse $x^2 + 2y^2 = 1$ is zero.

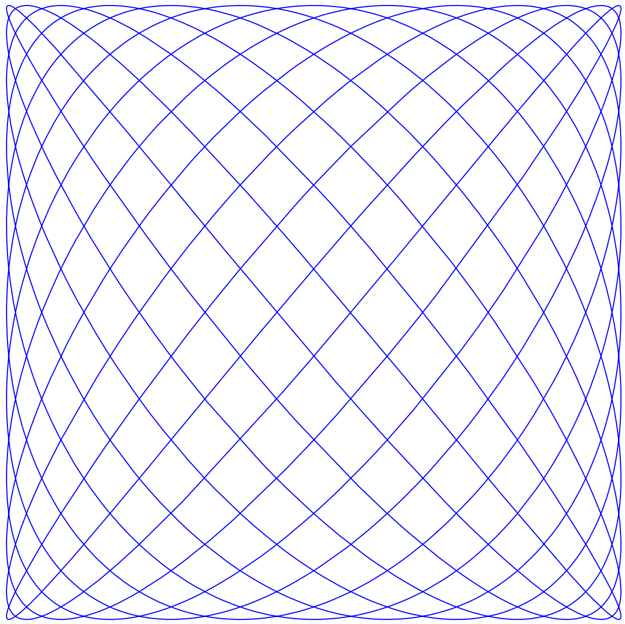
Solution:
The curl $Q_x - P_y$ of the vector field $\vec{F}(x,y) = \langle P, Q \rangle$ is 0. By Green's theorem, the line integral is zero. An other way to see this is that F is a gradient field $F = \nabla f$ with $f(x,y) = (x^2 + y^2)/2$. Therefore F is conservative: the line integral along any closed curve in the plane is zero.

20) | T | **F** | The identity $\text{div}(\text{grad}(f)) = 0$ is always true.

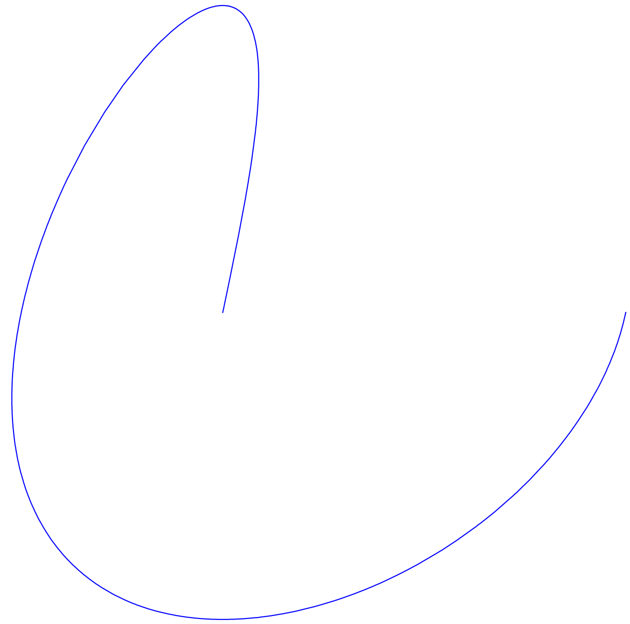
Solution:
$\text{div}(\text{grad}(f)) = \Delta f$.

Problem 2) (10 points)

