Math21a First Midterm Review

Plan:

Lines and Planes

Distance formulas

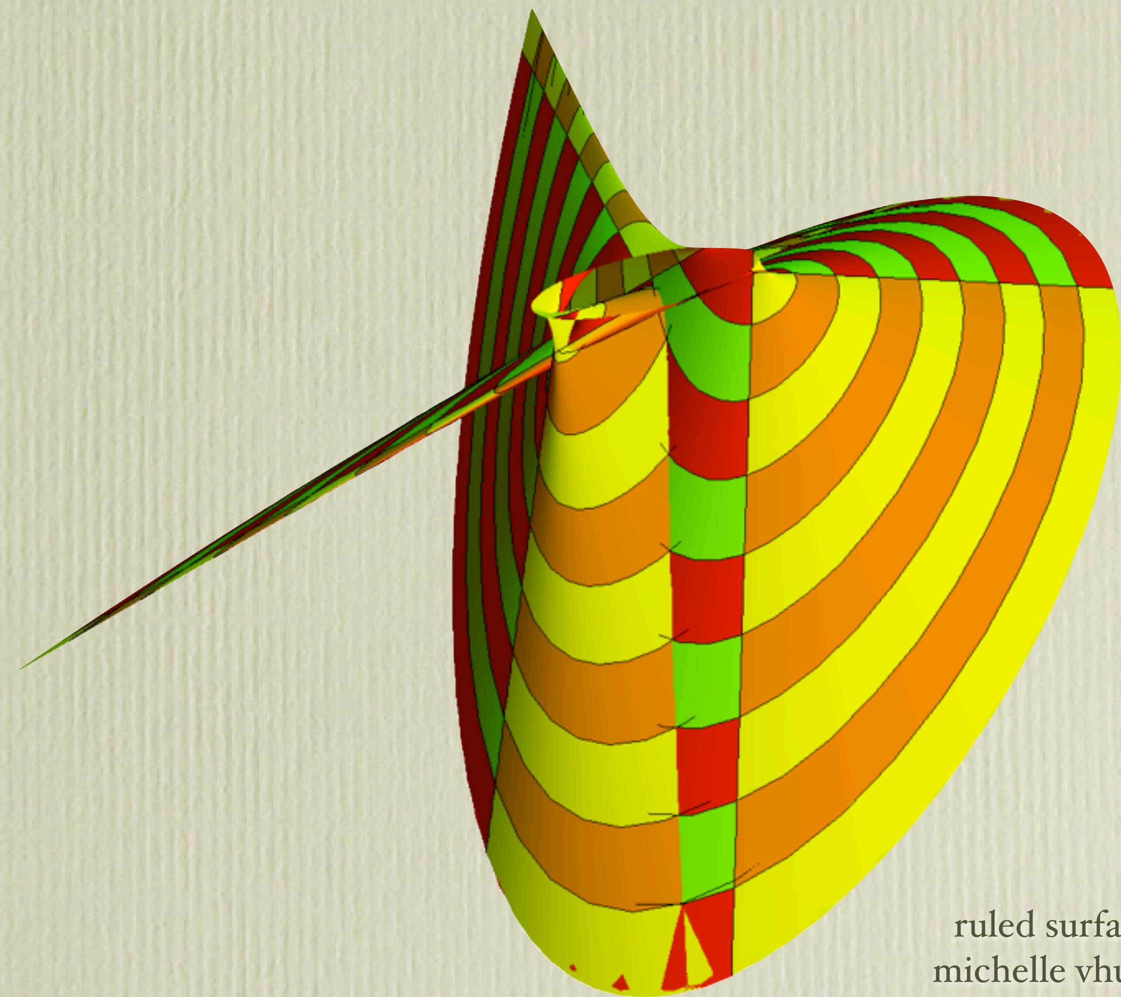
Parametrized curves

Surfaces Quadrics

Other coordinates

Parametric Surfaces

Continuity



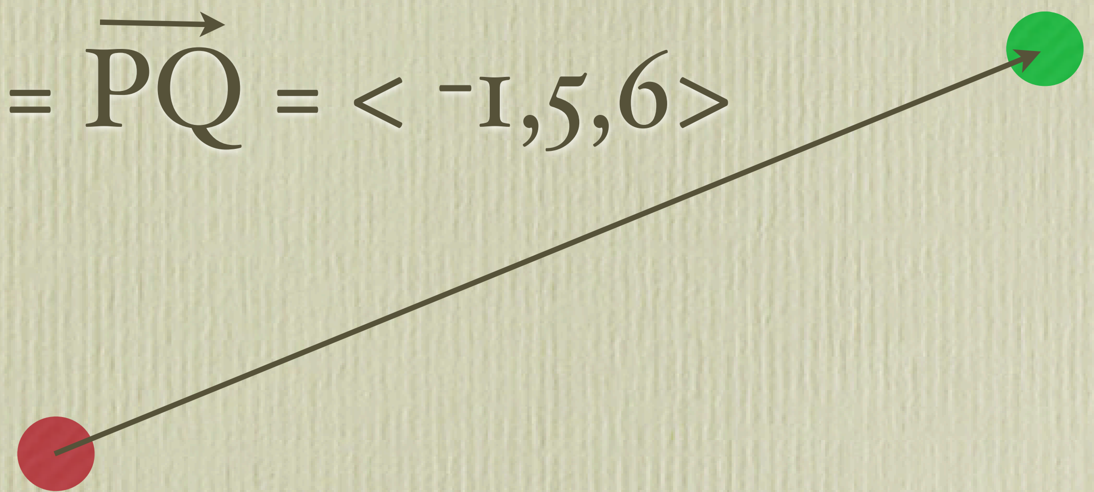
ruled surface by michelle vhuzijena

Oliver Knill, October 4, 2009

Geometry

Points and Vectors

$\vec{v} = \overrightarrow{PQ} = < {}^{-}1,5,6 >$

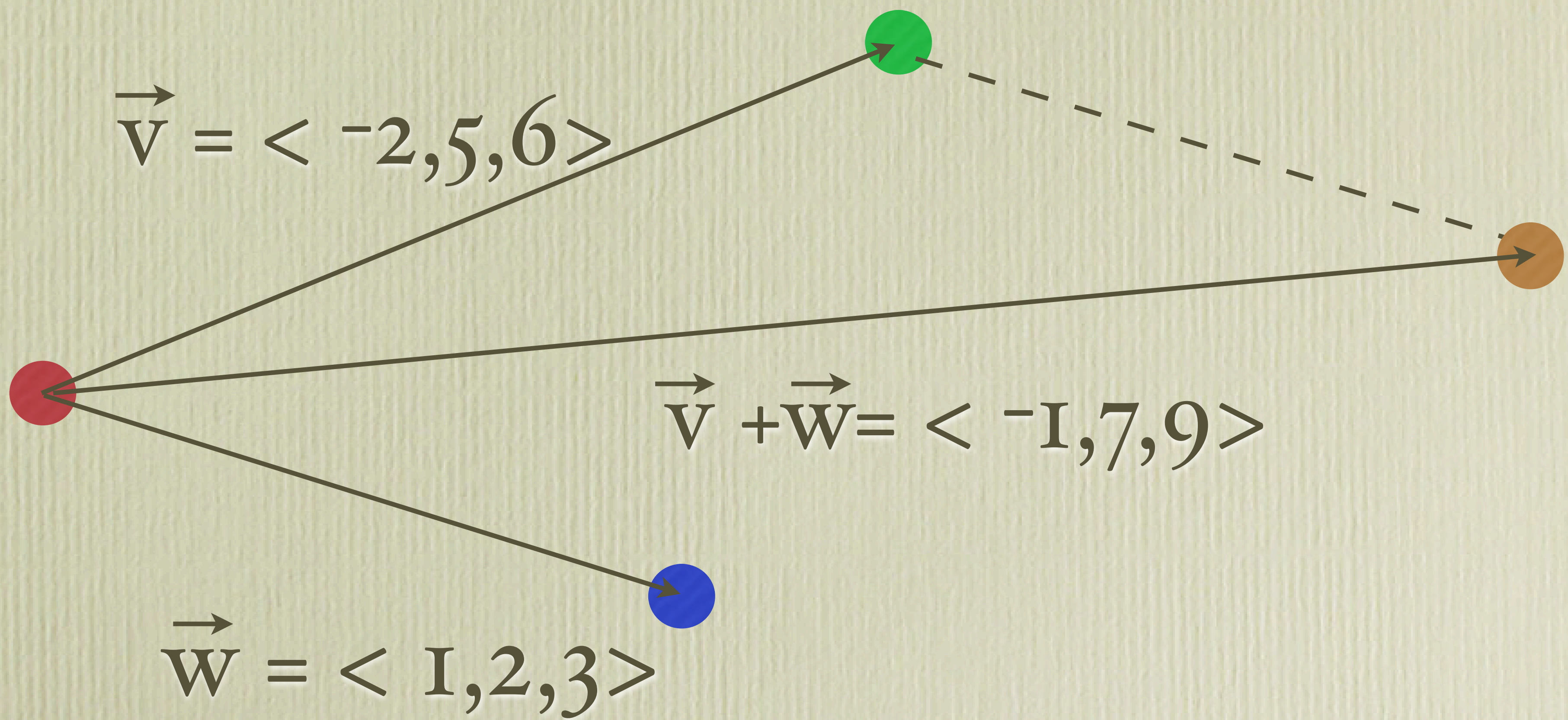


Q = (2,5,7)

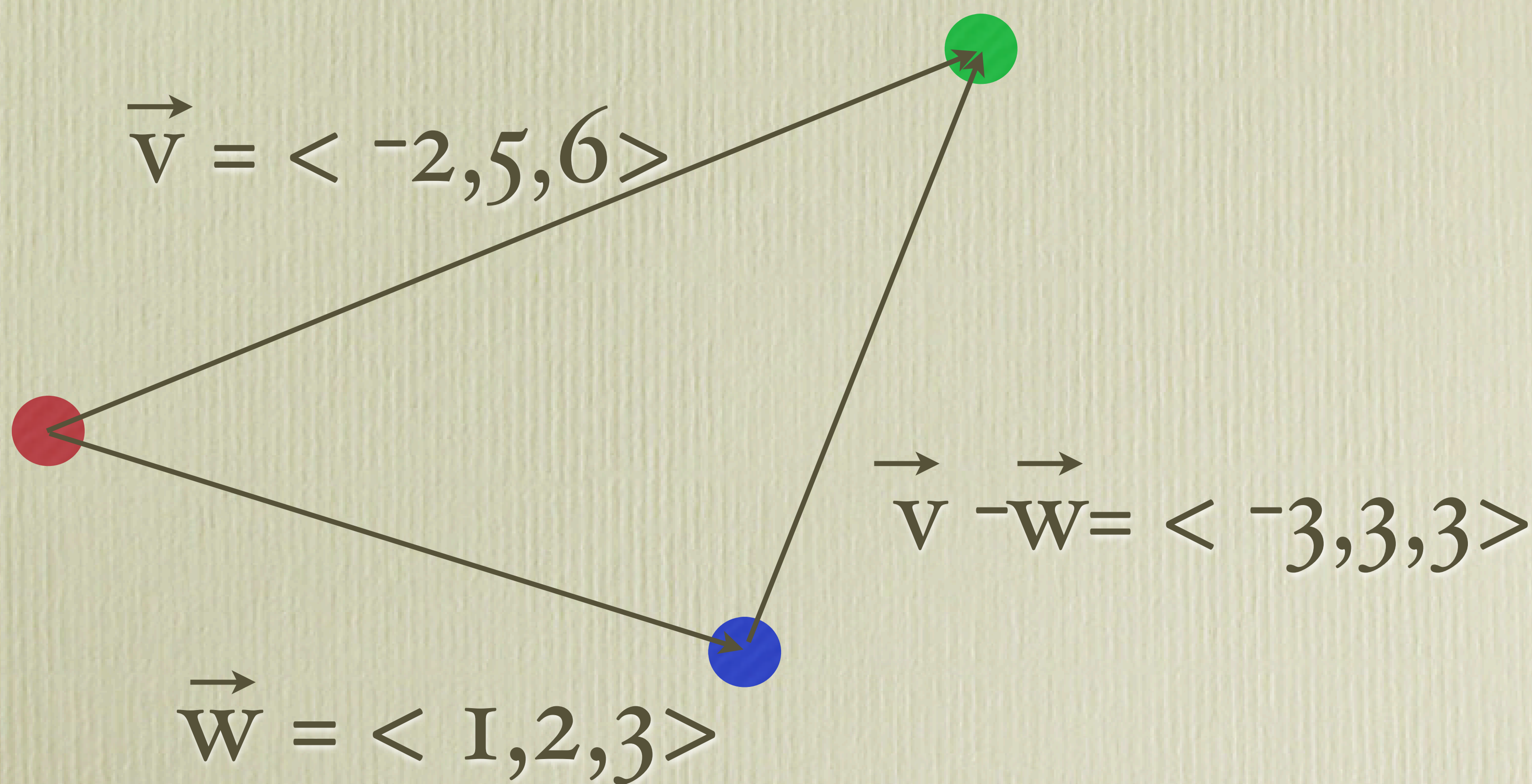
P = (3,0,1)

The components of $\vec{v}$
are the differences between
the coordinates of Q and P.

Addition

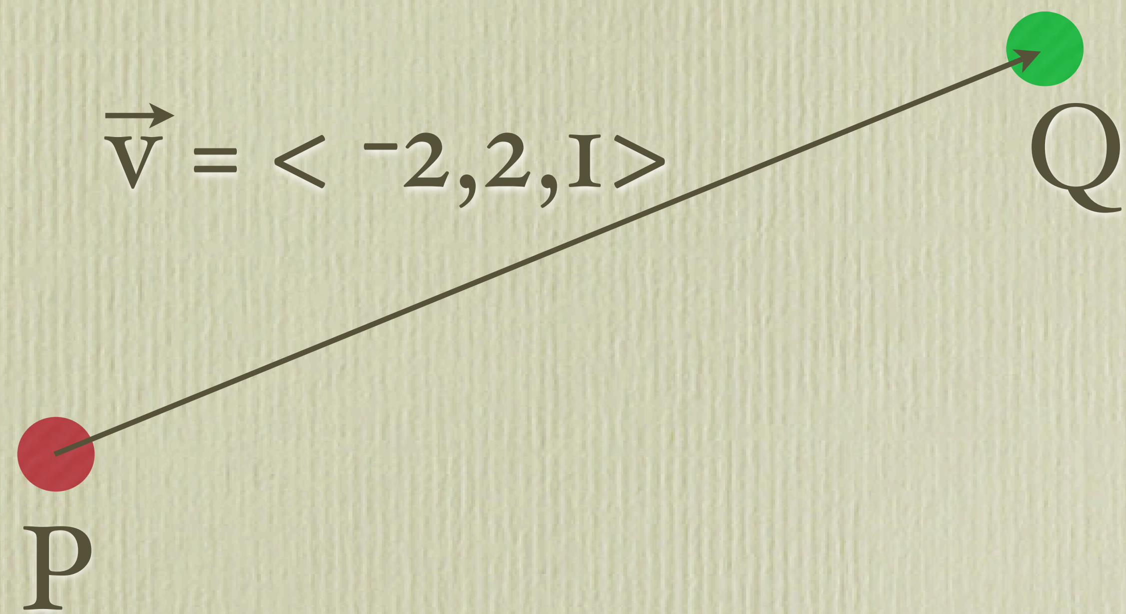


Subtraction



Distances

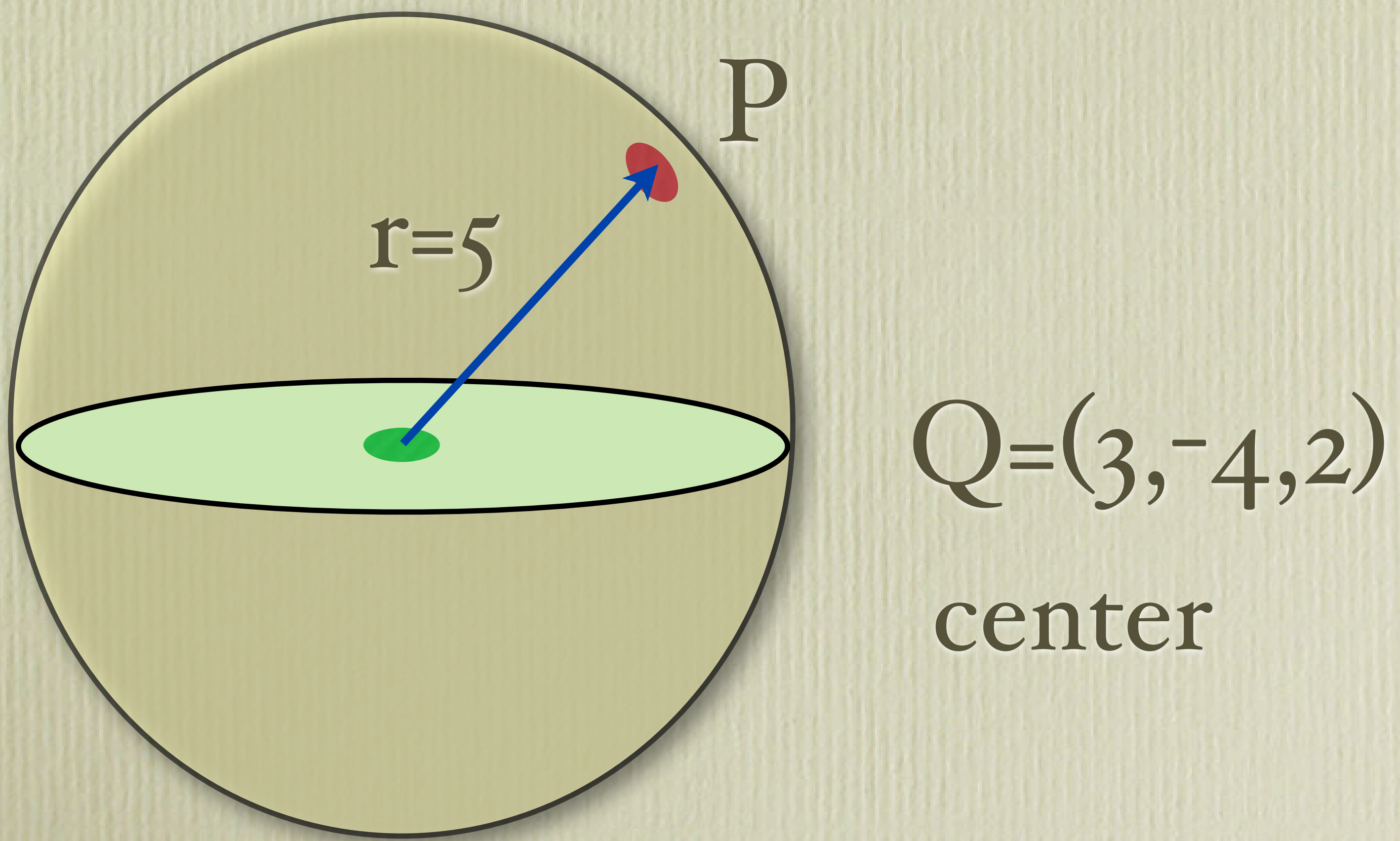
$\vec{v} = \langle -2, 2, 1 \rangle$



$$d(P,Q) = |\vec{v}|$$

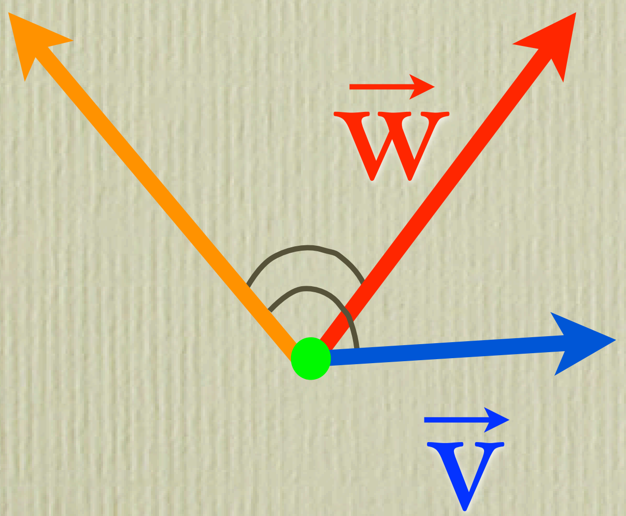
Spheres

$$(x-3)^2 + (y+4)^2 + (z-2)^2 = 25$$



Dot and Cross product

$\vec{v} = \langle 3,4,1 \rangle$ $\vec{w} = \langle 2,-1,1 \rangle$



$$\vec{v} \cdot \vec{w} = 6-4+2 = 4$$

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 3 & 4 & 1 \\ 2 & -1 & 2 \end{vmatrix}$$

$$= \langle 9,-4,-11 \rangle$$

Two important formulas

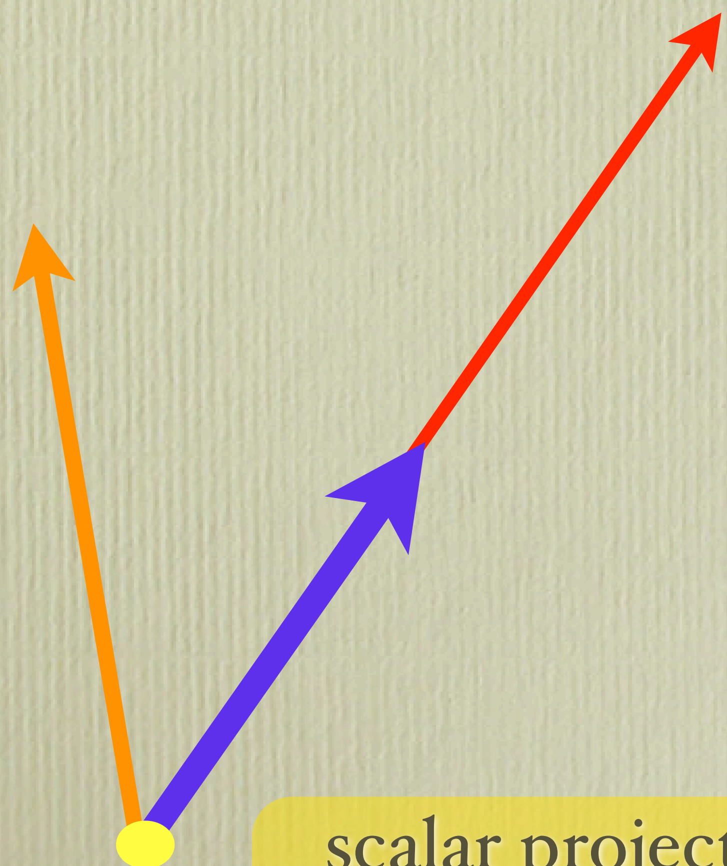
$$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}|\cos(\alpha)$$

$$|\vec{v} \times \vec{w}| = |\vec{v}|\,|\vec{w}|\sin(\alpha)$$

Projection

$\vec{v} = \langle 2, {}^{-}3, 4 \rangle$ $\vec{w} = \langle 4, 0, 1 \rangle$

Project $\vec{v}$ onto $\vec{w}$:

