Extended hour to hour syllabus for Math21a

Course head: Oliver Knill

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Abstract

This is a more detailed plan for the lectures in the fall 2009 semester. Each section covers enough material so that students are exposed to the necessary terminology and examples to solve the homework. Homework is synchronized for all sections. Tuesday/Thursday sections assign the first homework set during the first lecture and two thirds of a weekly homework set during the second class. The homework in this document is still very preliminary. Section numbers after each lecture title point belong to the book Stewart: Concept and Context, the 4'th edition.

0. Week: (9/1-9/6)

The intro lecture of September 3 should give the students most details about the syllabus. The website also lists these details. We do not have to repeat them later in each section. On Thursday afternoon, we will see how many students signed up for each section and can make a lineup plan. The course assistants are determined on Friday (sometimes Saturday). You will get the name of your CA by email. Arrange for a quick meeting before the first lecture (just before the lecture is a possibility).

1. Week: (9/8-9/12)

1. Lecture: Space, coordinates, distance (9.1)

After introducing ourselves, we use **coordinates** to describe space, as it was promoted by **Descartes** in the 16'th century. A fundamental notion is the **distance** between two points. The formula uses **Pythagoras** theorem. In order to get a feel about space, we look at some geometric objects defined through coordinates. We will focus first on **circles** and **spheres** and learn how to find the midpoint and radius of a sphere given as a quadratic expression in x, y, z. This method is called the **completion of the square**. We introduce the course assistant and reserve 10 minutes of the class for the course assistant to organize problem sessions.

2. Lecture: Vectors, dot product, projections, (9.2-9.3)

Two points P, Q define a **vector** $\vec{PQ}$ with head at Q and tail at P. The vector connects the initial point P with the end point Q. Vectors can be attached everywhere in space, but they are identified if they have the same length and direction. Vectors can describe **velocities**, **forces** or **color** or **data**. We introduce **addition**, **subtraction** and **scaling** both graphically as well as algebraically. The **dot product** $\vec{v} \times \vec{w}$ between two vectors, which results in a scalar, allows to compute **length**, **angles** and **projections**. By assuming the trigonometric cos-formula, we prove the important formula $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\alpha$, which relates length and angle with the dot product. This formula has some consequences like the **Cauchy-Schwartz inequality**. The book uses this geometric description as a definition and derives from it the algebraic sum of products of components relation. We mention the notation $\vec{i}, \vec{j}, \vec{k}$ for the unit vectors.

2. Week: (9/14-9/19)

1. Lecture: The cross product and planes (9.4)

We review the dot product and introduce the **cross product** of two vectors in space which is a new vector perpendicular to both. The product can be used for many things: it is useful for example to compute areas, or the distance between a point and a line, to construct a plane through three points or to find the line which is in the intersection of two planes. The cross product can be computed as a determinant. The formula $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$ can be interpreted geometrically as an area of the parallelepiped spanned by $\vec{v}$ and $\vec{w}$. The **triple scalar product** $(\vec{u} \times \vec{v}) \cdot \vec{w}$ is the signed volume of the parallelepiped spanned by $\vec{u}, \vec{v}$ and $\vec{w}$. Interpret $\vec{u} \times \vec{v} \cdot (\vec{x} - \vec{x}_0 = 0$ as the property that $\vec{x} - \vec{x}_0$ is in the plane spanned by $\vec{u}$ and $\vec{v}$. This means that $\langle a, b, c \rangle = \vec{n} = \vec{u} \times \vec{v}$ is perpendicular to $\vec{x} - \vec{x}_0$ which is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, the equation of a plane. Planes can best be drawn best using **traces** and **intercepts**.

2. Lecture: Lines and planes and distance formulas (9.5)

A good problem to start the lecture is to find a plane which passes through three points P,Q, and R. This or a similar problem allows to recall dot and cross product. We introduce lines by the parameterization $\vec{r}(t) = P + t\vec{v}$. If $\vec{v} = \langle a, b, c \rangle$ and $P = (x_0, y_0, z_0)$, then $(x - x_0)/a = (y - y_0)/b = (z - z_0)/c$ is the **Cartesian equation** or **symmetric equation** of a line. It can be interpreted as the intersection of two planes. As an application of dot and cross products, we look at **distance formulas** (featured in a separate course wide handout), the distance from a point to a plane, the distance from a point to a line or the distance between two lines.

