Extended hour to hour syllabus for Math21a

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Abstract

Here is a more detailed plan for the lectures in the fall 2007 semester. We will discuss the material more during our weekly course meetings. Tuesday/Thursday sections usually assign the first hour homework during the first lecture and two thirds during the second class. It is your responsibility to cover the material so that students can solve the homework, which is fixed for all sections.

1. Week: Geometry (week 9/24-9/28)

1. Lecture: Space, coordinates, distance (9.1)

After introducing ourselves, we use **coordinates** to describe space, as it was promoted by **Descartes** in the 16'th century. A fundamental notion is the **distance** between two points uses **Pythagoras** theorem. In order to get a feel about space, we look at some geometric objects defined through coordinates. We will focus on **circles** and **spheres** and learn how to find the midpoint and radius of a sphere given as a quadratic expression in x, y, z. This method is called the **completion of the square**. Discussion point: what distinguishes Eucledian distance from other distances? Introduce the course assistant and plan 10 minutes for the course assistant to organize problem sessions.

Homework: 1. Class 9.1: 8,10,14,16,18

2. Lecture: Vectors, dot product, projections, (9.2-9.3)

Two points P, Q define a **vector** $\vec{PQ}$ with head at Q and tail at P. The vector connects the initial point P with the end point Q. Vectors can be attached everywhere in space, but they are identified if they have the same length and direction. Vectors can describe **velocities**, **forces** or **color** or **data**. We introduce **addition**, **subtraction** and **scaling** both graphically as well as algebraically. The **dot product** $\vec{v} \times \vec{w}$ between two vectors, which results in a scalar, allows to compute **length**, **angles** and **projections**. By assuming the trigonometric cos-formula, we prove the important formula $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos \alpha$, which relates length and angle with the dot product. This formula has some consequences like the **Cauchy-Schwartz inequality** or the **Pythagoras theorem**. We mention the notation $\vec{i}, \vec{j}, \vec{k}$ for the unit vectors. Discussion point: why does the commonly seen definition: "a vector is an entity with length and direction" not work?

Homework: 2. Class 9.2: 16,22,38 9.3: 34,38

3. Lecture: The cross product and planes (9.4)

We review the dot product and introduce the **cross product** of two vectors in space which is a new vector perpendicular to both. The product can be used for many things. It is useful for example to compute areas, or the distance between a point and a line, to construct a plane through three points or to find the line which is in the intersection of two planes. The cross product can be computed as a determinant. The formula $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\alpha)$ can be interpreted geometrically as an area of the parallelepiped spanned by $\vec{v}$ and $\vec{w}$. The **triple scalar product** $(\vec{u} \times \vec{v}) \cdot \vec{w}$ is the signed volume of the parallelepiped spanned by $\vec{u}, \vec{v}$ and $\vec{w}$. Interpret $\vec{u} \times \vec{v} \cdot (\vec{x} - \vec{x}_0 = 0$ as the property that $\vec{x} - \vec{x}_0$ is in the plane spanned by $\vec{u}$ and $\vec{v}$. This means that $\langle a, b, c\rangle = \vec{n} = \vec{u} \times \vec{v}$ is perpendicular to $\vec{x} - \vec{x}_0$ which is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$. This is the equation of a plane. Planes can best be drawn best using **traces** and **intercepts**. Discussion point: what is $\vec{i} \times \vec{j} \times \vec{k}$? Is the cross product associative?

Homework: 3. Class 9.4: 4,14,16,26,30

2. Week: Functions (week 10/8-10/12)

2. Lecture: Lines and planes and distance formulas (9.5)

I good problem to start the lecture is to find a plane which passes through three points P,Q, and R. This or a similar problem allows to recall dot and cross product. We introduce now lines by the parameterization $\vec{r}(t) = P + t\vec{v}$. If $\vec{v} = \langle a, b, c\rangle$ and $P = (x_0, y_0, z_0)$, then $(x - x_0)/a = (y - y_0)/b = (z - z_0)/c$ is the **cartesian equation** or **symmetric equation** of a line. It can be interpreted as the intersection of two planes. As an application of dot and cross products, we look at **distance formulas**, the distance from a point to a plane, the distance from a point to a line or the distance between two lines. Discussion point: if we do not take absolute values of distances to planes, we often get negative numbers. What does the sign mean?