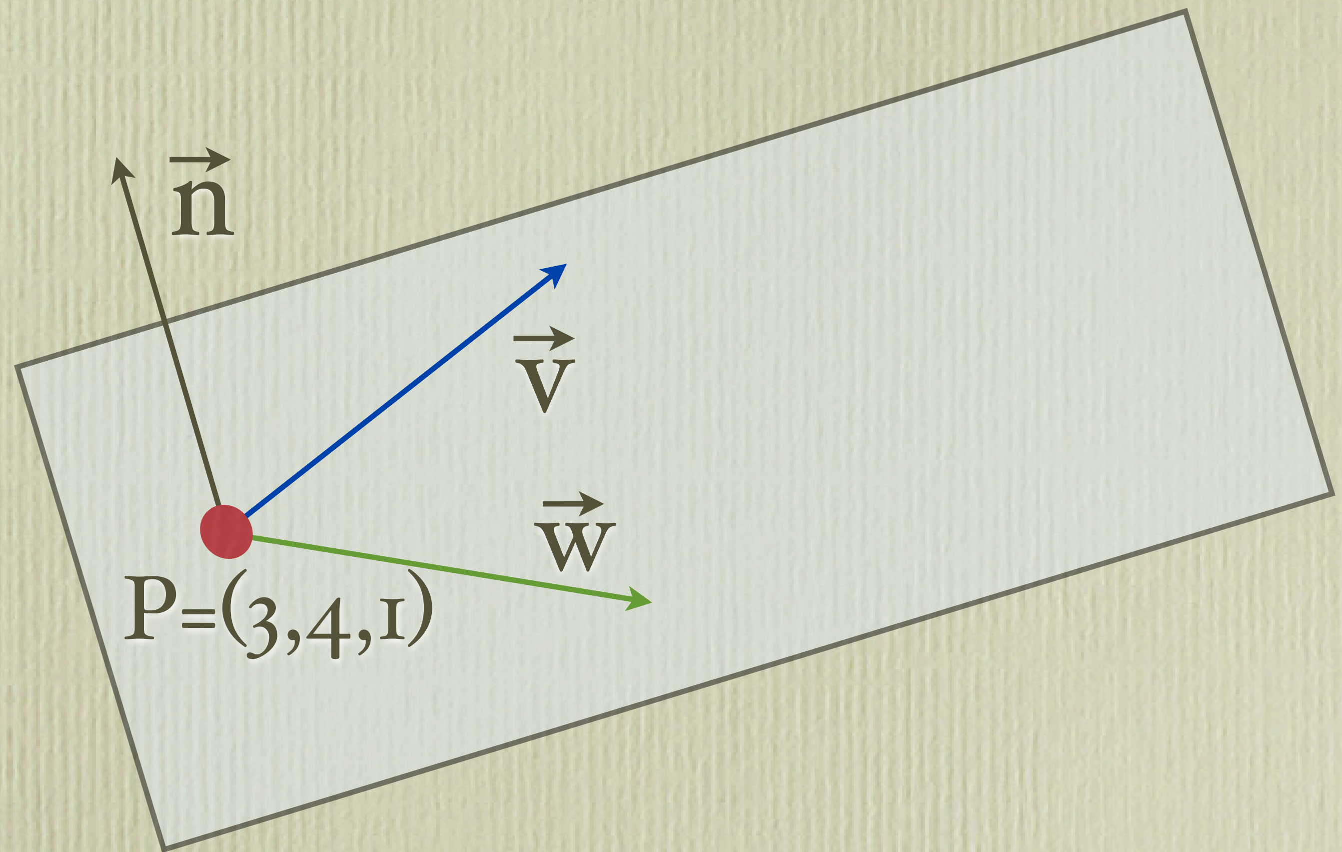


$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|} \frac{\vec{w}}{|\vec{w}|}$$

$$= \frac{12}{17} \langle 4, 0, 1 \rangle$$

scalar projection = component

Lines and Planes



$\overrightarrow{OP} = \langle 3,4,1\rangle$

$\vec{v} = \langle 1,1,-3\rangle$

$\vec{w} = \langle 1,-2,1\rangle$

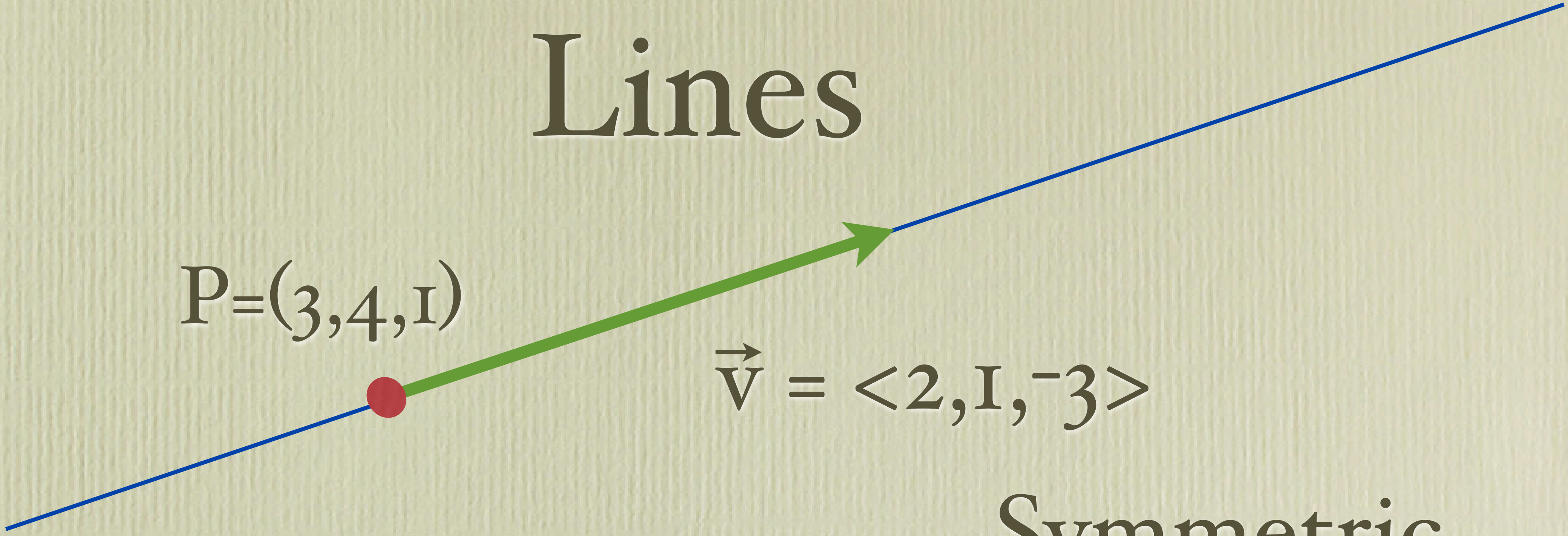
Parametrization

$\vec{n} = \langle 7,-4,-3\rangle$

$\vec{r}(t,s) = \langle\ 3+t+s, 4+t-2s, 1-3t+s\rangle$

$7x-4y-3z=2$

Lines



Parametrization

$$\vec{r}(t) = \langle 3+2t, 4+t, 1-3t \rangle$$

$$= \langle x, y, z \rangle$$

Symmetric equations

$$\frac{x-3}{2} = \frac{y-4}{1} = \frac{z-1}{-3}$$

(solve for t)

Problem

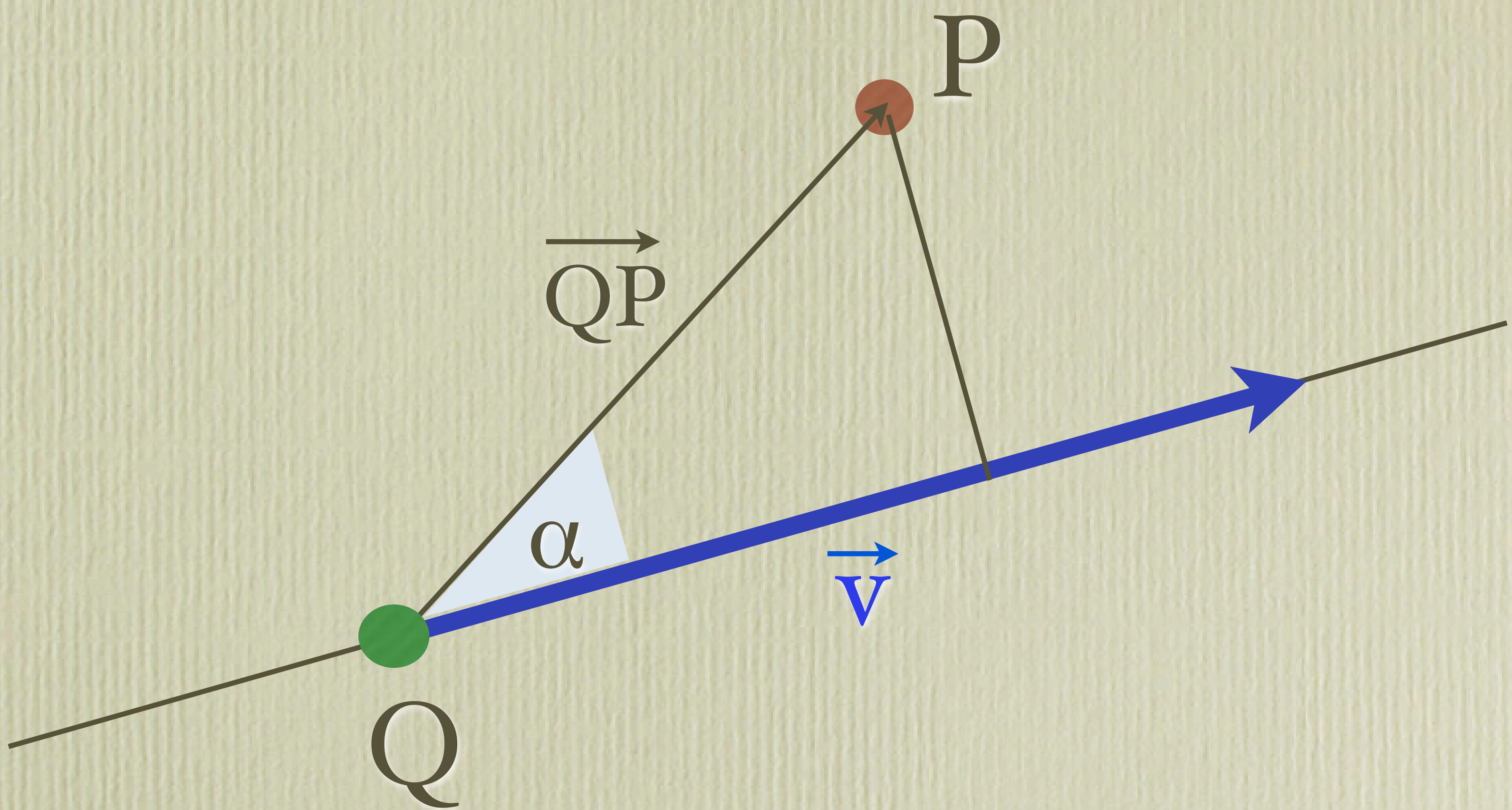
Find the equation of the plane passing through the points

$A=(0,1,1), B=(2,2,2), C=(5,5,4)$

the symmetric equation of the normal line through A and the area of the triangle ABC.

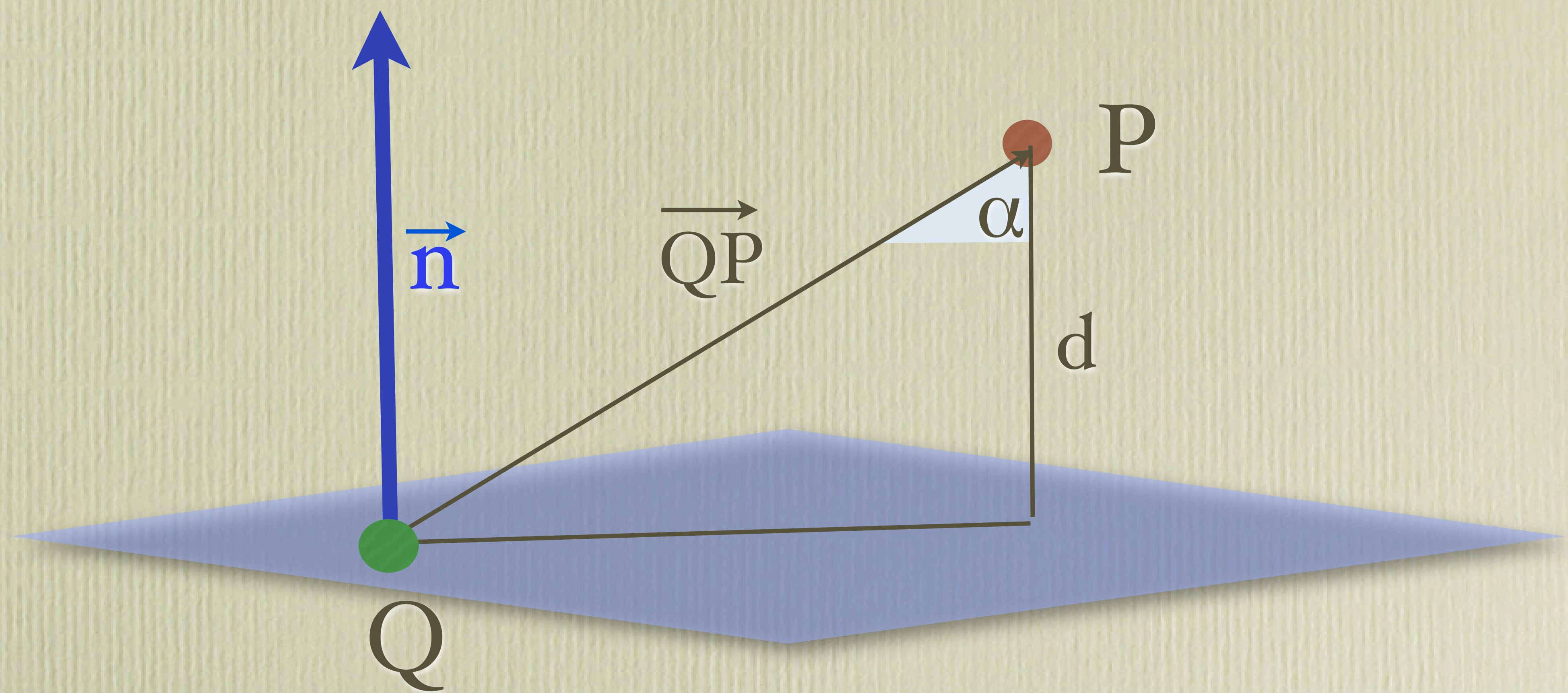
Distances

Distance Point/Line



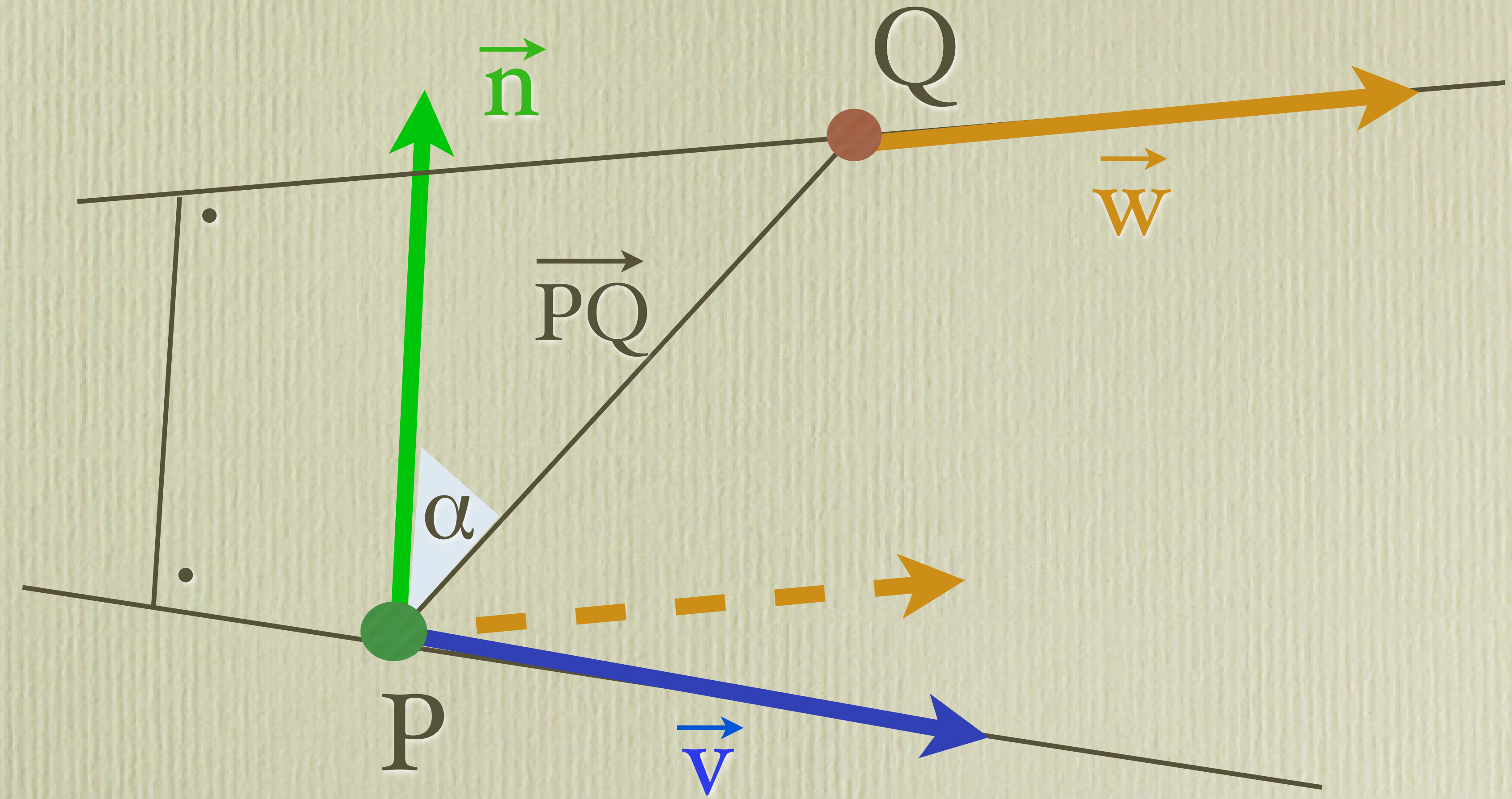
$$\frac{|\overrightarrow{QP}|\sin(\alpha)\ |\vec{v}|}{|\vec{v}|} = \frac{|\overrightarrow{QP} \times \vec{v}|}{|\vec{v}|}$$

Distance Point-Plane



$$d = \frac{|\overrightarrow{PQ}| \cos(\alpha)\, |\vec{n}|}{|\vec{n}|} = \frac{|\overrightarrow{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

Distance Line/Line



$$\frac{|\overrightarrow{PQ}|\cos(\alpha)\,|\vec{n}|}{|\vec{n}|} = \frac{|\overrightarrow{QP}\cdot\vec{v}\times\vec{w}|}{|\vec{v}\times\vec{w}|}$$

Problem

Problem

Starship "enterprise" officer Spock, beams from (1,1,1) to (3,4,5). The clingons can modify everything in distance 1 from the x axes. Will Spock be safe?

Reformulate:

Is the distance between the x axis and the line connecting (1,1,1) with (3,4,5) larger than 2?

EXPENDABILITY

KIRK,SPOCK,MCCOY AND ESNSIGN RICKY ARE BEAMING DOWN TO THE PLANET. GUESS WHO'S NOT COMING BACK.

you might think now....

Since Knill used this distance example, it will not appear in the test. However remember that he knows this too. To make a surprise, we could have put this example in the test. But because he mentioned this now, it most likely will not. But then again, it would be a surprise if it would appear ....

Similar reasoning proves:

There are no surprise exams!

Proof: assume we announce a surprise exam until Oct. 31.

October 31 is excluded:

You would know the night before that the exam has to take place the next day and it would not be a surprise exam.

October 30 is excluded:

The night before, because the 31st is excluded, you know it would be on the 30'th. So, Oct 30 is out.

etc...

