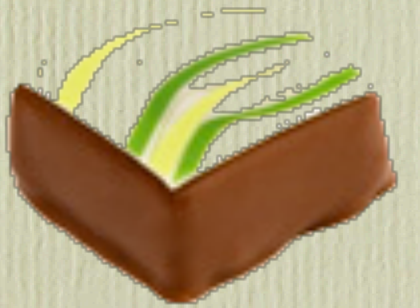
Math21a Review

Oliver Knill, November 1, 2009

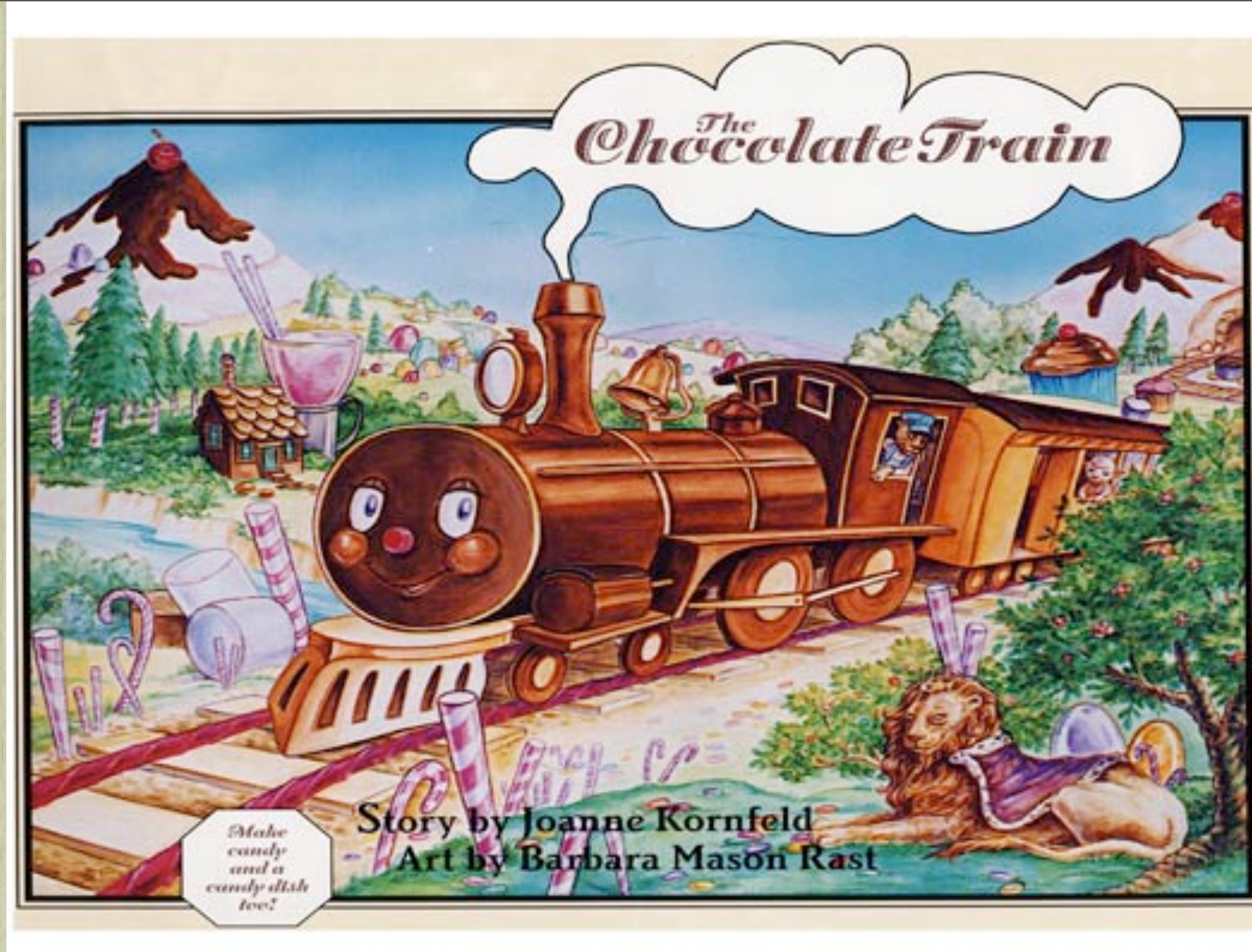
Multivariable Calculus Film

THX

Partial differential equations



5 equations



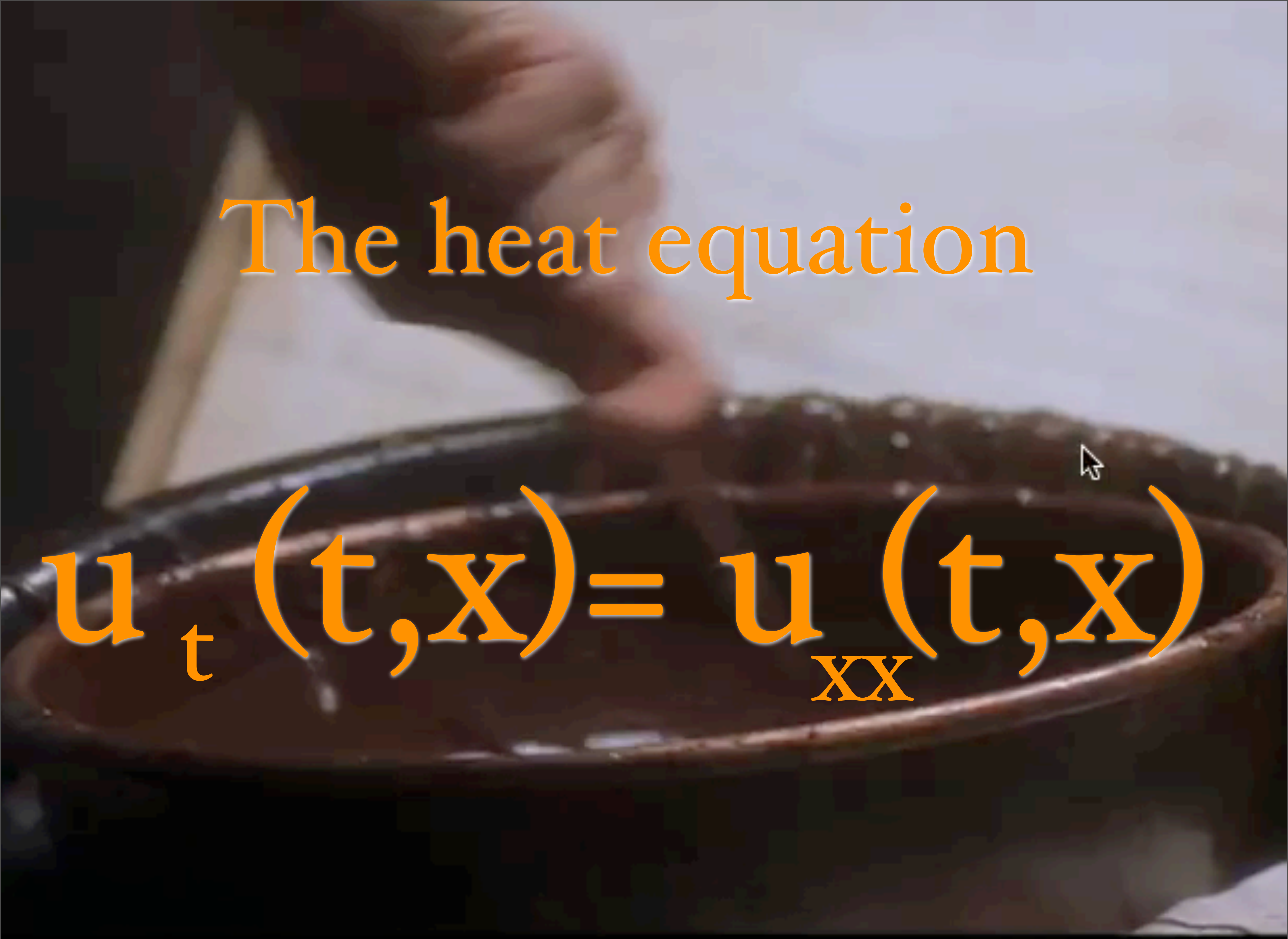
$$u_t(t,x) = u_x(t,x)$$

The transport equation


THE
TRANSPORTER
RULES ARE MADE TO BE BROKEN OCT. 11

The Laplace equation

$$u_{tt}(t,x) + u_{xx}(t,x) = 0$$



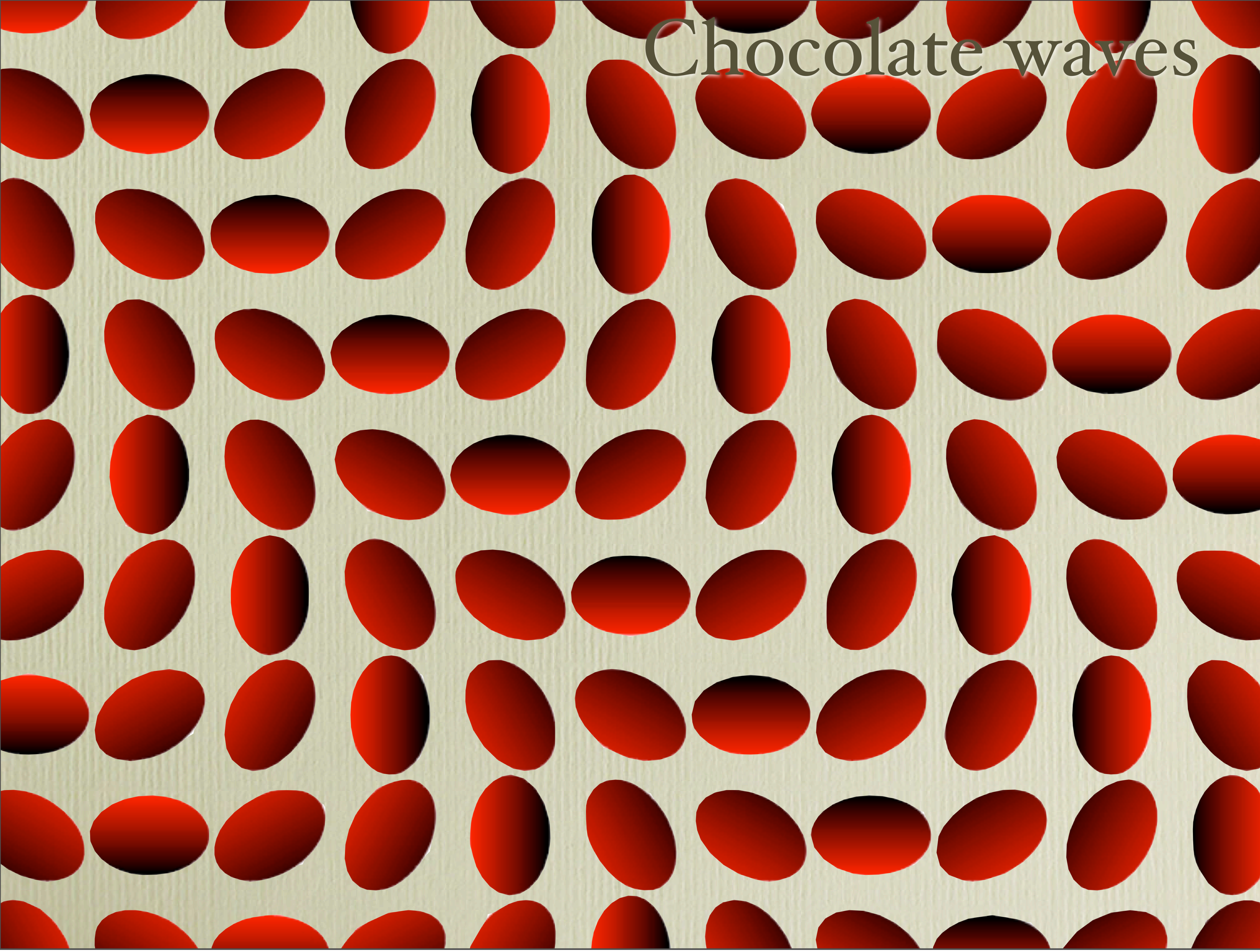
The heat equation

$$u_t(t,x) = u_{xx}(t,x)$$

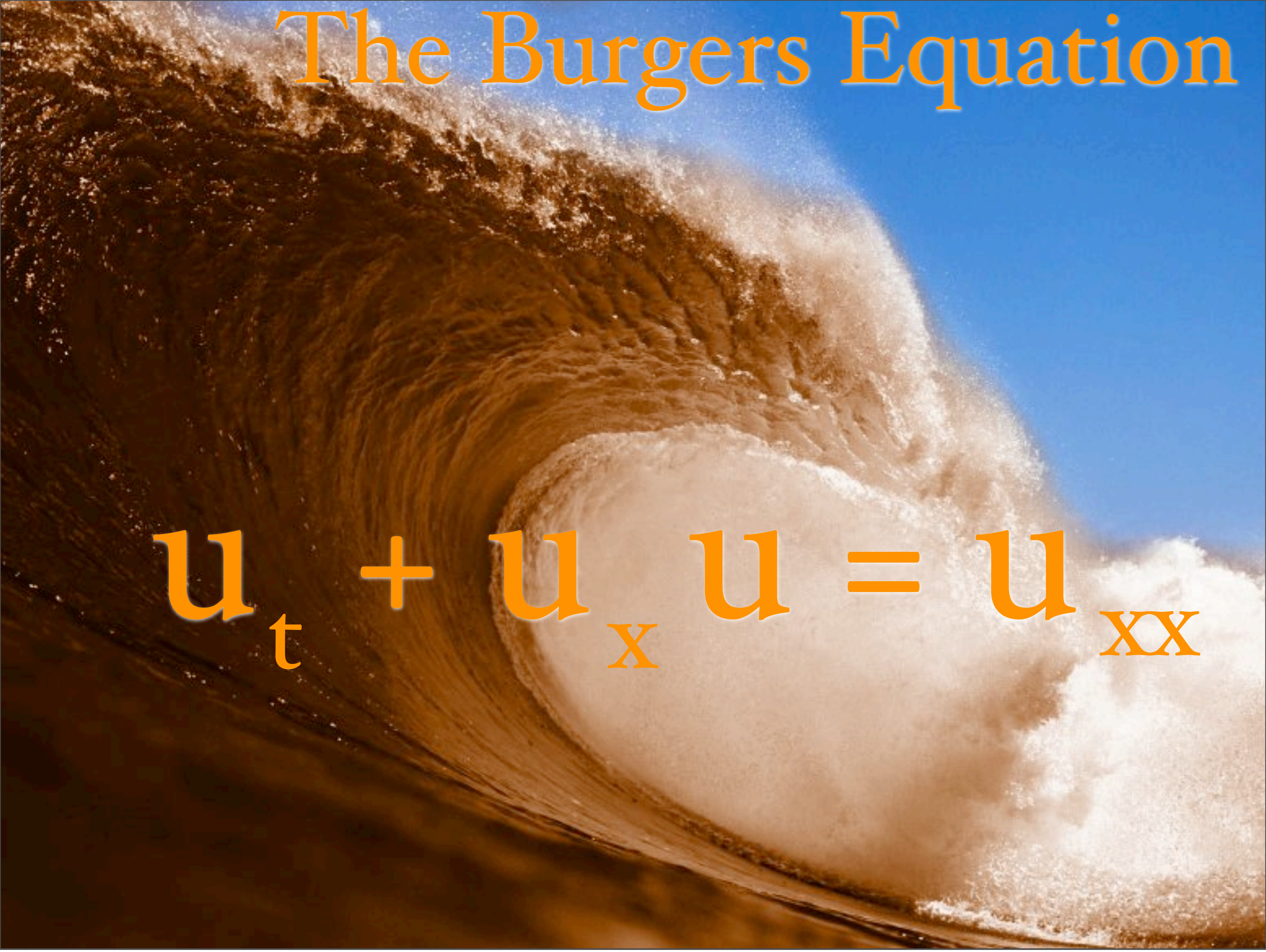
The wave equation

$$u_{tt}(t,x) = u_{xx}(t,x)$$

Chocolate waves

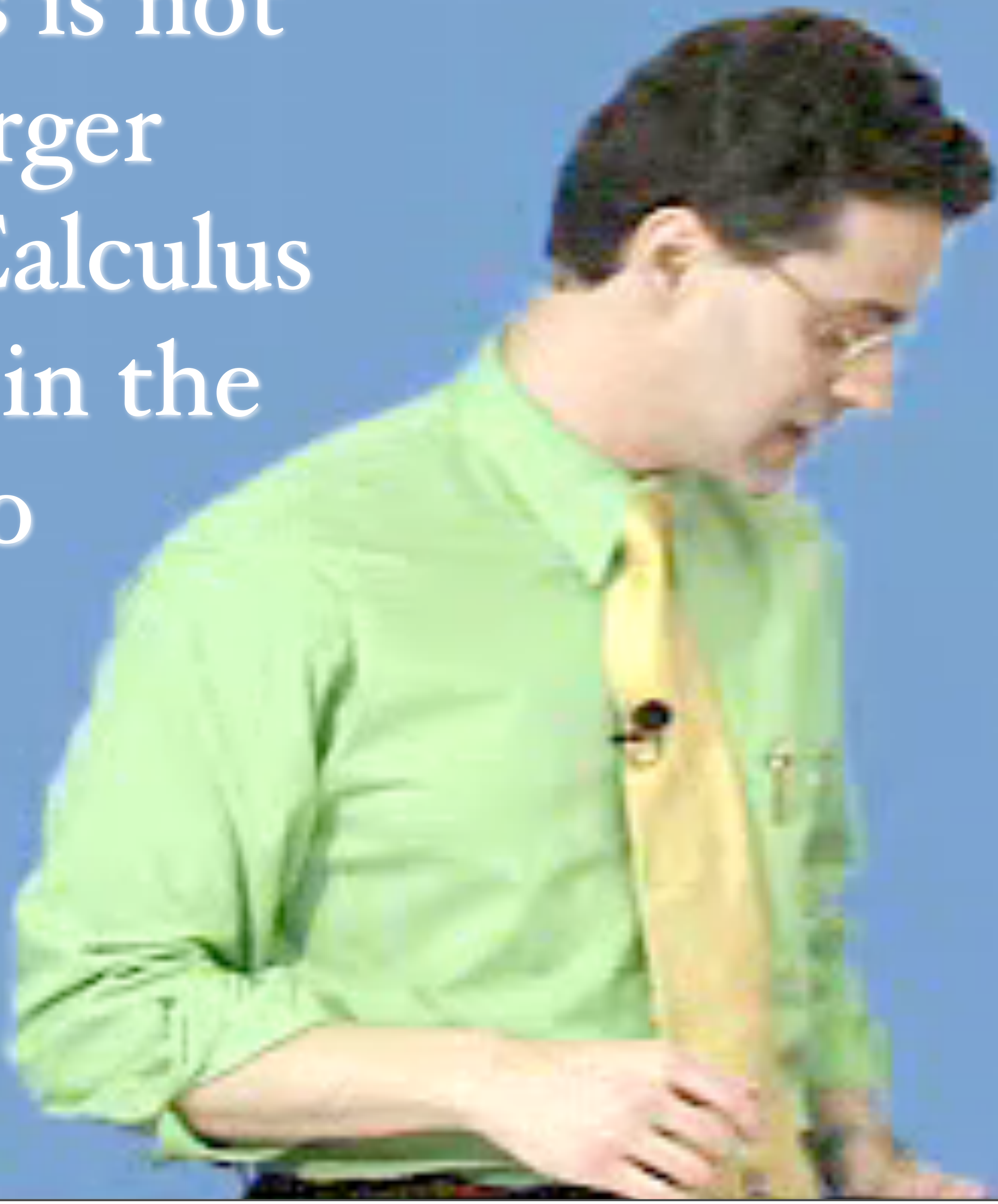


The Burgers Equation



$$u_t + u_x u = u_{xx}$$

Note: this is not the Burger Youtube Calculus star seen in the intro




5 equations

We set up even a Facebook page for these 5 equations facebook Home Profile Friends Inbox Partial-differential Equation Settings Logout Sea

Edit My Profile

Partial-differential Equation

Wall | Info | Photos | +

Edit Information

Basic Information

Sex: Male

Birthday: November 1, 1991

Hometown: Auxerre, France

Political Views: Shocks in conservation laws

Personal Information

Edit

Activities: Listening to the waves, feel the heat

Interests: Heat equation
Wave equation
Transport equation
Burger equation
Laplace equation

Favorite Music: Fourier transform

Favorite Movies: Heat

Favorite Quotations: The deep study of nature is the most fruitful source of mathematical discoveries.

