10/6/2009 FIRST HOURLY PRACTICE IV **Math 21a, Fall 2009**

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- Start by printing your name in the above box and check your section in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.
- The hourly exam itself will have space for work on each page. This space is excluded here in order to save printing resources.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) | T | **F** | The point $(x, y, z) = (-1, -1, -1)$ is in spherical coordinate described as $(\rho, \theta, \phi) = (\sqrt{3}, 5\pi, 3\pi/4)$

Solution:
Make a picture and look at the angles. The θ angle is false.

2) | T | **F** | If $|\vec{v} \times \vec{w}| = 0$ then $\vec{v} = \vec{0}$ or $\vec{w} = \vec{0}$.

Solution:
No, the vectors can be parallel without beeing zero.

3) | T | **F** | The surface $z^2 + 4y^2 = x^2 + 1$ is a two sheeted hyperboloid.

Solution:
It is a deformed one-sheeted hyperboloid.

4) | **T** | F | The surface $4x^2 - 4x + y^2 - 2y - 120 = -z^2$ is an ellipsoid.

Solution:
Complete the square

5) | **T** | F | The parametrized lines $\vec{u}(t) = \langle 1 + 2t, 2 - 5t, 1 + t\rangle$ and $\vec{v}(t) = \langle 3 - 4t, -3 + 10t, 2 - 2t\rangle$ are the same line.

Solution:
The vectors are parllel, and both lines go through the same point.

6) | **T** | F | The surface $\sin(x) = z$ contains lines which are parallel to the y-axis.

Solution:
One can translate the surface in the y direction.

7) | T | **F** |

If $\vec{u} \cdot \vec{v} = 0$, $\vec{v} \cdot \vec{w} = 0$ and $\vec{v}$ is not the zero vector, then $\vec{u} \cdot \vec{w} = 0$.

Solution:
The assumption means that $\vec{v}$ is perpendicular to $\vec{u}$ and $\vec{w}$. But that does not mean that $\vec{u}$ and $\vec{w}$ are perpendicular.

8) | T | **F** |

The curvature of a curve depends upon the speed at which one travels upon it.

Solution:
The curvature does not depend on the parametrization.

9) | T | **F** |

Two lines in space that do not intersect must be parallel.

Solution:
They can be skew.

10) | T | **F** |

A line in space can intersect an elliptic paraboloid in 4 points.

Solution:
It can only intersect it in 2 points or 1 point or avoid it at all.

11) | **T** | F |

If $\vec{u} \times \vec{v} = 0$ and $\vec{u} \cdot \vec{v} = 0$, then one of the vectors $\vec{u}$ and $\vec{v}$ is zero.

Solution:
A vector which is both parallel and perpendicular to an other vector can only be the zero vector.

12) T F

If the velocity vector $\vec{r}\,'(t)$ and the acceleration vector $\vec{r}\,''(t)$ of a curve are parallel at time $t = 1$, then the curvature $\kappa(t)$ of the curve is zero at time $t = 1$.

Solution:
You can see this from the formula $\kappa = |r'(t) \times r''(t)|/|r'(t)|^3$. You can also think about it as follows. Assume the curvature were $\kappa = 1/r$. Then you as well locally move on a circle with radius r. But the acceleration has now a component perpendicular to your velocity vector. But we assumed there is no such acceleration.

13) T F

If the speed of a parametrized curve is constant over time, then the curvature of the curve $\vec{r}(t)$ is zero.

Solution:
It would be true if the velocity would be constant over time. But we can move on a circle with constant speed.

14) T F

The length of the vector projection of a vector $\vec{v}$ onto a vector $\vec{w}$ is always equal to the length of the vector projection of $\vec{w}$ onto $\vec{v}$.

Solution:
If the lengths of $\vec{v}$ and $\vec{w}$ are the same, then the statement is true. In general, it is not.

15) T F

A quadric $ax^2 + by^2 + cz^2 = 1$ is contained in the interior of a sphere $x^2 + y^2 + z^2 < 100$, then the constants a, b, c are all positive and the quadric is an ellipsoid.

Solution:
If any of the constants would become negative, the quadric becomes unbounded.

16) T F

There is a hyperboloid of the form $ax^2 + by^2 - cz^2 = 1$ which has a trace which is a parabola.

Solution:
Traces are either hyperbola or ellipses.

17) T F

The set of points in space which have distance 1 from the line $x = y = z$ form a cylinder.