Area and Volume

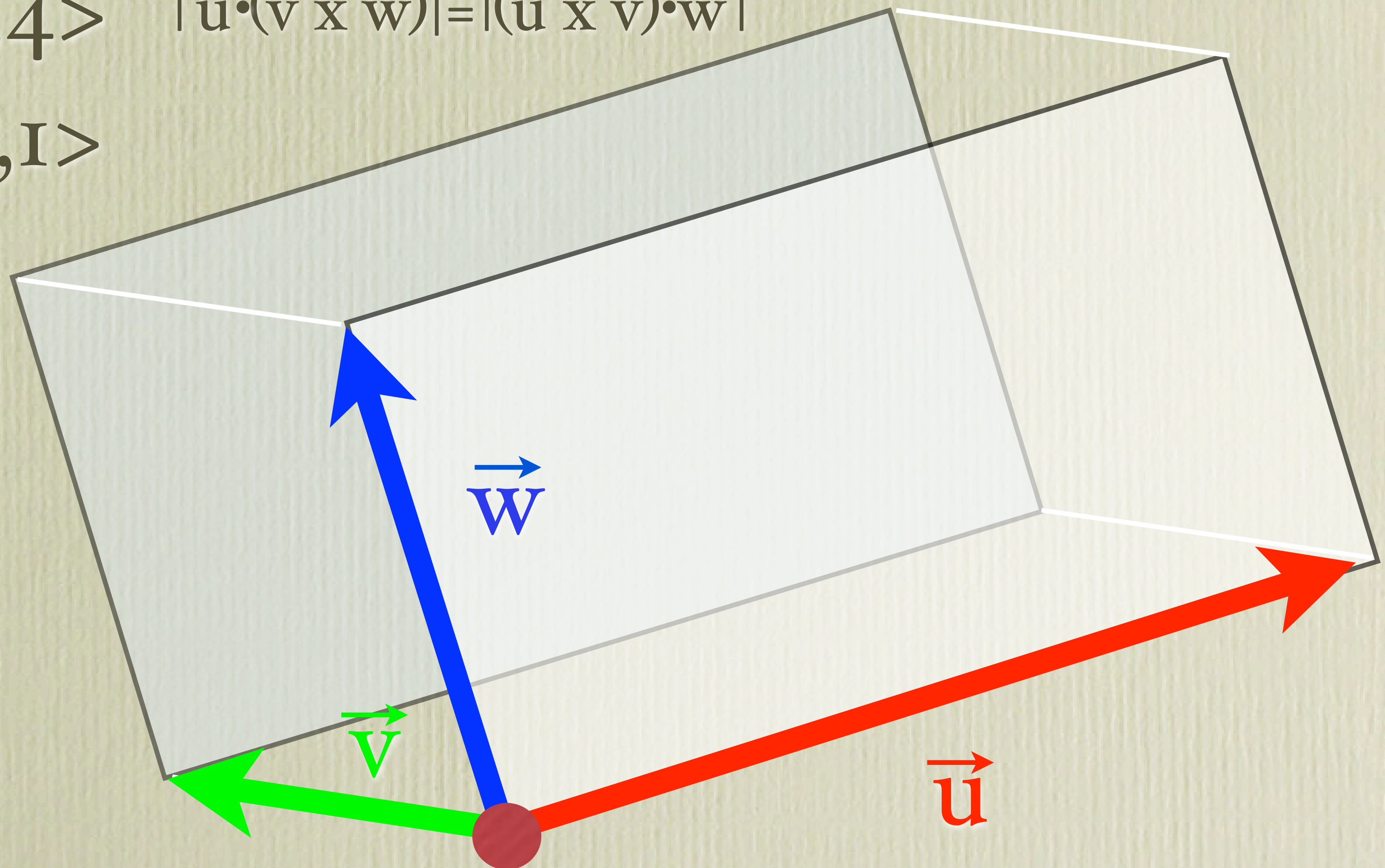
$\vec{u} = \langle 1,2,3 \rangle$

$\vec{v} = \langle 3,1,4 \rangle$

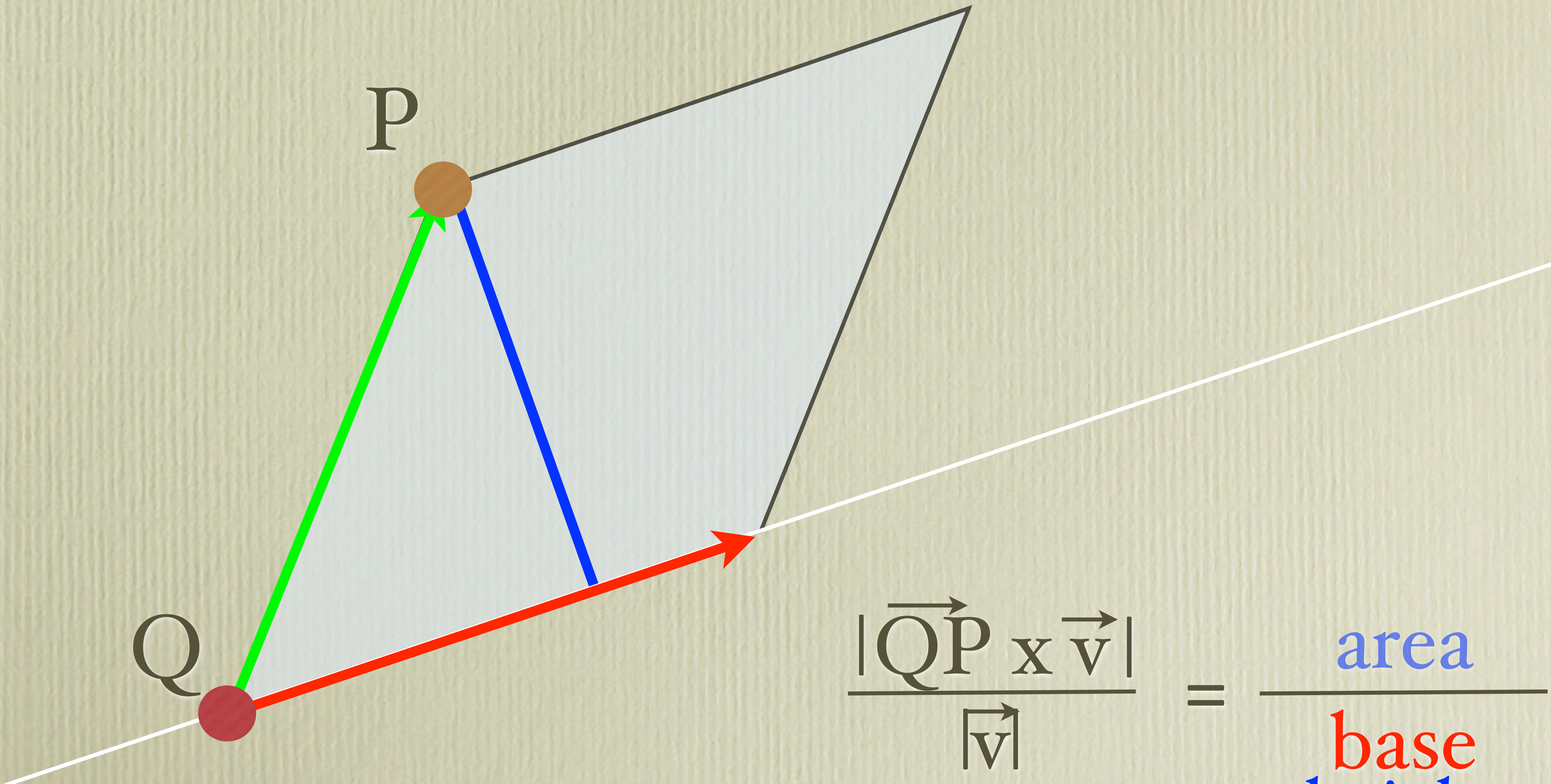
$\vec{w} = \langle 1,1,1 \rangle$

The volume of the parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$ is

$$|\vec{u} \cdot (\vec{v} \times \vec{w})| = |(\vec{u} \times \vec{v}) \cdot \vec{w}|$$



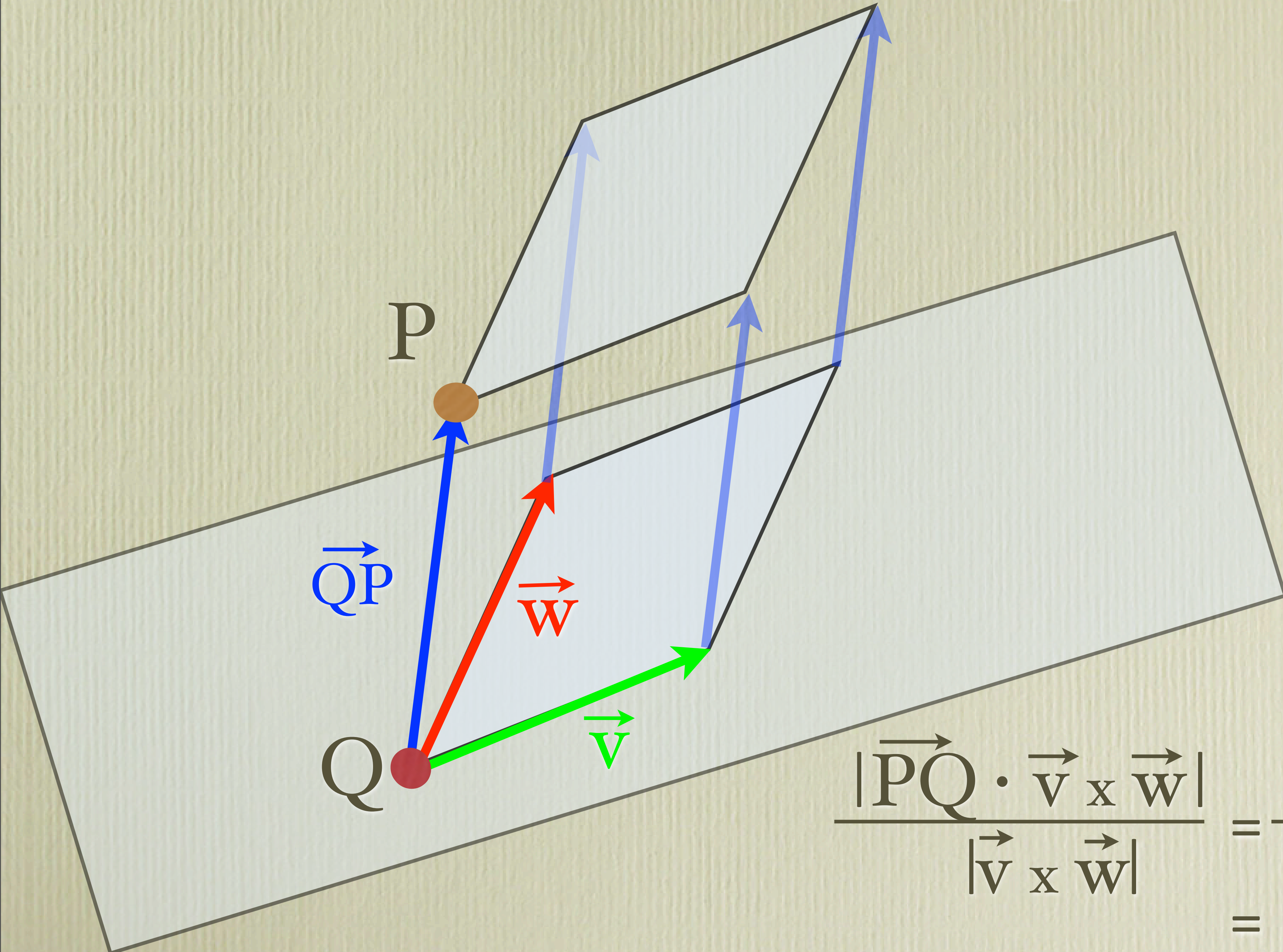
Distance Point-Line again



$$\frac{|\overrightarrow{QP} \times \vec{v}|}{|\vec{v}|} = \frac{\text{area}}{\text{base}}$$

$$= \text{height}$$

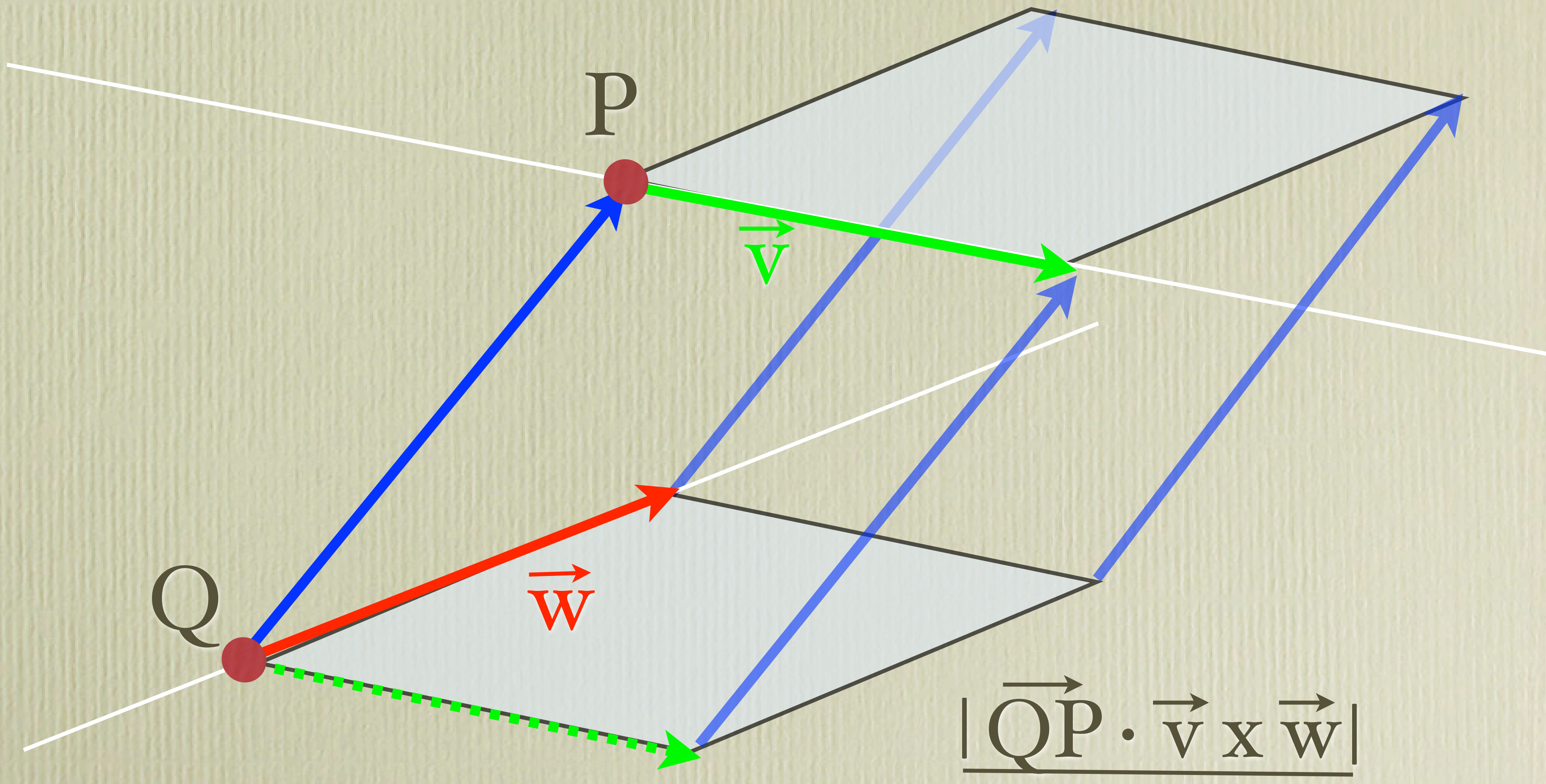
Distance Point-Plane again



$$\frac{|\overrightarrow{PQ} \cdot \vec{v} \times \vec{w}|}{|\vec{v} \times \vec{w}|} = \frac{\text{volume}}{\text{area}}$$

$$= \text{height}$$

Distance Line-Line



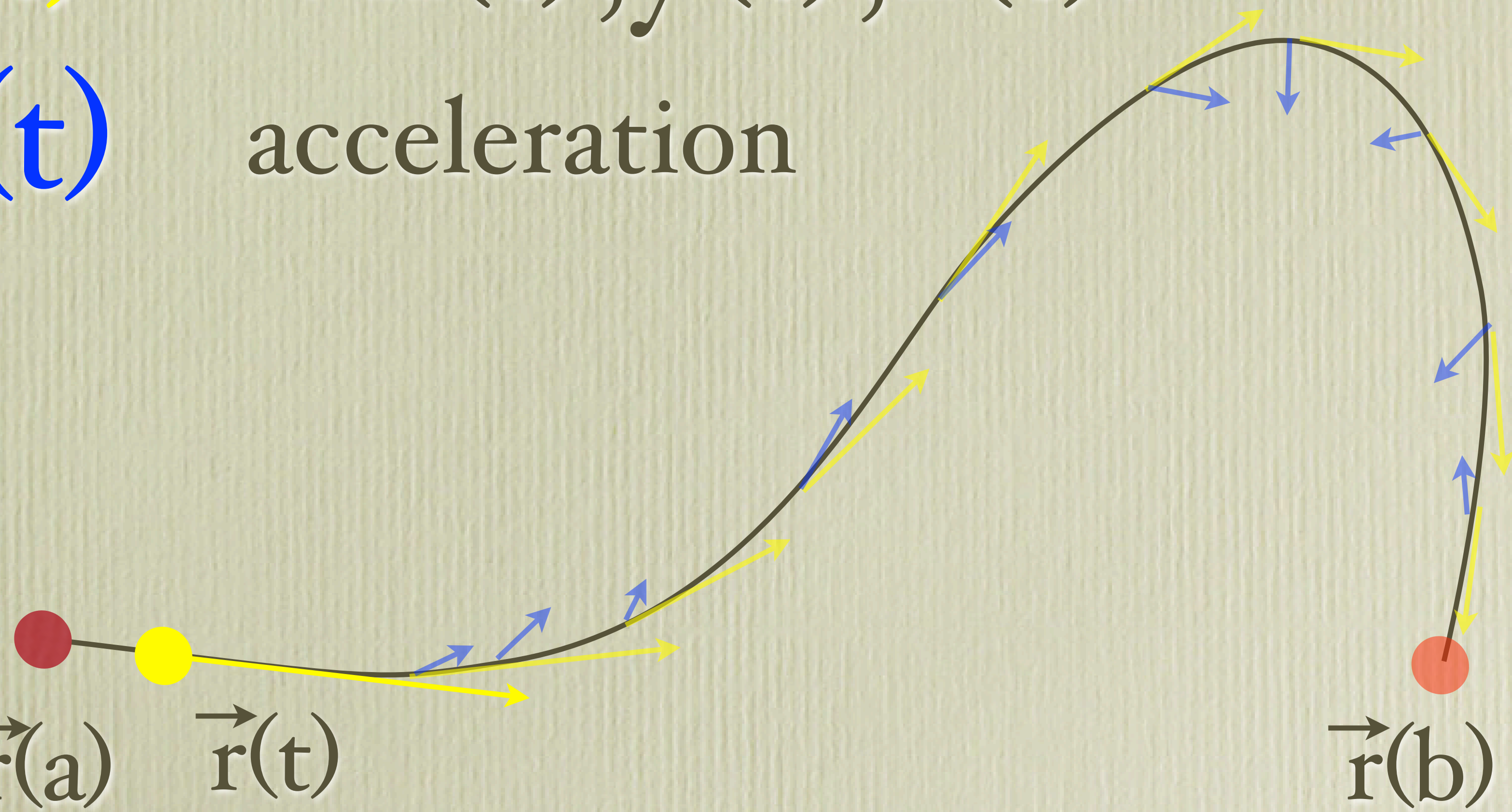
$$= \frac{\text{Volume}}{\text{Area}} = \frac{|\overrightarrow{QP} \cdot \vec{v} \times \vec{w}|}{|\vec{v} \times \vec{w}|}$$

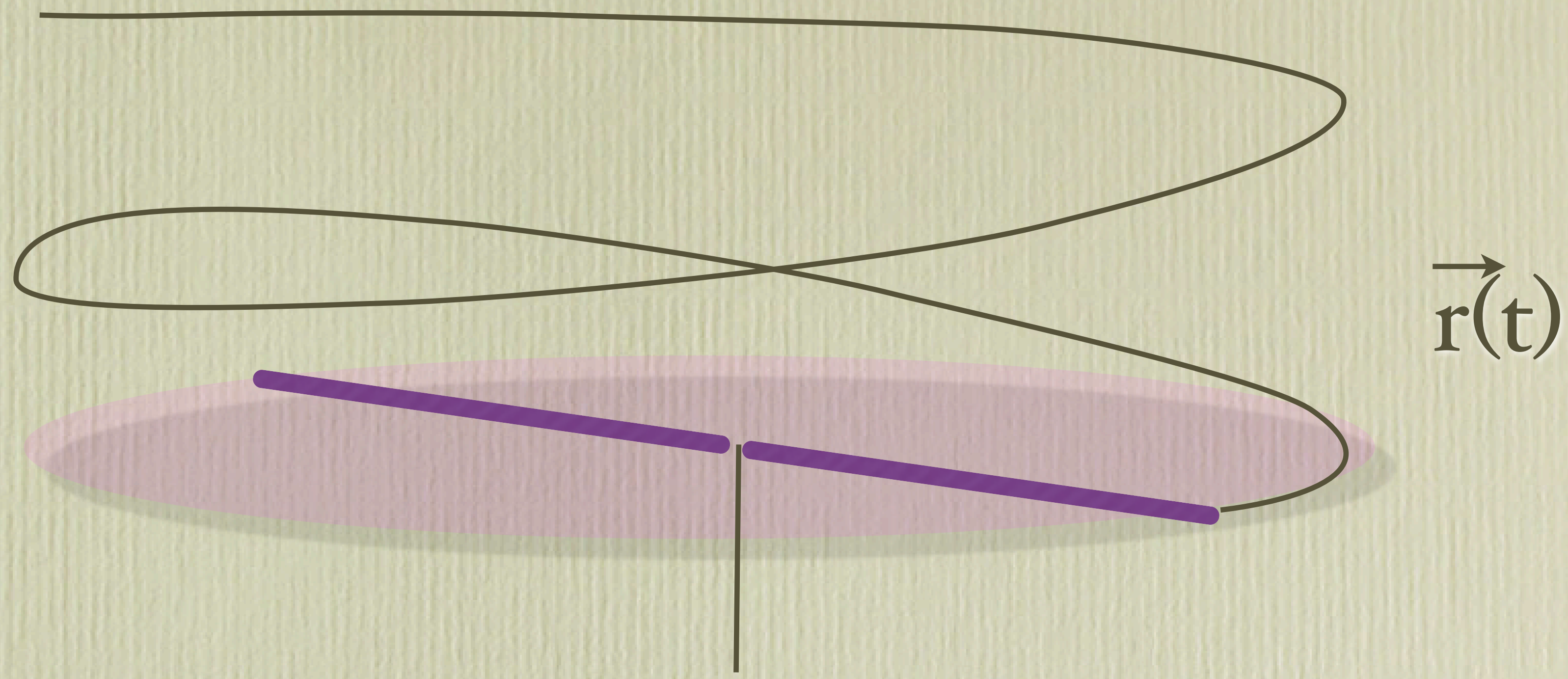
Parametrized Curves

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$$\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$$ velocity

$$\vec{r}''(t)$$ acceleration





Problem: a helicopter blade moves on the curve

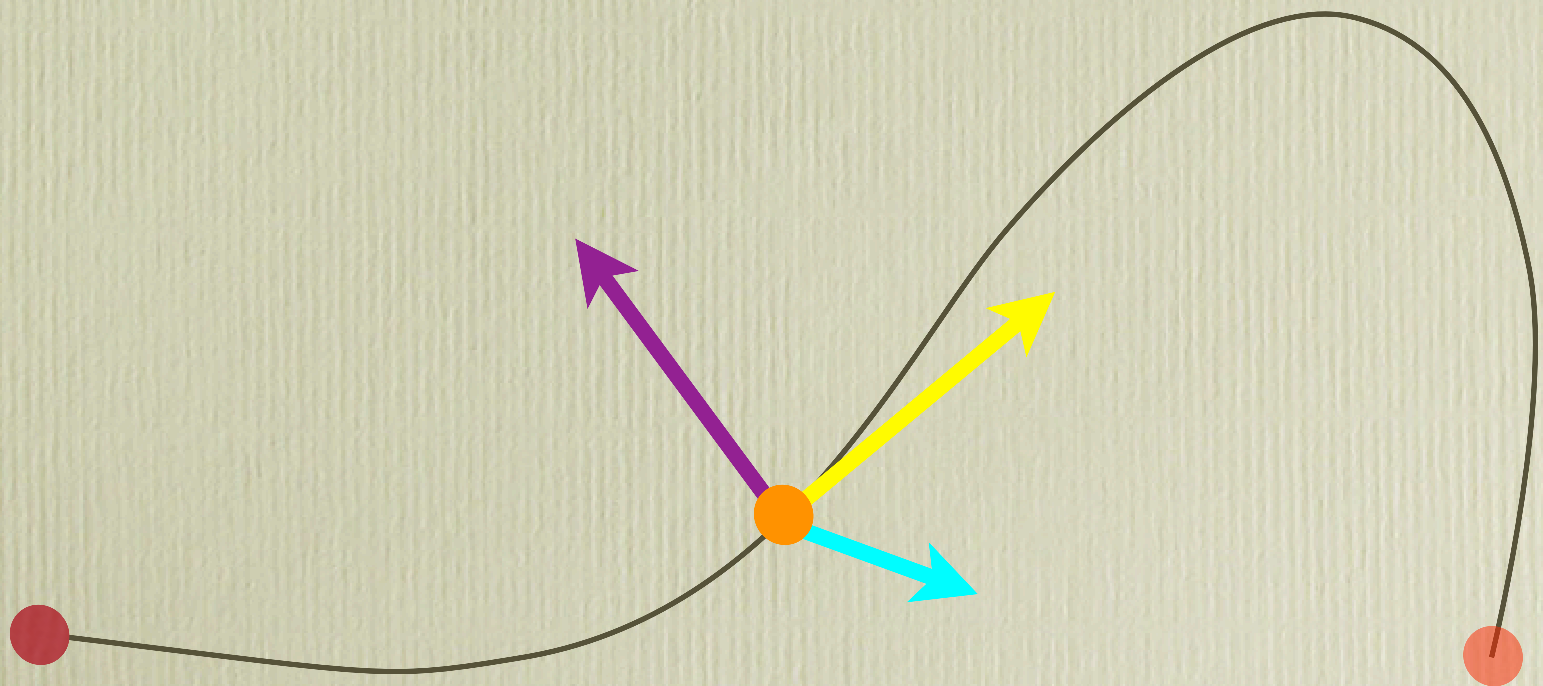
$\vec{r}(t) = < \cos(t), \sin(t), t >$

and hits the ceiling z=10. Under which angle does it hit?