About Me: I'm not unique but properly defined, I have a unique solution.

Contact Information

Email: oliver.knill@gmail.com

Current Address: One Oxford Street

Website: http://www.courses.fas.harvard.edu/~math...

Education and Work

Write something about yourself.

Information

Birthday:
November 1, 1991

Friends

0 friends

Find people you know

Photos

1 album See All

mathematicians
Created 21 minutes ago

Options

Information

Birthday:
November 1, 1991

Friends

0 friends

Find people you know

Photos

2 albums · See All

Wall Photos
Created 3 minutes ago

mathematician:
Created 37 minutes ago

Links

1 link · See All

mathematicians
3:46pm Nov 1

Create a Profile Badge

Partial-differential Equation

59 seconds ago · Comment · Like · Share

Partial-differential Equation

3 minutes ago · Comment · Like · Share

Partial-differential Equation Know 5 partial differential equations!

8 minutes ago · Comment · Like

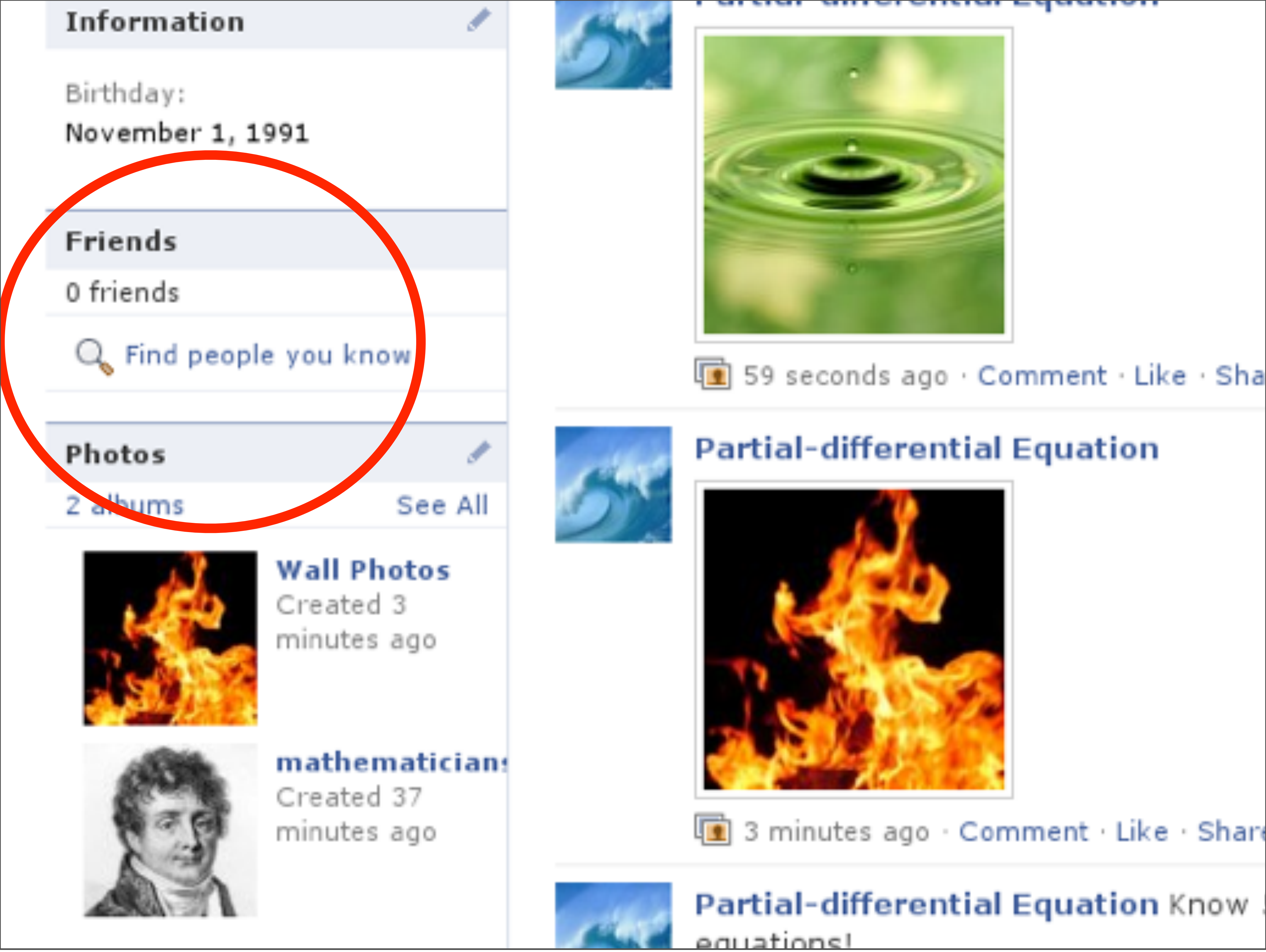
Partial-differential Equation Mathematicians

mathematicians

mathematicians related to partial differential equations

By: Partial-differential Equation

Photos: 5


Information
Birthday:
November 1, 1991
Friends
0 friends
Find people you know
Photos
2 albums
See All
Wall Photos
Created 3 minutes ago
mathematicians
Created 37 minutes ago
59 seconds ago · Comment · Like · Sha
Partial-differential Equation
3 minutes ago · Comment · Like · Share
Partial-differential Equation Know
equations!