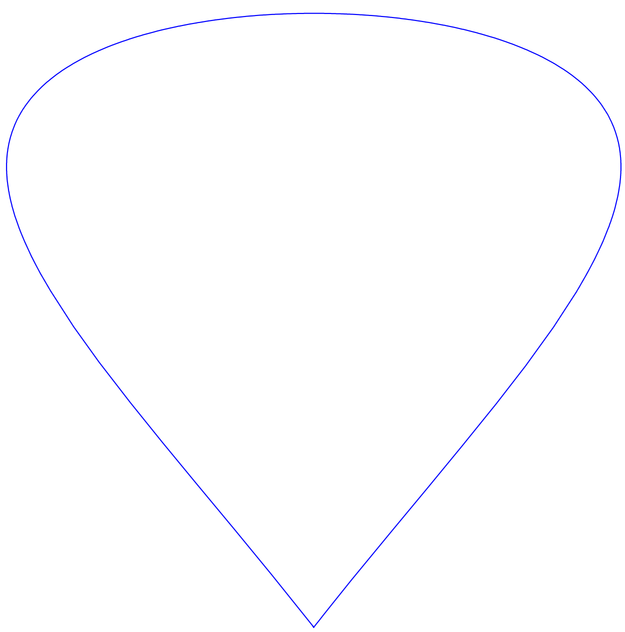
a) Match the equations with the curves. No justifications are needed.



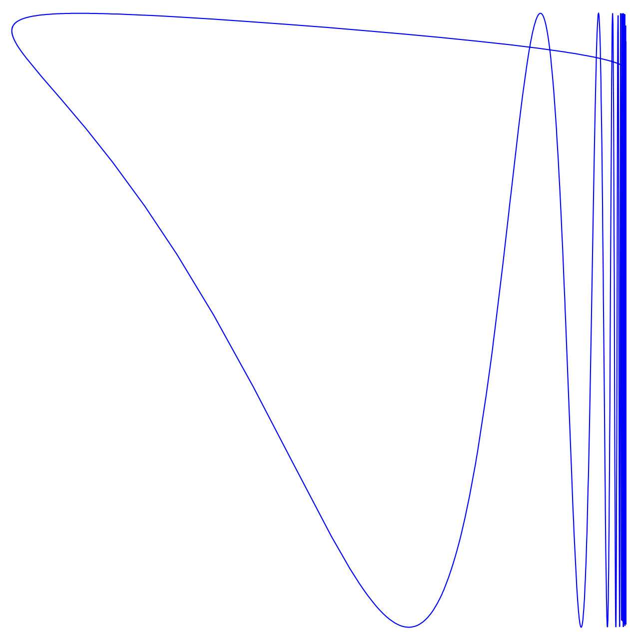
I



II



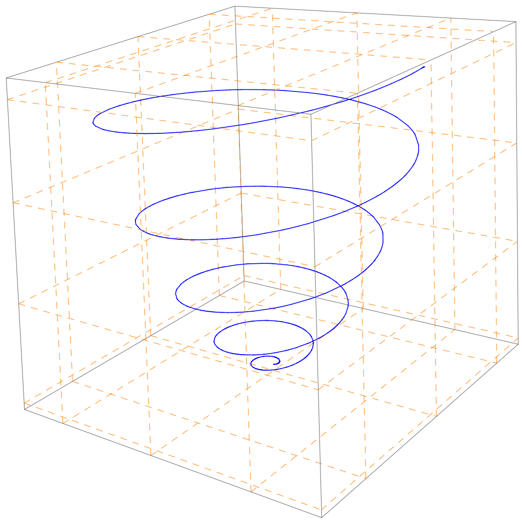
III



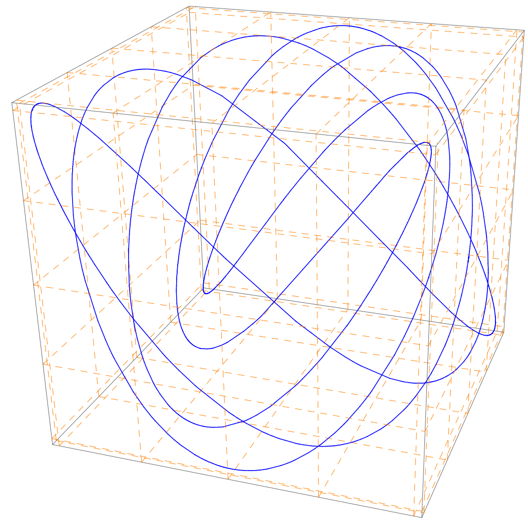
IV

Enter I,II,III,IV here	Equation
	$\vec{r}(t) = \langle \sin(t), t(2\pi - t) \rangle$
	$\vec{r}(t) = \langle \cos(11t), \sin(13t) \rangle$
	$\vec{r}(t) = \langle t\cos(t), \sin(t) \rangle$
	$\vec{r}(t) = \langle \cos(t), \sin(6/t) \rangle$