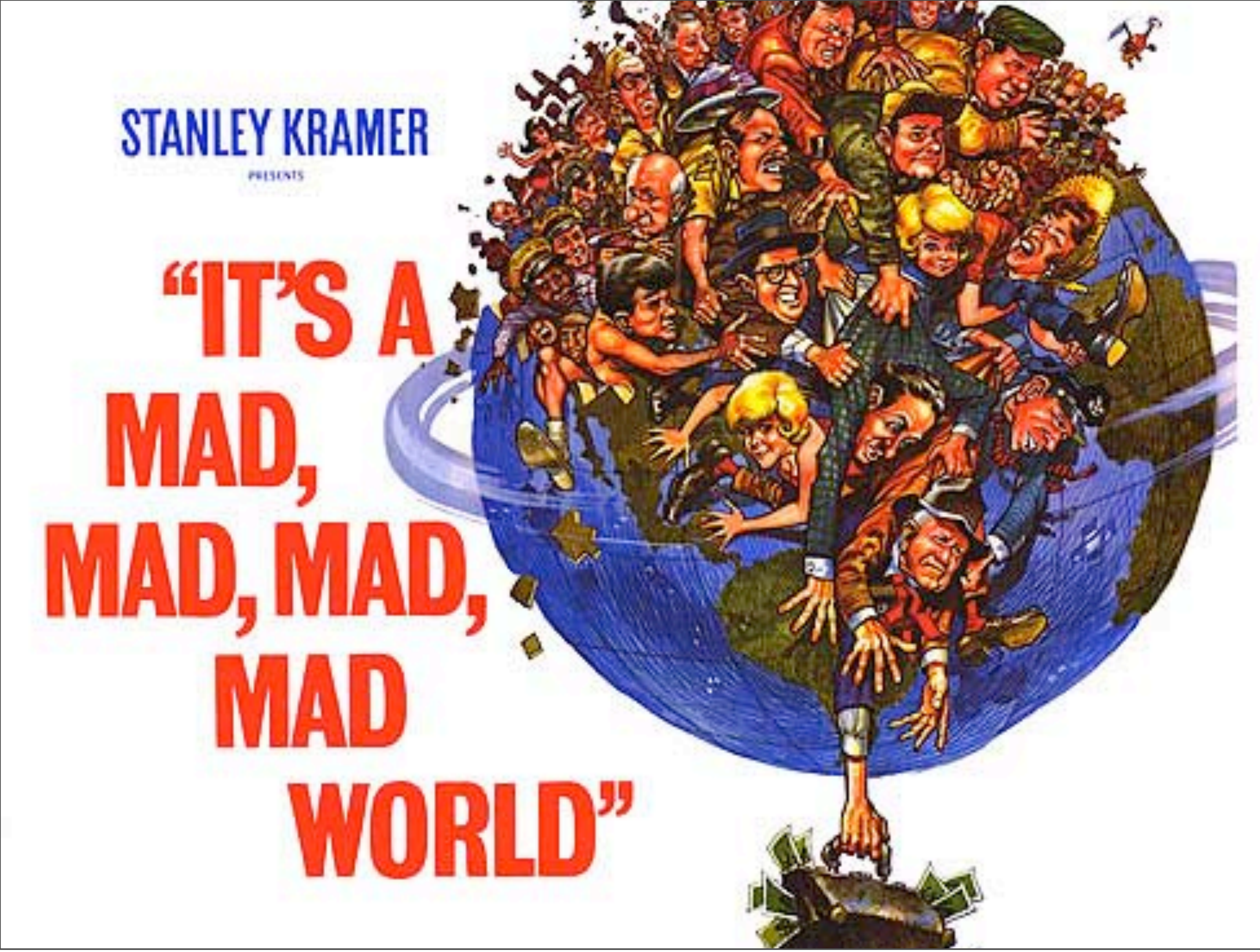
$$\vec{T}(t) = \vec{r}'(t)/|\vec{r}'(t)|$$

$$\vec{N}(t) = \vec{T}'(t)/|\vec{T}'(t)|$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$



DVD to win!


STANLEY KRAMER
PRESENTS
"IT'S A
MAD,
MAD, MAD,
MAD
WORLD"

Ready-Steady

The second next slide shows the question

Question:

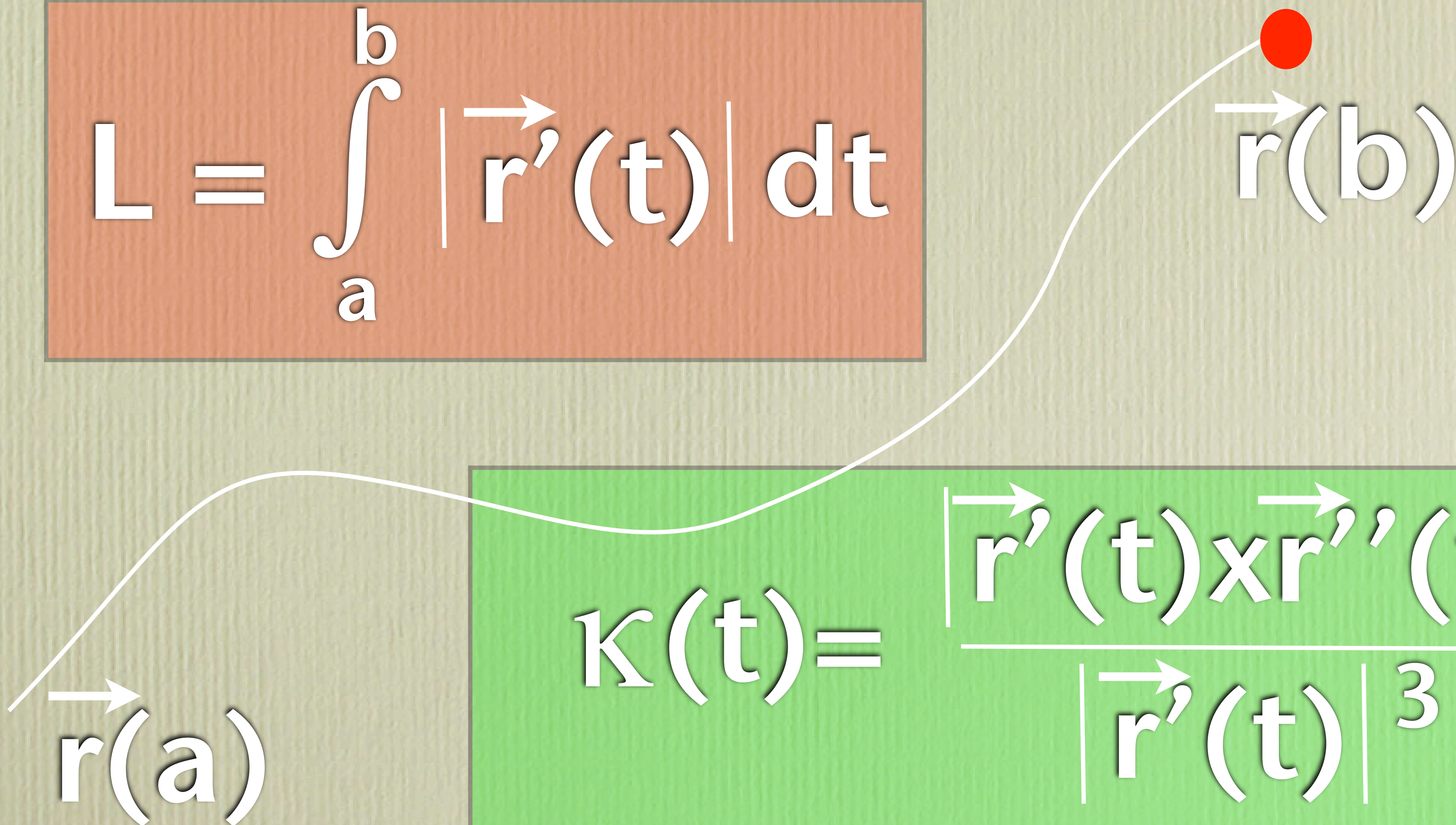
Oliver (virtually) flies over Harvard in the plane

$$4x + 2y - 4z = 12$$

What is the z component of the binormal vector B ?

Arc Length and Curvature

$$L = \int_a^b \left|\vec{r}'(t)\right| dt$$



$$\kappa(t) = \frac{\left|\vec{r}'(t) \times \vec{r}''(t)\right|}{\left|\vec{r}'(t)\right|^3}$$

Independent of parametrization

Problem 3

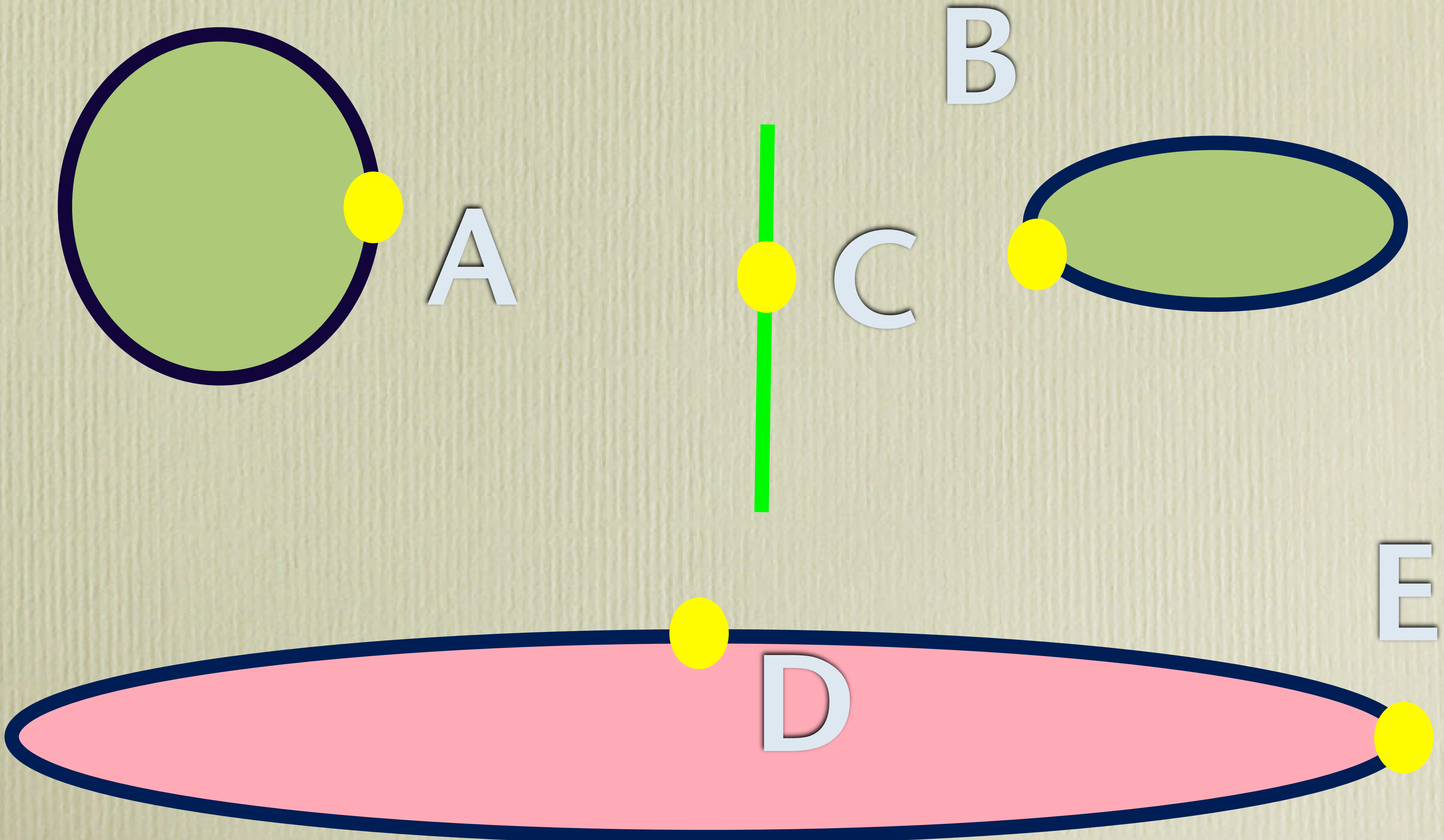
The helicopter crashes along the curve

$$\vec{r}(t) = \langle t^2/2, 1, -t^3/3 \rangle$$

From t=0 to t=1. Find the arc length of that path.

Curvature problem.

Sort the points in decreasing curvature. Which is the point with the largest, which is the point with smallest curvature



Problem 4

The helicopter is subject to a wind force:

$$\vec{r}''(t) = < t^2, 1, -t^3 >$$

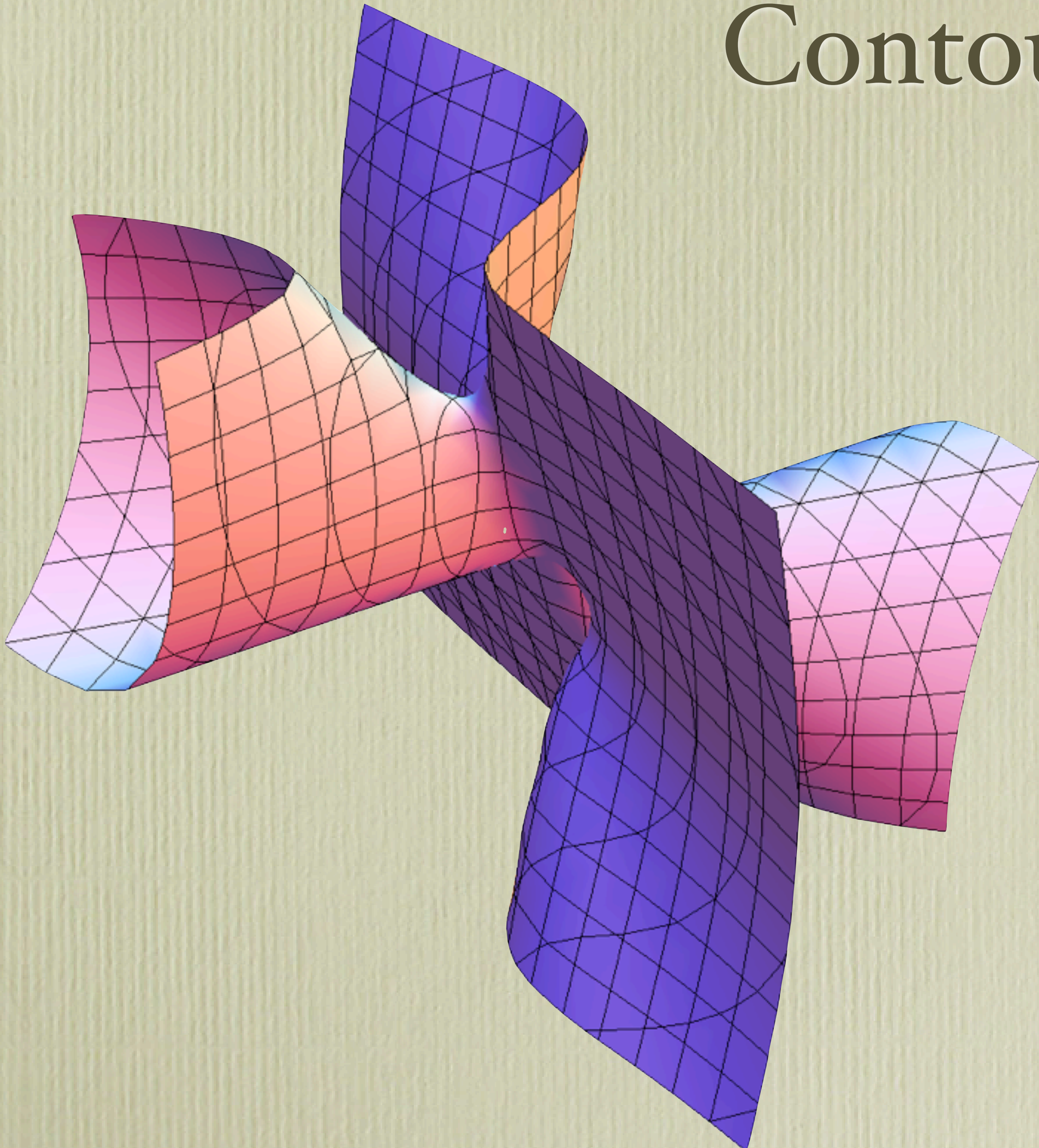
Where is it at time t=1 if

$$\vec{r}(0) = < 0,1,0 >$$

$$\vec{r}'(0) = < 1,0,0 >$$ (hit by the cat)

Level Surfaces

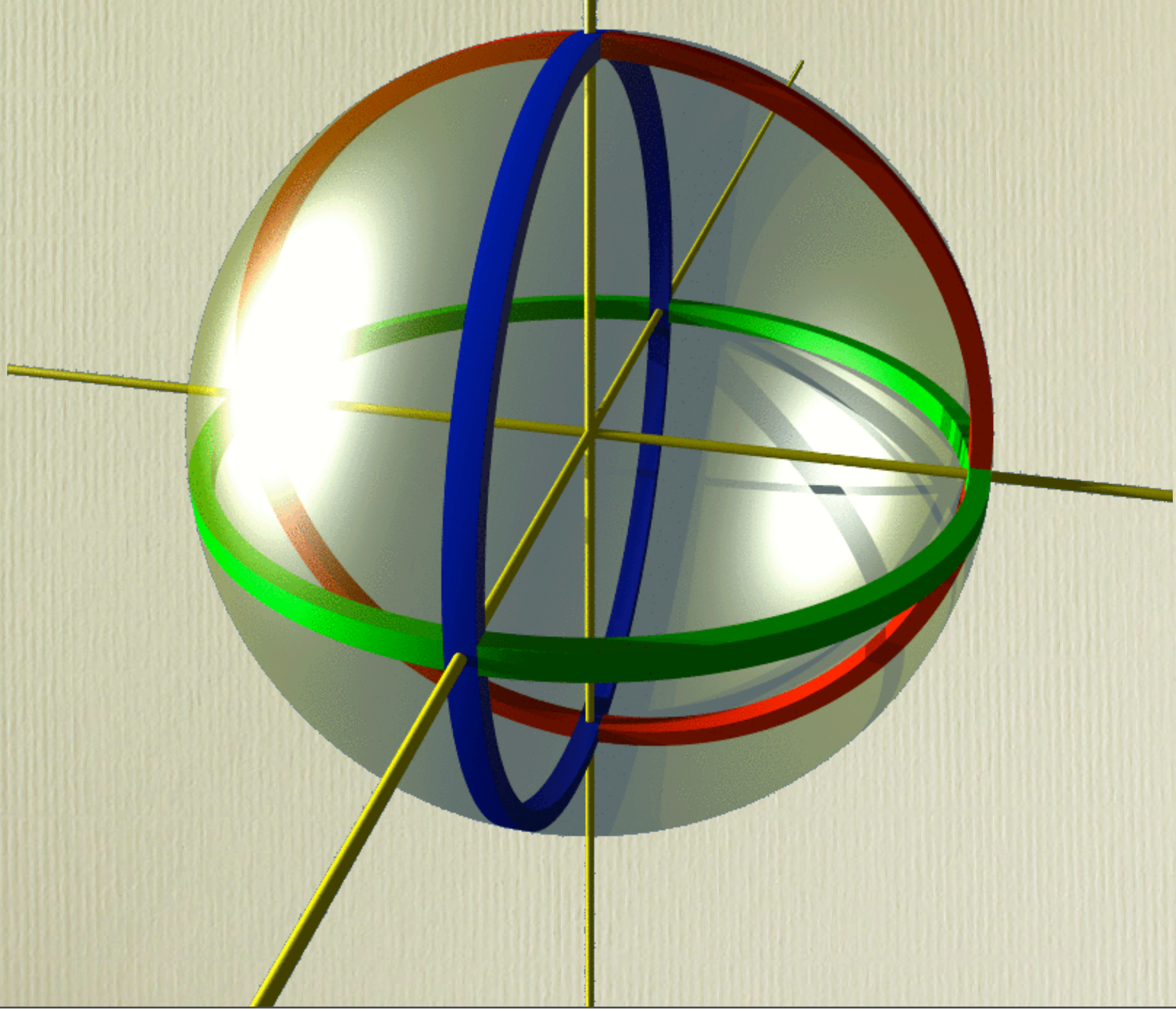
Contour Surfaces



$g(x,y,z) = c$

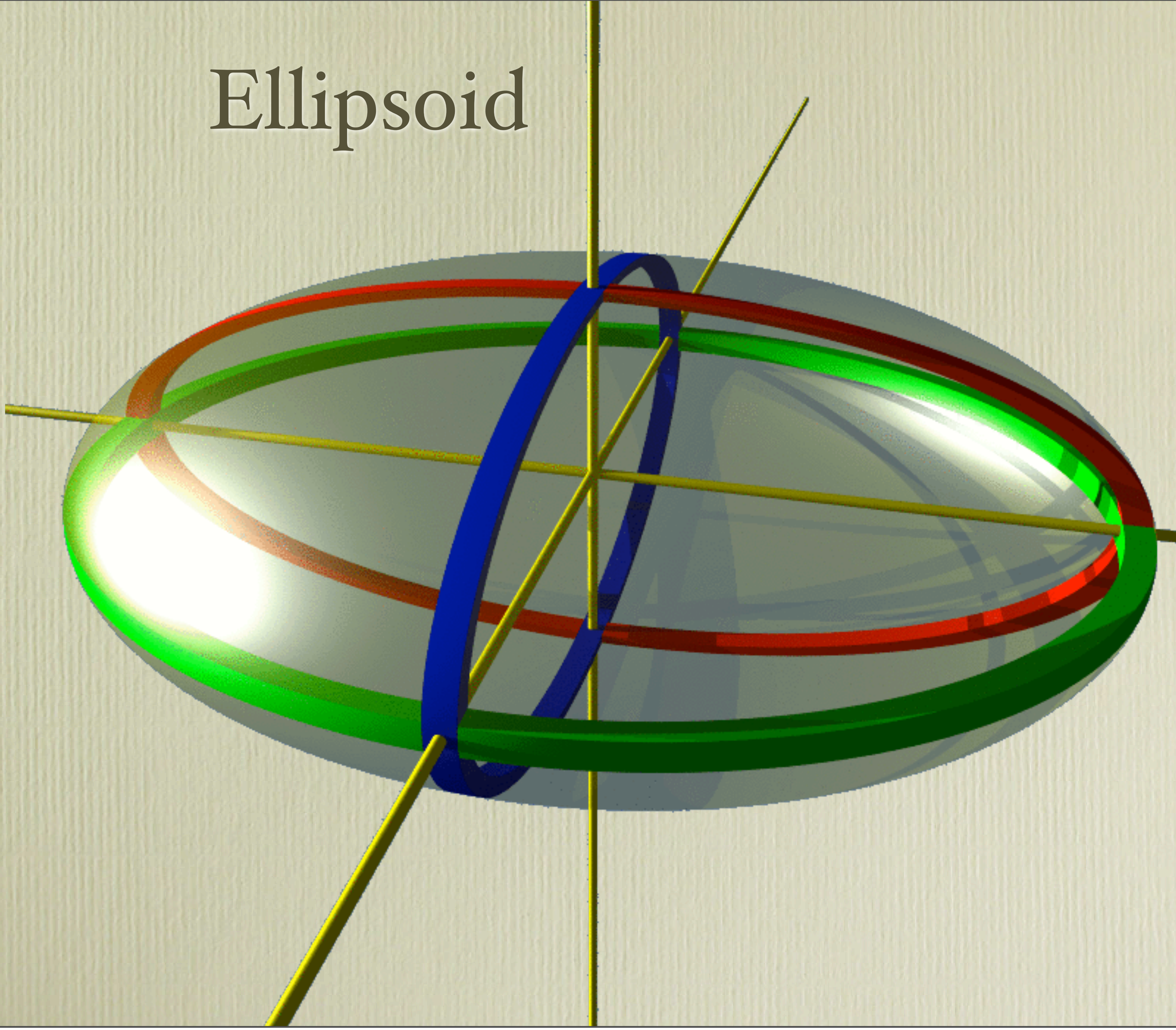
ContourPlot3D[x^3+y^3+x z^2-2x y^2+z+x^2 z+x y z-3, {x,-6,6}, {y,-6, 6}, {z,-6, 6}]

