Name:

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| MWF 10 Oliver Knill |
| MWF 10 HT Yau |
| MWF 11 Ana Caraiani |
| MWF 11 Chris Phillips |
| MWF 11 Ethan Street |
| MWF 12 Toby Gee |
| MWF 12 Xinwen Zhu |
| TTH 10 Jack Huizenga |
| TTH 10 Fred van der Wyck |
| TTH 11:30 Ming-Tao Chuan |
| TTH 11:30 Fred van der Wyck |

- Start by printing your name in the above box and **check your section** in the box to the left.

- Do not detach pages from this exam packet or unstaple the packet.

- Please write neatly. Answers which are illegible for the grader cannot be given credit.

- **Show your work**. Except for problems 1-2, and 8, we need to see the details of your computation.

- No notes, books, calculators, computers, or other electronic aids can be allowed.

- You have 90 minutes time to complete your work.

| 1 | | 20 |
| --- | --- | --- |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| Total: | | 100 |

Problem 1) True/False questions (20 points), no justifications needed

1) T F — If $f(x,y) = 1$ is a curve, and near $(2,3)$ one can write $y$ as a function of $x$, then $y' = -f_y(2,3)/f_x(2,3)$.

2) T F — If $\iint_R f(x,y)\,dA = 0$, then the function $f(x,y)$ is everywhere zero on $R = \{x^2 + y^2 \leq 1\}$.

3) T F — The directional derivative in the direction of the gradient is $|\nabla f|$.

4) T F — The linearization of $f(x,y) = x^3 + y^3$ at $(1,1)$ is the quadratic function $L(x,y) = 3x^2 + 3y^2$.

5) T F — The function $f(x,y) = x^2 + y^2$ satisfies the partial differential equation $D = f_{xx}f_{yy} - f_{xy}^2 = 1$.

6) T F — The function $x^2y^2$ has no local minimum at $(0,0)$ because the discriminant function $D$ is zero there.

7) T F — The double integral $\int_0^{\pi/4} \int_0^2 r^3\,drd\theta$ is the volume of the part of a solid cylinder $x^2 + y^2 \leq 4$ which is below the paraboloid $z = x^2 + y^2$ and above the $xy$ plane.

8) T F — The gradient of $f(x,y,z)$ at $(x_0, y_0, z_0)$ is perpendicular to the level surface of $f$ through $(x_0, y_0, z_0)$.

9) T F — If $f(x,y,z) = 3x - 4z$, then the minimal possible directional derivative $D_{\vec{u}}f$ at any point in space is $-5$.

10) T F — If $(x,y)$ is not a critical point, then the directional derivative $D_{\vec{v}}f$ can take both positive and negative values for different choices of $\vec{v}$.

11) T F — Using linearization of $f(x,y) = x/y$ we can estimate $1.01/1.001 = f(1.01, 1.001) \sim 1 + 0.01 - 0.001 = 1.009$.

12) T F — If $(0,0)$ is a critical point of $f(x,y)$ with nonzero discriminant $D = f_{xx}f_{yy} - f_{xy}^2$, we know that it is either a saddle, a global maximum or a global minimum.

13) T F — For a rectangular region $R$, Fubini tells that $\int_0^2 \int_0^3 f(x,y)\,dxdy = \int_0^2 \int_0^3 f(x,y)\,dydx$ for any continuous function $f(x,y)$.

14) T F — If a function $f(x,y)$ has only one critical point $(0,0)$ in $G = \{x^2 + y^2 \leq 1\}$ which is a local maximum and $f(0,0) = 1$, then $\iint_G f(x,y)\,dxdy > 0$.

15) T F — If $\vec{r}(t)$ is a curve in space for which the speed is 1 at all times and $f(x,y,z)$ is a function of three variables, then $d/dt f(\vec{r}(t)) = D_{\vec{r}\,'(t)}(f)$.

16) T F — $\int_0^1 \int_0^1 f_{xy}(x,y)\,dydx = f(1,1) - f(1,0) - f(0,1) + f(0,0)$.

17) T F — If $f_{yy}(x,y) > 0$ everywhere, then $f$ can not have any local maximum.

18) T F — The double integral $\int_0^1 \int_0^1 x^2 - y^2\,dxdy$ is the volume of the solid below the graph of $f(x,y) = x^2 - y^2$ and above the square $0 \leq x \leq 1, 0 \leq y \leq 1$ in the $xy$-plane.