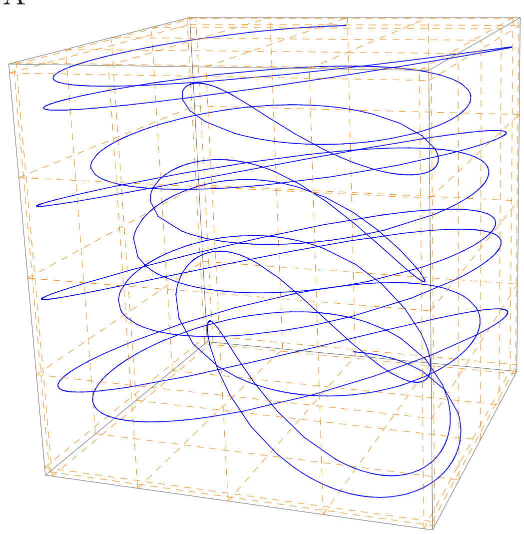
b) Match the parametrizations with the space curves. No justifications are needed.



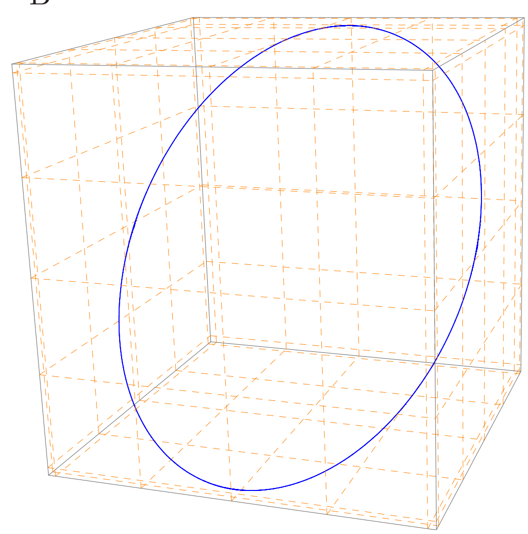
A



B



C



D

Enter A),B),C),D) here	Equation
	$\vec{r}(t) = \langle \cos(3t), \sin(3t), \sin(3t) \rangle$
	$\vec{r}(t) = \langle t\cos(t), t\sin(t), t^2 \rangle$
	$\vec{r}(t) = \langle \cos(5t), \sin(3t), \sin(7t) \rangle$
	$\vec{r}(t) = \langle \cos(13t), \sin(17t), t \rangle$

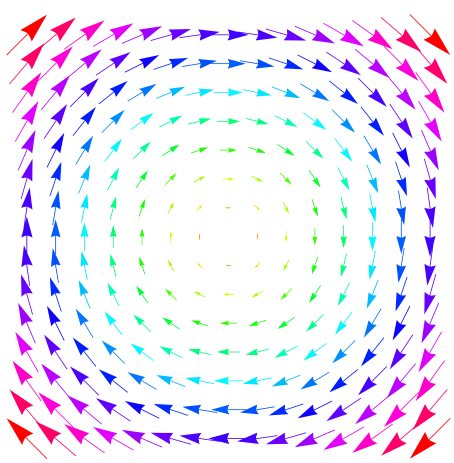
Solution:
a) III,I,II,IV
b) D), A),B),C)

Problem 3) (10 points)