Solution:
Yes, if the the line is the z-axis, then $x^2 + y^2 = 1$ is the equation of the cylinder.

18) | T | **F** | The velocity vector of a parametric curve $\vec{r}(t)$ always has constant length.

Solution:
This is only true for an arc length parametrization.

19) | T | **F** | The volume of a parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$ is $|(\vec{u} \times \vec{v}) \times \vec{w}|$.

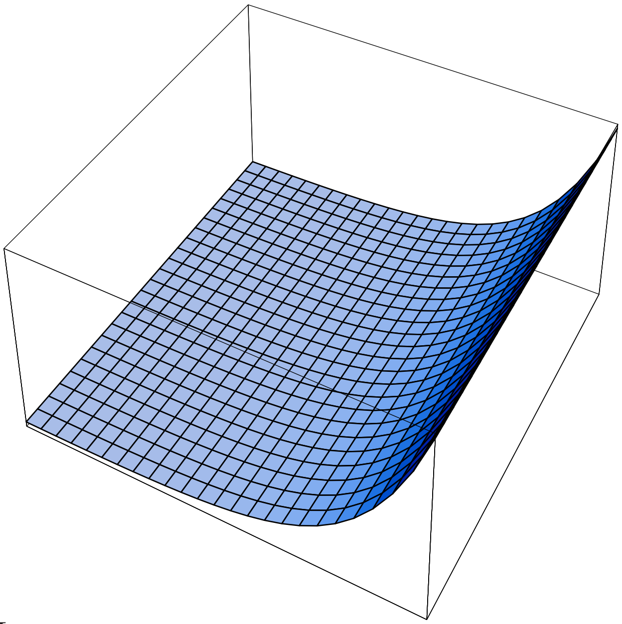
Solution:
The triple scalar product contains also a dot product.

20) | T | **F** | The equation $x^2 + y^2/4 = 1$ in space describes an ellipsoid.

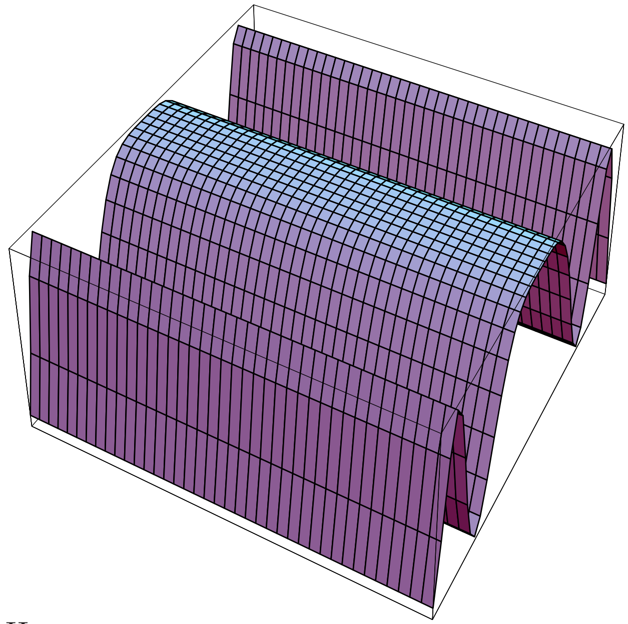
Solution:
The equation describes an elliptical cylinder.

Problem 2a) (3 points)

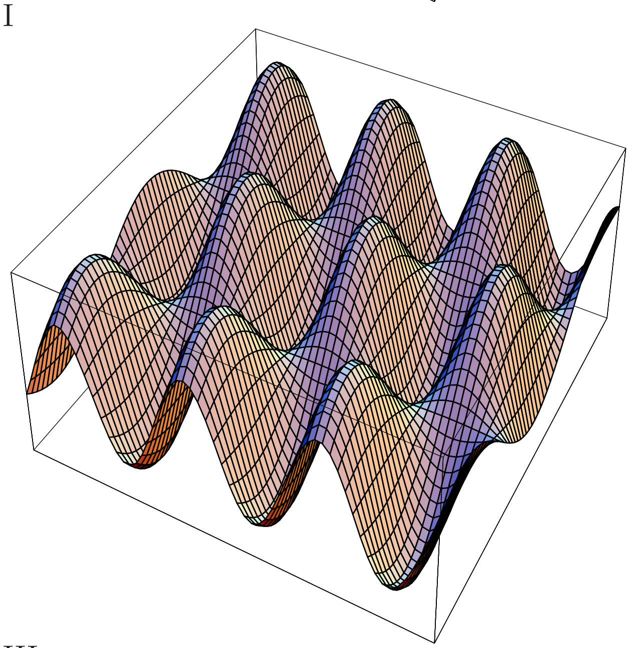
Match the equation with their graphs. No justifications are needed.