19) T F — For any unit vector $\vec{v}$ and any differentiable function $f$, one has $D_{\vec{v}}(f) + D_{-\vec{v}}(f) = 0$.

20) T F — The surfaces $x + y + z = 0$ and $x^2 + y^2 + z^2 + x + y + z = 0$ have the same tangent plane at $(0,0,0)$.

Problem 2) (10 points)

a) (6 points) Match the regions with the corresponding polar double integrals



A

B

C

D

E

F
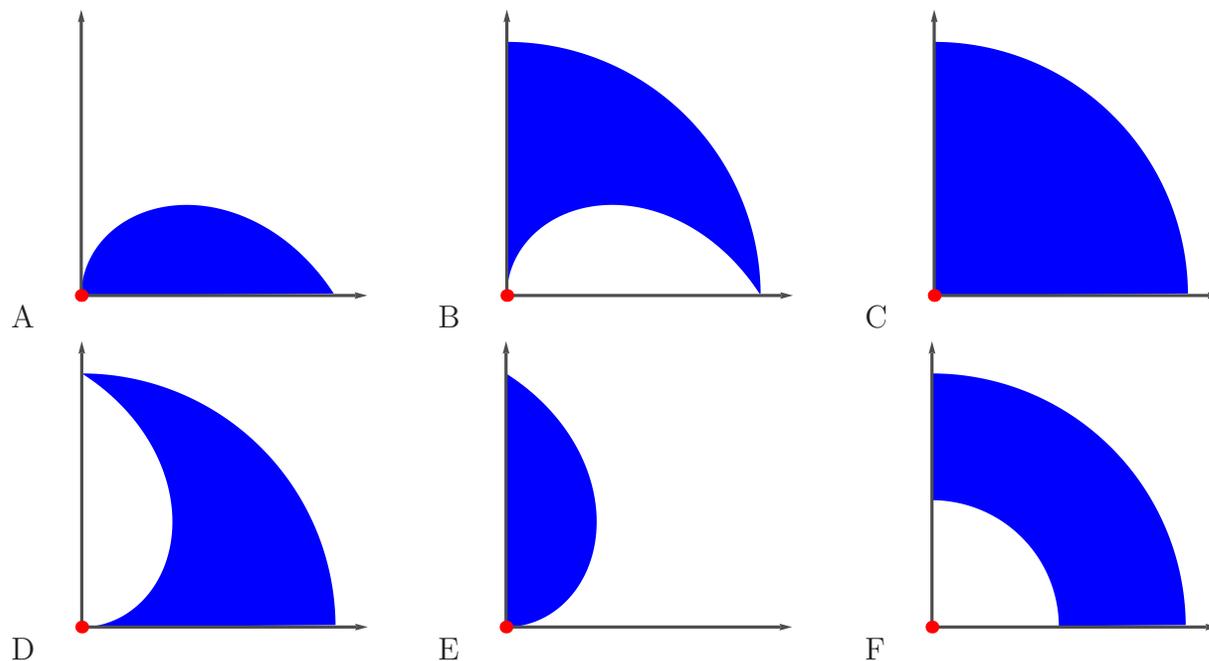
| Enter A-F | Integral of $f(r,\theta)$ | Enter A-F | Integral of $f(r,\theta)$ |
|---|---|---|---|
| | $\int_0^{\pi/2} \int_0^{\pi/2} f(r,\theta)r\ drd\theta$ | | $\int_0^{\pi/2} \int_\theta^{\pi/2} f(r,\theta)r\ drd\theta$ |
| | $\int_0^{\pi/2} \int_0^{\theta} f(r,\theta)r\ drd\theta$ | | $\int_0^{\pi/2} \int_{\pi/2-\theta}^{\pi/2} f(r,\theta)r\ drd\theta$ |
| | $\int_0^{\pi/2} \int_0^{\pi/2-\theta} f(r,\theta)r\ drd\theta$ | | $\int_0^{\pi/2} \int_{\pi/4}^{\pi/2} f(r,\theta)r\ drd\theta$ |

b) (4 points) Match the partial differential equations (PDE's) for the functions $u(t,s)$ with their names. No justifications are needed.