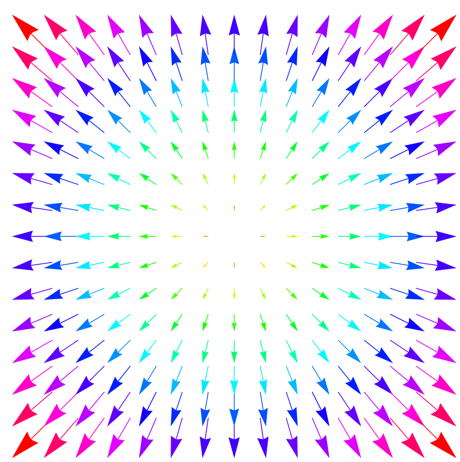
In this problem, vector fields $\vec{F}$ are written as $\vec{F} = \langle P, Q \rangle$. We use abbreviations $\text{curl}(\vec{F}) = Q_x - P_y$ and $\text{div}(\vec{F}) = P_x + Q_y$. When stating $\text{curl}(\vec{F})(x,y) = 0$ we mean that $\text{curl}(\vec{F})(x,y) = 0$ vanishes for **all** (x,y). The statement $\text{curl}(\vec{F}) \neq 0$ means that $\text{curl}(\vec{F})(x,y)$ does not vanish for at least one point (x,y).
The same notation applies if curl is replaced by div. Check the box which match the formulas of the vector fields with the corresponding picture I,II,III or IV. Check also the boxes where curl and div are zero. In each of the four rows, you will need to check three boxes. No justifications are needed.

Vectorfield	I	II	III	IV	$\text{curl}(F) = 0$	$\text{curl}(F) \neq 0$	$\text{div}(F) = 0$	$\text{div}(F) \neq 0$
$\vec{F}(x,y) = \langle 0, 5 \rangle$								
$\vec{F}(x,y) = \langle y, -x \rangle$								
$\vec{F}(x,y) = \langle x, y \rangle$								
$\vec{F}(x,y) = \langle 2, x \rangle$								

I

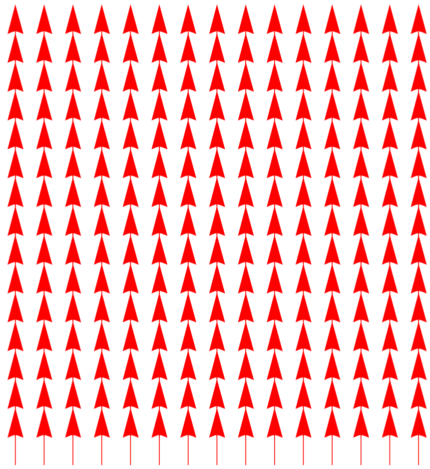


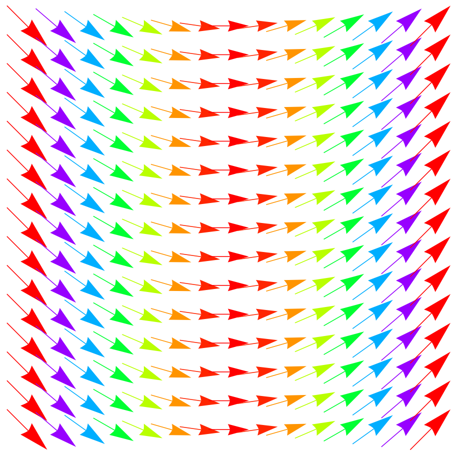
II



III

IV





Solution:

Vectorfield	I	II	III	IV	curl$(F) = 0$	curl$(F) \neq 0$	div$(F) = 0$	div$(F) \neq 0$
$\vec{F}(x,y) = \langle 0,5 \rangle$			X		X		X	
$\vec{F}(x,y) = \langle y,-x \rangle$	X					X	X	
$\vec{F}(x,y) = \langle x,y \rangle$		X			X			X
$\vec{F}(x,y) = \langle 2,x \rangle$				X		X	X	