The gradient

The gradient

- Nabla

$$\nabla f = < f_x, f_y, f_z >$$

The gradient

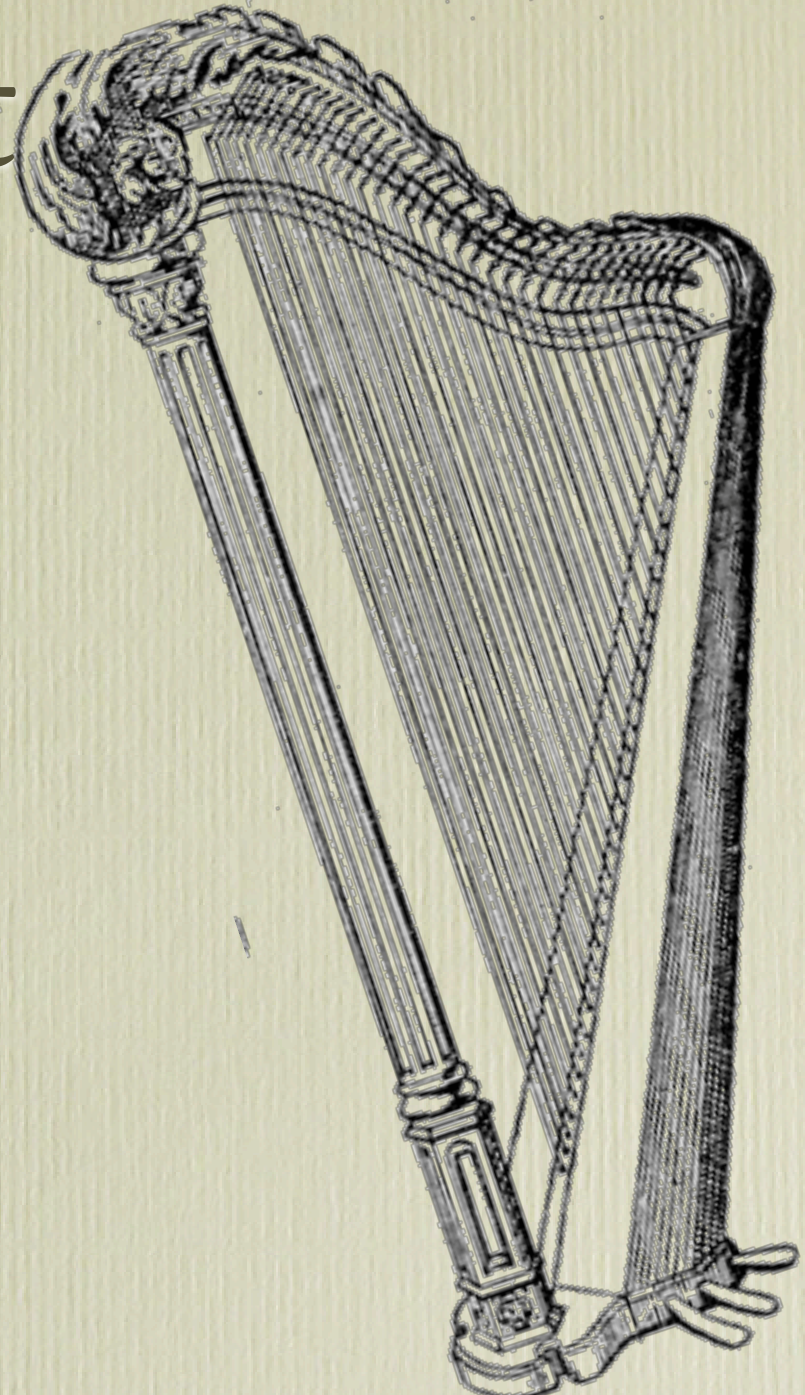
- Nabla ∇

$$\nabla f = < f_x, f_y, f_z >$$

The gradient

- Nabla ∇

$$\nabla f = \langle f_x, f_y, f_z \rangle$$



The gradient

- Nabla ∇

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

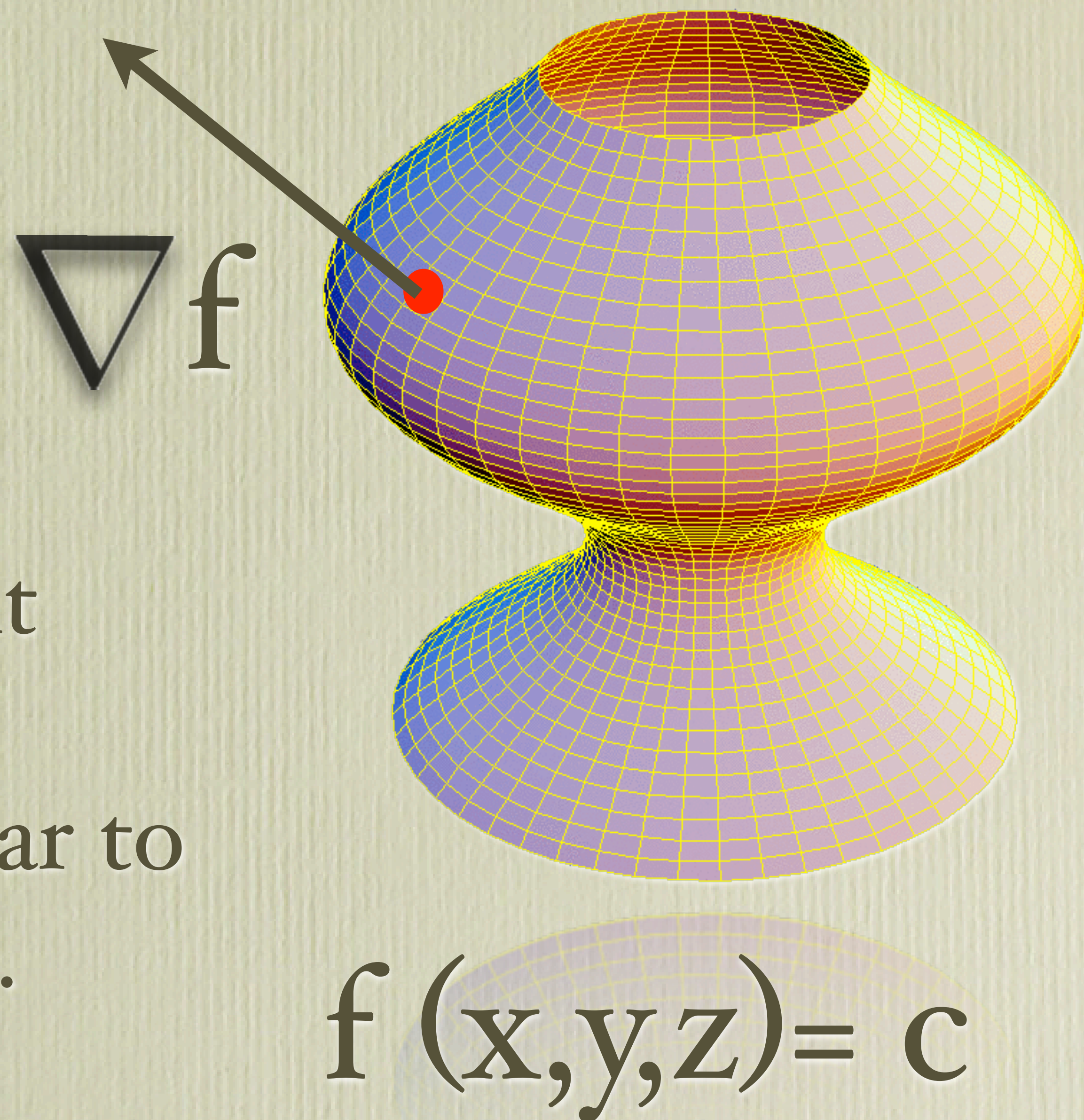
The gradient

- Nabla ∇

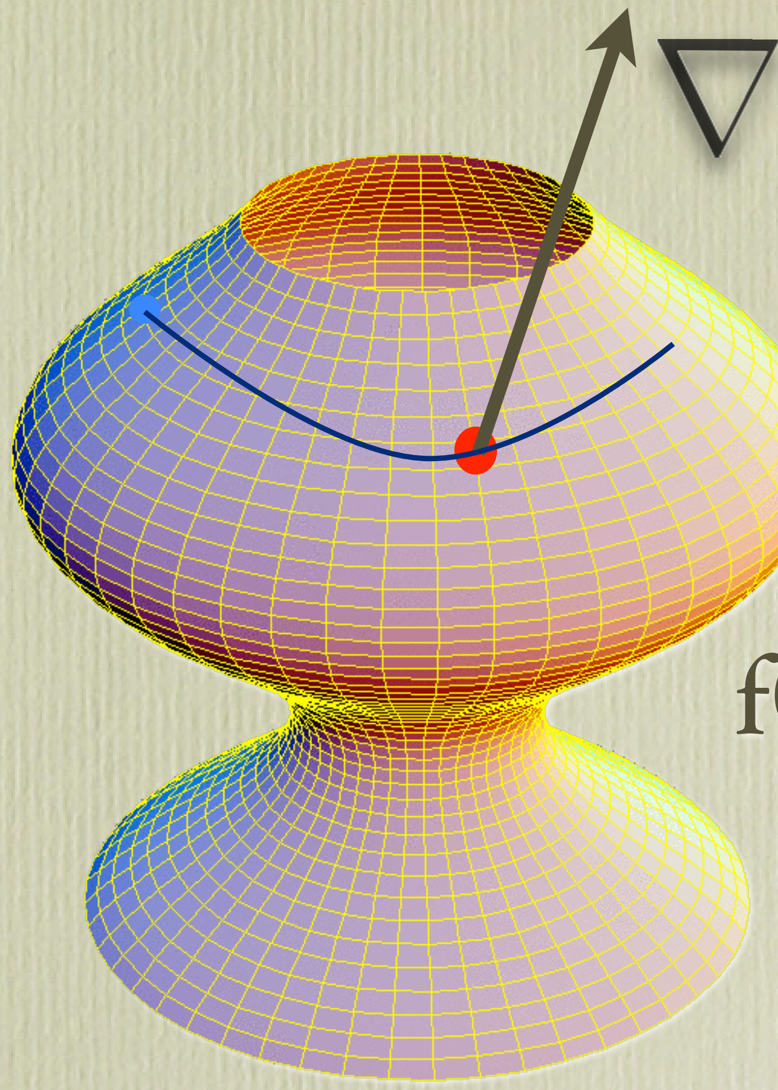
$$\nabla f = < f_x, f_y, f$$

Important Fact

- The gradient vector is perpendicular to the level set.



Proof

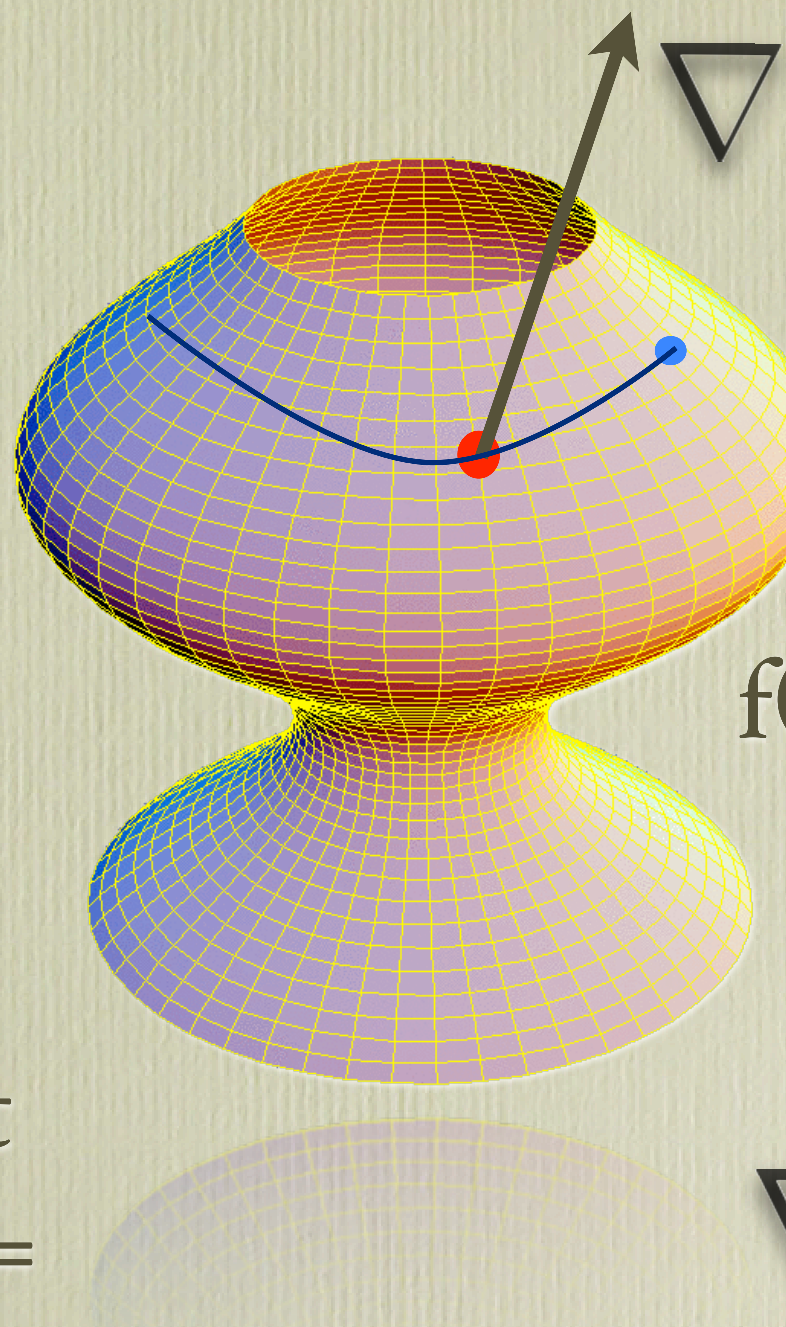


$f(x,y,z)=c$

$f(r(t))=\text{const}$

$0 = dt\ f(r(t)) = \nabla f(r(t)) \, . \, r'(t)$

Proof

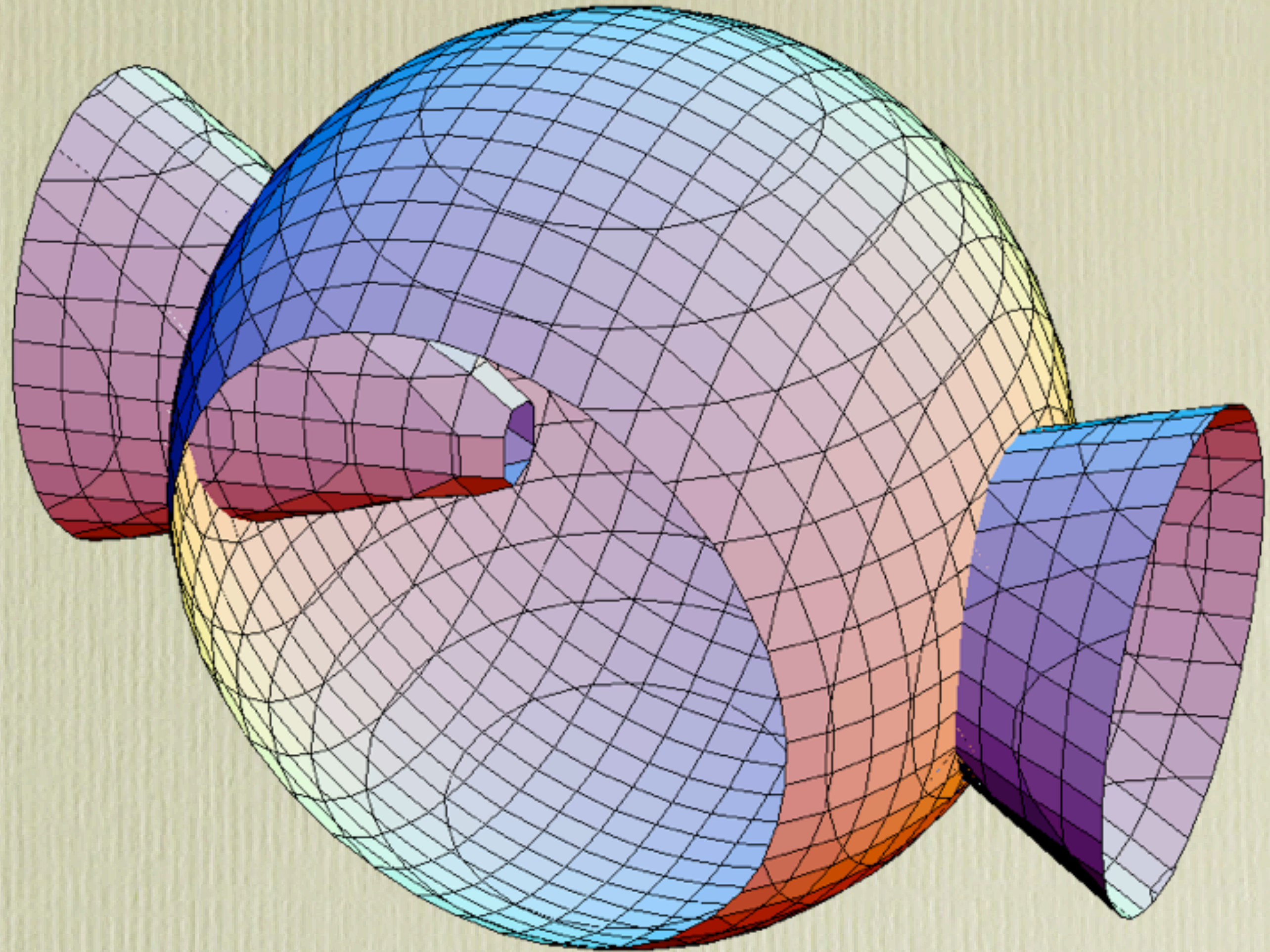


$f(x,y,z)=c$

$f(r(t))=const$

$0 = dt\ f(r(t)) = \nabla f(r(t)) \, . \, r'(t)$

Problem



- Show that the sphere $x^2 + y^2 + z^2 = r^2$ and the elliptic cone $y^2 = x^2 + z^2$ are perpendicular at every point of their intersection.