2. Lecture: Functions and graphs (9.5)

As the name "multi-variable calculus" suggests, functions of several variables play an essential role in this course. We first focus on functions of two variables. The **graph** of a function $f(x, y)$ of two variables is defined as the set of points (x, y, z) for which $z - f(x, y) = 0$. We look at many examples and let the class match graphs with functions. There are several techniques which students can learn here: traces, the intersection of the graph with the coordinate planes, generalized traces like $f(x, y) = c$, which define **level curves**. In this lecture, one could let the class match surfaces with formulas. Discussion point: are there surfaces which can not be written as graphs?

3. Lecture: Level curves, level surfaces quadrics (9.5-9.6)

After a short review of **conic sections** like ellipses, parabola and hyperbola, we look at surfaces of the form $g(x, y, z) = 0$. Start with known examples like the sphere and the plane. If $g(x, y, z)$ is a function which only involves quadratic terms, the level surface is called a **quadric**. Important quadrics are **spheres**, **ellipsoids**, **cones**, **cylinders** as well as various **hyperboloids**. Give an overview over all quadrics as well as degenerate cylindrical cases. Discussion point: what happens at the transition, when a hyperboloid is deformed from a one sheeted hyperboloid to a two sheeted hyperboloid?

3. Week: Curves (week 10/1-10/5)

1. Lecture: Columbus day, no class

2. Lecture: Curves, velocity, acceleration (10.1, 10.2)

Curves are one-dimensional objects in the plane or in space. They take many different forms. Closed curves in space are called **knots**. We define curves by parameterization $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$. Differentiation leads to the **velocity** $\vec{r}\,'(t)$, a vector which is tangent to $\vec{r}(t)$ and an other differentiation to the **acceleration** vector $\vec{r}\,''(t)$. The **speed** $|r'(t)|$ is a scalar. We also can reverse differentiation. Given $\vec{r}'(t)$ and $= \vec{r}(0)$, it is possible to find $\vec{r}(t)$. A nice application is the **free fall**. Discussion points: where could the third derivative of a curve play a role. Can a curve have corners if x(t),y(t),z(t) are smooth (Example: $(t^2, t^3, 0)$). Can a curve look smooth even so $(x(t), y(t), z(t)$ are not? (Example: $(\sin(1/t), \sin(1/t), t)$).

3. Lecture: Arc length and curvature (10.3, 10.4)

The **curvature** of a curve measures how much a curve is bent. Both acceleration and curvature involve second derivatives, but curvature is an intrinsic quantity of the curve which does not depend on the parameterization. One "feels" the acceleration and "sees" the curvature. Discussion point: can you imagine curves for which arc lenght is infinite? There is a formula for the **arc length** of a curve. It can be motivated by computing the length for a polygon and passing to the limit which leads to the one-dimensional integral $\int_a^b |r'(t)| \, dt$. A reparametrization of a curve does not change the arc length. We introduce the unit normal vector $\vec{T}$, the normal vector $\vec{N}$ as well as the **binormal vector** $\vec{B}$. Discussion point: what can happen with the Frenet frame at points, where the speed is zero?

4. Week: Surfaces (week 10/15-10/19)

1. Lecture: Review for the first hourly

The first hourly covers the material from the first 3 weeks. Possible review styles are answering questions, discussion of the practice exam or to go through some True/False questions.

2. Lecture: Cylindrical and spherical coordinates (9.7)

First we review **polar coordinates** (r, θ) in the plane. Most students will have seen polar coordinates already. Stress that the radius r is always nonnegative and that the formula $\theta = \arctan(y/x)$ is ambiguous and only determine ϕ up to π. Next we introduce cylindrical coordinates, which is just adding a third component z to polar coordinates in the plane. Finally, we introduce the Euler angles θ, ϕ to describe points in space using **spherical coordinates**. To avoid confusion, always use ρ as the distance to the origin and r as the distance to the z axes. Note that many students are not familiar with the Greek alphabet. Discussion point: can you come up with other coordinate systems in space? Maybe somebody know about toral coordinates.