Problem 4) (10 points)

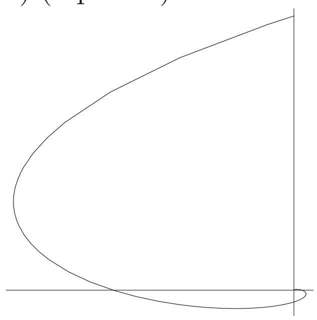
a) Find the scalar projection of the vector $\vec{v} = (3,4,5)$ onto the vector $\vec{w} = (2,2,1)$.

b) Find the equation of a plane which contains the vectors $\langle 1,1,0 \rangle$ and $\langle 0,1,1 \rangle$ and contains the point $(0,1,0)$.

Solution:
a) $|\vec{v} \cdot \vec{w}|/|\vec{w}| = (6+8+5)/3 = \boxed{19/3}$
b) $(1,1,0) \times (0,1,1) = (1,-1,1)$. The plane has the form $x - y + z = d$ and $d = -1$ is obtained by plugging in the point $(0,1,0)$. The solution is $\boxed{x - y + z = -1}$

Problem 5) (10 points)

a) (5 points) Find the surface area of the ellipse cut from the plane $z = 2x + 2y + 1$ by the cylinder $x^2 + y^2 = 1$.
b) (5 points) Find the arc length of the plane curve $\vec{r}(t) = (\sin(t)e^t, \cos(t)e^t)$ for $t \in [0, 2\pi]$.



Solution:
Parametrize the surface $r(u,v) = \langle u, v, 2u + 2v + 1\rangle$ on the disc $R = \{u^2 + y^2 \leq 1\}$. We get $r_u \times r_v = \langle 1,0,2\rangle \times \langle 0,1,2\rangle = \langle -2,-2,1\rangle$ and $|r_u \times r_v| = 3$. The surface integral $\int\int_R |r_u \times r_v|\ dudv = \int\int_R 3\ dudv = 3\int\int_R\ dudv$ which is 3 times the area of the disc R: Solution: $\boxed{3\pi}$.
b) $\vec{r}\ '(t) = \langle \cos(t)e^t + \sin(t)e^t, -\sin(t)e^t + \cos(t)e^t\rangle$ satisfies $|\vec{r}\ '(t)| = \sqrt{2}e^t$ so that $\int_0^{2\pi} |\vec{r}\,'(t)|\ dt = \boxed{\sqrt{2}(e^{2\pi} - 1)}$.

Problem 6) (10 points)

a) Verify that if $u(x,y)$ and $v(x,y)$ are two functions, then $(uv)_{xx} = u_{xx}v + 2u_xv_x + uv_{xx}$.
b) The identity $\Delta(uv) = (\Delta u)v + u(\Delta v) + 2\nabla u \cdot \nabla v$ holds.
c) Assume u and v satisfy the Laplace equation $\Delta u = u_{xx} + u_{yy} = 0$ and $\nabla u \cdot \nabla v = 0$ then uv satisfies the Laplace equation.

Solution:
a) This is a direct computation. Just differentiate the identity $(uv)_x = u_xv + uv_x$ again.
b) Add up the identity you got in a) with the corresponding identity with $(uv)_{yy}$.
c) In the identity b), all terems to the right are zero under the assumption $\Delta u = 0, \Delta v = 0$ and $\nabla u \cdot \nabla v = 0$.

Problem 7) (10 points)

Let $f(x,y,z) = 2x^2 + 3xy + 2y^2 + z^2$ and let R denote the region in space, where $2x^2 + 2y^2 + z^2 \leq 1$. Find the maximum and minimum values of f on the region R and list all points, where the maximum and minimum values are achieved. Distinguish between local extrema in the interior and extrema on the boundary.