Spheres

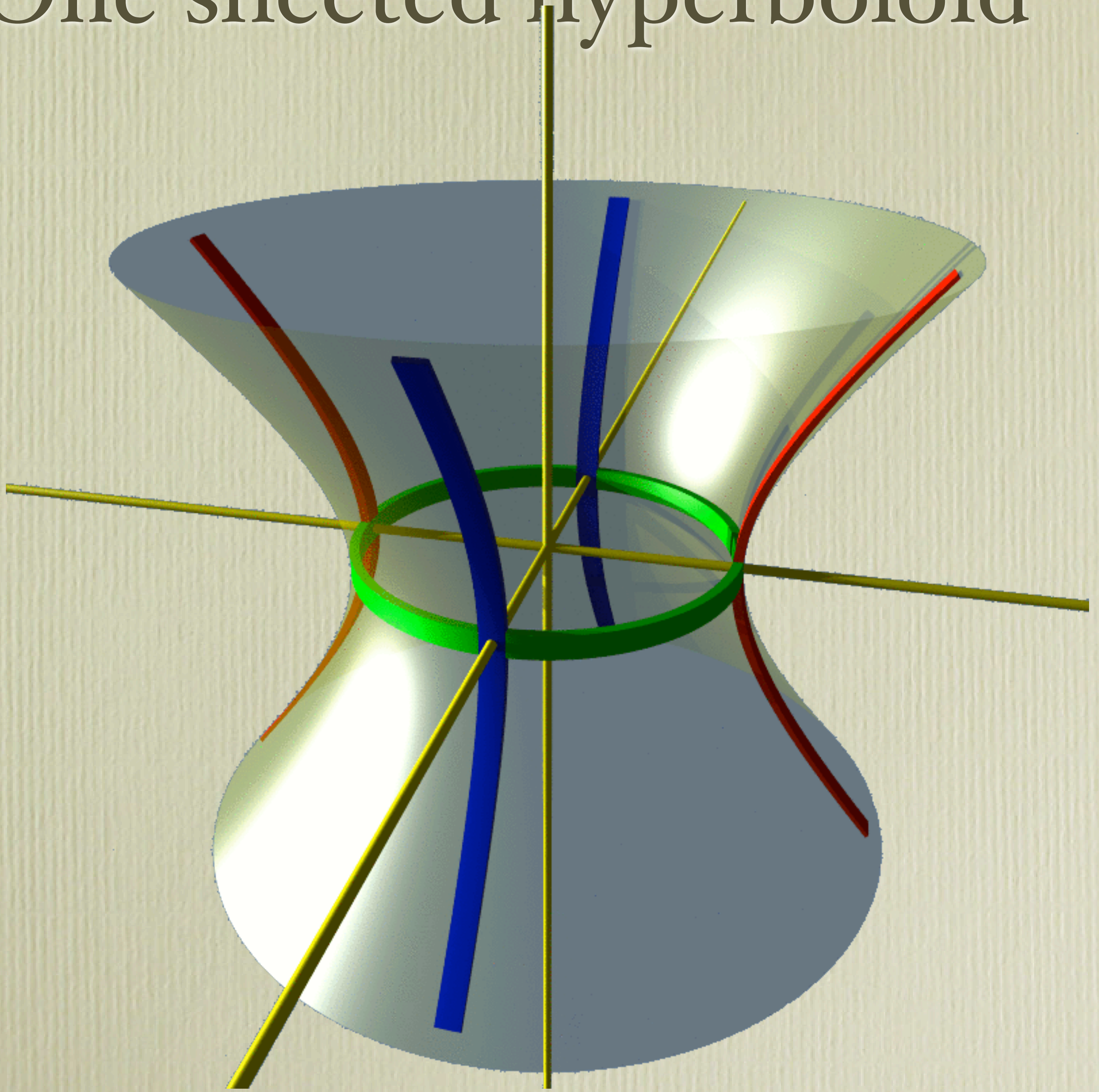


some have even fun with spheres....

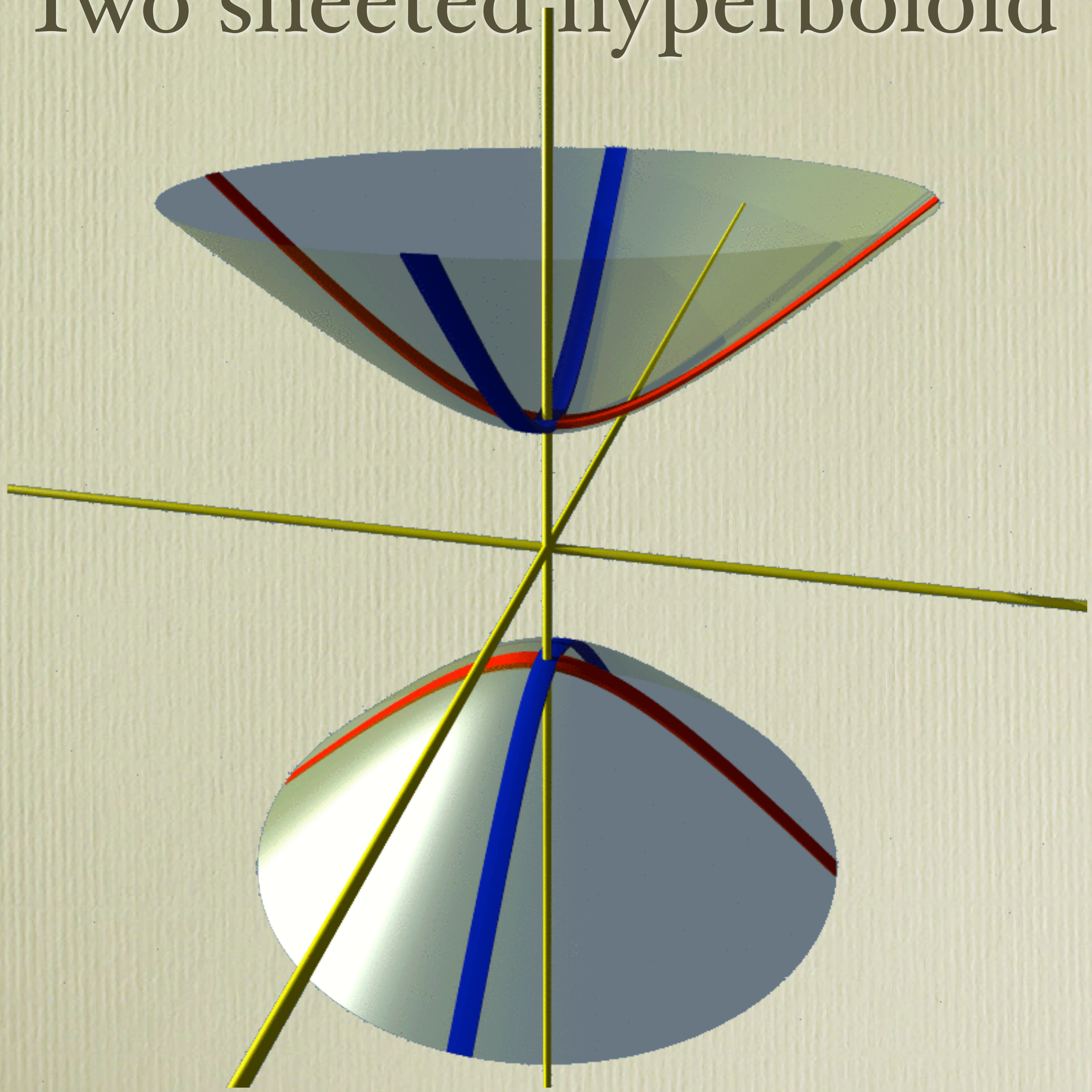
Ellipsoid



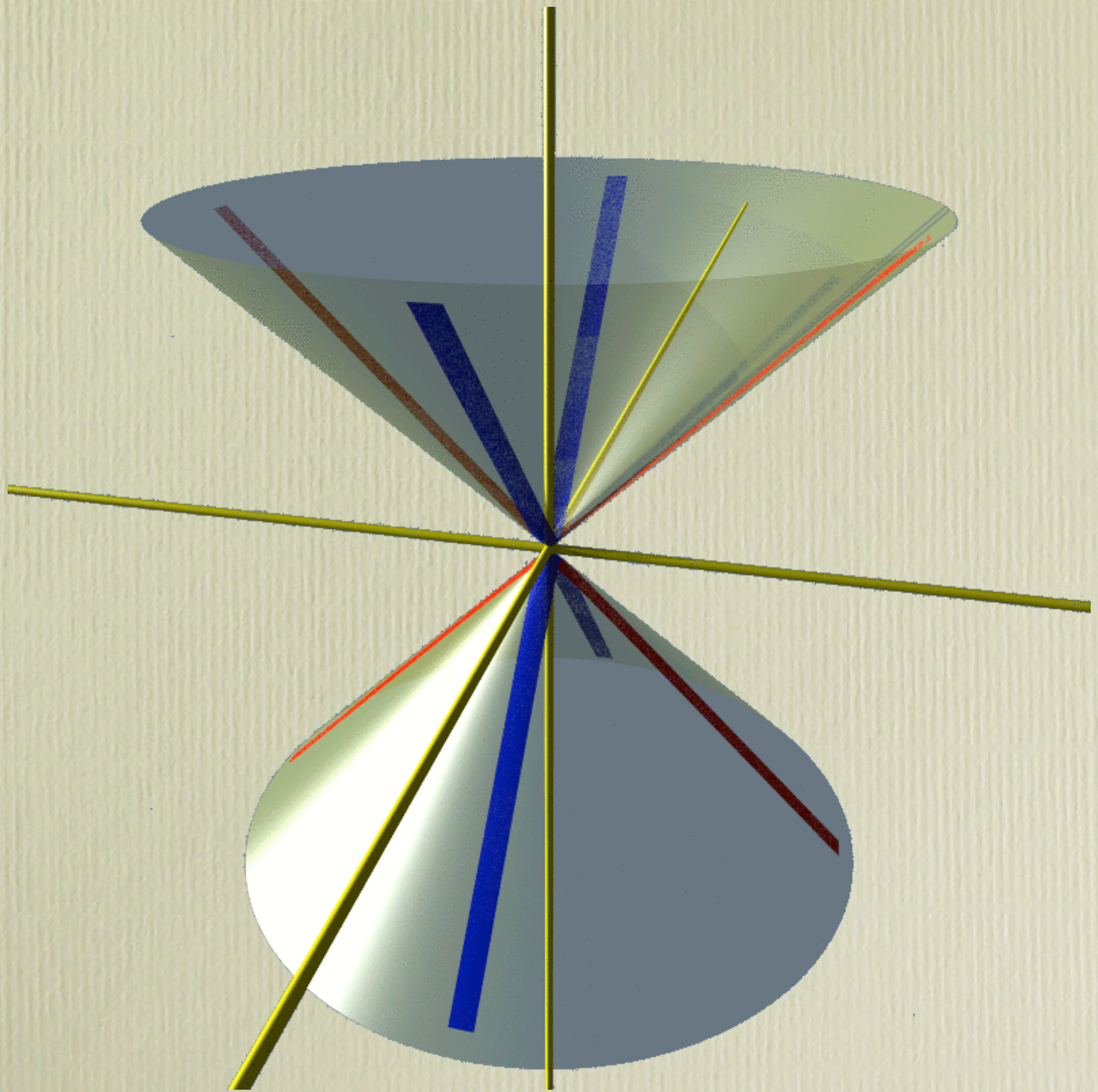
One sheeted hyperboloid



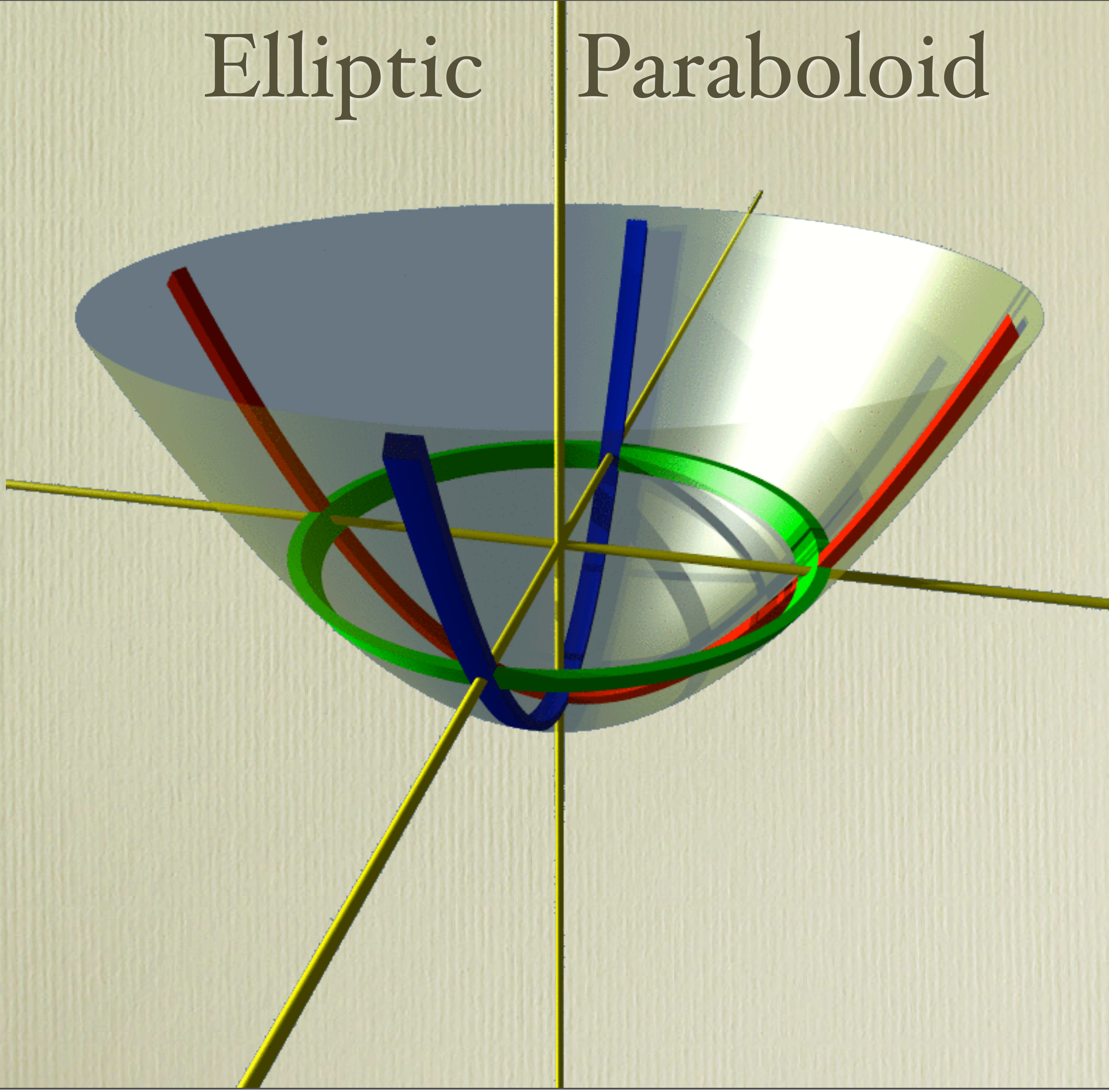
Two sheeted hyperboloid



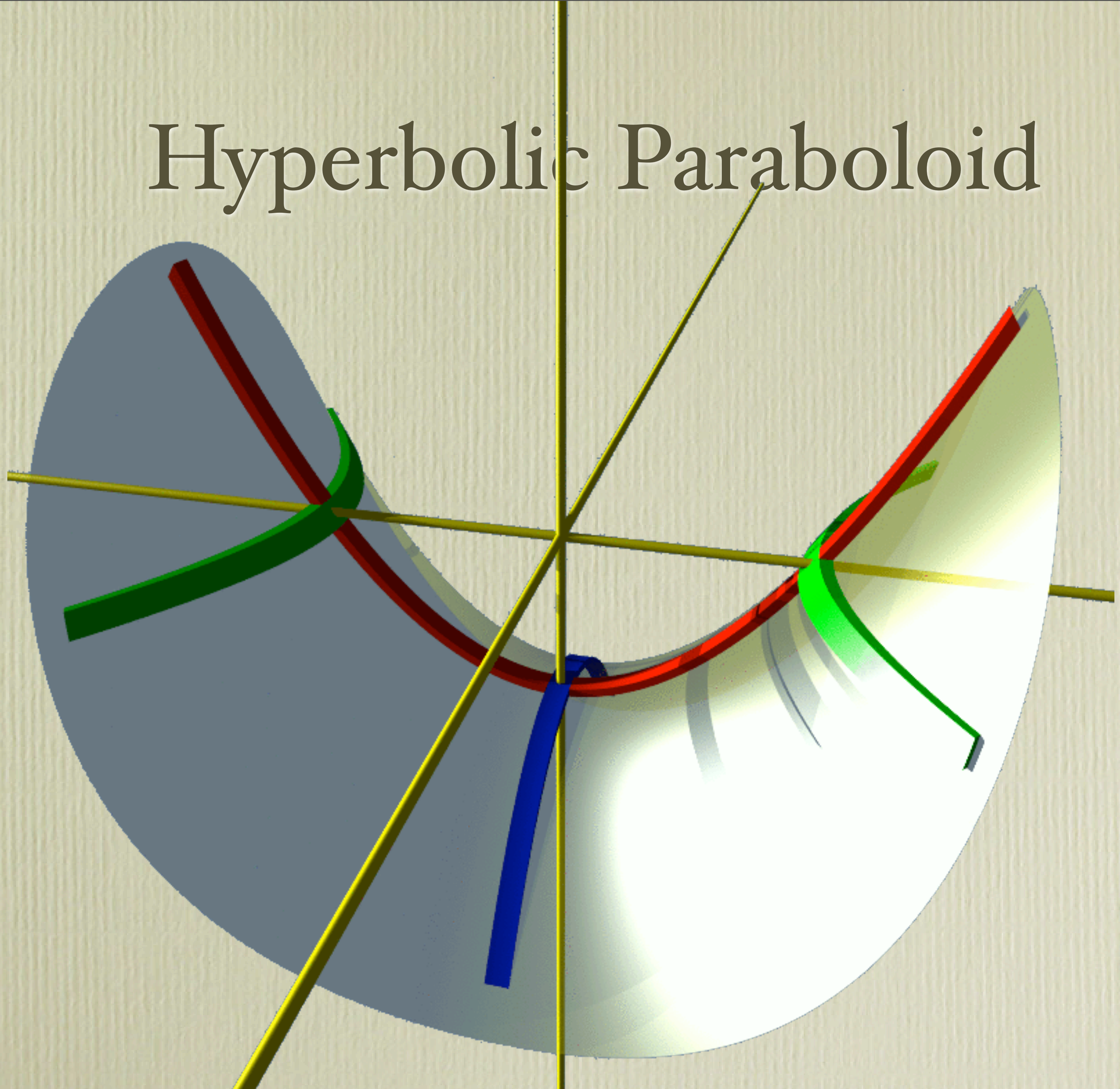
Cone



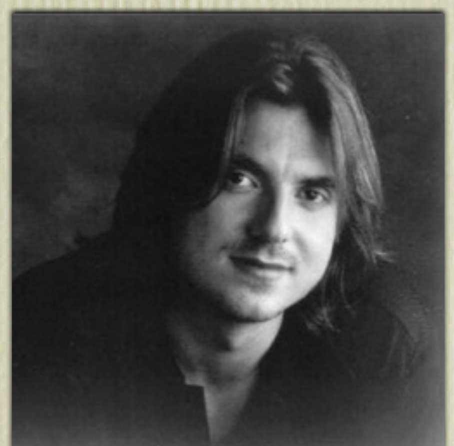
Elliptic Paraboloid



Hyperbolic Paraboloid



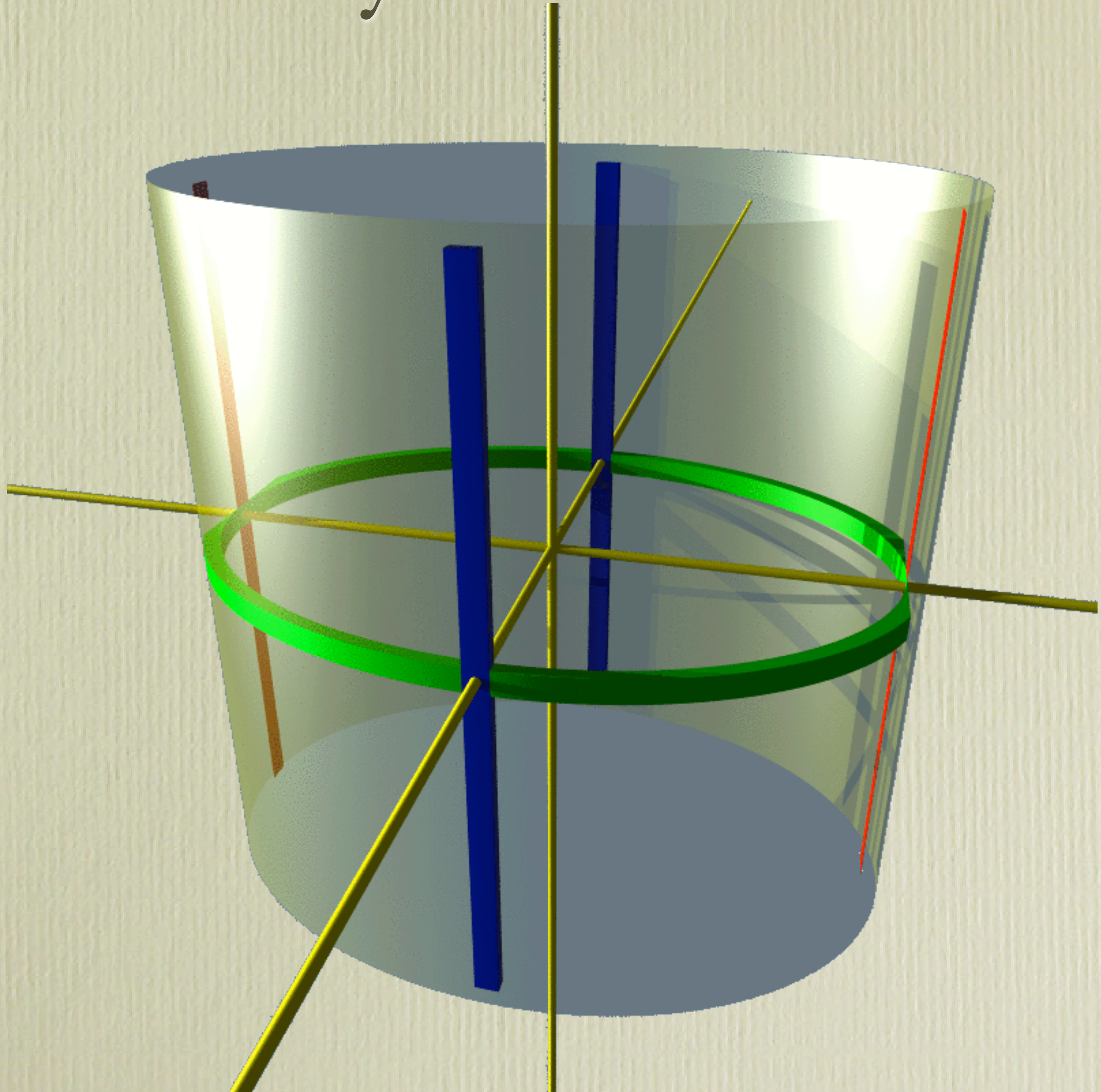
I think Pringles initial intention was to make tennis balls. But on the day that the rubber was supposed to show up, a big truckload of potatoes arrived. But Pringles was a laid-back company. They said "Screw it. Cut them up".