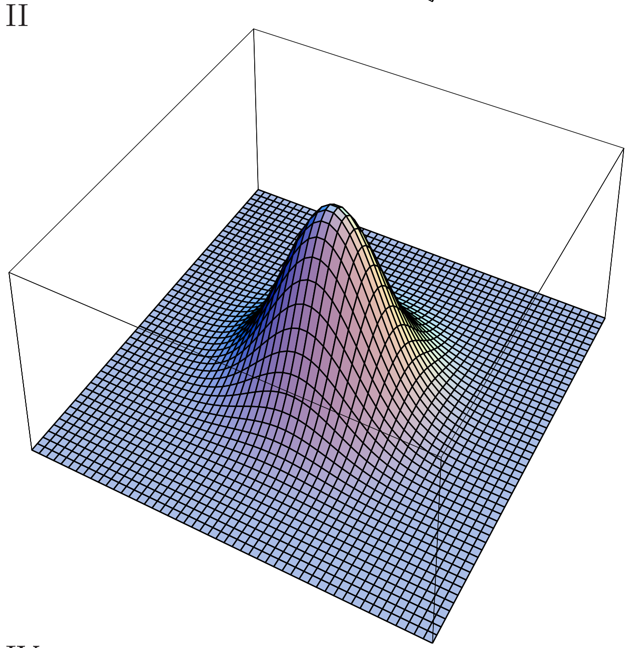
I



II



III



IV

Enter I,II,III,IV here	Equation
	$z = \sin(5x)\cos(2y)$
	$z = \cos(y^2)$
	$z = e^{-x^2-y^2}$
	$z = e^x$

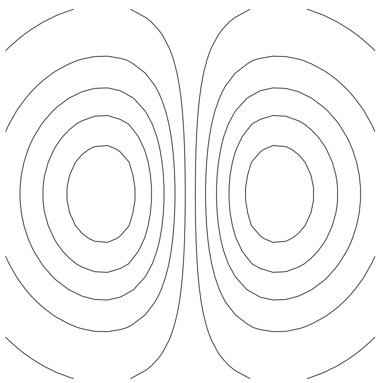
Solution:

Enter I,II,III,IV here	Equation	Justification
III	$z = \sin(5x)\cos(2y)$	two traces show waves
II	$z = \cos(y^2)$	no x dependence, periodic in y
IV	$z = e^{-x^2-y^2}$	has a maximum at (0,0)
I	$z = e^x$	no y dependence, monotone in x

Problem 2b) (4 points)

Match the contour maps with the corresponding functions $f(x, y)$ of two variables. No justifications are needed.

I



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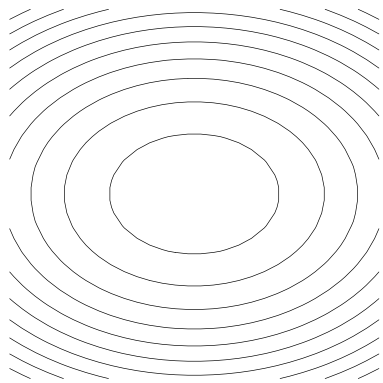

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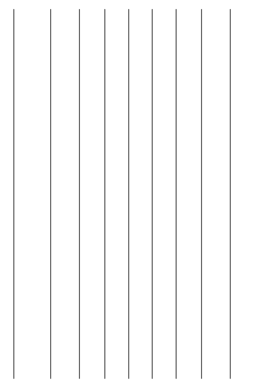