Problem

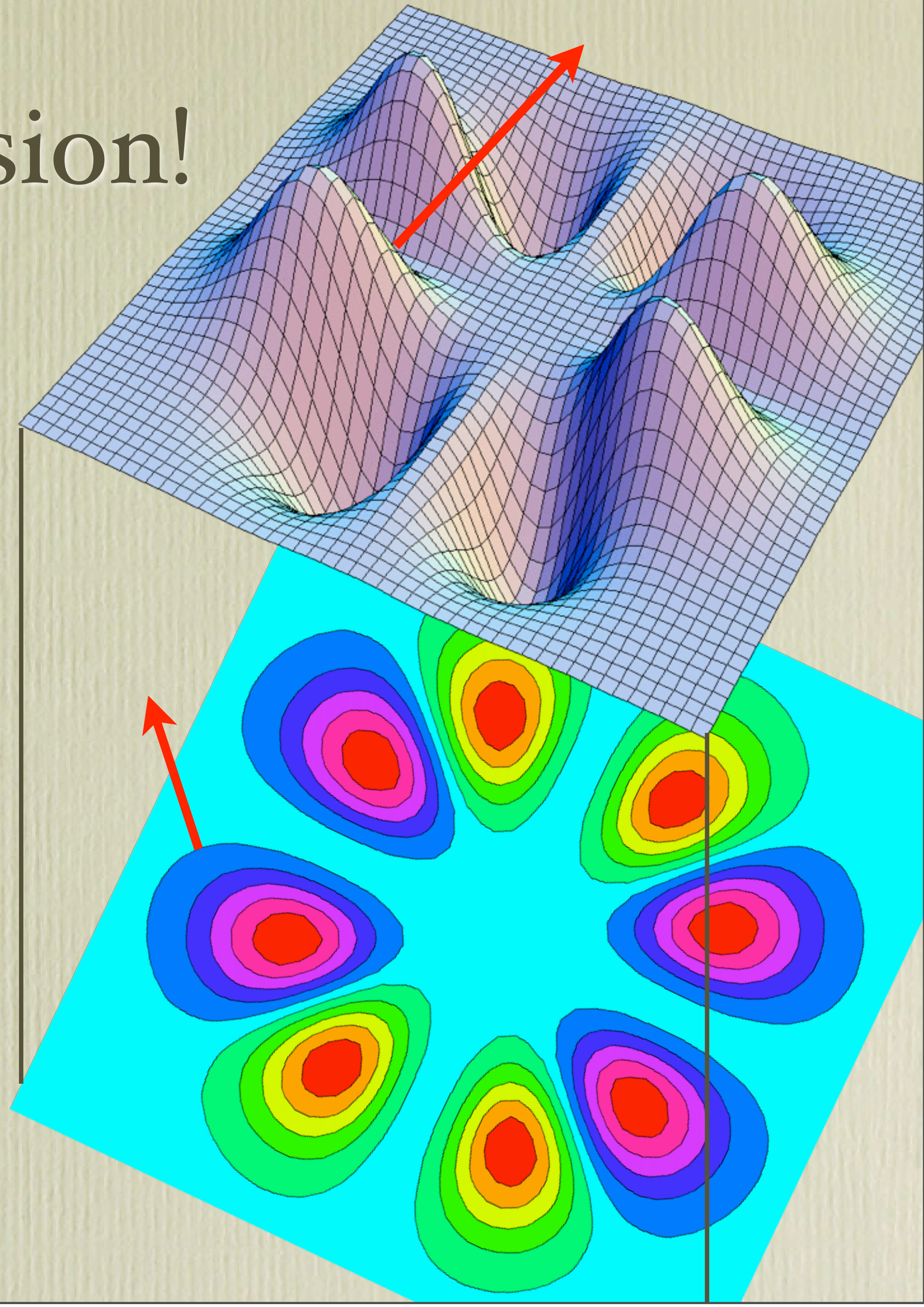
Find a vector normal to

$$2xy + e^{x-1}\cos(y) = 1$$

at (1,0).

Mind the dimension!

- Watch out for the dimensions.
- The gradient of a function of 2 variables is a vector in the plane
- The gradient of a function of 3 variables is a vector in space.




Surprise
aux noix

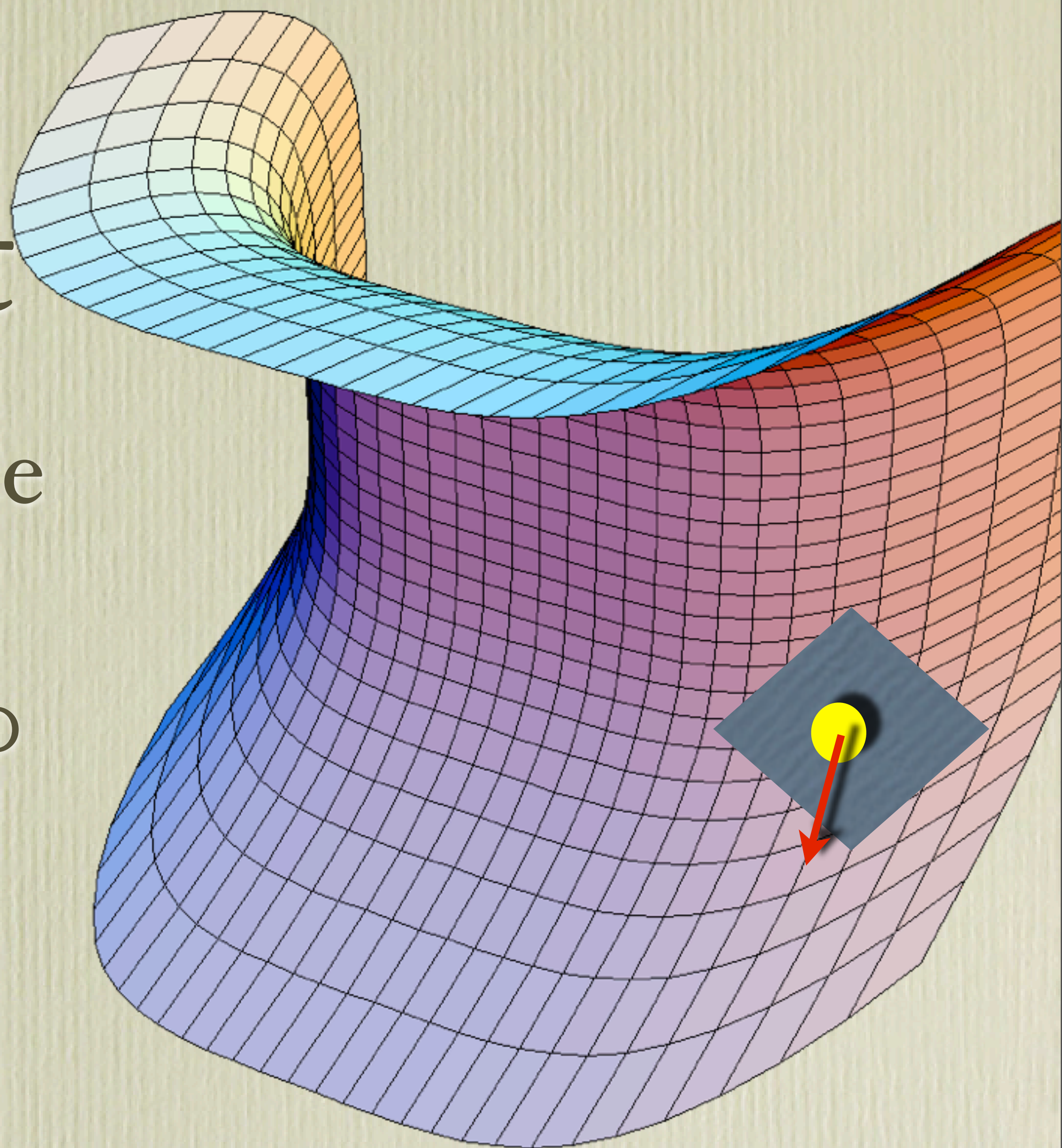
Tangent spaces

Problem

Find the **tangent plane** to the surface

$$f(x,y,z) = x^4 - 2z^4 - y = 0$$

at the point (1,0,1)

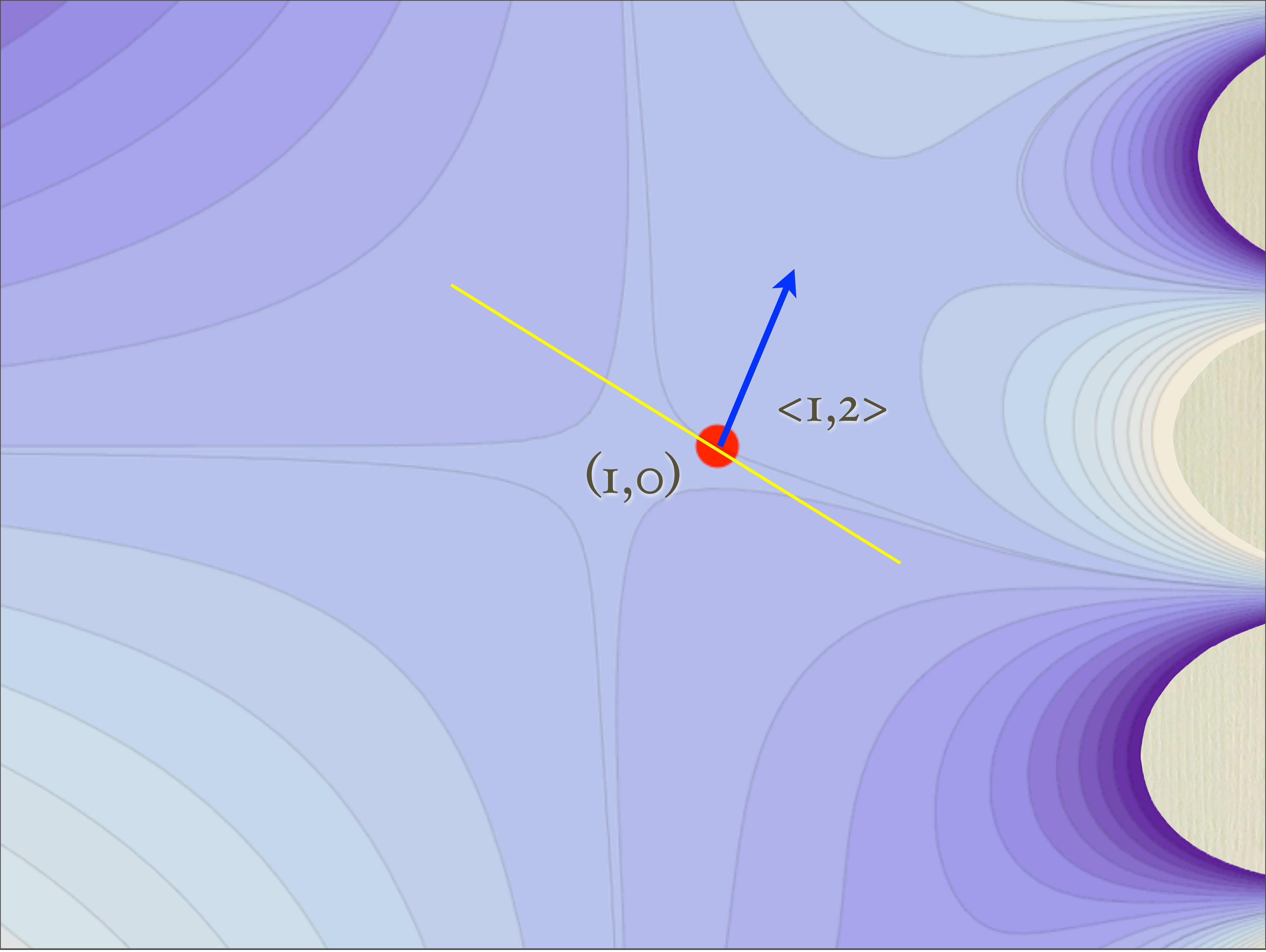


Tangent line

Find the tangent line to the curve

$$2xy + e^{x-1}\cos(y) = 1$$

at (1,0).


<1,2>
(1,0)

Linearization, estimation

Linearization, estimation

$$L(x,y) = f(a,b) + \nabla f(a,b) \cdot (x-a, y-b)$$

this function is close to f(x,y) near (a,b).

Problem

Can you estimate

$f(0.9999999, 0.9999, 0.99999)$ for

$f(x,y,z) = x^{-1} + 2y^{-1} + 3z^{-1}$

My main man:

Can you check this out without your brain blowing up?

Ali G Science

Ali G Estimation

Can you estimate

$f(0.9999999, 0.9999, 0.99999)$ for

$f(x,y,z) = x^{-1} + 2y^{-1} + 3z^{-1}$

without your brain blowing up?

Quiz coming up!

There will be Questions:

The first person who shouts the correct answer with a good explanation wins some Swiss chocolate.

Swiss Chocolate is even known to Bond:

Ready

Ready

Steady

Ready

Steady

Go

Problem

You know that f(x,y) satisfies the transport PDE

$$f_x = f_y$$

What can you say about the critical points of f?

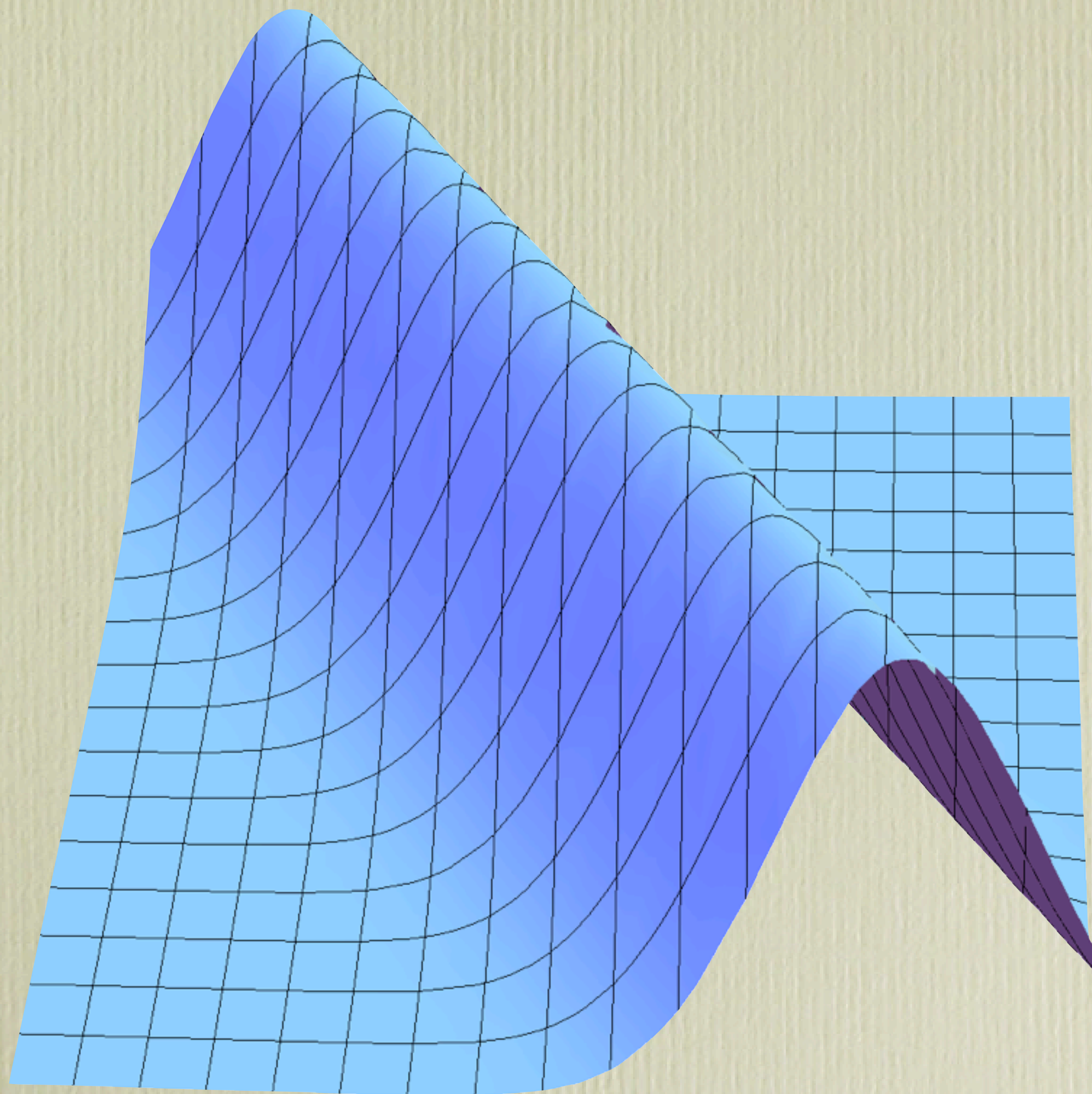
Problem

You know that f(x,y) satisfies the transport PDE

$$f_x = f_y$$

What can you say about the critical points of f?