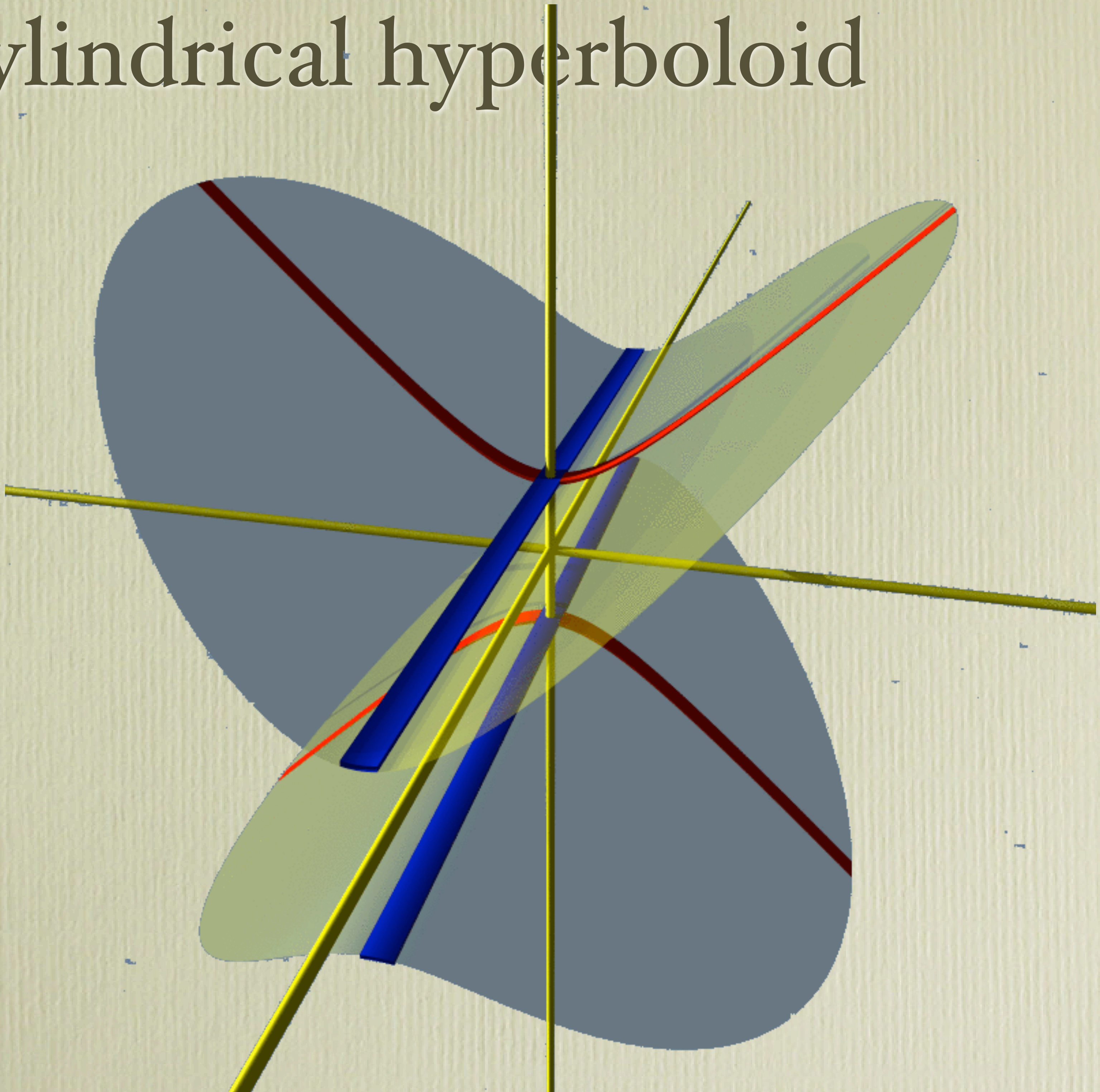
Mitch Hedberg, 1968-2005

Mitch Hedberg,
1968-2005

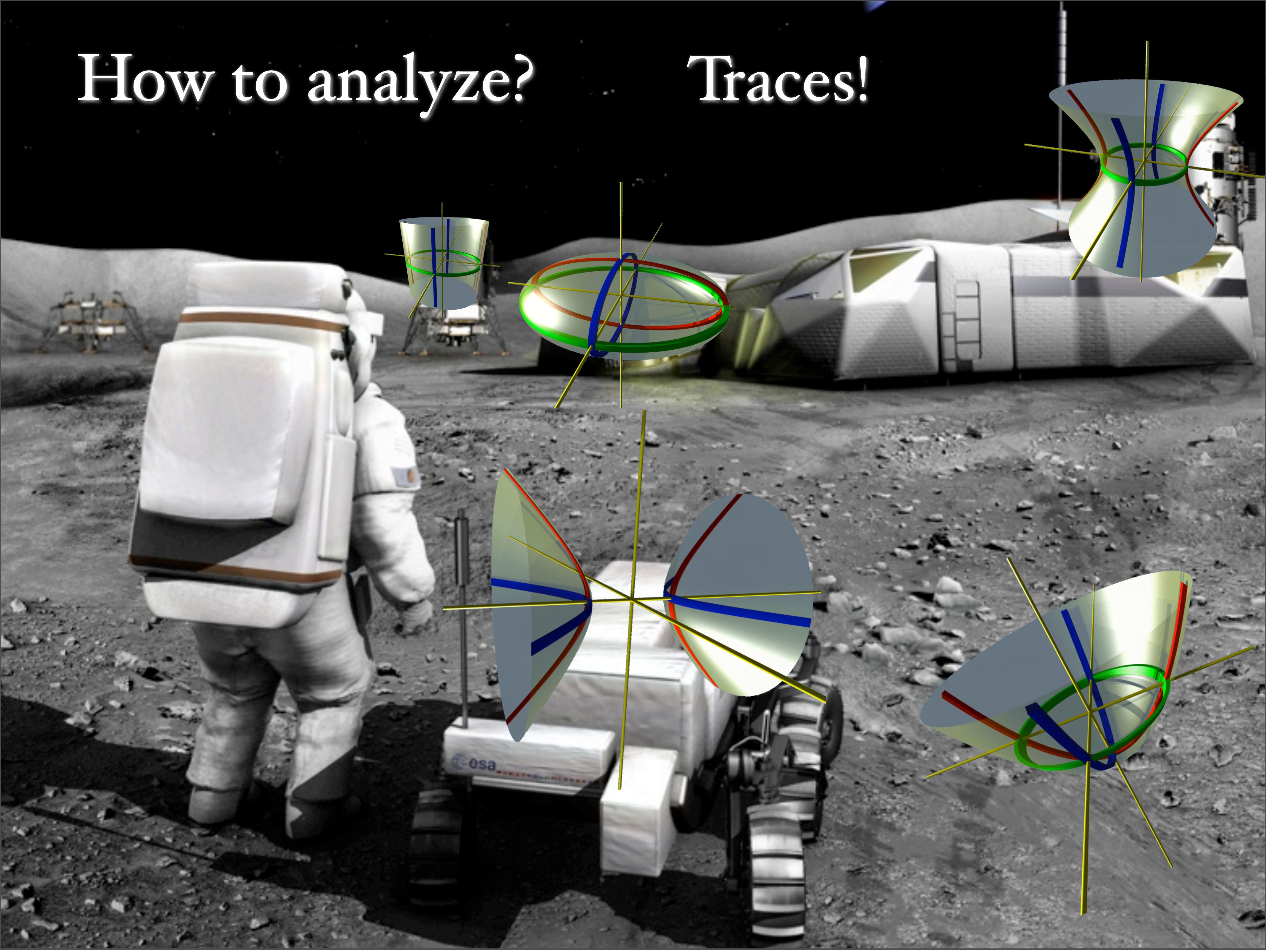
Cylinder



Cylindrical hyperboloid



How to analyze? Traces!



Problem:

What surface is this:

$$x^2 - y^2 - z^2 = 2y$$

Other coordinate systems

Polar coordinates

$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

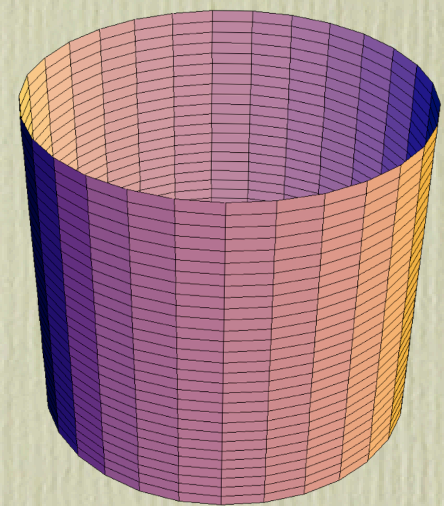
Cylindrical coordinates

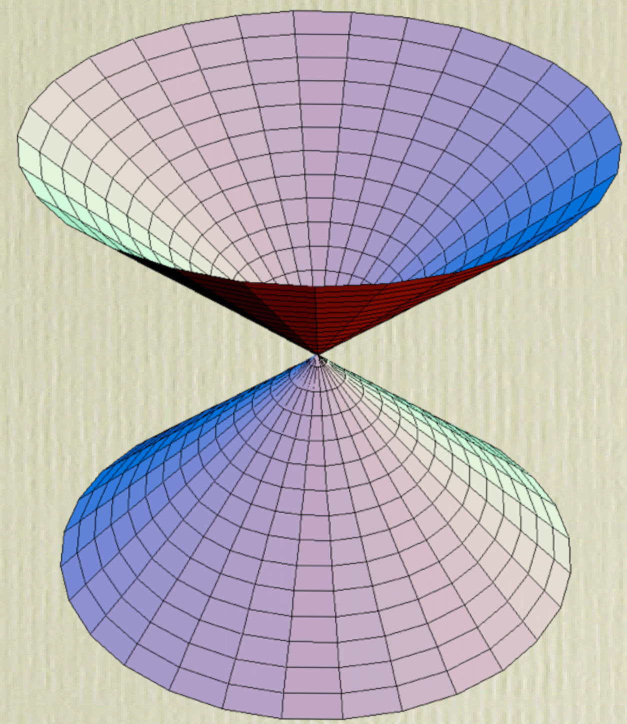
$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

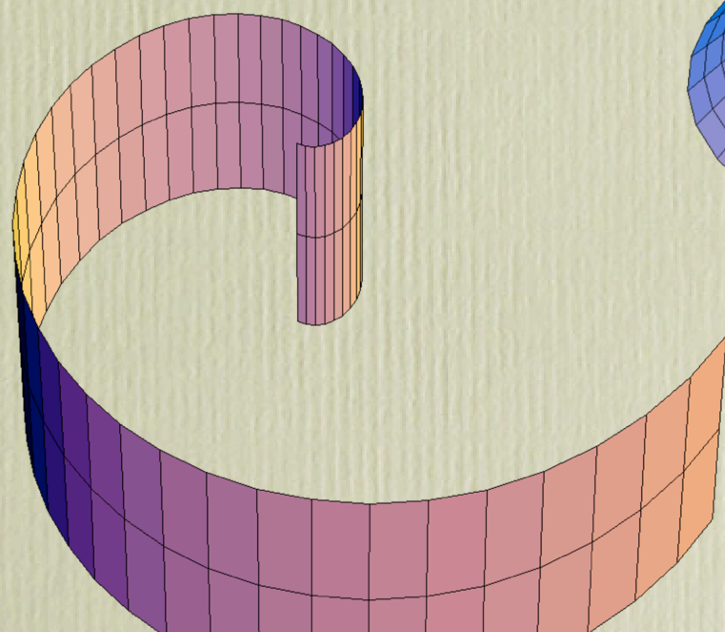
$$z = z$$

Surfaces in Cylindrical coordinates





- $r=1$
- $r=z$
- $r = \theta$



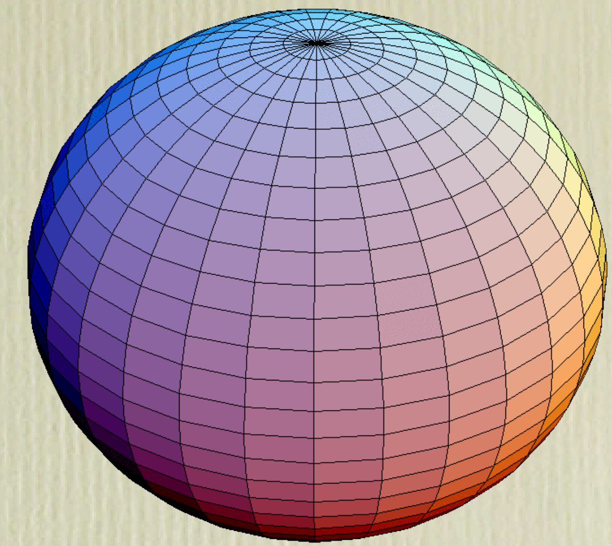
Spherical coordinates

$$x = \varrho \sin(\phi) \cos(\theta)$$

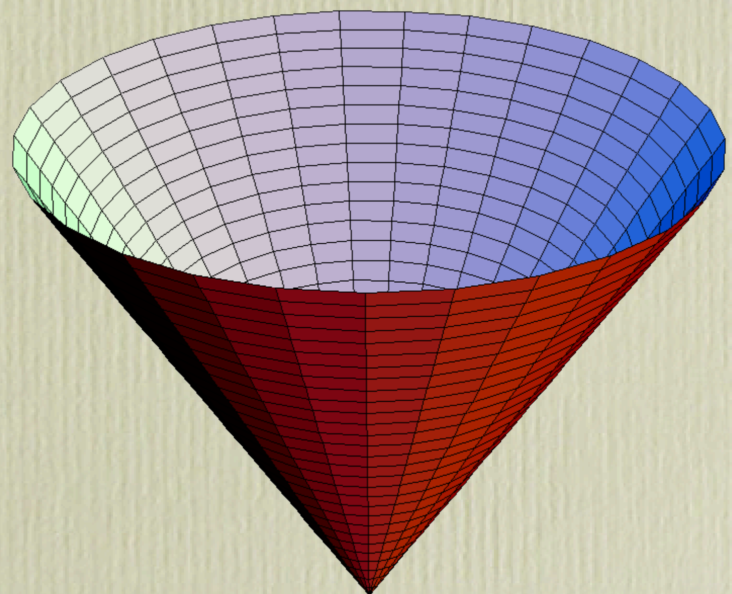
$$y = \varrho \sin(\phi) \sin(\theta)$$

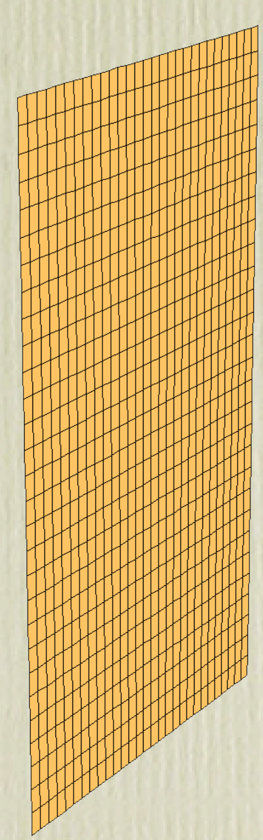
$$z = \varrho \cos(\phi)$$

Surfaces in Spherical coordinates



- $\rho = 1$
- $\phi = 1$
- $\theta = 1$





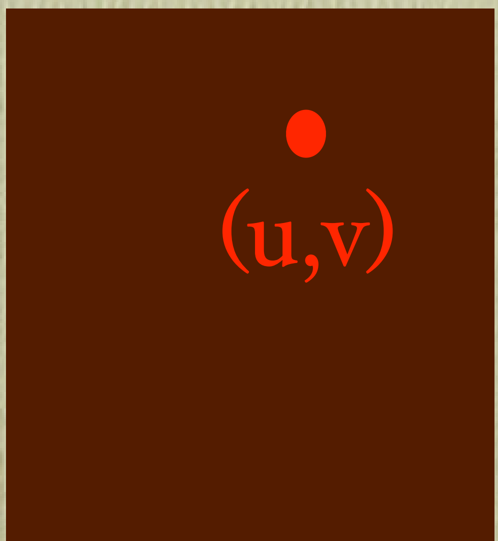
Problem:

What surface is this:

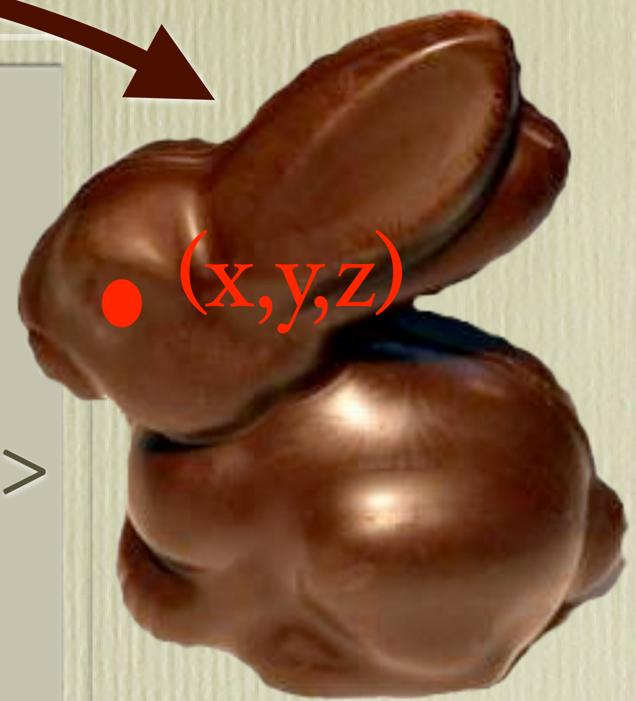
$$r \cos(2\theta) = z$$

Parametrized Surfaces

Parametrization of surfaces



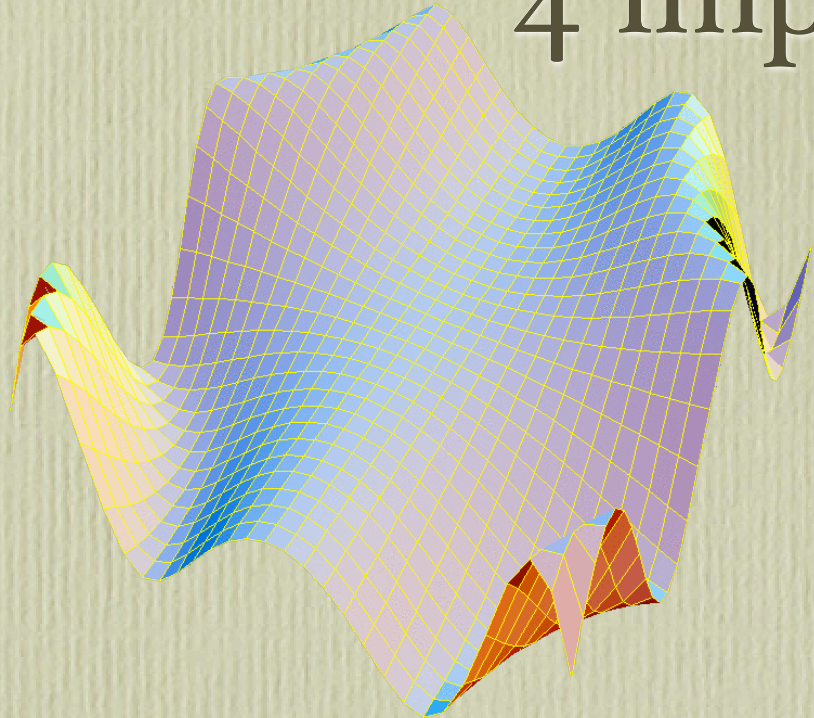
$$\vec{r}(u,v) = \langle\, x(u,v), y(u,v), z(u,v) \,\rangle$$

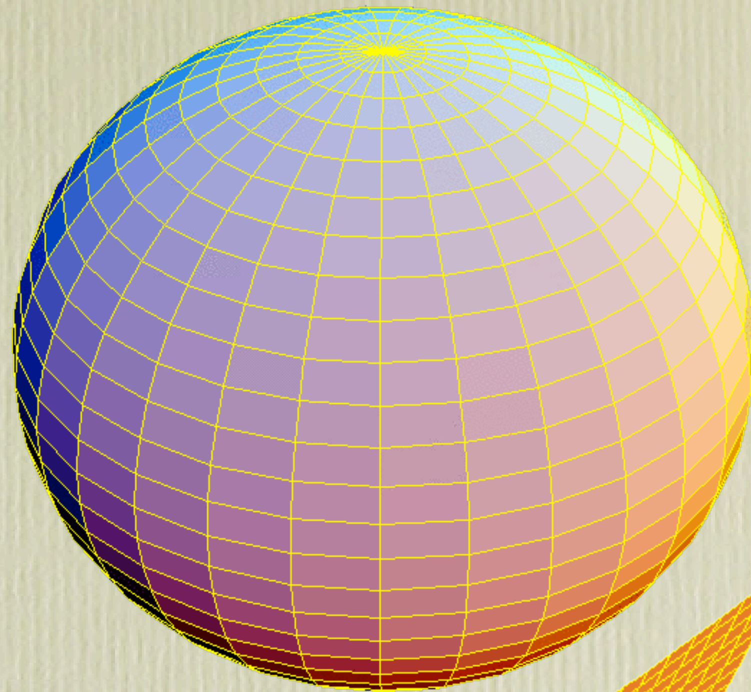


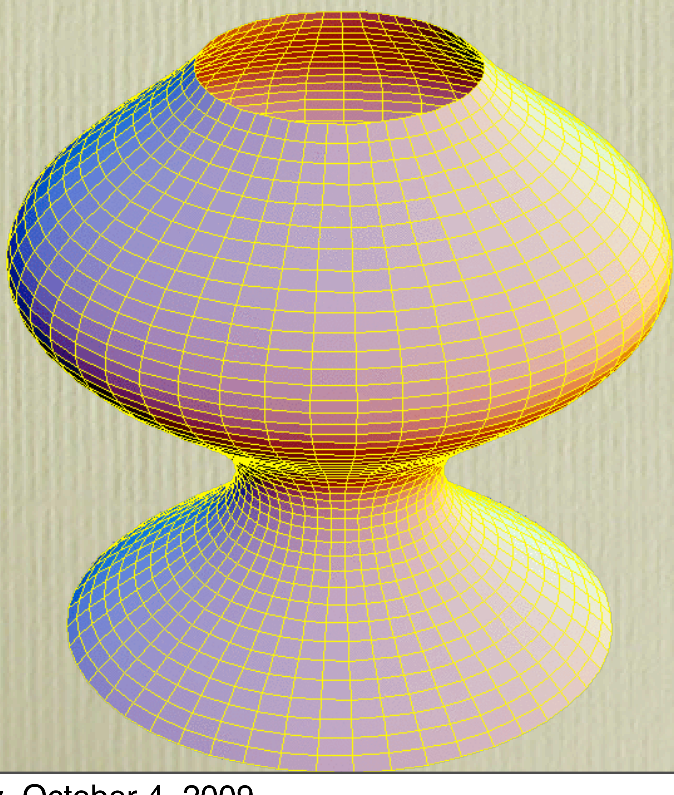
We can of course use different variables.