3. Lecture: Functions, graphs, quadrics (9.5-9.6)

As the name "multi-variable calculus" suggests, functions of several variables play an essential role in this course. We first focus on functions of two variables. The **graph** of a function $f(x, y)$ of two variables is defined as the set of points (x, y, z) for which $z - f(x, y) = 0$. We look at many examples and let the class match graphs with functions. There are several techniques to learn here: traces, the intersection of the graph with the coordinate planes, generalized traces like $f(x, y) = c$, which define **level curves**. In this lecture, one could let the class match surfaces with formulas. After a short review of **conic sections** like ellipses, parabola and hyperbola, we look at surfaces of the form $g(x, y, z) = 0$. Start with known examples like the sphere and the plane. If $g(x, y, z)$ is a function which only involves quadratic terms, the level surface is called a **quadric**. Important quadrics are **spheres**, **ellipsoids**, **cones**, **cylinders** as well as various **hyperboloids**. Give an overview over all quadrics as well as degenerate cylindrical cases.

3. Week: (9/21-9/26)

1. Lecture: Curves, velocity, acceleration (10.1, 10.2)

Curves are one-dimensional objects in the plane or in space. They take many different forms. Closed curves in space are called **knots**. We define curves by parameterization $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$. Differentiation leads to the **velocity** $\vec{r}\,'(t)$, a vector which is tangent to $\vec{r}(t)$ and an other differentiation to the **acceleration** vector $\vec{r}\,''(t)$. The **speed** $|r'(t)|$ is a scalar. We also can reverse differentiation. Given $\vec{r}\,'(t)$ and $= \vec{r}(0)$, it is possible to find $\vec{r}(t)$. A nice application is the **free fall**.

2. Lecture: Arc length and curvature (10.3, 10.4)

The **curvature** of a curve measures how much a curve is bent. Both acceleration and curvature involve second derivatives, but curvature is an intrinsic quantity of the curve which does not depend on the parameterization. One "feels" the acceleration and "sees" the curvature. There is a formula for the **arc length** of a curve. It can be motivated by computing the length for a polygon and passing to the limit which leads to the one-dimensional integral $\int_a^b |r'(t)|\, dt$. A re-parametrization of a curve does not change the arc length. We introduce the unit normal vector $\vec{T}$, the normal vector $\vec{N}$ as well as the **bi-normal vector** $\vec{B}$.

3. Lecture: Cylindrical and spherical coordinates (9.7)

First we review **polar coordinates** (r, θ) in the plane, where $r \geq 0$ and $0 \leq \theta < 2\pi$. We stress that the radius r is always nonnegative and that the formula $\theta = \arctan(y/x)$ is ambiguous and only determine ϕ up to π. Next we introduce cylindrical coordinates, which is just adding a third component z to polar coordinates in the plane. Finally, we cover more Greek alphabet by introducing Euler angles θ, ϕ to describe points in space using **spherical coordinates**. To avoid confusion, we always use ρ as the distance to the origin and r as the distance to the z-axes.

4. Week: (9/28-10/3)

1. Lecture: Parametric surfaces (10.5)

Surfaces can be described in two fundamental ways: implicitly or parametrically. Implicit descriptions are $g(x, y, z) = 0$ or $x^2 + y^2 + z^2 - 1 = 0$, parametric descriptions are $r(u, v) = (x(u, v), y(u, v), z(u, v))$ like $\vec{r}(\theta, \phi) = \langle \rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)\rangle$. In many cases, it is possible to switch from a parametric description to an implicit and back. Examples are the plane, spheres, graphs of functions of two variables or **surfaces of revolution**. Using a computer, one can **visualize** complicated surfaces. It is worth mentioning that in computer applications, the parameterization $r(u, v)$ is called the **"uv-map"**. More surfaces will be explored in the Mathematica project.

2. Lecture: Functions (11.1)

We might review a bit parametric surfaces here first, since this is a difficult topic. We then look at functions of several variables. For functions of one variable, continuity can fail with jump discontinuities, infinities or singular oscillations. In higher dimensions, this is more interesting. Rather than giving an $\epsilon - \delta$ definition for continuity, we look at one or two examples which illustrates how continuity can fail in two dimensions. We also look at contour maps of functions and see the relation with continuity. For example, if contours to different levels intersect, the function can not be continuous.