The function f(x,y)



Ready

Ready

Steady

Ready

Steady

Go

You know that f(x,y) satisfies the Laplace equation

$$f_{xx} + f_{yy} = 0$$

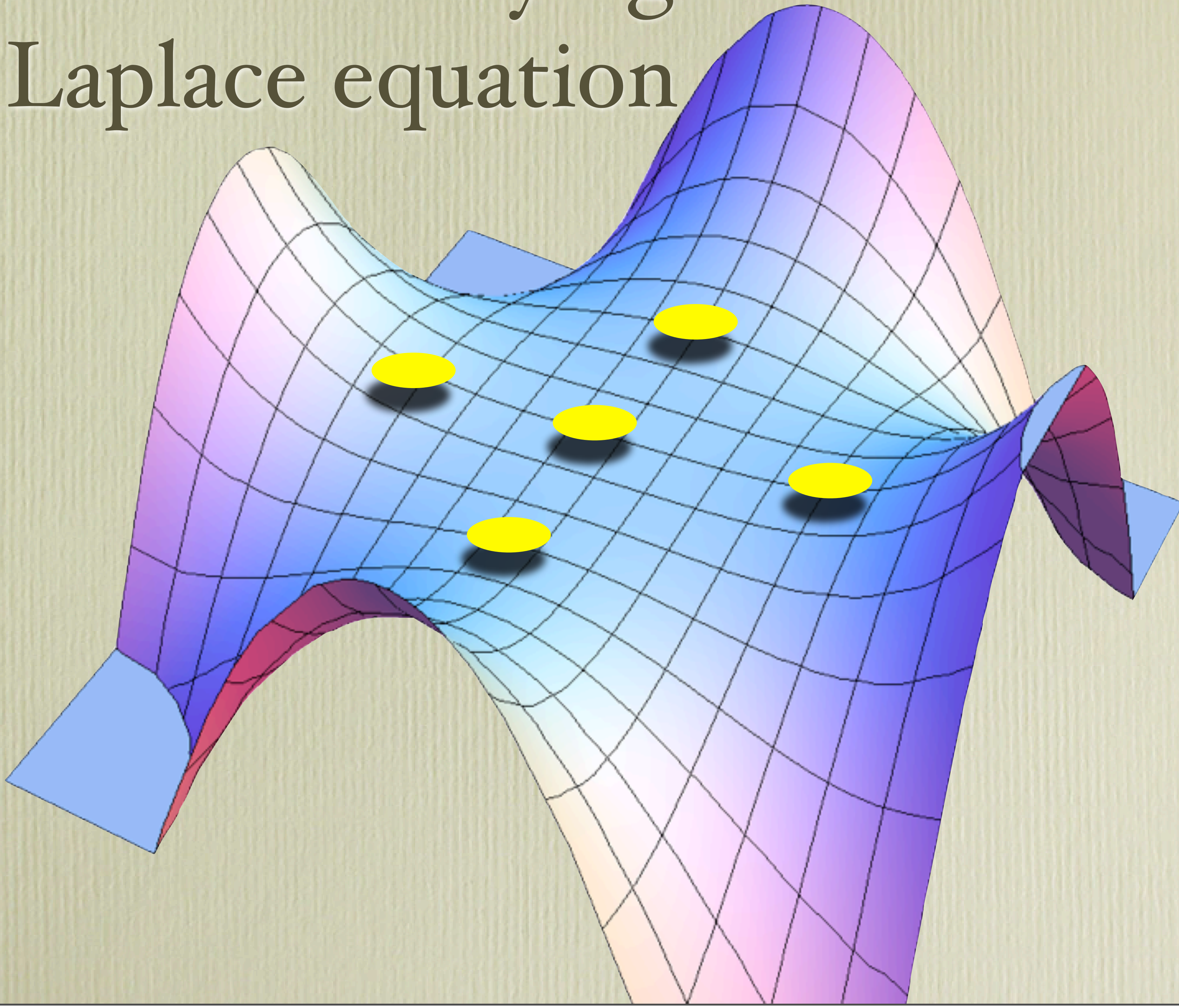
What can you say about the critical points of f(x,y) for which D is different from zero?

You know that f(x,y) satisfies the Laplace equation

$$f_{xx} + f_{yy} = 0$$

What can you say about the critical points of f(x,y) for which D is different from zero?

A function satisfying the Laplace equation



Ready

Ready

Steady

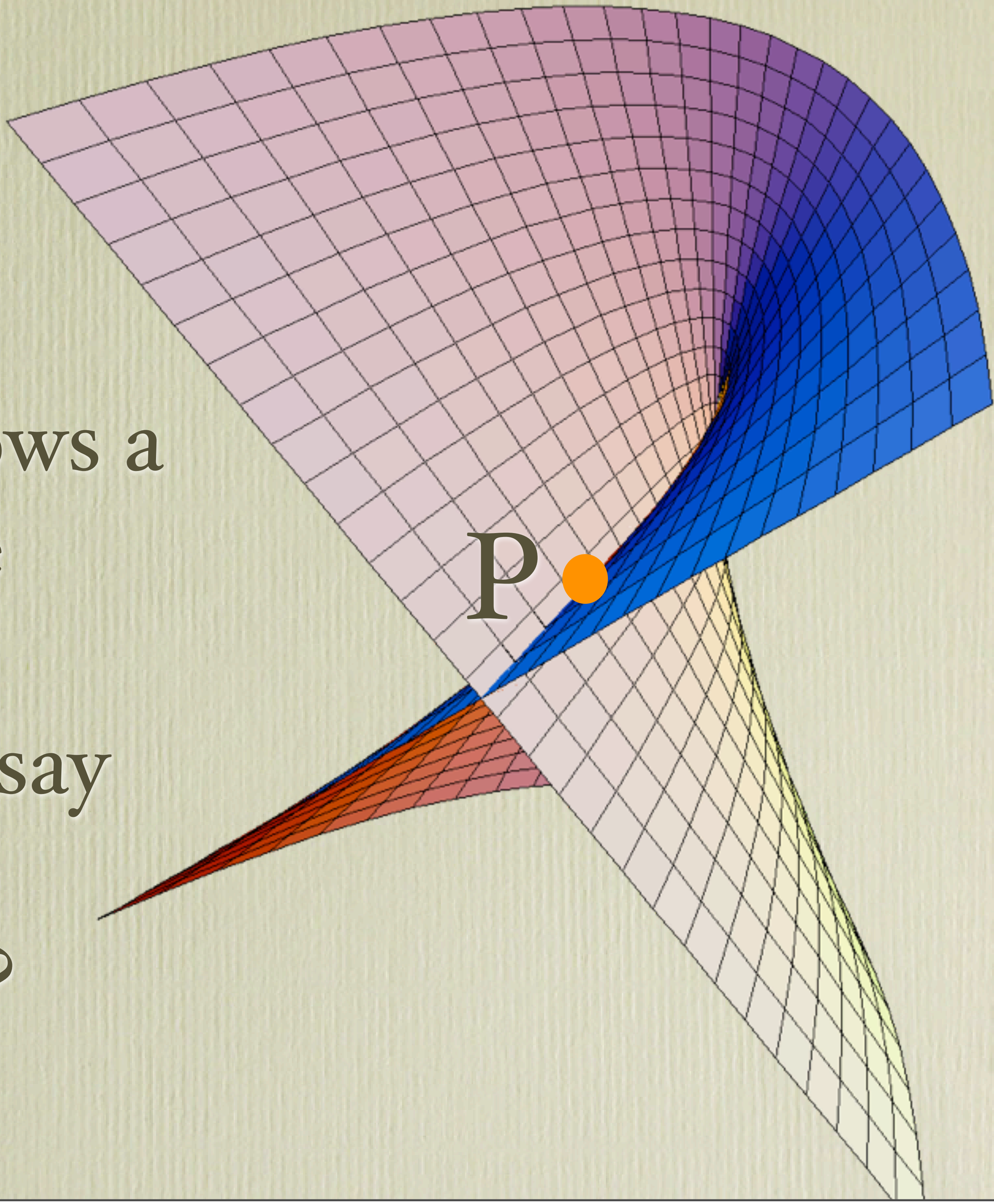
Ready

Steady

Go

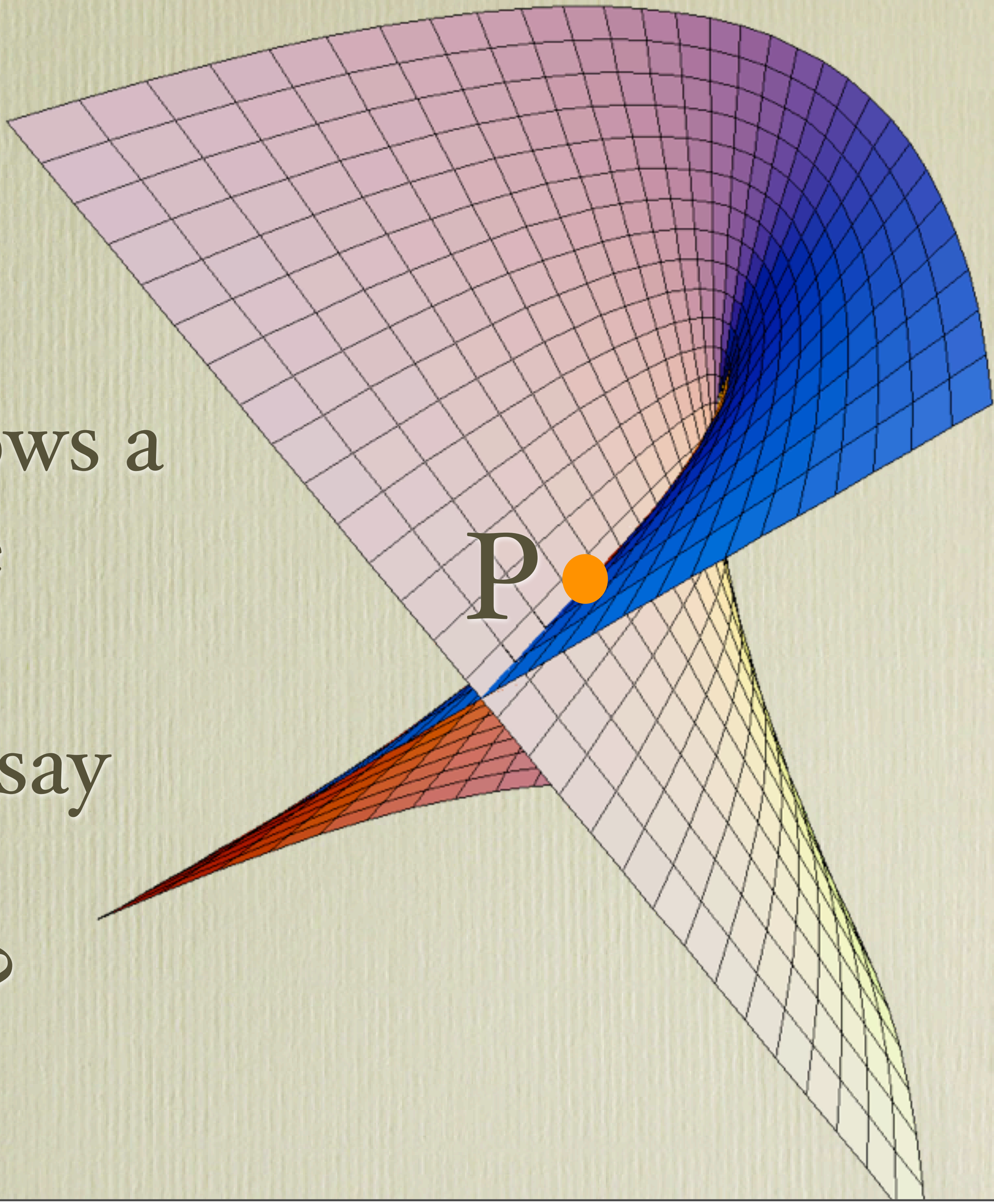
Problem:

The picture shows a level surface $f(x,y,z)=c$.
What can you say about the point P ?

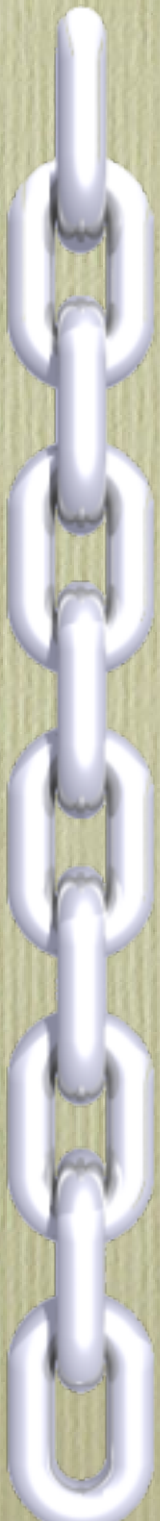


Problem:

The picture shows a level surface $f(x,y,z)=c$. What can you say about the point P ?