Solution:
a) Extrema in the interior of the ellipsoid $2x^2 + 2y^2 + z^2 < 1$.
$\nabla f(x,y,z) = \langle 4x+3y, 3x+4y, 2z \rangle = \langle 0,0,0 \rangle$ for $(x,y,z) = (0,0,0)$. One has $f(0,0,0) = 0$ as a critical point. It is a candidate for the minimum.
b) To get the extrema on the boundary $g(x,y,z) = 2x^2 + 2y^2 + z^2 - 1 = 0$ we solve the Lagrange equations $\nabla f = \lambda \nabla g, g = 0$. They are

$$\begin{aligned} 4x + 3y &= \lambda 4x \\ 3x + 4y &= \lambda 4y \\ 2z &= \lambda 2z \\ 2x^2 + 2y^2 + z^2 &= 1 \end{aligned}$$

We obtain $z = 0, x = \pm y$ or $z = \pm 1, x = y = 0$ giving 6 critical points $(1/2,1/2,0), (-1/2,1/2,0), (1/2,-1/2,0), (-1/2,-1/2,0), (0,0,1), (0,0,-1)$.
c) Comparing the values $f(0,0,0) = 0$, $f(1/2,1/2,0) = f(-1/2,-1/2,0) = 7/4$ and $f(1/2,-1/2,0) = f(-1/2,1/2,0) = 1/4$ and $f(0,0,\pm 1) = 1$ shows that
$(1/2,1/2,0)$ and $(-1/2,-1/2,0)$ are maxima and that $(0,0,0)$ is the minimum.

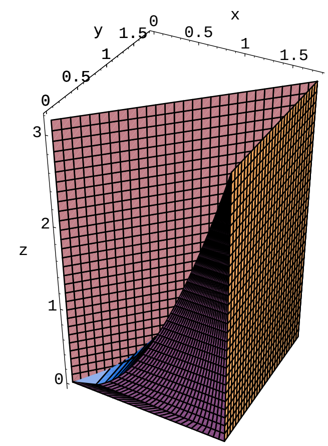
Problem 8) (10 points)

Sketch the region of integration of the following iterated integral and then evaluate the integral:

$$\int_0^{\pi} \left(\int_{\sqrt{z}}^{\sqrt{\pi}} \left(\int_0^x \sin(xy)\, dy \right) dx \right) dz \; .$$

Solution:
The region is contained inside the cube $[0, \sqrt{\pi} \times [0, \sqrt{\pi}] \times [0, \pi]$. It is bounded by the surfaces $x = \sqrt{z}, x = y, z = 0, y = \sqrt{\pi}$ (see picture). The integral can not be solved in the given order. Using the picture as a guide, we write the integral as

$$\int_0^{\sqrt{\pi}} \int_0^{x^2} \int_0^x \sin(xy) \, dydzdx$$



Solve the most inner integral:

$$\int_0^{\sqrt{\pi}} \int_0^{x^2} -\frac{\cos(xy)}{x}|_0^x \, dzdx = \int_0^{\sqrt{\pi}} \int_0^{x^2} (1 - \cos(x^2))/x \, dzdx$$

Now solve the z integral:

$$= \int_0^{\sqrt{\pi}} x^2(1 - \cos(x^2))/x \, dx = \int_0^{\sqrt{\pi}} x(1 - \cos(x^2)) \, dx$$

to finally get

$$= (-\frac{\sin(x^2)}{2} + \frac{x^2}{2})|_0^{\sqrt{\pi}} = \frac{\pi}{2} \, .$$

The answer is $\boxed{\frac{\pi}{2}}$.