| Enter A,B,C,D here | PDE | Enter A,B,C,D here | PDE |
|---|---|---|---|
| | $u_t + uu_s - u_{ss} = 0$ | | $u_{tt} - u_{ss} = 0$ |
| | $u_{tt} + u_{ss} = 0$ | | $u_t - u_{ss} = 0$ |

| A) Wave equation | B) Heat equation | C) Burgers equation | D) Laplace equation |
|---|---|---|---|

3

Problem 3) (10 points)

a) (7 points) Find and classify all the critical points of the function

$$f(x, y) = 5 + 3x^2 + 3y^2 + y^3 + x^3 .$$

b) (3 points) Is there a global maximum or a global minimum for $f(x, y)$?
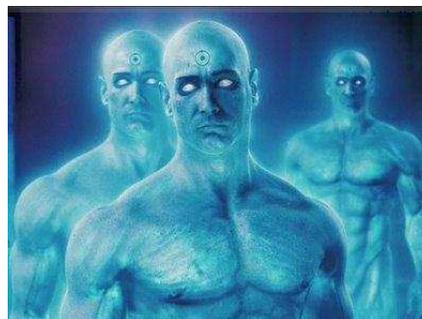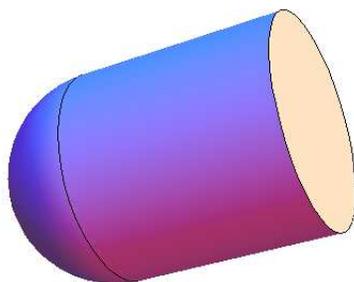
Problem 4) (10 points)

A **solid bullet** made of a half sphere and a cylinder has the volume $V = 2\pi r^3/3 + \pi r^2 h$ and surface area $A = 2\pi r^2 + 2\pi r h + \pi r^2$. Doctor Manhatten designs a bullet with fixed volume and minimal area. With $g = 3V/\pi = 1$ and $f = A/\pi$ he therefore minimizes

$$f(h, r) = 3r^2 + 2rh$$

under the constraint

$$g(h, r) = 2r^3 + 3r^2 h = 1 .$$

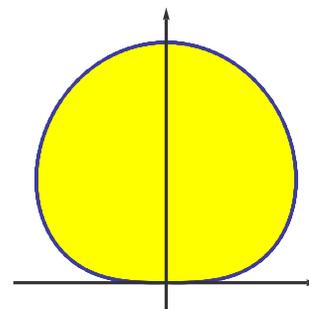Use the Lagrange method to find a local minimum of $f$ under the constraint $g = 1$.

Problem 5) (10 points)

A region $R$ in the plane shown to the right is called the **"blob of nothingness"**. It does not have any purpose nor meaning. It just sits there. The region is given in polar coordinates as $0 \le r \le \theta(\pi - \theta)$ for $0 \le \theta \le \pi$. Find the area

$$\int \int_R 1 \, dxdy$$

of this nihilistic object.

Problem 6) (10 points)

a) (4 points) If
$$f(x, y) = y \cos(x - y) ,$$
find equation of plane tangent to $z = f(x, y)$ at the point $(2, 2, 2)$.

b) (3 points) Find the equation of the tangent line to $f(x, y) = 2$ at $(2, 2)$.

c) (3 points) Estimate $f(2.1, 1.9)$ using linear approximation.

Problem 7) (10 points)

A **Harvard robot bee** flies along the curve
$$\vec{r}(t) = \langle t - t^3, 3t^2 - 3t \rangle$$
and measures the temperature $f(x, y)$. It flies over the target point $(0, 0)$ at time $t = 0$ and time $t = 1$. At each time, its sensor measures the temperature change $g'(t)$ where $g(t) = f(\vec{r}(t))$.

a) (5 points) Assume you knew that the gradient of $f$ at $(0, 0)$ is $\langle a, b \rangle$. What are the values of $g'(t) = d/dt f(\vec{r}(t))$ at $t = 0$ and $t = 1$ in terms of $a$ and $b$?

b) (5 points) The bee measures $g'(0) = 3$ and $g'(1) = 3$. What is the gradient $\nabla f(0, 0) = \langle a, b \rangle$ of $f$ at $(0, 0)$?