The chain rule



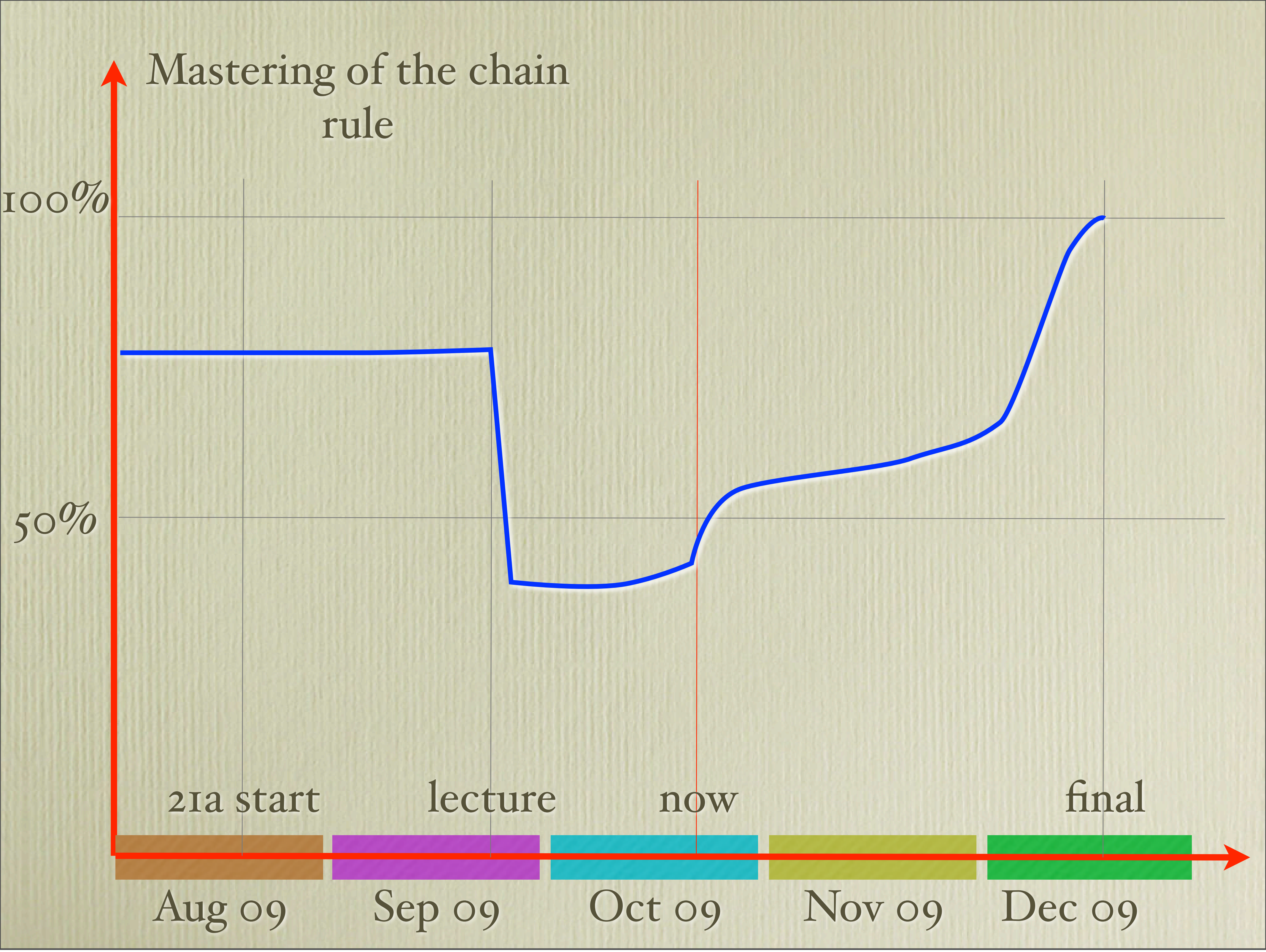
The chain rule

$$\frac{d}{dt} f(x(t),y(t)) \quad =$$

$$f_x(x(t),y(t))x'(t) \quad +$$

$$f_y(x(t),y(t))y'(t)$$

$$\frac{d}{dt}\, f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$$



Mastering of the chain rule
100%
50%
21a start
lecture
now
final
Aug 09
Sep 09
Oct 09
Nov 09
Dec 09

The good news:
you only need to know the 1D
chain rule:

$$d/dt\ \ f(g(t)) = f'(g(t))\, g'(t)$$

and replace the derivative of f
with the gradient and the
derivative of g with the velocity

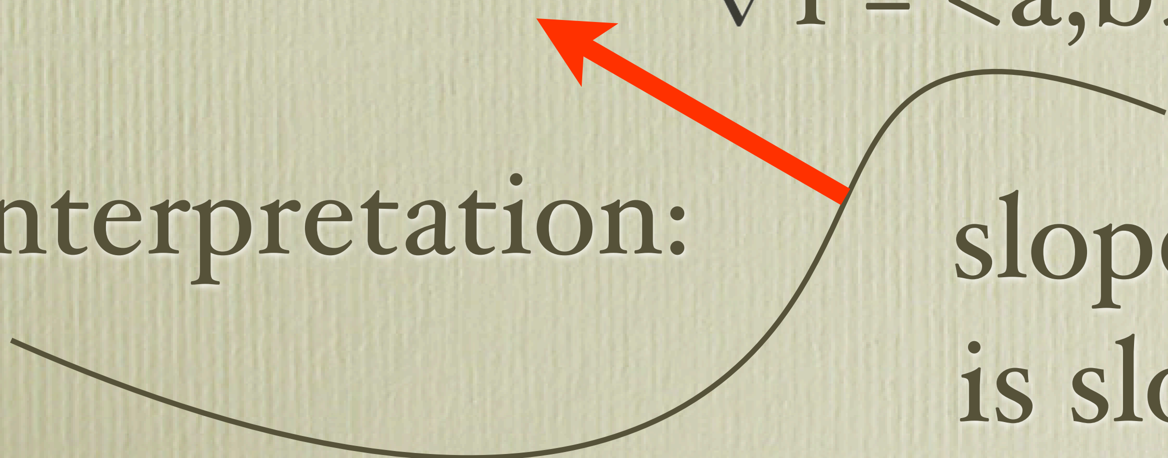
Implicit differentiation 2D

$$\frac{d}{dt} f(t,g(t)) = f_x(t,g(t))\, 1 + f_y(t,g(t))\, g'(t)$$

$$g'(t) = -f_x(t,g(t))/f_y(t,g(t))$$

$$\nabla f = \langle a,b \rangle$$

interpretation:



slope: b/a of gradient
is slope -a/b of graph

Problem:

$\sin(x\,y) + x\,y = 0$

Find y'(x) at x=1.

Additionally: Find the tangent line.

Implicit differentiation 3D

$$\frac{d}{dx}\, f(x,y(x,y),z(x,y)) \;=\; f_x(x,y,z)\ 1$$

$$+f_y(x,y,z)\ 1 \ +f_z(x,y,z)\ z_x(x,y)$$

$$z_x(x,y) = -f_x(x,y,z)/f_z(x,y,z)$$

similarly

$$z_y(x,y) = -f_y(x,y,z)/f_z(x,y,z)$$

Directional Derivative

$$D_{\vec{v}} f = \nabla f(x,y) \cdot \vec{v}$$

Rate of change of f in the direction $\vec{v}$.

The vector $\vec{v}$ is a unit vector.


The Rollerman Problem
• Jean-Yves Blondeau
• aka Rollerman

Problem

The mountain has the height

$$f(x,y) = x^4 + y^2$$

Jean-Yves drives along the path

$$\vec{r}(t) = <1-t, -\sin(t)>.$$

What is the rate of change of $f(\vec{r}(t))$?

in other words:

What actual slope does he experience at time t=0?

At time t=0, we have

$\vec{r}(0) = \langle 1, 0 \rangle$

and

$\vec{r}\,'(0) = \langle -1, -1 \rangle$.

The gradient vector at the point (x,y) is $\langle 4x^3, 2y^2 \rangle$.

At (1,0), it is $= \langle 4, 0 \rangle$

The slope is the directional derivative in the direction $\langle -1, -1 \rangle / \sqrt{2}$.

Which is $-4/\sqrt{2}$

Full body rollerblading also works in town:

Steepest Ascent

$$\frac{\nabla f(x,y)}{|\nabla f(x,y)|}$$

is the direction in which f increases most.

the slope in that direction is:

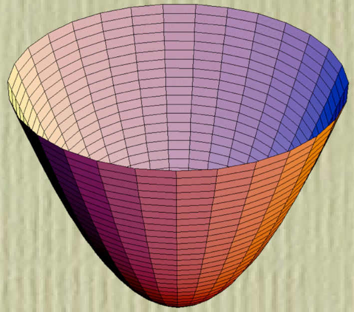
$$|\nabla f(x,y)|$$

An Illustration

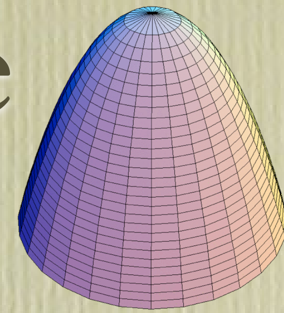

Alcohol
& calculus
don't mix.
Never drink
& derive.

Sweet extrema

Extrema without constraints



Second derivative test



Second derivative test

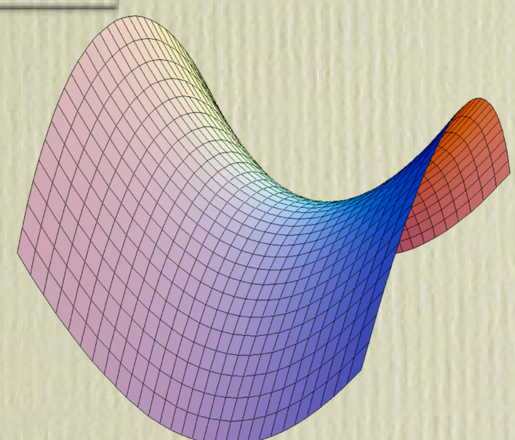
$D > 0$, $f_{xx} > 0$ Min

$D > 0$, $f_{xx} < 0$ Max

$D < 0$, Saddle

$D = f_{xx} \cdot f_{yy} - f_{xy}^{2}$

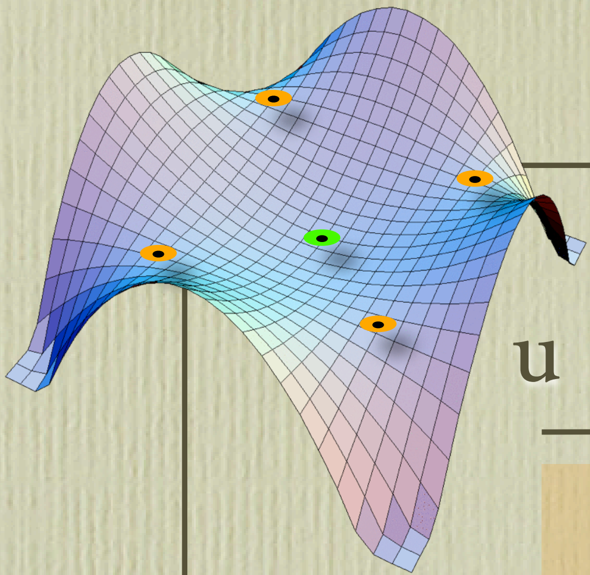
D Discriminant at the critical point (x,y)



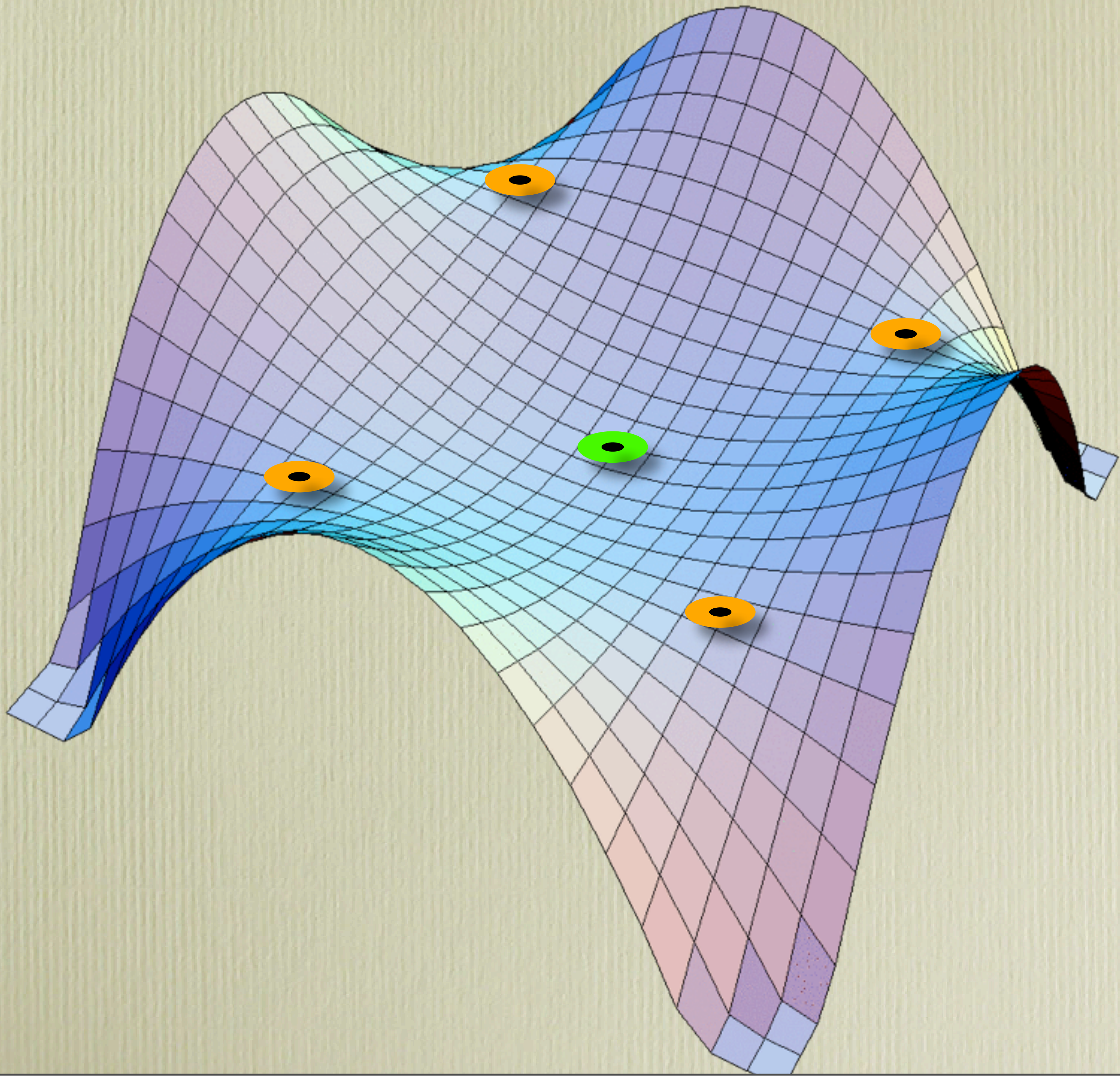
Problem 11

Find the extrema of the function

$$f(x,y) = x^2 + y^2 - x^2y^2$$



u	v	D	f_{uu}	Type	f
-1	-1	-16	0	saddle	1
-1	1	-16	0	saddle	1
0	0	4	2	min	0
1	-1	-16	0	saddle	1
1	1	-16	0	saddle	1