3. Lecture: Parametric surfaces (10.5)

Surfaces can be described in two fundamental ways: implicitly or parametrically. Implicit descriptions are $g(x, y, z) = 0$ or $x^2 + y^2 + z^2 - 1 = 0$, parametric descriptions are $r(u, v) = (x(u, v), y(u, v), z(u, v))$ like $\vec{r}(\theta, \phi) = \langle \rho \cos(\theta) \sin(\phi), \rho \sin(\theta) \sin(\phi), \rho \cos(\phi) \rangle$. In many cases, it is possible to switch from a parametric description to an implicit and and back. Examples are the plane, spheres, graphs of functions of two variables or **surfaces of revolution**. Using a computer, one can **visualize** complicated surfaces. It is worth mentioning that in computer applications, the parameterization $r(u, v)$ is called the **"uv-map"**. More surfaces will be explored in the Mathematica project. A discussion point: can you parameterize the Moebius strip?

5. Week: Functions (week 10/22-10/26)

1. Lecture: Functions, partial derivatives (11.1-11.2)

For functions of one variable, continuity can fail with jump discontinuities, infinities or singular oscillations. You do not need to give an ϵ, δ definition for continuity. The book hits the right level for our students. Next we look at functions of several variables. Finally, we introduce **partial derivatives** $f_x = \partial_x f = \frac{\partial f}{\partial x}$. Promote the index notation for its simplicity. Explain Clairots theorem. Maybe already start with partial differential equations. Discussion point: give an example, where Clairot theorem fails.

2. Lecture: Partial differential equations (11.3)

To practice differentiation and to get a glimpse of how calculus is used in science, we check whether functions are solutions to **partial differential equations**, abbreviated PDE. More precisely, we look at the transport equation $f_x(t,x) = f_t(t,x)$, the wave equation $f_{tt}(t,x) = f_{xx}(t,x)$ and the heat equation $f_t(t,x) = f_{xx}(t,x)$. There will be a handout on differential equations. Discussion point: how can you derive the heat or wave equation? What do these equations mean?

3. Lecture: Linearization and tangents (11.4)

Linearization is an important concept in science because many physical laws are linearization of more complicated laws. This allows us to compute **tangent planes** and **tangent lines** as well as to approximate a linear function by a linear function near a point. Many physical laws are actually just linearization of more complicated nonlinear laws. We treat linearization in two and three dimensions. The linearization of the function $f(x,y)$ at the point (x_0, y_0) is $L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$. Discussion point: estimate the 10'th root of 2.

6. Week: Gradient (week 10/29-11/2)

1. Lecture: Chain rule and implicit differentiation (11.5)

Most students will remember chain rule $d/dt f(g(t)) = f'(g(t))g'(t)$ in one dimensions. In several dimensions, we introduce the chain rule $d/dt f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$. If you do not want to introduce the gradient yet, write it out as $d/dt f(x(t),y(t),z(t)) = f_x(x(t),y(t),z(t))x'(t) + f_y(x(t),y(t),z(t))y'(t) + f_z(x(t),y(t),z(t))z'(t)$. This is the only chain rule we introduce in this course. (The most general case is obtained by taking this chain rule coordinate wise for a vector valued function). An application of the chain rule is **implicit differentiation**: if $f(x,y,z) = 0$ defines locally $z = g(x,y)$ and because $f_x + f_z z' = 0$ we can compute z'. Discussion point: one can derive other differentiation laws from the chain rule from $f(x,y) = xy$, we get $d/dt x(t)y(t) = yx'(t) + xy'(t)$.

1 1. Class: 11.5: 2,32,28,26,36

2. Lecture: Gradient and level surfaces (11.6)

The **gradient** of a function is an important tool to describe the geometry of surfaces. One of the most central facts students should know about is that the gradient of a function f(x,y) at a point is perpendicular to the level curve $f(x,y) = c$ passing through that point and that the gradient of a function $g(x,y,z)$ at a point is perpendicular to the level surface $f(x,y,z) = c$ which passes through that point. Prove this using the chain rule introduced in the last lecture: for a curve $\vec{r}(t)$ on the surface $f(\vec{r}(t)) = 0$. Differentiation leads to the fact. A special case is $g(x,y,z) = ax + by + cz = d$, where $\nabla g = \langle a,b,c \rangle$. Discussion point: Can you write the Möbius strip as a level surface?

3. Lecture: Directional derivative (11.6)

We introduce the **directional derivative** $D_v(f)$ as $D_v f = \nabla f \cdot v$ for **all** vectors v, not only for unit vectors. Discuss, that partial derivatives are special directional derivatives and that the direction of the normal vector gives a nonnegative partial derivative. If you go into the direction of the normal vector, you go up because $D_{\nabla f} f = |\nabla f|^2$. Discussion point: how many directional derivatives does one have to know to find the gradient?