Problem 9) (10 points)

Evaluate the line integral

$$\int_C \vec{F} \cdot \vec{dr} \, ,$$

where C is the planar curve $\vec{r}(t) = \langle t^2, t/\sqrt{t+2} \rangle$, $t \in [0, 2]$ and $\vec{F}$ is the vector field $\vec{F}(x, y) = \langle 2xy, x^2 + y \rangle$. Do this in two different ways:

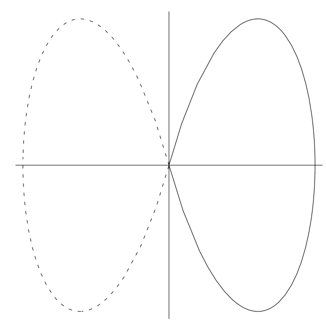
a) by verifying that $\vec{F}$ is conservative and replacing the path with a different path connecting $(0, 0)$ with $(4, 1)$,

b) by finding a potential function $f(x, y)$ which satisfies $\nabla f(x, y) = \vec{F}(x, y)$.

Solution:
a) To verify that $\vec{F} = \langle P, Q \rangle$ is conservative, it is enough to verify that $\text{curl}(\vec{F}) = Q_x - P_y = 0$ everywhere in the plane. This is actually the case. To calculate the line integral, we can by the path independence property of gradient fields, replace the given path with a straight line $C : \vec{r}(t) = \langle 4t, t \rangle$ which has the velocity $\langle 4, 1 \rangle$ and calculate $\int_C \vec{F}\ d\vec{r} = \int_0^1 \langle 8t^2, 16t^2 + t \rangle \cdot \langle 4, 1 \rangle\ dt = (48t^2 + t)|_0^1 = \boxed{16 + 1/2}$.
b) A function is $f(x, y) = x^2y + y^2/2$. The value of f at $(4, 1)$ is $16 + 1/2$. The value of f at $(0, 0)$ is 0. The difference between the potential values is $\boxed{16 + 1/2}$ again.

Problem 10) (10 points)

Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = (x + e^x \sin(y), x + e^x \cos(y))$ and C is the right handed loop of the lemniscate described in polar coordinates as $r^2 = \cos(2\theta)$. The lemniscate part on the positive x half-plane is oriented counterclockwise.



Solution:
By Green's theorem, the integral is $\int\int_R \text{curl}(F)\ dA$, where R is the region enclosed by C. From $\vec{F} = \langle P, Q \rangle = \langle x + e^x \sin(y), x + e^x \cos(y) \rangle$, we calculate $\text{curl}(\vec{F}) = Q_x - P_y = 1 + e^x \cos(y) - \cos(y)e^x = 1$ so that the result is the area of R which is $\int_{-\pi/4}^{\pi/4} \cos(2\theta)/2\ d\theta = \frac{\sin(2\theta)}{4}|_{-\pi/4}^{\pi/4} = \boxed{1/2}$.

Problem 11) (10 points)

Let $f(x, y, z)$ be the distance to the surface $x^4 + 2y^4 + z^4 = 1$. Show that f is a solution of the partial differential equation

$$f_x^2 + f_y^2 + f_z^2 = 1$$

outside the curve.

Hint: no computations are needed here. The shape of the surface pretty much irrelevant. What does the PDE say about the gradient ∇f?

Solution:
This is just a three dimensional version of the problem in the second midterm practice exam. The partial differential equation is the **eiconal equation** of optics. The level surfaces of g are surfaces containing all points for which the distance to the curve is constant. Let's look at the level surfaces of f, if f is the solution to the PDE. The PDE tells us $|\nabla f|^2 = 1$. Which means that the gradient of f is a unit vector everywhere. This shows that the directional derivative in the gradient direction is 1 everywhere. If we imagine f to be a height, and if we move in the gradient direction, we experience a slope which is 1. This implies that the level curves of f are equi distributed too. The level curves of f and g are the same. Because f and g are both zero at the curve, the two functions must be the same.