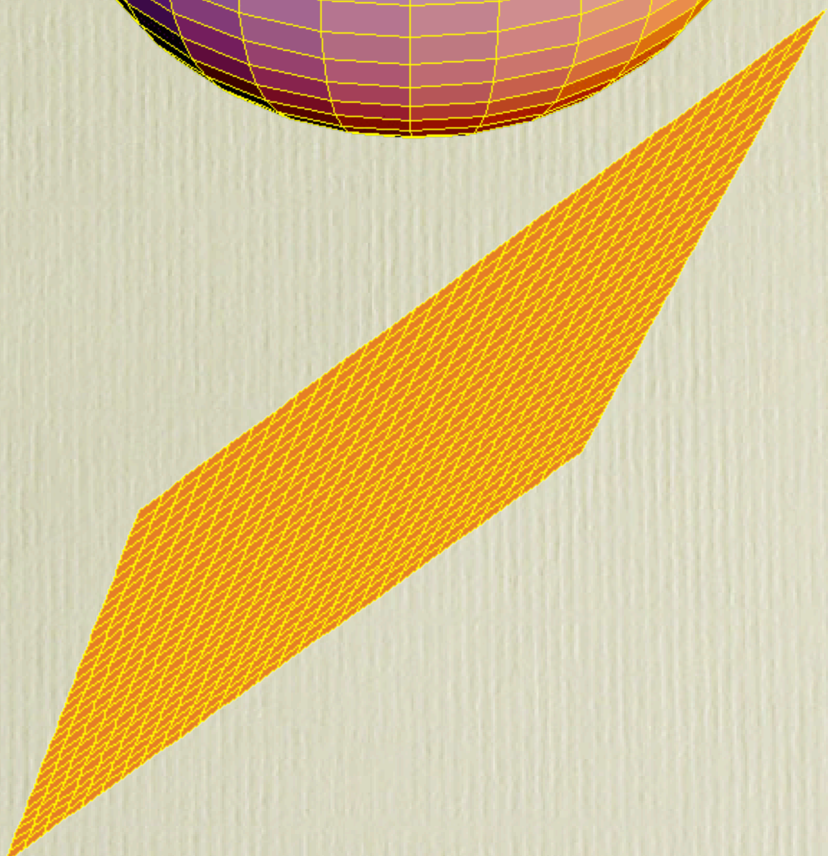
We can have r(s,t), r(θ , ϕ) .

4 important classes:



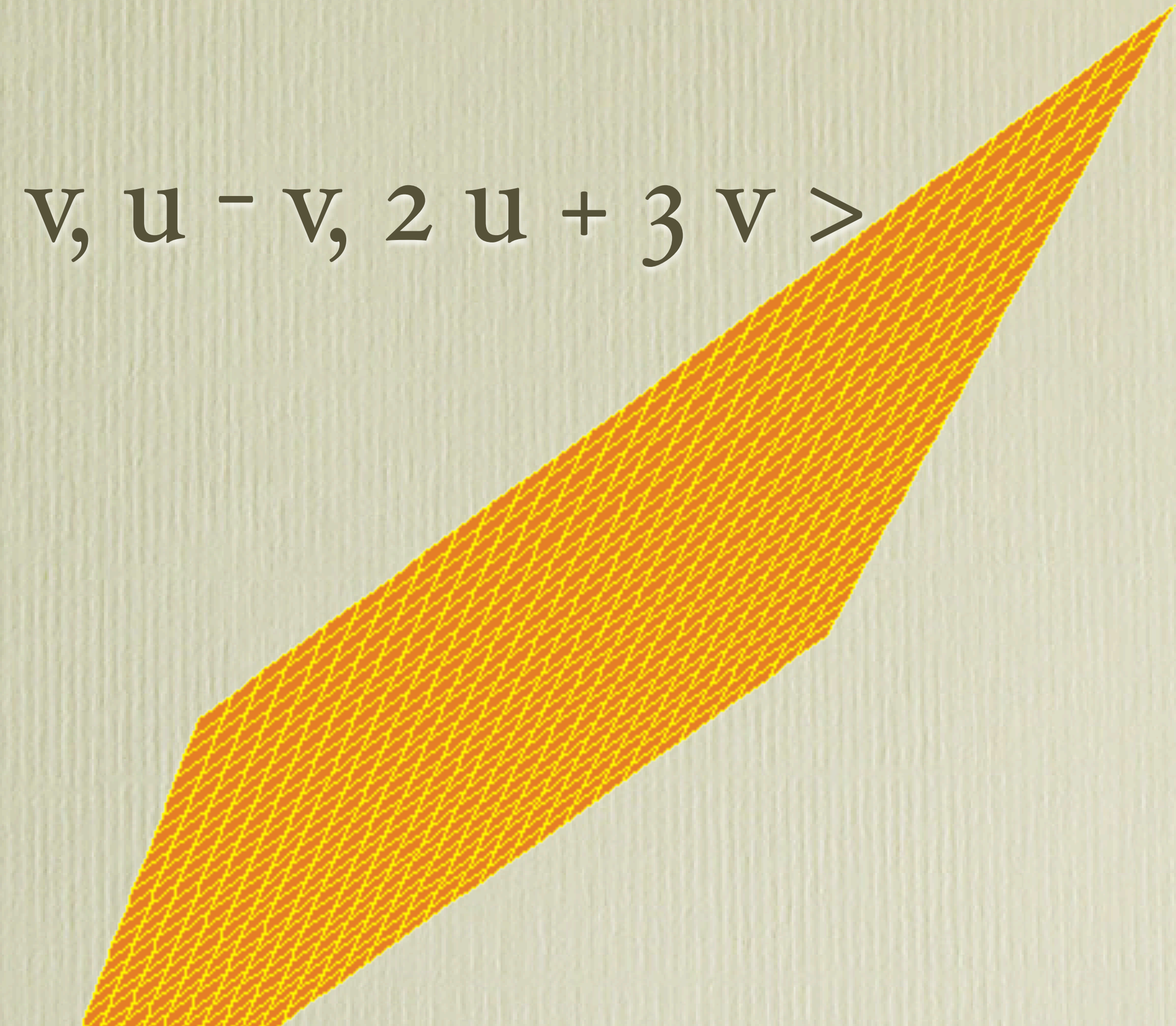




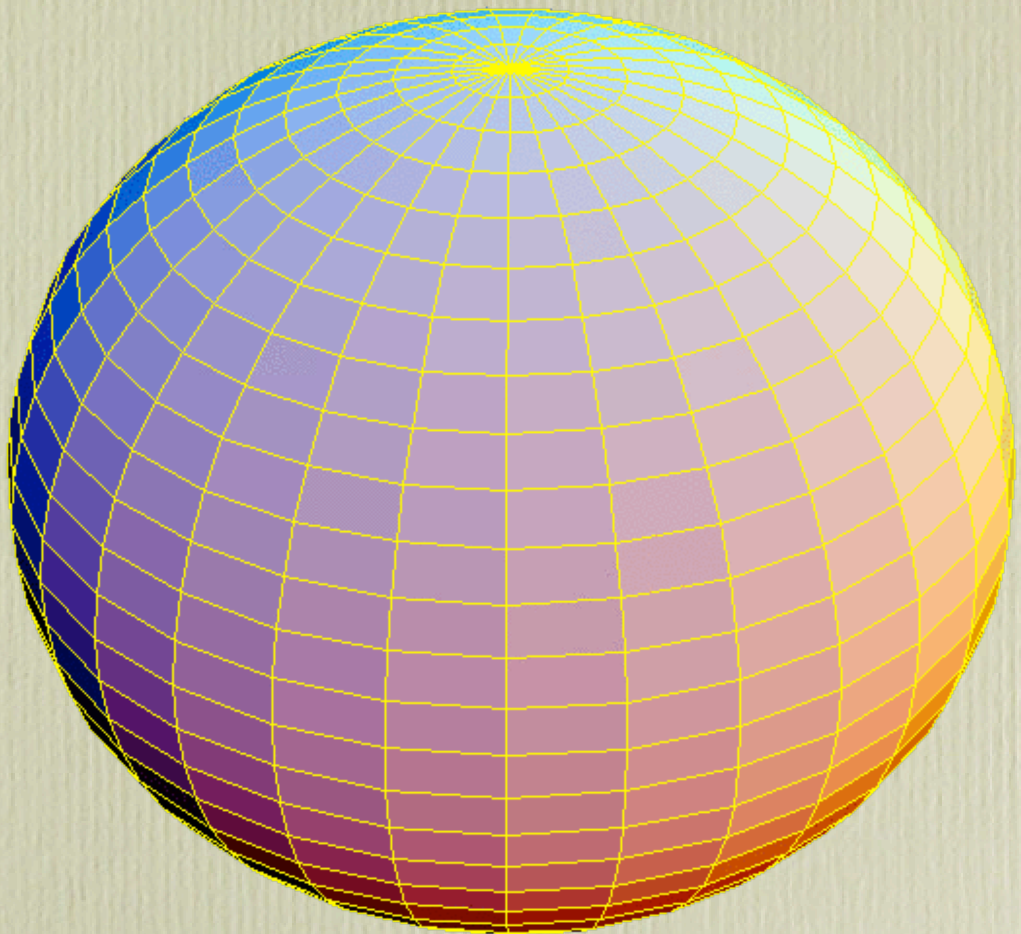


Planes

$\vec{r}(u,v) = \langle 1 + u + v,\ u - v,\ 2u + 3v \rangle$



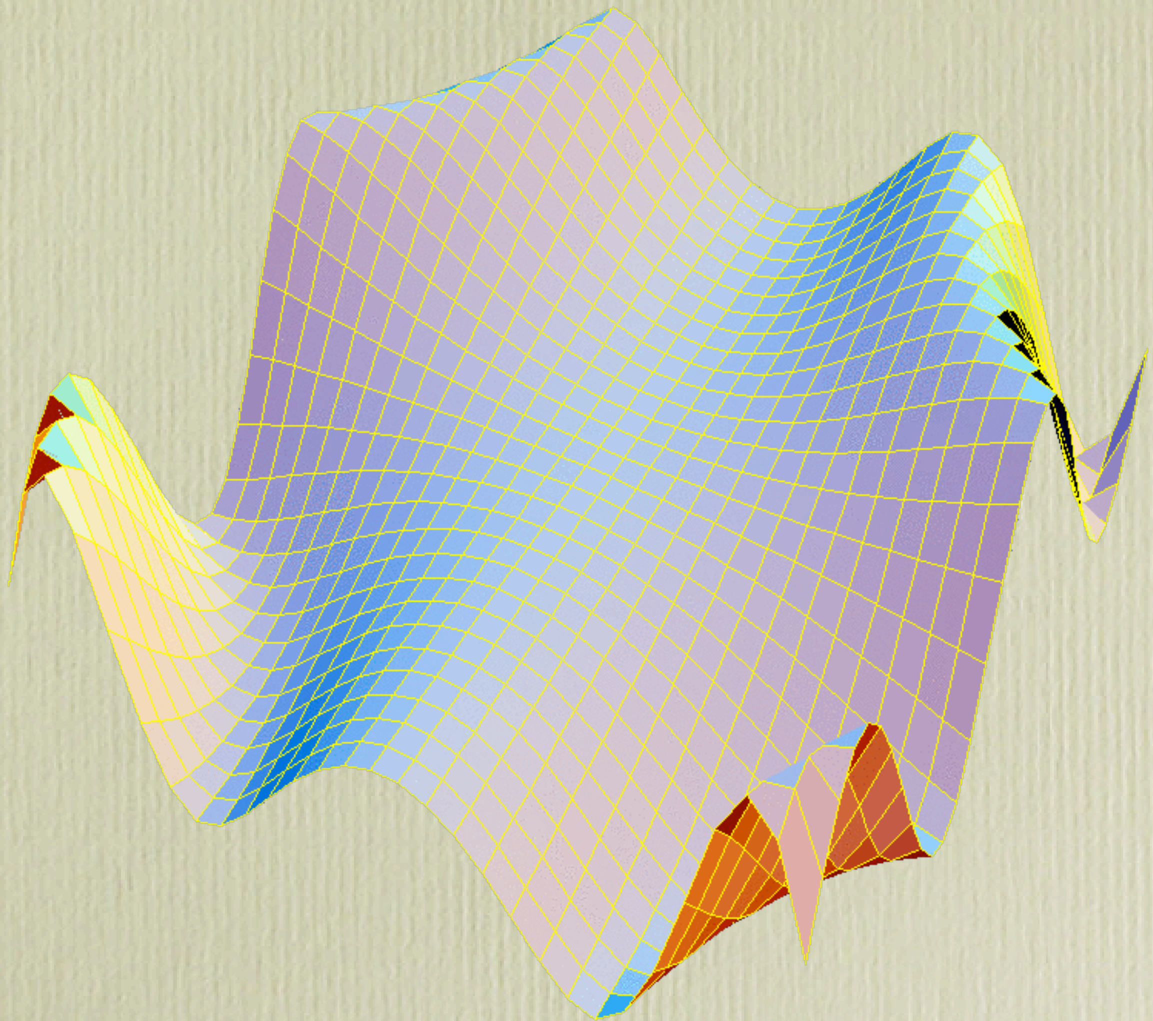
Spheres



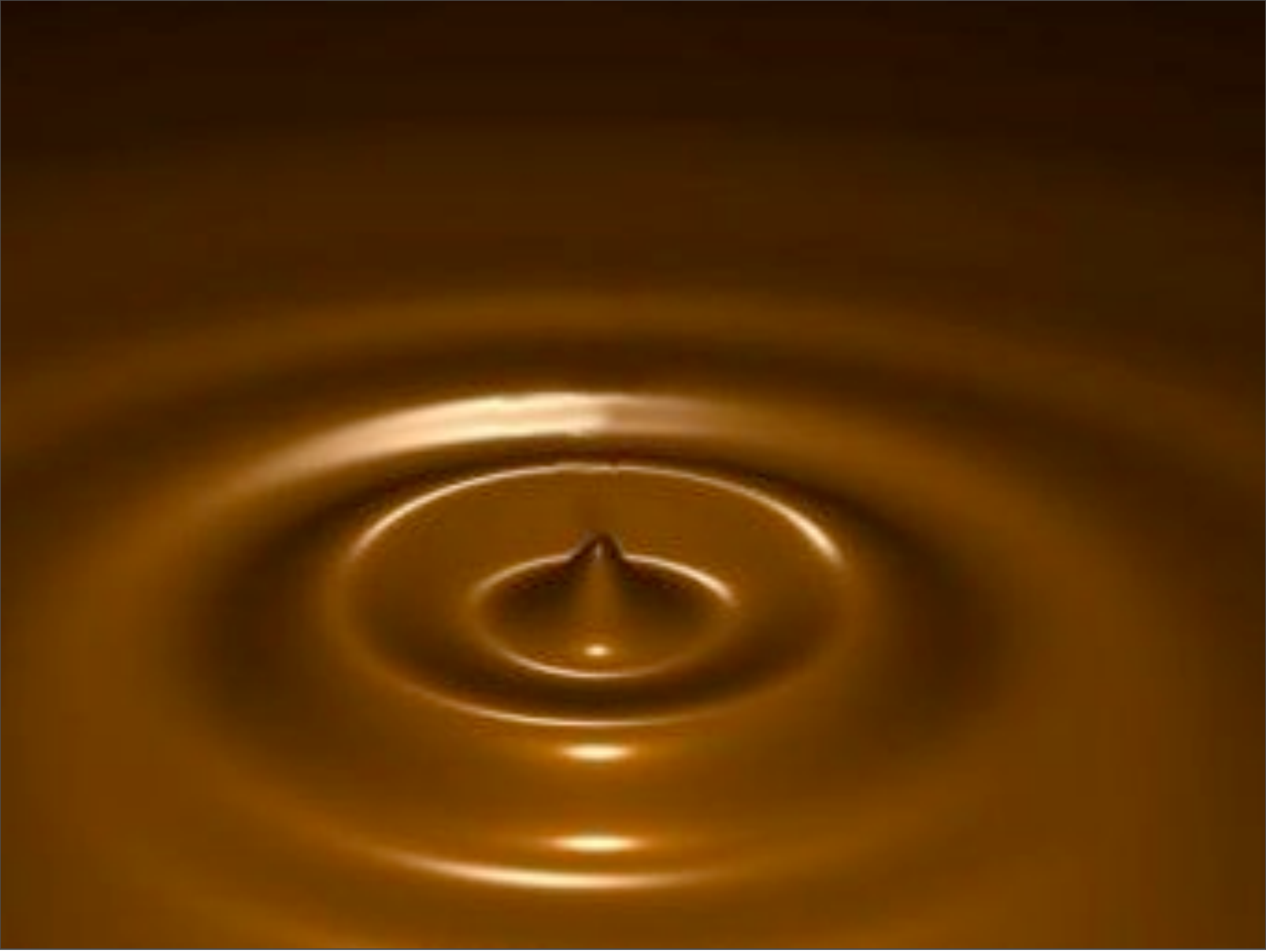
$\vec{r}(u,v) = \langle \cos(u)\sin(v), \sin(u)\sin(v), \cos(v) \rangle$

Sunday, October 4, 2009

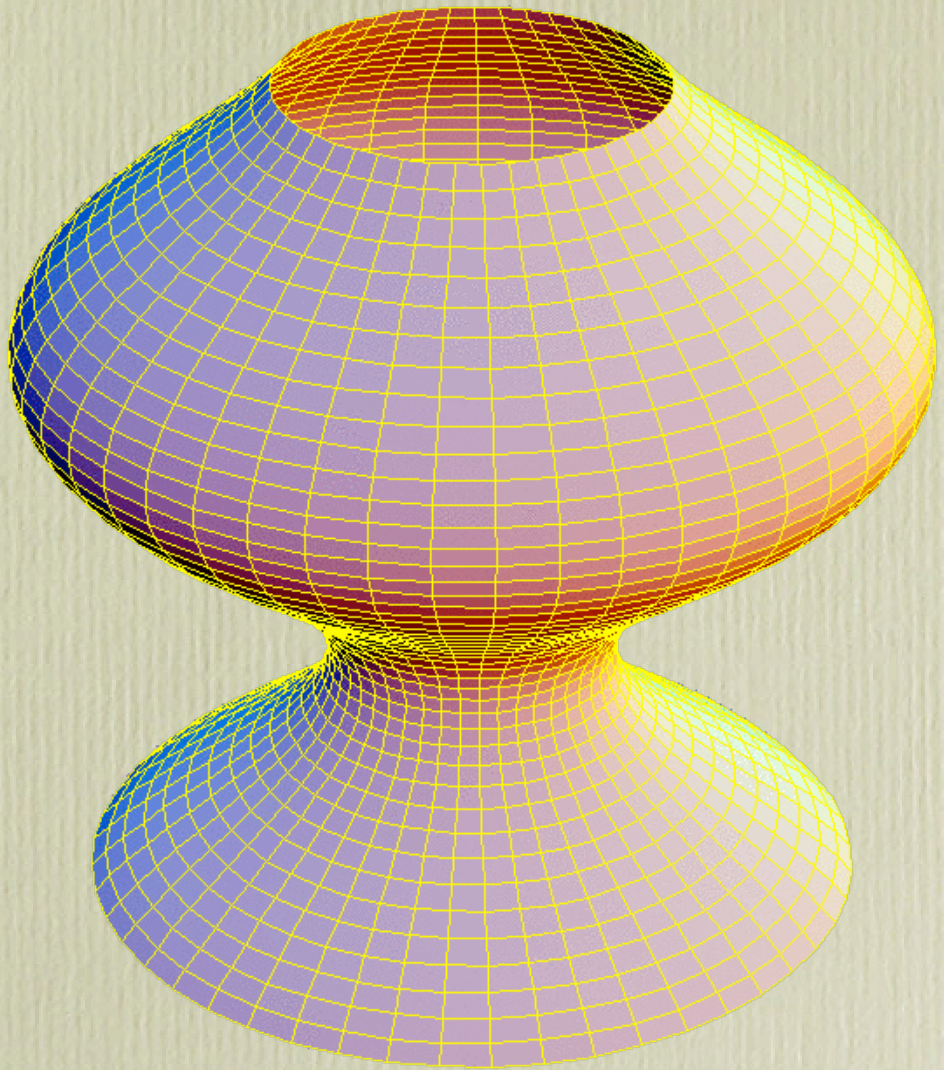
Graphs



$$\vec{r}(u,v) = \langle u, v, f(u,v) \rangle$$

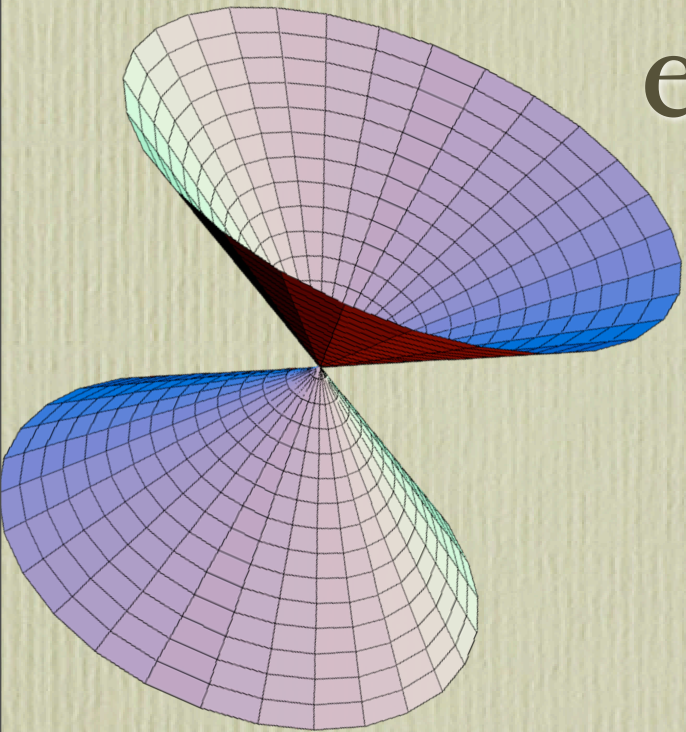


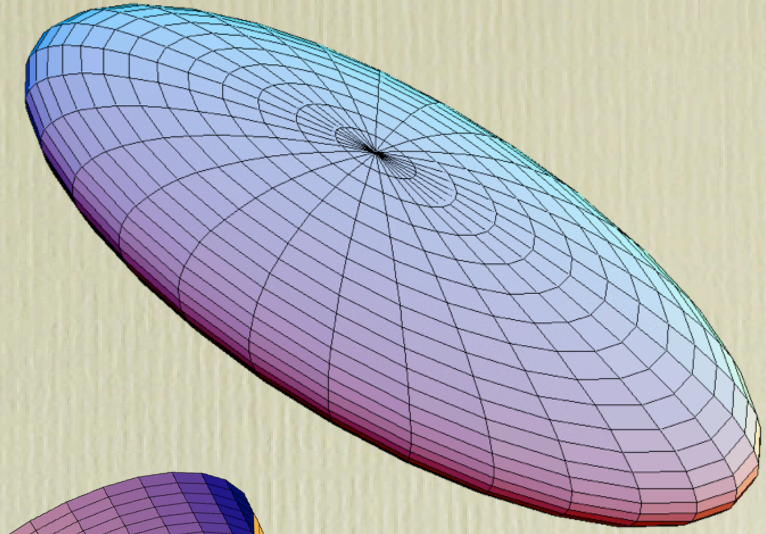
Surfaces of revolution



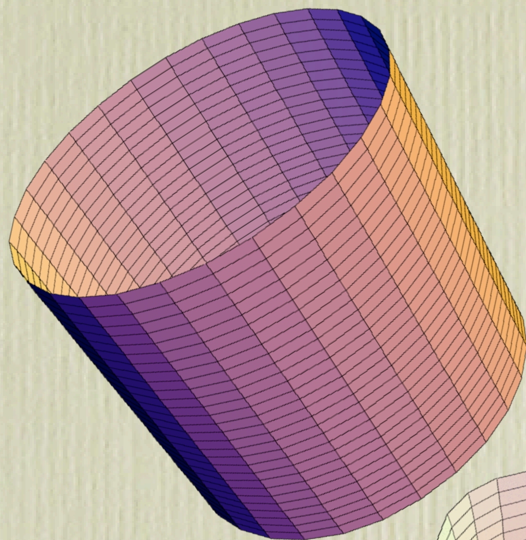
$$\vec{r}(u,v) = \langle r(v)\cos(u),\ r(v)\sin(u),\ v \rangle$$