7. Week: Extrema (week 11/5-11/9)

1. Lecture: Extrema, second derivative test (11.7)

A central application of multi-variable calculus is to **extremize** functions of two variables. One first identifies **critical points**, points where the gradient vanishes. The nature of these critical points can be established using the **second derivative test**. There will be three fundamentally different cases: **local maxima**, **local minima** as well as **saddle points**. Discussion point: can you find a function with exactly one local min, exactly one local max on the plane and no other critical points?

2. Lecture: Extrema with constraints (11.8)

We extremize a function $f(x, y)$ in the presence of a **constraint** $g(x, y) = 0$. A necessary condition for a critical point is that the gradients of f and g are parallel. This leads to equations called the **Lagrange equation**. When extremizing a function on a bounded domain in the plane, we have to look at extrema in the interior as well as extrema on the boundary. This lecture allows to review and practice both extremization problems with and without constraints. Discussion point: look at an example, where one does not find critical points as usually described, like when $\nabla g = 0$.

3. Lecture: Mixed problems (11.8)

When extremizing functions on a domain bounded by a curve $g(x, y) = 0$, we have to solve two problems: find the extrema in the interior and the extrema on the boundary.

8. Week: Double Integrals (week 11/12-11/16)

1. Lecture: Veteransday no class

2. Lecture: Review for second midterm

3. Lecture: Double integrals (12.1,12.2)

Integration in two dimensions is first done on rectangles, then on regions bound by graphs of functions. Similar than in one dimension, there is a **Riemann sum approximation** of the integral. This allows us to prove results like **Fubinis theorem** on the change of the integration order. An application of double integration is the computation of **area**, for which $f(x, y) = 1$. We especially practice translating back and forth from a region to the integral. A good example is the change of order of integration in regions which are both type I and type II. A double integral can be interpreted as the signed volume under a surface. Discussion point: what do you do if a region is not type I or type II?

9. Week: Surface area (week 11/19-11/23)

1. Lecture: Integration in polar coordinates (12.4,12.5)

Many regions can be described better in **polar coordinates**. There are beautiful regions, which can be defined in polar coordinates. Examples are **roses** which trace flower-like shapes in the plane but are graphs in polar coordinates. There are examples, where one has to change the order of integration for double integrals. Discussion point: integrate $\exp(-x^2 - y^2)$ over the plane. We look at applications of double integrals from statistics or physics.

2. Lecture: Surface area (12.6)

We derive the formula $\int \int_R |r_u \times r_v| \; dudv$ and give examples like graphs, surfaces of revolution and especially the sphere. Similar as for arc length, it is easy to give examples, where the surface area can not be computed in closed form.

3. lecture Thanksgiving holiday

10. Week: Tirple integrals (week 11/26-11/30)

1. Lecture: Triple integrals, (12.7)

Triple integrals allow the computation of volumes, moment of inertias or centers of masses of solids. First introduced for cubes it is then extended to more general regions which are bound by graphs of functions of two variables. Discussion point how come that some double integrals describe volume, but which is actually a triple integral. Discussion point: Monte Carlo method to integrate.

2. Lecture: Spherical and cylindrical coordinates (12.8)

Applications make the topic of integration more interesting. It also allows to practice how to set up integrals. Applications are computations of **mass** $\int \int \int_E \delta(x, y, z) \; dxdydz$, **moment of inertia** $\int \int \int_E (x^2 + y^2 + z^2) \; dxdydz$, **center of mass**, $\int \int \int \langle x, y, z \rangle \; dV$ the **expectation** $\mathrm{E}[X] = \int \int \int X(x, y, z) \; dV / \int \int \int \; dV$ of a random variable $X(x, y, z)$ on a region Ω. Discussion point: can you find a body, where the center of mass is outside the body? Some regions can be described better in **cylindrical coordinates**, the analogue of polar coordinates in space. **Spherical coordinates** allow an even more elegant computation of triple integrals for certain regions like cones or spheres.