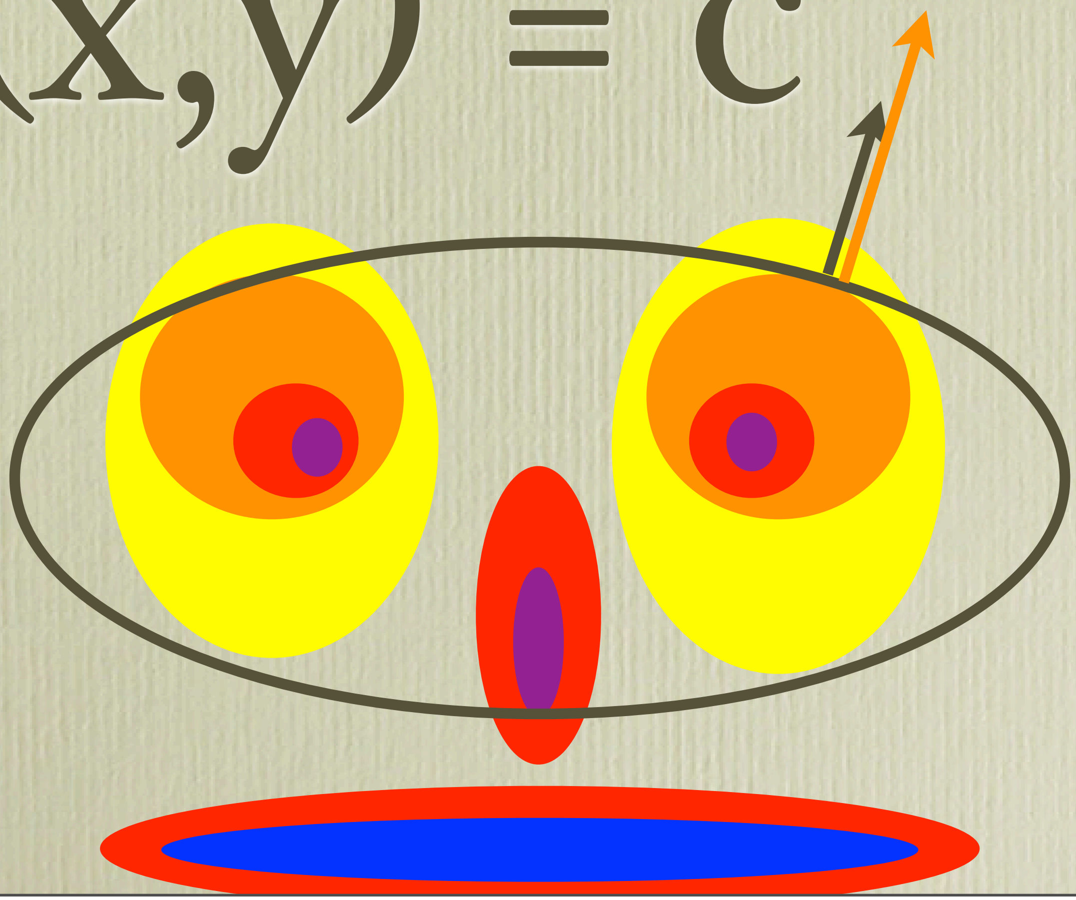


Extrema with constraints

Lagrange Equations

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

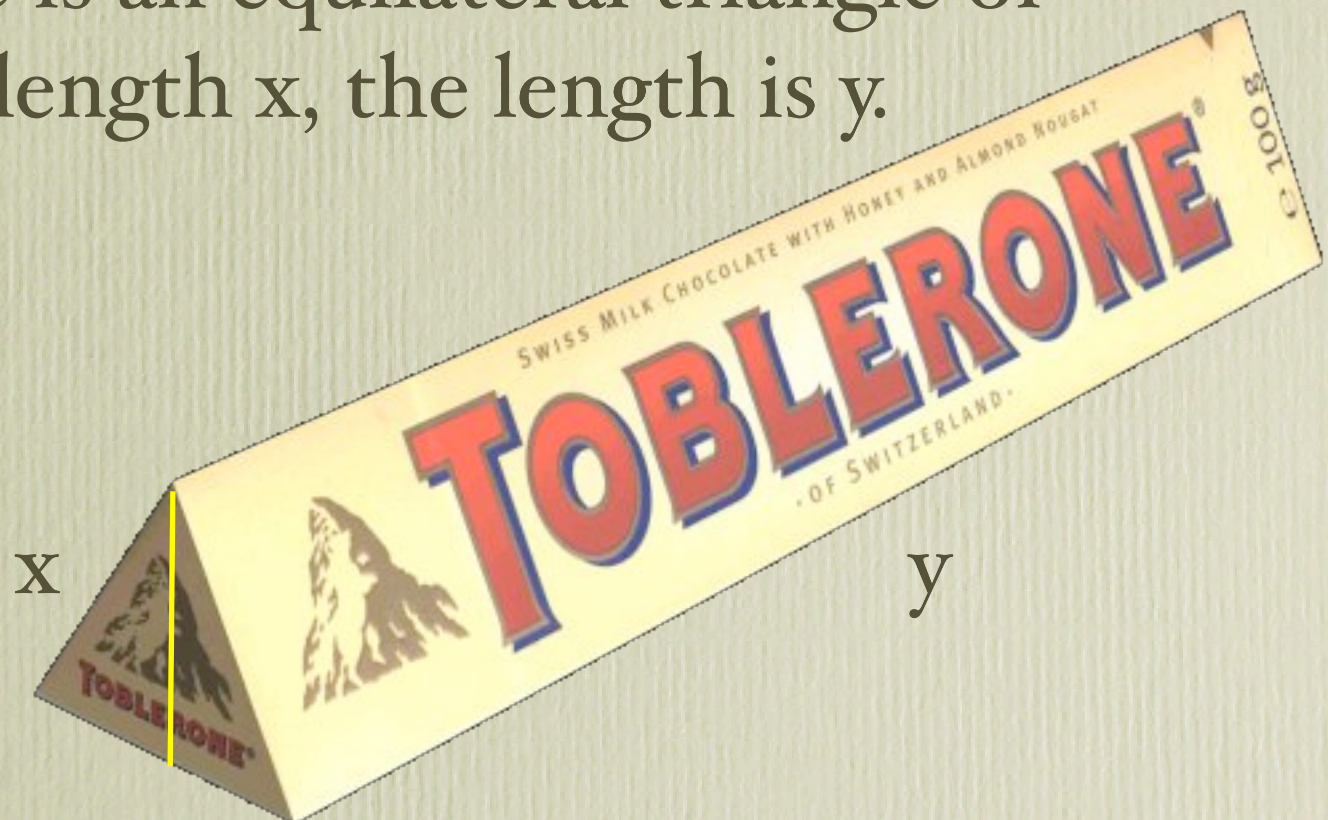
$$g(x,y) = c$$



Toblerone Problem

Toblerone Problem

Find the toblerone chocolate shape which has maximal volume if the surface area is constant 1.
The base is an equilateral triangle of side length x, the length is y.



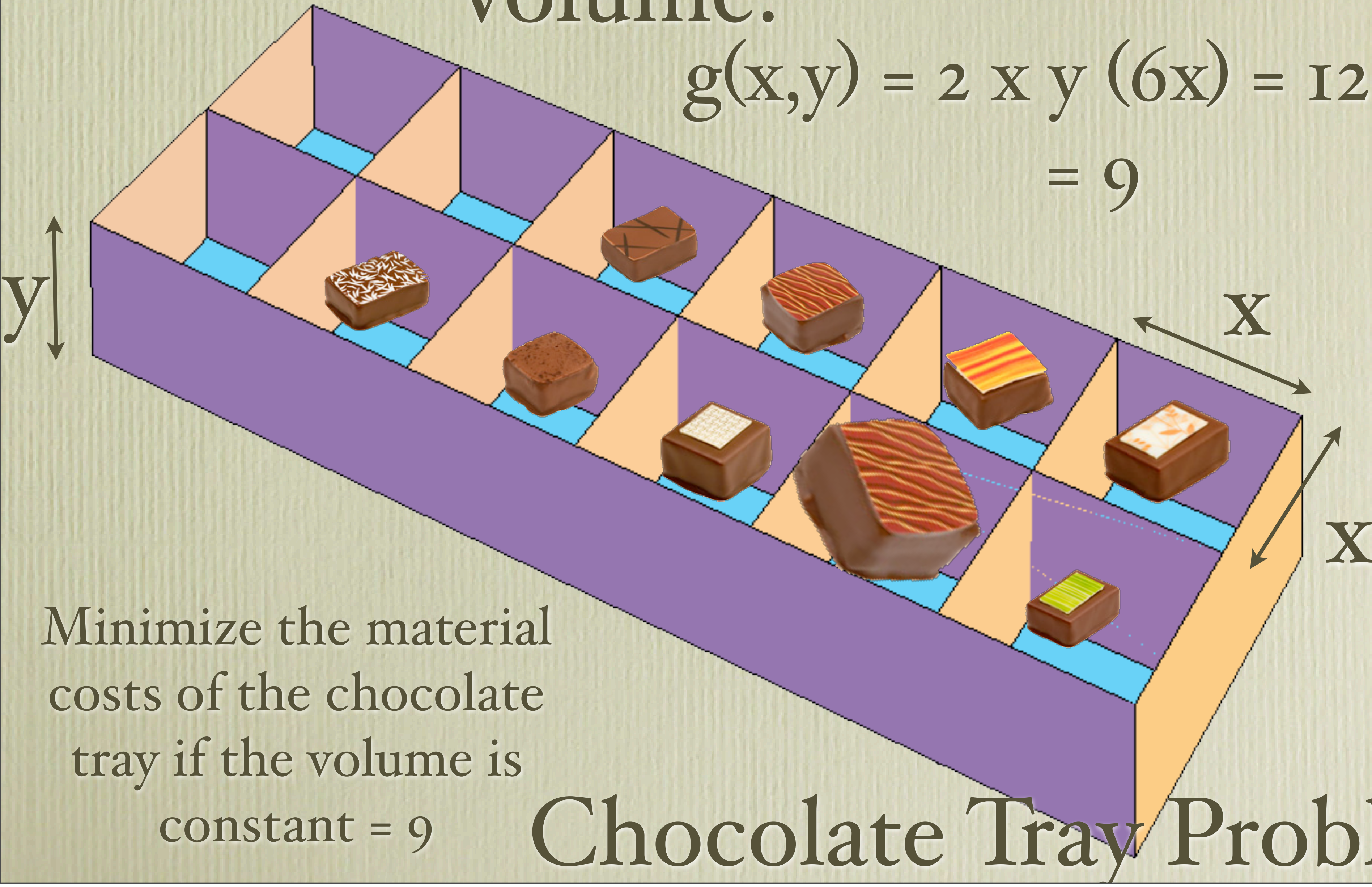
Area: $f(x,y) = (2x)(6x) + 3(6xy) + 7(2xy)$

$$= 12x^2 + 32xy$$

Volume:

$$g(x,y) = 2xy(6x) = 12x^2y$$

$$= 9$$



Minimize the material costs of the chocolate tray if the volume is constant = 9

Chocolate Tray Problem

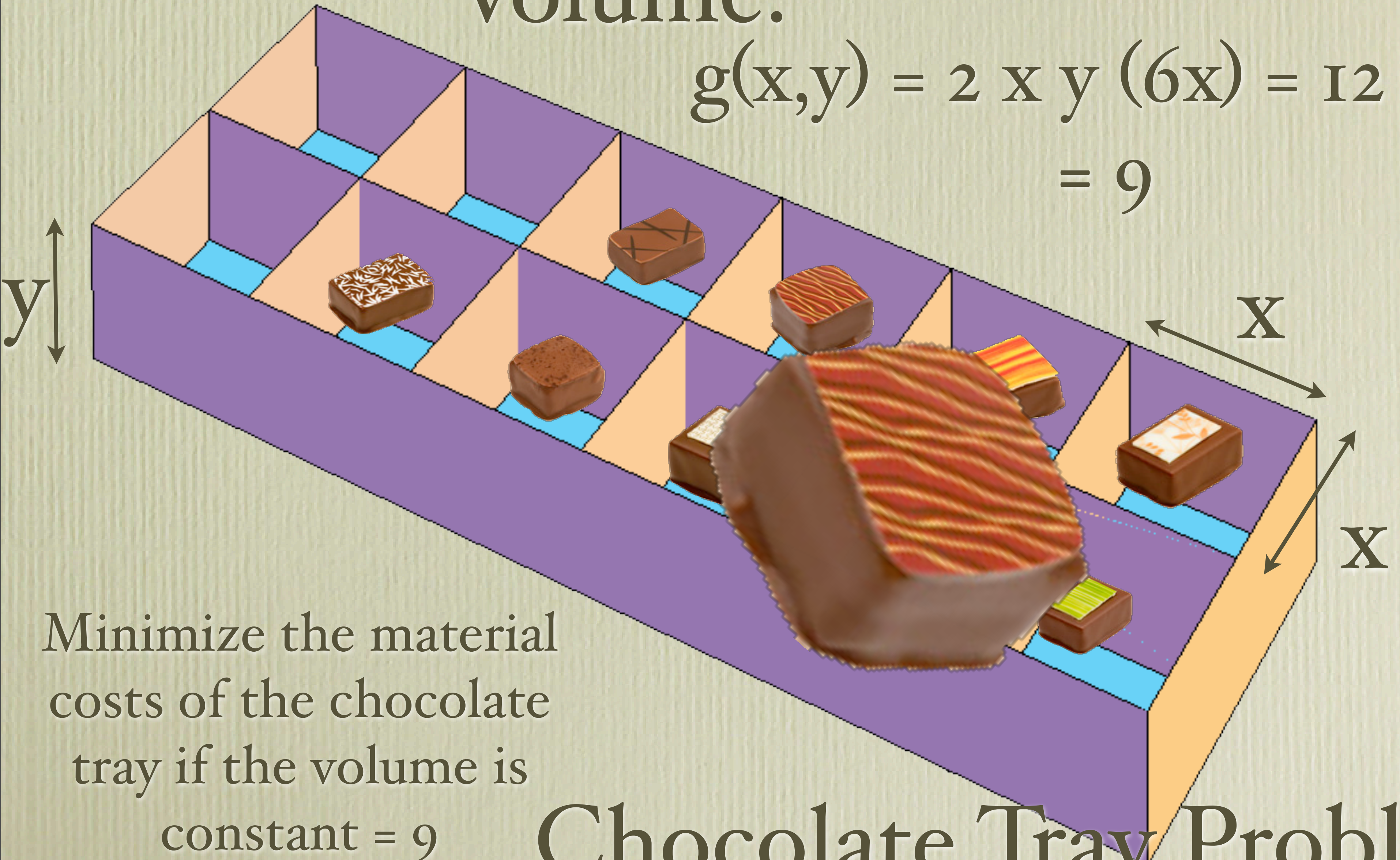
Area: $f(x,y) = (2x)(6x) + 3(6xy) + 7(2xy)$

$$= 12x^2 + 32xy$$

Volume:

$$g(x,y) = 2xy(6x) = 12x^2y$$

$$= 9$$



Minimize the material costs of the chocolate tray if the volume is constant = 9

Chocolate Tray Problem

$$g = 3xy + x^2\frac{\sqrt{3}}{2} = 1$$

$$f = x^2 y \frac{\sqrt{3}}{4}$$

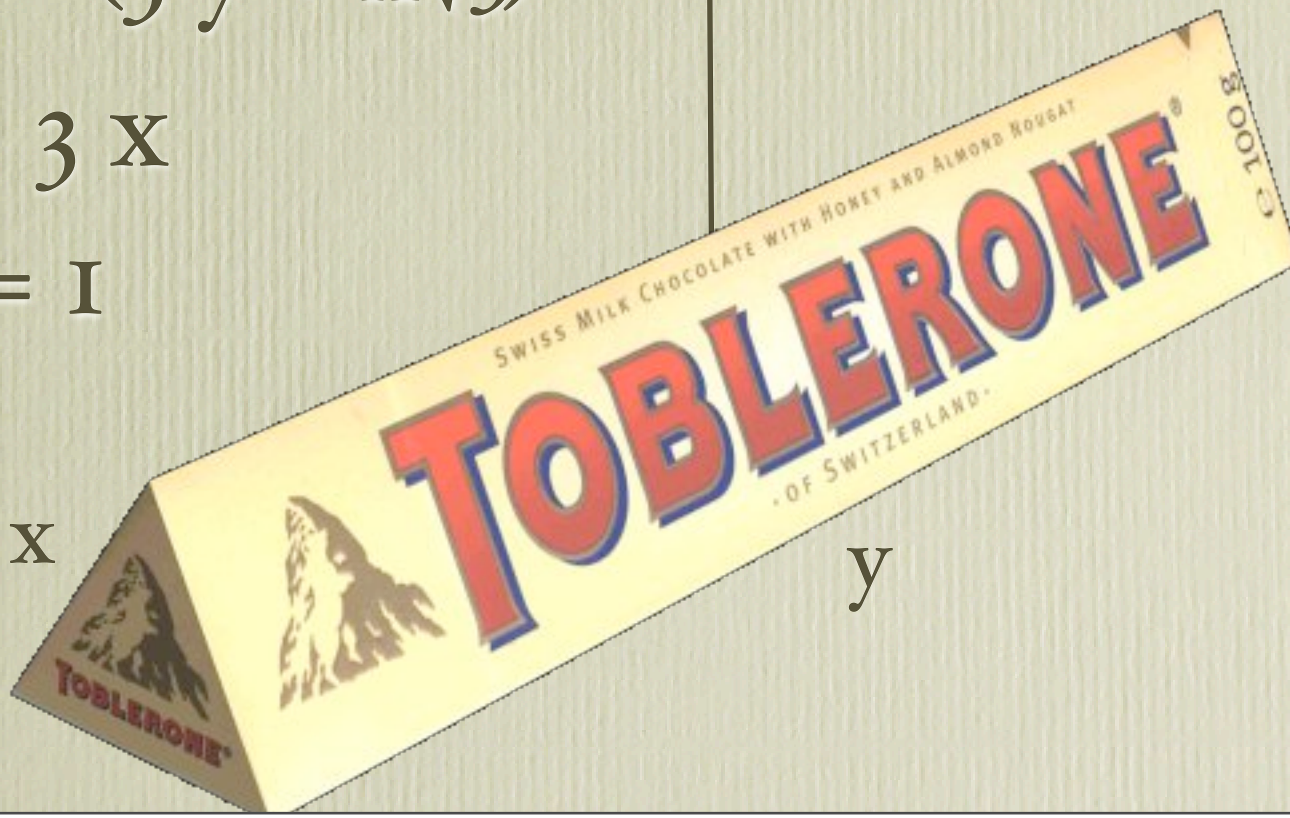
1 | $xy\frac{\sqrt{3}}{2} = \lambda\,(3y + x\sqrt{3})$