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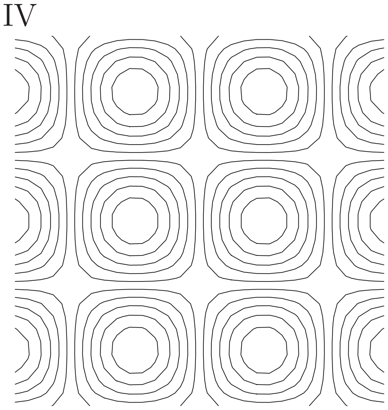
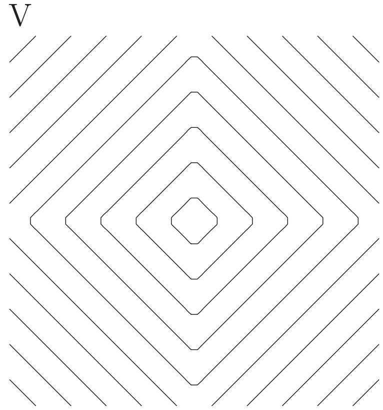
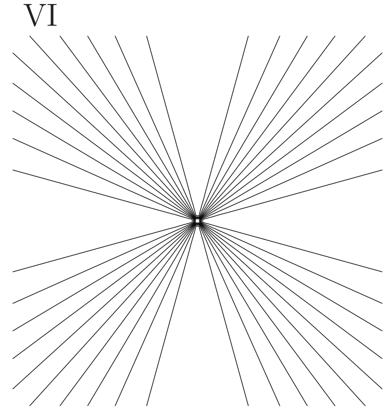


V



VI



Enter I,II,III,IV,V or VI here	Function $f(x, y)$
	$f(x, y) = \sin(x)$
	$f(x, y) = x^2 + 2y^2$
	$f(x, y) = \lvert x\rvert + \lvert y\rvert$
	$f(x, y) = \sin(x)\cos(y)$
	$f(x, y) = xe^{-x^2-y^2}$
	$f(x, y) = x^2/(x^2 + y^2)$

Solution:

Enter I,II,III,IV,V or VI here	Function $f(x,y)$
III	$f(x,y) = \sin(x)$
II	$f(x,y) = x^2 + 2y^2$
V	$f(x,y) = \|x\| + \|y\|$
I	$f(x,y) = xe^{-x^2-y^2}$
VI	$f(x,y) = x^2/(x^2+y^2)$

Problem 2c) (3 points)

Match the following points in cartesian coordinates with the points in spherical coordinates:

a) $(x, y, z) = (\sqrt{2}, 0, 0)$
b) $(x, y, z) = (0, \sqrt{2}, 0)$
c) $(x, y, z) = (0, 0, \sqrt{2})$
d) $(x, y, z) = (1, 1, 0)$
e) $(x, y, z) = (1, 0, 1)$
f) $(x, y, z) = (0, 1, 1)$

1) $(\rho, \phi, \theta) = (\sqrt{2}, 0, 0)$.
2) $(\rho, \phi, \theta) = (\sqrt{2}, \pi/2, \pi/4)$.
3) $(\rho, \phi, \theta) = (\sqrt{2}, \pi/2, 0)$.
4) $(\rho, \phi, \theta) = (\sqrt{2}, \pi/2, \pi/2)$.
5) $(\rho, \phi, \theta) = (\sqrt{2}, \pi/4, \pi/2)$.
6) $(\rho, \phi, \theta) = (\sqrt{2}, \pi/4, 0)$.

Solution:
a = 3, b = 4, c = 1, d = 2, e = 6, f = 5

Problem 3) (10 points)

a) (7 points) Find a parametric equation for the line which is the intersection of the two planes $2x - y + 3z = 9$ and $x + 2y + 3z = -7$.

b) (3 points) Find a plane perpendicular to both planes given in a) which has the additional property that it passes through the point $P = (1, 1, 1)$.

Solution:
a) We get the direction of the line by taking the cross product of $\langle 2, -1, 3 \rangle$ and $\langle 1, 2, 3 \rangle$ which is $\langle -9, -3, 5 \rangle$. To find a point in both lines, subtract one from the other to get $x - 3y = 16$. If $z = 0$, then $2x - y = 9$ and $x + 2y = -7$ so that $x = 11/5, y = -23/5$. The parametric equations are $(x, y, z) = (11/5, -23/5, 0) + t(-9, -3, 5)$.
b) Plug in the coordinates $(x, y, z) = (1, 1, 1)$ of the point to get the constant $-9x - 3y + 5z = -7$.

Problem 4) (10 points)

Given the vectors $\vec{v} = \langle 1, 1, 0 \rangle$ and $\vec{w} = \langle 0, 0, 1 \rangle$ and the point $P = (2, 4, -2)$. Let Σ be the plane which goes through the origin $(0, 0, 0)$ and which contains the vectors $\vec{v}$ and $\vec{w}$. Let S be the unit sphere $x^2 + y^2 + z^2 = 1$.

a) (6 points) Compute the distance from P to the plane Σ.

b) (4 points) Find the shortest distance from P to the sphere S.