3. Lecture: Partial derivatives (11.2- 11.3)

Finally, we introduce **partial derivatives** $f_x = \partial_x f = \frac{\partial f}{\partial x}$. It can help to promote the index notation f_x for its simplicity replacing the cumbersome partial signs. To explain Clairot's theorem, one can ask something like to compute f_{xxyx} for $f(x, y) = x^2\sin(\sin(y)) + x\tan(y)$.

5. Week: (10/5-10/10)

1. Lecture: Review for the first hourly

The first hourly covers the material from the first 3 weeks. Possible review strategies are answering questions, discussion of the practice exam or to go through some True/False questions.

2. Lecture: Partial differential equations (11.3)

To practice differentiation and to get a glimpse of how calculus is used in science, we check whether functions are solutions to **partial differential equations**, abbreviated PDE. More precisely, we look at the transport equation $f_x(t, x) = f_t(t, x)$, the wave equation $f_{tt}(t, x) = f_{xx}(t, x)$ and the heat equation $f_t(t, x) = f_{xx}(t, x)$. There will be a handout on differential equations. Partial differential equations are very important not only because they describe the fabric of the structure we are made of like the Einstein equations in cosmology, or Maxwell equations in electromagnetism. They also are very useful for us: Navier Stokes equations help in weather prediction and turbine design, the Black-Scholes equation in pricing financial options, the shallow water equation to model Tsunamis, the nonlinear wave equation to design optical fibers or the diffusion-migration equation to discover oil.

3. Lecture: Linear approximation and tangents (11.4)

Linearization is an important concept in science because many physical laws are linearization of more complicated laws. This will allow us to compute **tangent planes** and **tangent lines** as well as to approximate a linear function by a linear function near a point. Many physical laws are actually just linearization of more complicated nonlinear laws. We treat linearization in two and three dimensions. The linearization of the function $f(x, y)$ at the point (x_0, y_0) is $L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

6. Week: (10/12-10/17)

1. Lecture: Columbus day, no class

2. Lecture: Chain rule and implicit differentiation (11.5)

We recall the chain rule $d/dt f(g(t)) = f'(g(t))g'(t)$ in one dimensions. In several dimensions, we introduce the chain rule $d/dt f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$. If you do not want to introduce the gradient yet, write it out as $d/dt f(x(t), y(t), z(t)) = f_x(x(t), y(t), z(t))x'(t) + f_y(x(t), y(t), z(t))y'(t) + f_z(x(t), y(t), z(t))z'(t)$. This is the only chain rule we need and introduce in this course because the most general case can be obtained by taking this chain rule coordinate wise for a vector valued function. A nice application of the chain rule is **implicit differentiation**: if $f(x, y, z) = 0$ defines locally $z = g(x, y)$ and because $f_x + f_z z' = 0$ we can compute z'.

3. Lecture: Gradient, Directional derivative (11.6)

The **gradient** of a function is an important tool to describe the geometry of surfaces. One of the most central facts students should know about is that the gradient of a function $f(x, y)$ at a point is perpendicular to the level curve $f(x, y) = c$ passing through that point. Or that the gradient of a function $g(x, y, z)$ at a point is perpendicular to the level surface $f(x, y, z) = c$ which passes through that point. Prove this using the chain rule introduced in the last lecture: for a curve $\vec{r}(t)$ on the surface $f(\vec{r}(t)) = 0$. A special case is $g(x, y, z) = ax + by + cz = d$, where $\nabla g = \langle a, b, c\rangle$. In this lecture, we also look at tangent planes and tangent lines. Compute the gradient vector at a point $\nabla f(x0, y0, z0) = \langle a, b, c\rangle$, then form the plane ax+by+cz = d, where d is obtained by plugging in a constant. We introduce the **directional derivative** $D_v(f)$ as $D_v f = \nabla f \cdot v$ for unit vectors v. Some examples give a feel, what the directional derivative means, also geometrically. Partial derivatives are special directional derivatives. The direction of the normal vector gives a nonnegative partial derivative. Moving into the direction of the normal vector, increases f because $D_{\nabla f/||\nabla f}f = |\nabla f|$.