Problem 12) (10 points)

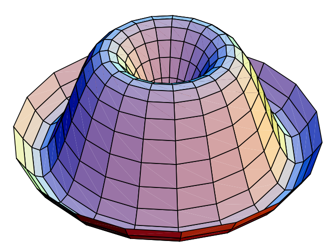
a) Find the line integral $\int_C \vec{F} \cdot \vec{dr}$ of the vector field $\vec{F}(x, y) = (xy, x)$ along the unit circle $C : t \mapsto \vec{r}(t) = (\cos(t), \sin(t)), t \in [0, 2\pi]$ by doing the actual line integral.

b) Find the value of the line integral obtained in part a) by evaluating a double integral.

Solution:
a) $\int_0^{2\pi} (\cos(t)\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t))\ dt = \int_0^{2\pi} \cos^2(t) - \sin^2(t)\cos(t)\ dt = \pi + \sin^3(t)/3|_0^{2\pi} = \boxed{\pi}$
b) $\text{curl}(F) = Q_x - P_y = 1 - x$. By Green's theorem, the line integral is a double integral which we evaluate using Polar coordinates
$\int\int_D (1 - x)\ dxdy = \int_0^1 \int_0^{2\pi} (1 - \cos(\theta)r)r\ d\theta dr = 2\pi/2 = \boxed{\pi}$.

Problem 13) (10 points)

Consider the surface given by the graph of the function $z = f(x, y) = \frac{100}{1+x^2+y^2} \sin\left(\frac{\pi}{8}(x^2 + y^2)\right)$ in the region $x^2 + y^2 \leq 16$. The surface is pictured to the right.



A magnetic field $\vec{B}$ is given by the curl of a vector potential $\vec{A}$. That is, $\vec{B} = \nabla \times \vec{A} = \text{curl}(\vec{A})$ and $\vec{A}$ is a vector field too. Suppose

$$\vec{A} = \langle z \sin(x^3), x\left(1 - z^2\right), \log(1 + e^{x+y+z}) \rangle .$$

Compute the flux of the magnetic field through this surface. The surface has an upward pointing normal vector.

Solution:
The surface S is bounded by the curve $\gamma : \vec{r}(t) = (4\cos(t), 4\sin(t), 0)$. By Stokes theorem, the flux of the curl of $\vec{A}$ through the surface S is the line integral of A along γ:

$$\int_\gamma \vec{A} \cdot d\vec{r} = \int_0^{2\pi} (0, 4\cos(t), \log(1 + e^{x+y})) \cdot (-4\sin(t), 4\cos(t), 0)\, dt = \int_0^{2\pi} 16\cos^2(t)\, dt =$$

$\boxed{16\pi}$.

Problem 14) (10 points)

Let S be the surface given by the equations $z = x^2 - y^2$, $x^2 + y^2 \leq 4$, with the upward pointing normal. If the vector field $\vec{F}$ is given by the formula $\vec{F}(x, y, z) = \langle -x, y, \sqrt{x^2 + y^2} \rangle$, find the flux of $\vec{F}$ through S.

Solution:
Parameterize the surface by $\vec{r}(u, v) = \langle u, v, u^2 - v^2 \rangle$. Then $r_u \times r_v = \langle 1, 0, 2u \rangle \times \langle 0, 1, -2v \rangle = \langle -2u, 2v, 1 \rangle$. The flux integral is

$$\begin{aligned} \int\int_D \vec{F} \cdot dS &= \int\int_D \langle -u, v, \sqrt{u^2 + v^2} \rangle \cdot \langle -2u, 2v, 1 \rangle\, dxdy \\ &= \int\int_D 2(u^2 + v^2) + \sqrt{u^2 + v^2}\, dA \\ &= \int_0^{2\pi} \int_0^2 (2r^2 + r) r\, dr d\theta \\ &= 2\pi(2 \cdot 2^4/4 + 2^3/3) = 64\pi/3 . \end{aligned}$$

The answer is $\boxed{64\pi/3}$.

Remark. One could be tempted to use the divergence theorem. The divergence of $\vec{F}$ is zero. One would have to close the surface and compute the flux through the other piece. This is possible but gives more work. There are problems which look like this, where one has to use such an indirect approach.