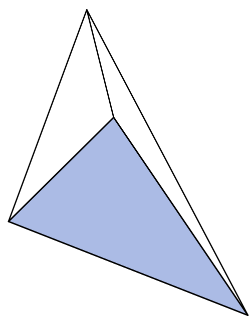
Solution:
a) $\Sigma : x - y = 0$, $n = (1, -1, 0)$. The point $Q = (0, 0, 0)$ is on the plane. $\vec{PQ} \cdot \vec{n}/|\vec{n}| = \langle 2, 4, -2 \rangle \cdot \langle 1, -1, 0 \rangle / \sqrt{2} = 2/\sqrt{2} = \sqrt{2}$ is the distance.
b) $\sqrt{4 + 16 + 4} = \sqrt{24} = 2\sqrt{6}$ is the distance to the origin. So the distance to the sphere is 1 less. The answer is $\sqrt{24} - 1$.

Problem 5) (10 points)

a) (6 points) Find an equation for the plane through the points $A = (0, 1, 0), B = (1, 2, 1)$ and $C = (2, 4, 5)$.

b) (4 points) Given an additional point $P = (-1, 2, 3)$, what is the volume of the tetrahedron which has A, B, C, P among its vertices.

A useful fact which you can use without justification in b): the volume of the tetrahedron is 1/6 of the volume of the parallelepiped which has AB, AC, and AP among its edges.



Solution:
a) The vectors $\vec{v} = \vec{AB} = \langle 1,1,1\rangle$ and $\vec{w} = \vec{AC} = \langle 2,3,5\rangle$ are in the plane. Their cross product is $\vec{n} = \langle 2,-3,1\rangle$. This vector is perpendicular to the plane. The equation of the plane is therefore $2x - 3y + z = d$. Plugging in one point like A, gives $d = -3$.
b) With the vector $\vec{u} = \vec{AP} = \langle -1,1,3\rangle$, one can express the volume of the parallelepiped as $|[\vec{u}, \vec{v}, \vec{w}]| = |\vec{u} \cdot \vec{n}| = |\langle -1,1,3\rangle \cdot \langle 2,-3,1\rangle| = |2| = 2$. The volume of the tetrahedron is $2/6 = 1/3$.

Problem 6) (10 points)

The parametrized curve $\vec{u}(t) =< t, t^2, t^3 >$ (known as the "twisted cubic") intersects the parametrized line $\vec{v}(s) = \langle 1+3s, 1-s, 1+2s\rangle$ at a point P. Find the angle of intersection.

Solution:
The curves intersect at $P = (1,1,1)$ with $t = 1$, $s = 0$ So, it remains to find the angle between the velocity vectors $\vec{u}\,'(= \langle 1,2,3\rangle$ and $\vec{v}\,'(0) = \langle 3,-1,2\rangle$, which is 60 degrees.

Problem 7) (10 points)

Let $\vec{r}(t)$ be the space curve $\vec{r}(t) = (\log(t), 2t, t^2)$, where $\log(t)$ is the natural logarithm (denoted by $\ln(t)$ in some textbooks).

a) What is the velocity and what is the acceleration at time $t = 1$?

b) Find the length of the curve from $t = 1$ to $t = 2$.

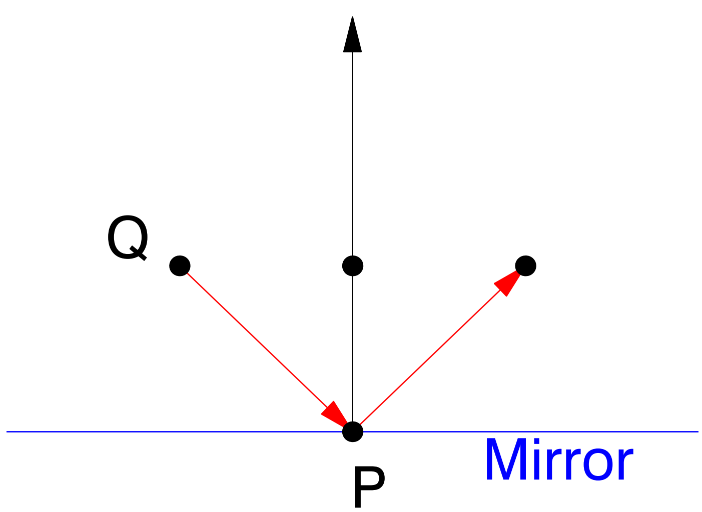
Solution:
a) $\vec{v}(t) = \vec{r}\,'(t) = \langle 1/t, 2, 2t \rangle$.
$\vec{v}(1) = \langle 1, 2, 2 \rangle$
$\vec{a}(t) = \vec{r}\,''(t) = \langle -1/t^2, 2, 2 \rangle$.
$\vec{a}(1) = \langle -1, 0, 2 \rangle$.
b) $\int_1^2 \sqrt{1/t^2 + 4 + 4t^2}\; dt = \int_1^2 1/t + 2t\; dt = \log(t) + t^2|_1^2 = \log(2) + 3$.