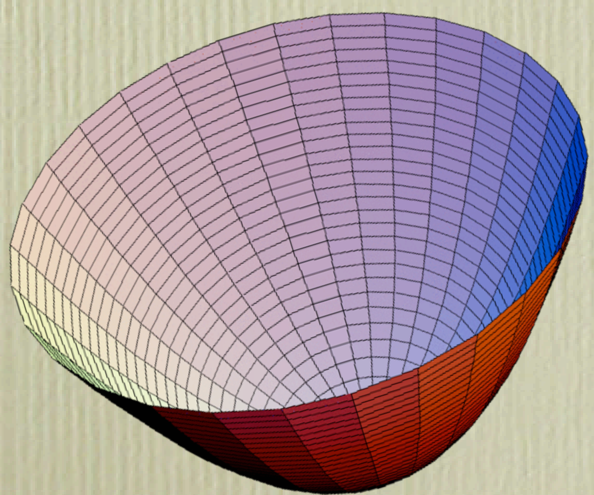
More examples

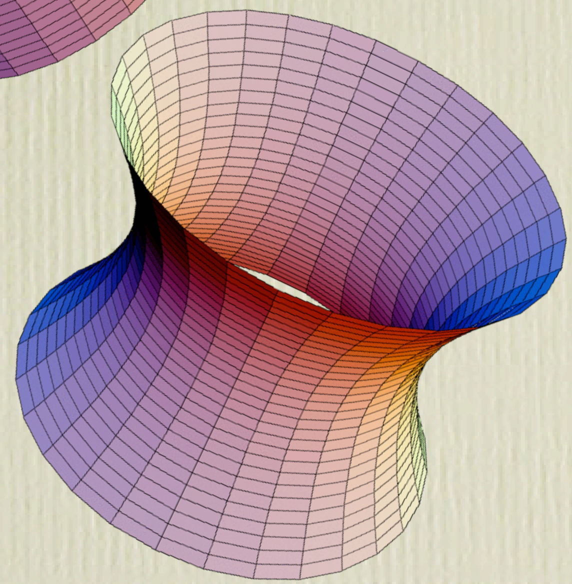




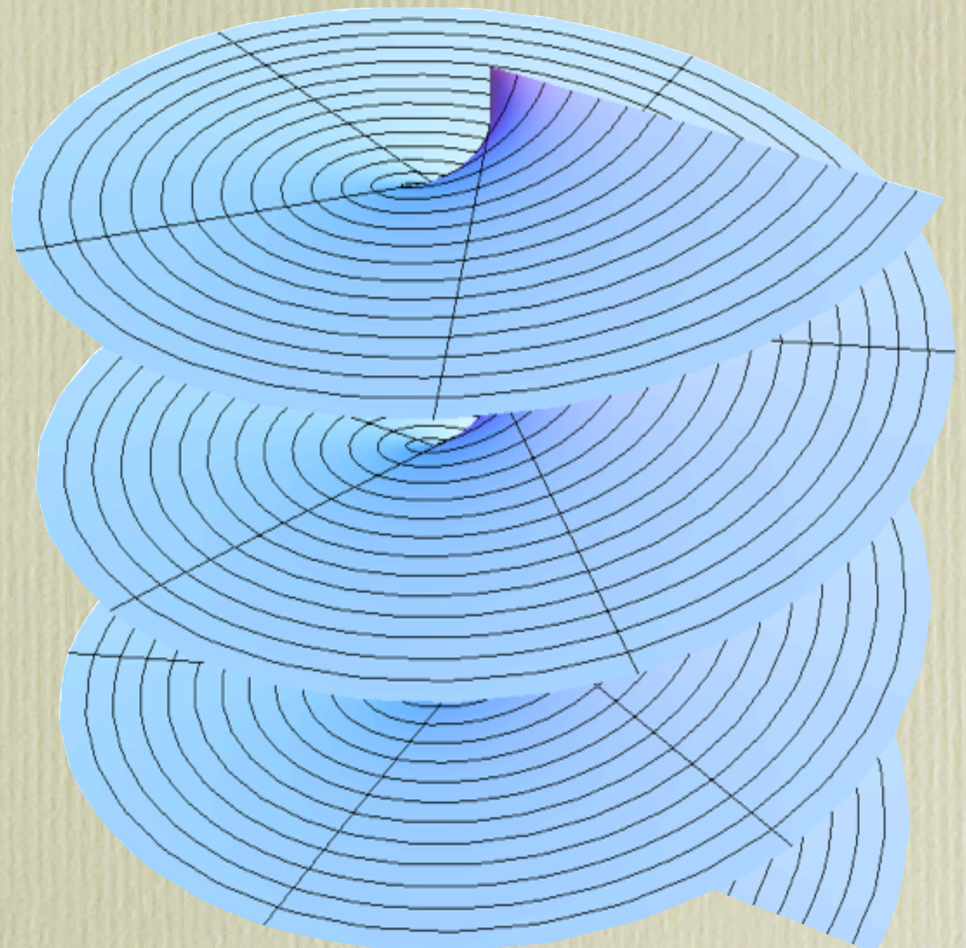
- Ellipsoids
- Cone
- Cylinder
- Paraboloids
- Hyperboloids







Problem: What is this?

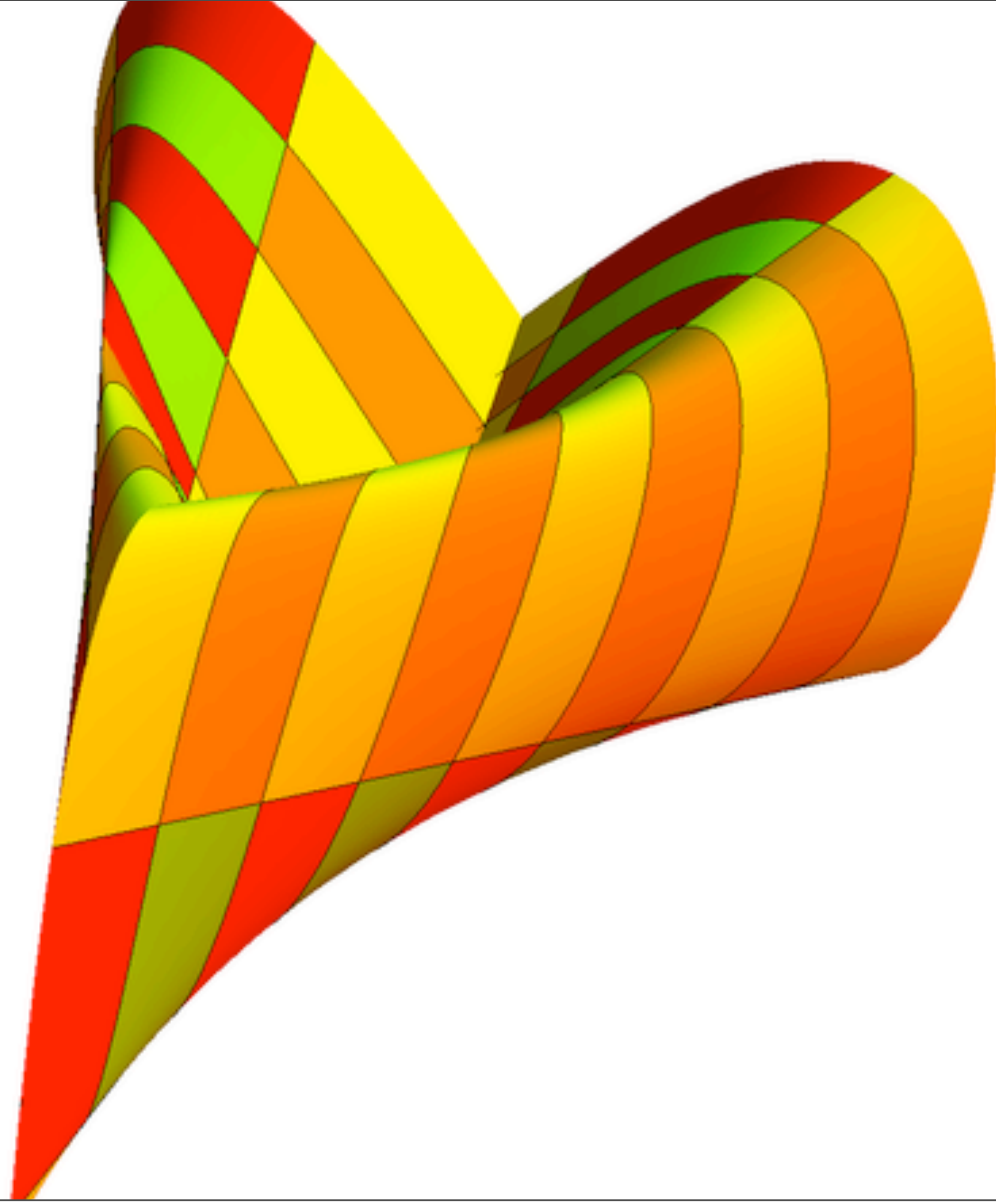


this is a ruled surface!

$$\vec{r}\,(u,v) = \langle v\cos(u),\ v\sin(u),\ u \rangle$$

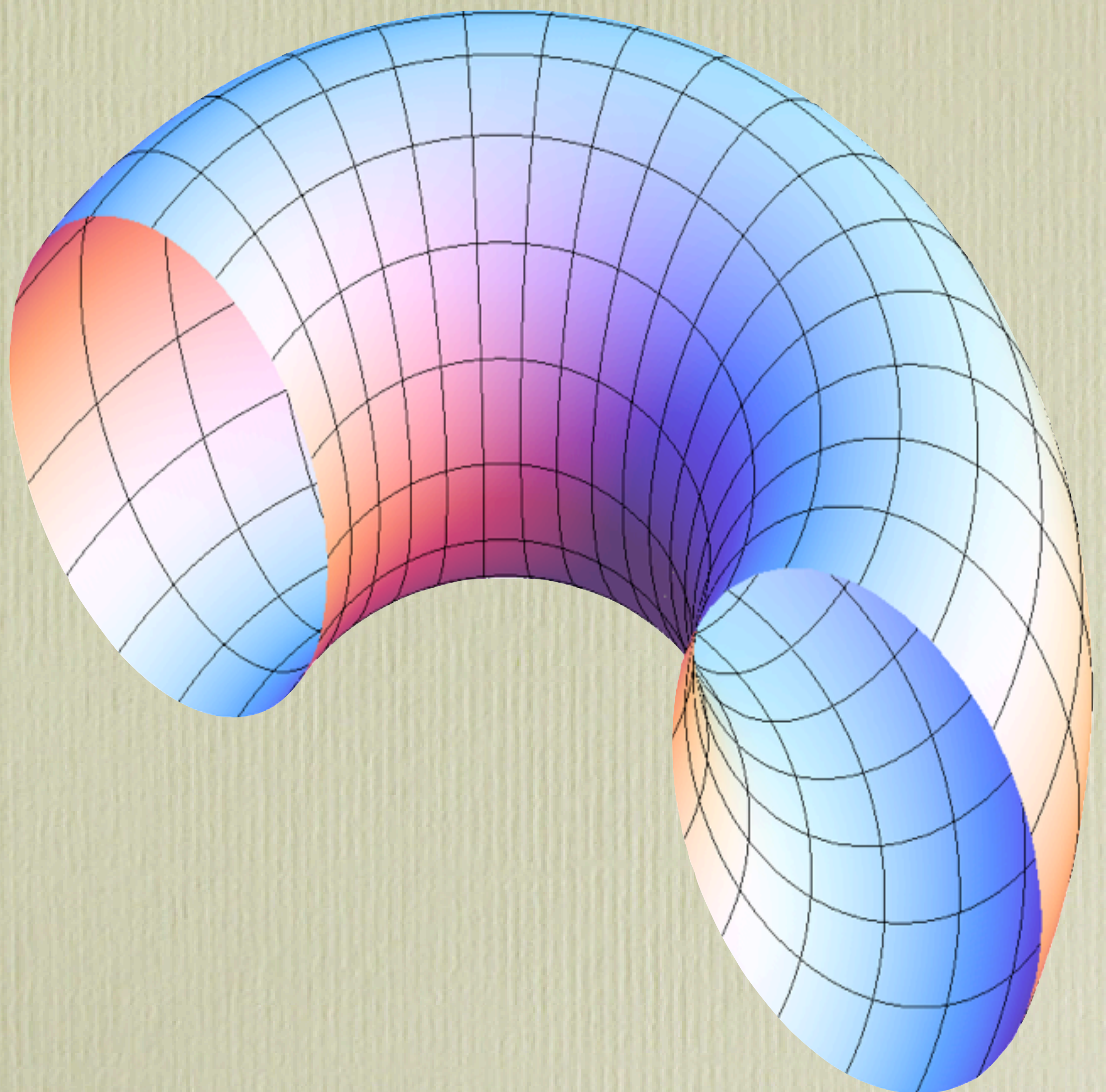
Winning surfaces of Problem A submission:

- Andrew BRODY
- Max SCHULMAN
- Ann MORGAN
- Christopher DEVINE
- George ZUO
- Caroline QUAZZO
- Jakob LINDAAS
- Ashlee ADAMS
- Carl DAHER
- Lauren FELDMAN
- Lisa WETSTONE
- Yasha IRAVANTCHI
- Magdalena KALA
- Emily ZHANG



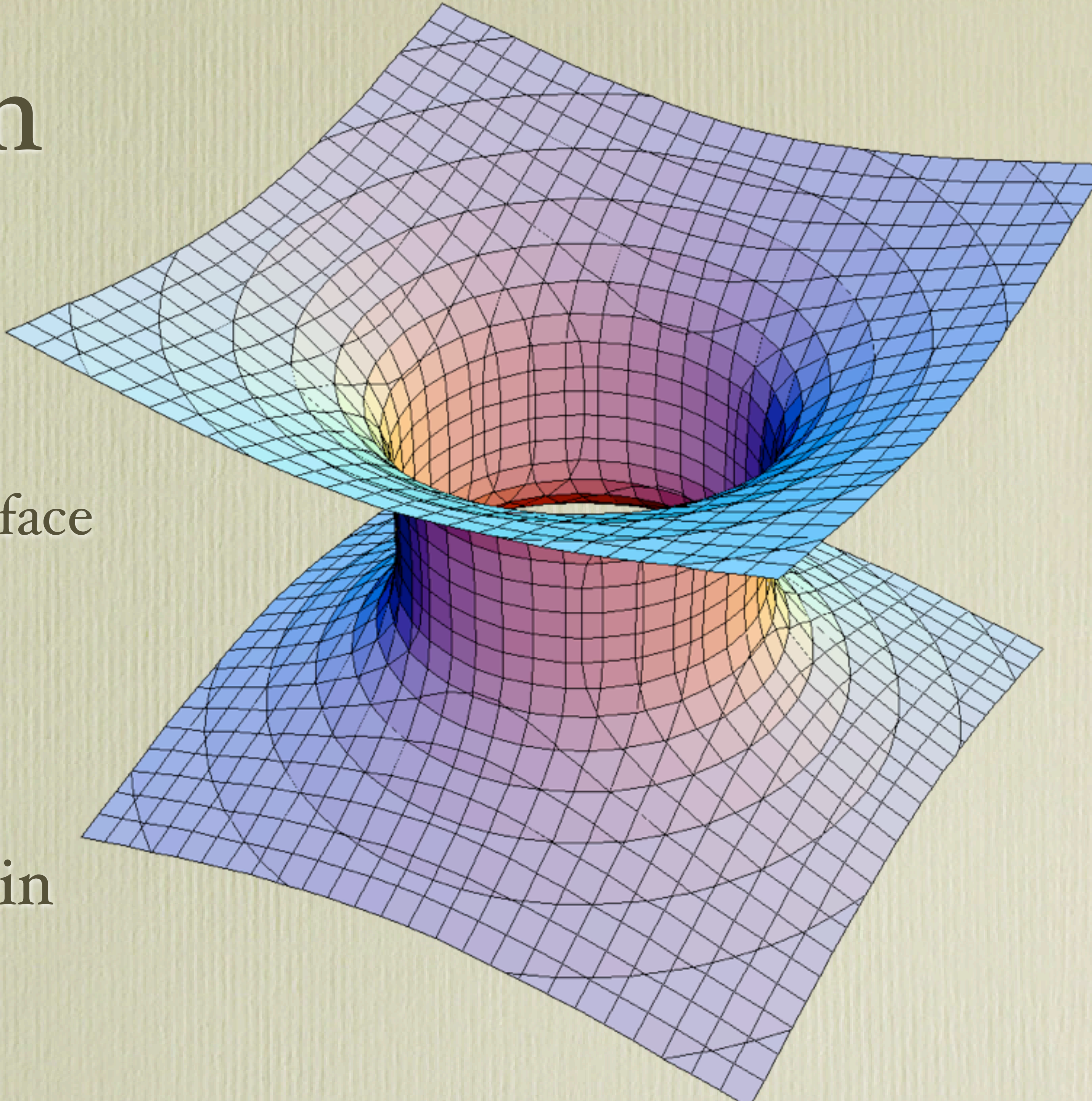

Bumpy sphere!

Example:



$\vec{r}(u,v) = <(2+\cos(v)) \cos(u), (2+\cos(v)) \sin(u), \sin(v)>$

Problem



Parametrize the surface

$$x^2 + y^2 - z^4 = 1$$

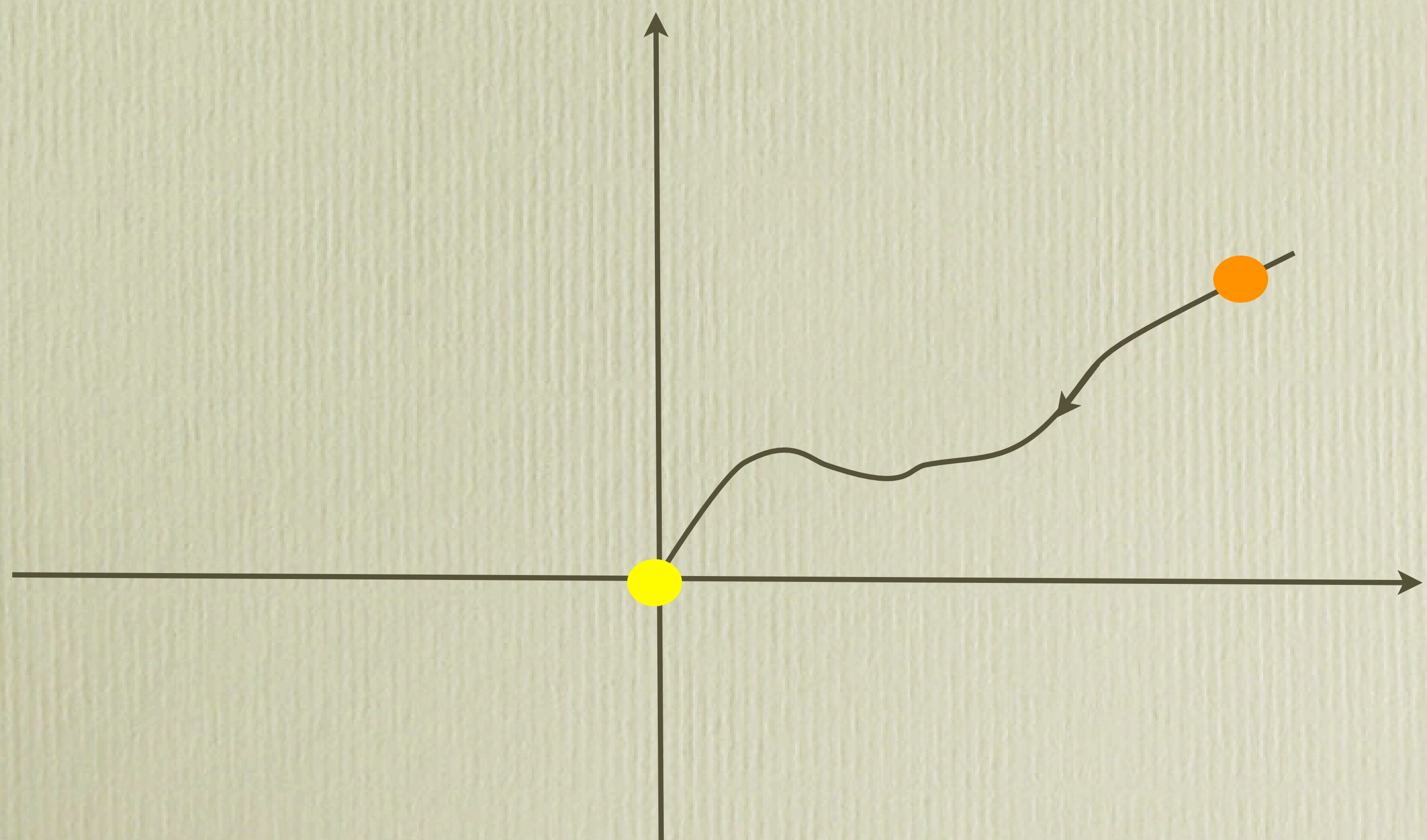
and describe it in cylindrical coordinates.

Continuity and derivatives

Continuity

f(x,y) continuous at (0,0) if

$f(x,y) \rightarrow f(0,0)$ if $(x,y) \rightarrow (0,0)$



Use Polar coordinates!!
Works in almost all cases!

Problem

Is there a value we can assign to f(0,0) so that

$$f(x,y) = \frac{\cos(x^2 + y^2) - 1}{x^2 + y^2}$$

is continuous at (0,0)?

Problem

What is the partial derivative of

$$f(x,y) = \frac{\cos(x^2 + y^2) - 1}{x^2 + y^2}$$

with respect to x? Is it continuous at (0,0)?