7. Week: (10/19-10/24)

1. Lecture: Extrema, second derivative test (11.7)

A central application of multi-variable calculus is to **extremize** functions of two variables. One first identifies **critical points**, points where the gradient vanishes. (We seldom focus on points where the derivative does not exist in this course). The nature of these critical points can be established using the **second derivative test**. There will be three fundamentally different cases: **local maxima**, **local minima** as well as **saddle points**.

2. Lecture: Extrema with constraints (11.8)

We extremize a function $f(x, y)$ in the presence of a **constraint** $g(x, y) = 0$. A necessary condition for a critical point is that the gradients of f and g are parallel. This leads to a system of equations called the **Lagrange equations** $\nabla f = \lambda \nabla g$. Additionally, one has to look at points $\nabla g = 0$.

3. Lecture: Global extrema on closed regions (11.8)

When extremizing functions on a domain bounded by a curve $g(x, y) = 0$, we have to solve two problems: find the extrema in the interior and the extrema on the boundary. This is an opportunity to practice both extremization problems again and give more examples.

8. Week: (10/26-10/31)

1. Lecture: Double integrals (12.1,12.2)

Integration in two dimensions is first done on rectangles, then on regions bound by graphs of functions. Similar than in one dimension, there is a **Riemann sum approximation** of the integral. This allows us to derive results like **Fubini's theorem** on the change of the integration order. An application of double integration is the computation of **area**, for which $f(x, y) = 1$.

2. Lecture: General regions (12.3)

We especially practice translating back and forth from a region to the integral. A good example is the change of order of integration in regions which are both type I and type II. A double integral can be interpreted as the signed volume under a surface.

3. Lecture: Integration in polar coordinates (12.4)

Many planar regions can be described better in **polar coordinates**. There are beautiful regions, which can be defined in polar coordinates. Examples are **roses** $r \leq g(\theta)$ which trace flower-like shapes in the plane but are graphs in polar coordinates. There are examples, where one has to change the order of integration for double integrals. We might already look at applications of double integrals from statistics or physics.

9. Week: (11/2-11/7)

1. Lecture: Review for second midterm

The second midterm covers the material from week 1-8.

2. Lecture: Application double integrals Surface area (12.5-12.6)

After looking at some more examples of double integrals, we derive the formula $\int\int_R |r_u \times r_v| \; dudv$ and give examples like graphs, surfaces of revolution and especially the sphere. Similar as for arc length, it is easy to give examples, where the surface area can not be computed in closed form, like triangles, parts of the sphere or cylinder or paraboloid.

3. Lecture: Triple integrals, (12.7)

Triple integrals allow the computation of volumes, moment of inertias or centers of masses of solids. First introduced for cubes it is then extended to more general regions which are bound by graphs of functions of two variables. It can be interesting to mention the Cavalieri method or Monte Carlo method to find volumes. The main point is to give a feel how to **set up a triple integral**. In practically all cases which appear in calculus text books, the region is contained between the graphs of two functions of two variables and if not, then this is an indication that a different coordinate system should be used.

10. Week: (11/9-11/14)

1. Lecture: Spherical and cylindrical coordinates (12.8)

Applications spice up the topic of integration. It also allows to practice how to set up integrals. Applications are computations of **mass** $\int\int\int_E \delta(x, y, z) \; dxdydz$, **moment of inertia** $\int\int\int_E (x^2 + y^2 + z^2) \; dxdydz$, **center of mass**, $\int\int\int \langle x, y, z\rangle \; dV$ the **expectation** $\mathrm{E}[X] = \int\int\int X(x, y, z) \; dV / \int\int\int \; dV$ of a random variable $X(x, y, z)$ on a region Ω. the analogue of polar coordinates in space. **Spherical coordinates** allow an even more elegant computation of triple integrals for regions like cones or spheres.

2. Lecture: Vector fields and line integrals (13.1)

Vector fields occur as force fields or velocity fields or mechanics. Most students have seen vector fields already in single variable calculus in the context of differential equations. It is helpful to discuss vector fields in concrete setups. We introduce **flow lines**, curves which satisfy $\vec{r}'(t) = \vec{F}(\vec{r}(t))$. If the vector field is a velocity field and an eddy is placed at some point, then it will move along a flow line.
Line integrals $\int_C F(\vec{r}(t)) \; \vec{r}\,'(t) \; dt$ along a curve in the presence of a vector field. An important example is if F is a **force field** and the line integral as **work**.