Problem 8) (10 points)

A planar mirror in space contains the point $P = (4, 1, 5)$ and is perpendicular to the vector $\vec{n} = \langle 1, 2, -3 \rangle$. The light ray $\vec{QP} = \vec{v} = \langle -3, 1, -2 \rangle$ with source $Q = (7, 0, 7)$ hits the mirror plane at the point P.

a) (4 points) Compute the projection $\vec{u} = \vec{\mathrm{P}}_{\vec{n}}(\vec{v})$ of $\vec{v}$ onto $\vec{n}$.

b) (6 points) Identify $\vec{u}$ in the figure and use it to find a vector parallel to the reflected ray.



Solution:
a) $P_{\vec{n}}(\vec{v}) = \frac{(\vec{n}\cdot\vec{v})}{|\vec{n}|^2}\vec{n} = (\langle 1,2,-3\rangle \cdot \langle -3,1,-2\rangle)/14\vec{n} = (5/14)\langle 1,2,-3\rangle$.
b) With $\vec{u}$ we can get the reflected vector $\vec{w}$ because $\vec{w}-\vec{v} = -2\vec{u}$ so that $\vec{w} = \vec{v} - 2\vec{u}$. Note that $\vec{u}$ points down towards the mirror.

Problem 9) (10 points)

We know the acceleration $\vec{r}\,''(t) = \langle 2,1,3\rangle + t\langle 1,-1,1\rangle$ and the initial position $\vec{r}(0) = \langle 0,0,0\rangle$ and initial velocity $\vec{r}\,'(0) = \langle 11,7,0\rangle$ of an unknown curve $\vec{r}(t)$. Find $\vec{r}(6)$.

Solution:

$$\vec{r}\,'(t) = \int_0^t \langle 2+t, 1-t, 3+t\rangle\ dt + \vec{r}\,'(0) = \langle 11+2t+t^2/2, 7+t-t^2/2, 3t+t^2/2\rangle$$

$$\vec{r}\,(t) = \int_0^t \langle 2t+t^2/2, t-t^2/2, 3t+t^2/2\rangle\ dt + \vec{r}(0) = \langle 11t+t^2+t^3/6, 7t+t^2/2-t^3/6, 3t^2/2+t^3/6\rangle$$

Plug in the time $t=6$ gives $= \boxed{\langle 138, 24, 90\rangle}$.

Problem 10) (10 points)

Intersecting the elliptic cylinder $x^2 + y^2/4 = 1$ with the plane $z = \sqrt{3}x$ gives a curve in space.

a) (3 points) Find the parametrization of the curve.

b) (3 points) Compute the unit tangent vector $\vec{T}$ to the curve at the point $(0,2,0)$.

c) (4 points) Write down the arc length integral and evaluate the arc length of the curve.

Solution:
a) With $x = \sin(t), y = 2\cos(t), z = \sqrt{3}\sin(t)$, we check $x^2 + y^2/4 = \sin^2(t) + \cos^2(t) = 1$. The parametrization is $\vec{r}(t) = \langle \sin(t), 2\cos(t), \sqrt{3}\sin(t)\rangle$.
b) Compute $\vec{r}\ '(t) = \cos(t), -2\sin(t), \sqrt{3}\cos(t)\rangle$, the speed $|\vec{r}\ '(t)| = 2$ and $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,')(t)| = \langle \cos(t)/2, -\sin(t), \sqrt{3}\cos(t)/2\rangle$.
c) $|\vec{r}(t)| = 2$. The length is $\int_0^{2\pi} |\vec{r}\,'(t)|\ dt = \int_0^{2\pi} 2\ dt = \boxed{4\pi}$.