3. Lecture: Vector fields and line integrals (13.1 and 13.2)

Next, we will introduce **vector fields**. They occur as force fields or velocity fields or mechanics and are closely related to the field of ordinary differential equations. Vector fields will occupy us until the end of the course. Discussion point: propose a perpetuum motion machine. Most students have seen vector fields already in single variable calculus in the context of differential equations. It is helpful to discuss vector fields in concrete setups. It helps to introduce **flow lines**. **Line integrals** $\int_C F(\vec{r}(t)) \; \vec{r}\,'(t) \; dt$ along a curve in the presence of a vector field. If the vector field is a force field, then the line integral has the interpretation work done, when walking along the path. It is helpful to introduce the curl by imagining F to be a **force field** and the line integral as **work**. Discussion point: if you lift a stone and keep it in fixed height, you do not do any work. Still you feel using power. Why?

Homework: 2. Class: 13.1: 24,34 13.2: 18,20,42

11. Week, Green (week 12/3-12/7)

1. Lecture: Fundamental theorem of line integrals (13.3)

For a class of vector fields which we call **conservative vector fields** one can evaluate the line integral easily using an identity called the **fundamental theorem of line integrals**. Distinguish first **path independencee** and **gradient fields**, the field is the gradient of a potential f. Discussion point: check whether a field in the plane is not conservative: $\text{curl}(F) = Q_x - P_y \neq 0$ at some point.

2. Lecture: Green theorem (13.4)

Greens theorem relates a line integral along a closed curve with a double integral of a derivative of the vector field in the region enclosed by the curve. One can formulate the theorem already using the two-dimensional curl, which is a scalar. The theorem is useful for example to **compute areas**. It also allows an easy computation of line integrals in certain cases. One can also mention that Greens theorem verifies that if $\text{curl}(F) = 0$ everywhere in the plane, then the field has the **closed loop property** and is therefore conservative. Discussion point: the vortex vector field $F(x, y) = (-y/(x^2 + y^2), x/(x^2 + y^2))$.

3. Lecture: Curl and divergence (13.5)

The **curl** of a vector field $\vec{F} = \langle M, N, R\rangle$ in three dimensions is a vector field itself. The three components give the vorticity of the vector field in the x,y and z direction. As in crossed products, there are different ways to introduce the curl. Most students prefer the matrix method: using $\nabla \times F$. In two dimensions, it is a scalar field $\text{curl}(M, N) = N_x - M_y$ which measures the vorticity of the vector field in the plane. There are many ways to visualize and motivate the curl. It is possible to introduce the **divergence**, a scalar field. Discussion point: in two dimensions, can you relate divergence and curl in some way, maybe be rotating the field?

2 13.5: 6,10,14,27,36

12. Week, Stokes and Gauss (week 12/10-12/14)

1. Lecture: Flux integrals (13.6)

Derive the formula $\int\int_R |\vec{r}_u \times \vec{r}_v|\ dudv$ and give examples like graphs, surfaces of revolution and especially the sphere. Similar as for arc length, it is easy to give examples, where the **surface area** can not be computed in closed form. One way to motivate the introduction of the flux is to have a membrane S and a fluid with velocity F and wanting to compute how much fluid passes through the membrane S in unit time. Obviously the angle between F and the normal vector $\vec{n} = \vec{r}_u \times \vec{r}_v$ will matter. Through a small flat square, the flux is $d\vec{S} = \vec{F} \cdot \vec{n}dudv$. Integrating this over the parameter domain R is called a **flux integral**. Discussion point: can you say something about the flux of the gradient field ∇f of a closed level surface $f = c$?

2. Lecture: Stokes theorem (13.7)

Stokes theorem is Greens theorem lifted into three dimensions, where the region is replaced by a surface. It tells that one can replace the line integral along the boundary of the surface by an integral of the "curl" of the field over the surface. It is good that students see different examples on how to use Stokes to compute line integrals or to compute flux integrals. There are many applications which can be mentioned here, like the dynamo. Discussion point: can one derive Greens theorem from Stokes theorem?

3. Lecture: Gauss theorem (13.8)

Finally, the **divergence** of a vector field inside a solid is related to the flux of the vector field through the boundary of the surface using the **divergence theorem** which is sometimes also called **Gauss theorem**. The divergence theorem relates the "local expansion rate" of a vector field with the flux through a closed surface and is useful for example to compute the gravitational field inside a solid. Discussion point. How would integral theorems look like in 4 dimensions.

13. Week, Last lecture (week 12/17-12/21)

1. Lecture: Applications 13.9 or free topic

The mathematica project is due.