11. Week, (11/16-11/21)

1. Lecture: Fundamental theorem of line integrals (13.3)

For a class of vector fields which we call **conservative vector fields** one can evaluate the line integral easily using an identity called the **fundamental theorem of line integrals**.

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \; dt \; .$$

We distinguish first between **path independence** (the line integral of any path connecting two points A and B are the same) and **gradient fields**, field for which there is a function f with $\vec{F} = \nabla f$. In other words, the field is the gradient of a **potential function** f. There is a third property, the **closed loop property** which is equivalent to both **path independence** and being a gradient field. Without any reservations, the three things are equivalent and summarized as **conservative**.

2. Lecture: Green theorem (13.4)

Greens theorem relates a line integral along a closed curve with a double integral of a derivative of the vector field in the region enclosed by the curve. One can formulate the theorem already using the two-dimensional curl $\text{curl}(F) = Q_x - P_y$, which is a scalar. The theorem is useful for example to **compute areas**. It also allows an easy computation of line integrals in certain cases. Greens theorem implies that if $\text{curl}(F) = 0$ everywhere in the plane, then the field has the **closed loop property** and therefore conservative. Vector fields which have zero curl are called **irrotational**. Greens theorem shows that if a region is simply connected, then the property of being irrotational is equivalent to being conservative.

3. Lecture: Curl and divergence (13.5)

The **curl** of a vector field $\vec{F} = \langle P, Q, R \rangle$ in three dimensions is a new vector field which can be computed as $\nabla \times \vec{F}$. The three components give the vorticity of the vector field in the x,y and z direction. In two dimensions, it is a scalar field $\text{curl}(P, Q) = Q_x - P_y$ which measures the vorticity of the vector field in the plane. There are many ways to visualize and motivate the curl. We also introduce the **divergence**, a scalar field, which is a scalar field. The divergence at a point measures "how much field is generated" at a point.

12. Week, (11/23-11/28)

1. Lecture: Flux integrals (13.6)

One way to motivate the introduction of the flux is to have a membrane S and a fluid with has velocity $\vec{F}(x, y, z)$ at the point (x, y, z). If we compute how much fluid passes through a tiny membrane in unit time, we are led to the quantity $\vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\ dudv$. Obviously the angle between F and the normal vector $\vec{n} = \vec{r}_u \times \vec{r}_v$ will matter. Through a small flat square, the flux is $\vec{dS} = \vec{F} \cdot \vec{n}\ dudv$. Integrating this over the parameter domain R is called a **flux integral** $\int \int_S \vec{F} \vec{dS}$. We do not introduce a scalar surface integral. The later should be seen as a generalization of the surface area integral and not a new concept.

2. Lecture: Stokes theorem (13.7)

Stokes theorem is Greens theorem lifted into three dimensions, where the region is replaced by a surface. It tells that one can replace the line integral along the boundary C of the surface by an integral of the "curl" of the field over the surface. $\int_C \vec{F} \cdot \vec{dr} = \int \int_S \text{curl}(F) \cdot \vec{dS}$. We see different examples on how to use Stokes theorem to compute line integrals or to compute flux integrals. There are many applications which can be mentioned here, like the dynamo. While Green's theorem can be derived from Stokes theorem, it is not a special case. The theorems live in different dimensions.

3. Lecture: Thanksgiving, no class

13. Week, (11/30-12/3)

1. Lecture: Gauss theorem (13.8)

The **divergence** of a vector field in the interior of a solid E is related to the flux of the vector field $\vec{F}$ through the boundary of the surface using the **divergence theorem**. It relates the "local expansion rate" integrated over the solid $\int \int \int_E \text{div}(\vec{F})\ dV$ of a vector field $\vec{F}$ with the flux $\int \int_S \vec{F} \cdot \vec{dS}$ through the boundary surface S of E. It is useful for example to compute the gravitational field inside a solid like the sphere or a hollow sphere.

2. Lecture: Overview over integral theorems

In this lecture the Mathematica project is due. We give an overview over all integral theorems. It turns out that in d-dimensions there are d theorems. We have here seen the situation in dimension 2 and 3:

dimension	1D objects	2D objects	3D objects
1	FTC	-	-
2	FTL	Green	-
3	FTL	Stokes	Gauss

but we could continue in higher dimensions too.