Image source: Harvard Press release on
robobees.seas.harvard.edu

Problem 8) (10 points)

A function $f(x, y)$ of two variables has level curves as shown in the picture. The function values at neighboring level curves differ by 1. [No justifications are needed in this problem. Naturally, since there are less points then boxes, some of the points A-G will appear more than once, but each box will only be filled with one letter.]

| Enter A-G | is a point, where ... |
|---|---|
| | $f_x(x, y) = 0$ and $f_y(x, y) \neq 0$. |
| | $f_y(x, y) = 0$ and $f_x(x, y) \neq 0$. |
| | $f(x, y)$ has either a max or a min. |
| | $f(x, y)$ has a saddle point. |
| | $f(x, y)$ has no max nor min but is extremal under a constraint $y = c$ for some $c$. |
| | $f(x, y)$ has no max nor min but is extremal under a constraint $x = c$ for some $c$. |
| | the length of the gradient vector of $f$ is largest among all points A-G. |
| | $D_{\langle 1/\sqrt{2}, 1/\sqrt{2} \rangle} f(x, y) = 0$ and $D_{\langle 1/\sqrt{2}, -1/\sqrt{2} \rangle} f(x, y) \neq 0$. |
| | $D_{\langle 1/\sqrt{2}, -1/\sqrt{2} \rangle} f(x, y) = 0$ and $D_{\langle 1/\sqrt{2}, 1/\sqrt{2} \rangle} f(x, y) \neq 0$. |
| | the tangent line to the curve is $x + y = d$ for some constant $d$. |
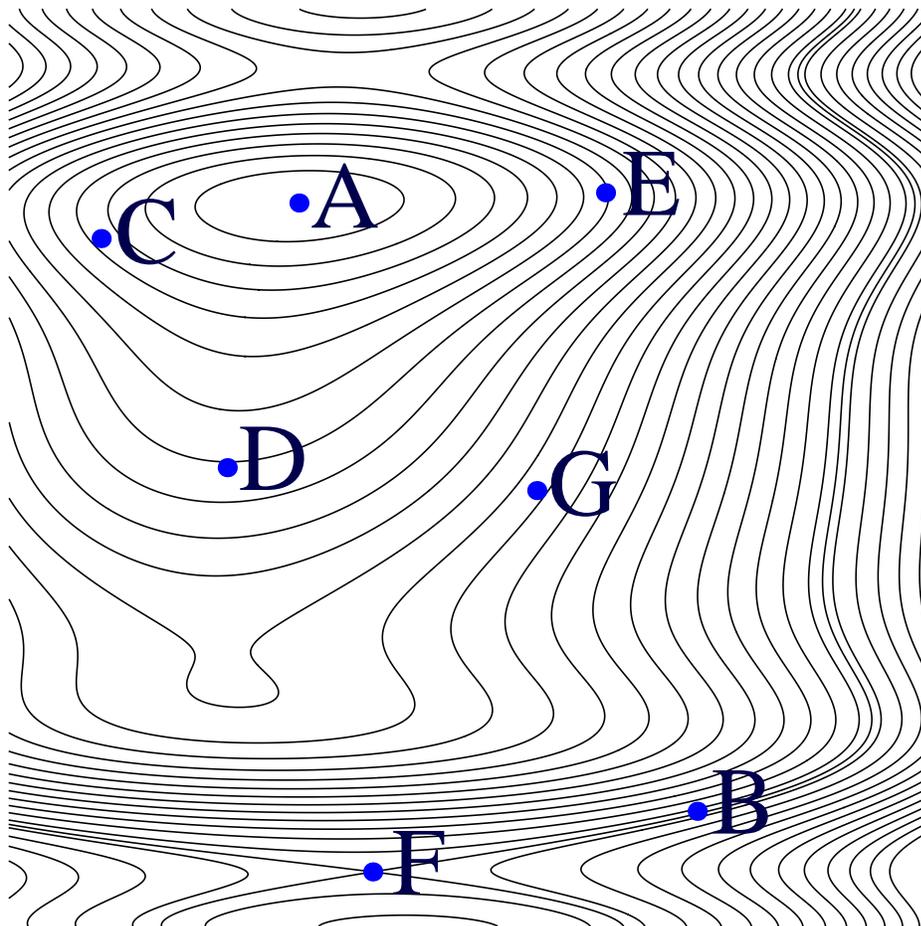


Problem 9) (10 points)

Evaluate the following double integral

$$\int_0^1 \int_0^{(1-x)^2} \frac{x^3}{(1 - \sqrt{y})^4} \, dy \, dx \ .$$