2 | $x^2\frac{\sqrt{3}}{4} = \lambda\, 3x$

3 | $g = 1$

$1 / 2$ $\quad x/y = \sqrt{3}$

plug into 3



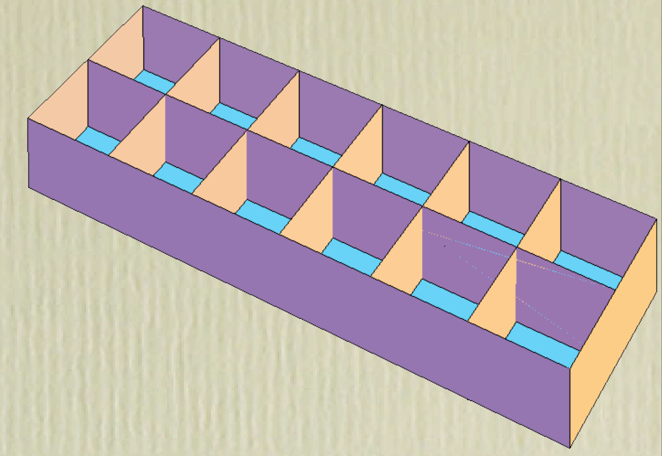
x

y

Exams are like box of chocolates:

Area: $f(x,y) = 12\,x^2 + 32\,x\,y$

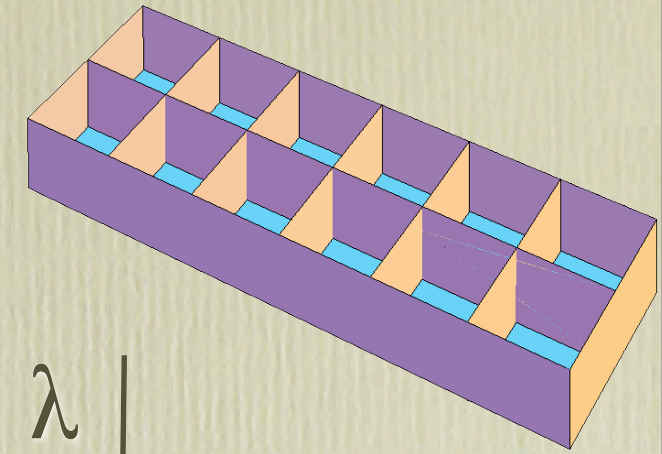
Volume: $g(x,y) = 12\,x^2 y = 9$



Lagrange equations:

$$\left|\begin{array}{l} 24\,x + 32\,y = 24\,x\,y \quad \lambda \\ 32\,x = 12\,x^2 \quad \lambda \\ 12\,x^2 y = 9 \end{array}\right|$$

Solving the Lagrange equations:



$$
\begin{array}{ll}
① & 24x + 32y = 24xy\,\lambda \\
② & 32x = 12x^2\,\lambda \\
③ & 12x^2 y = 9
\end{array}
$$

$①/②$

$$\frac{24}{32} + \frac{y}{x} = \frac{2y}{x}$$

$$4y = 3x$$

$$9x^3 = 9$$

$x=1,$

$y=3/4$

Global extrema

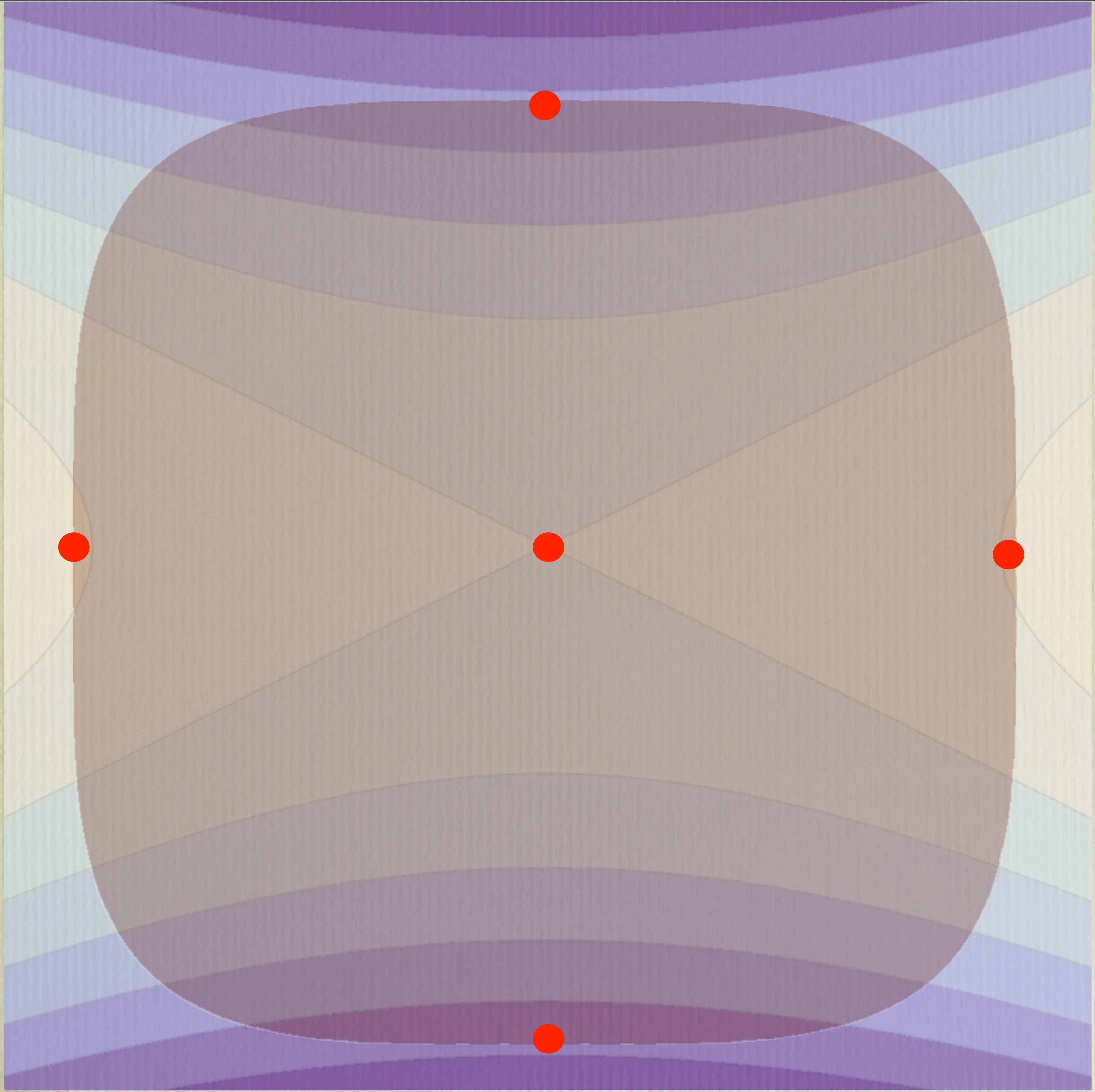
Problem:

Sugar concentration on the top of a cake is

$$f(x,y) = x^2 - 4y^2$$

The cake surface is the domain $x^4 + y^4 \leq 1$

Where is the sugar concentration maximal?



Extrema:

$$2x-2 = 0 \qquad 2y-4 = 0$$

$$x=1, \quad y=2$$

Extrema with constraints:

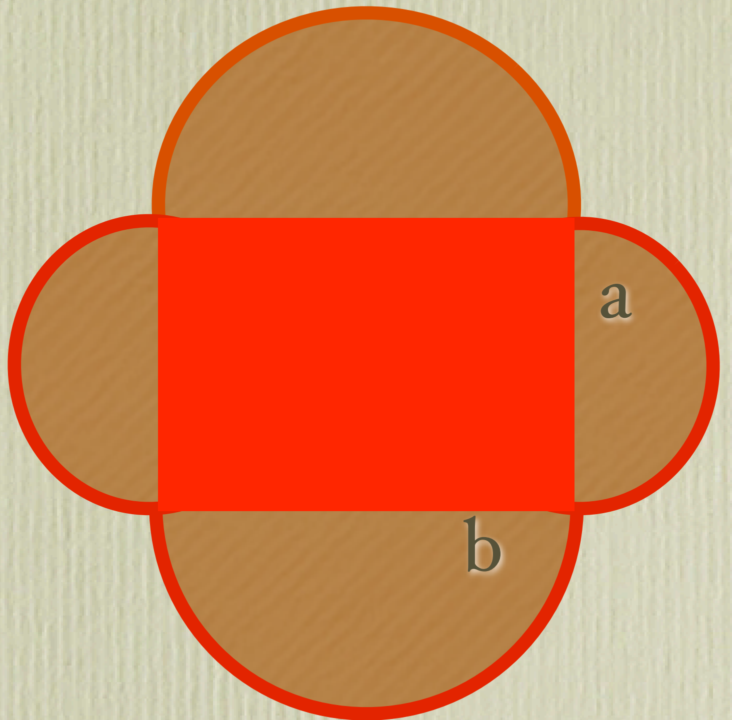
$$\begin{cases} (1) & 2x-2 = -2x\lambda \\ (2) & 2y-4 = \lambda \\ (3) & y - x^2 = 0 \end{cases}$$

$$(1)/(2): \quad \frac{2x-2}{2y-4} = -2x \qquad y - x^2 = 0$$

solutions are:

$$(-1,1),\ ((1\pm\sqrt{3})/2,(2\pm\sqrt{3})/2)$$

Halloween Candy problem



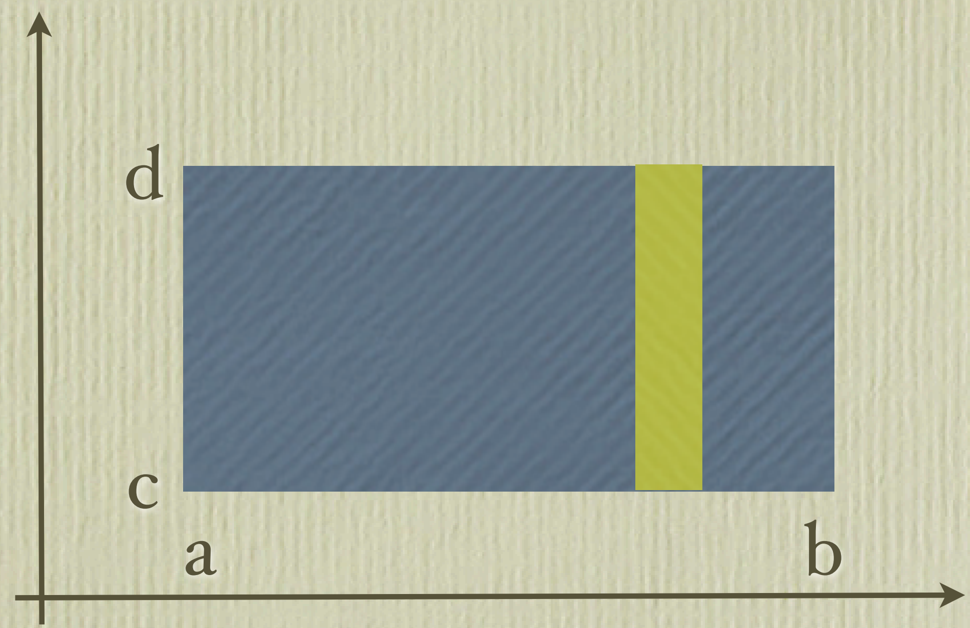
Extremize the area under the constraint that

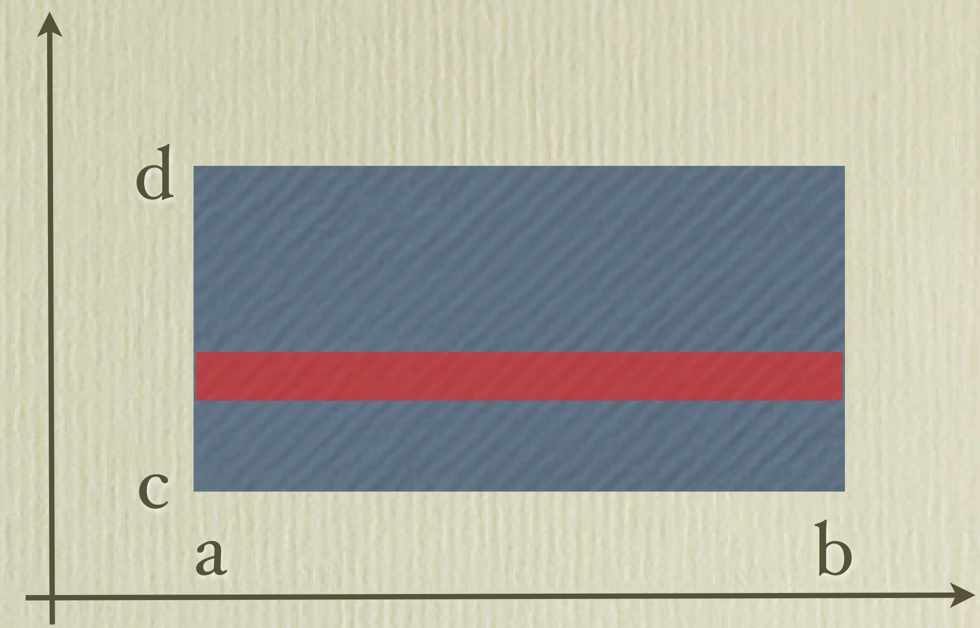
$$2\pi a + 2\pi b + 2\pi a b = 6\pi$$

rim sugar coating

Integration of functions of two variables.

Fubini Theorem

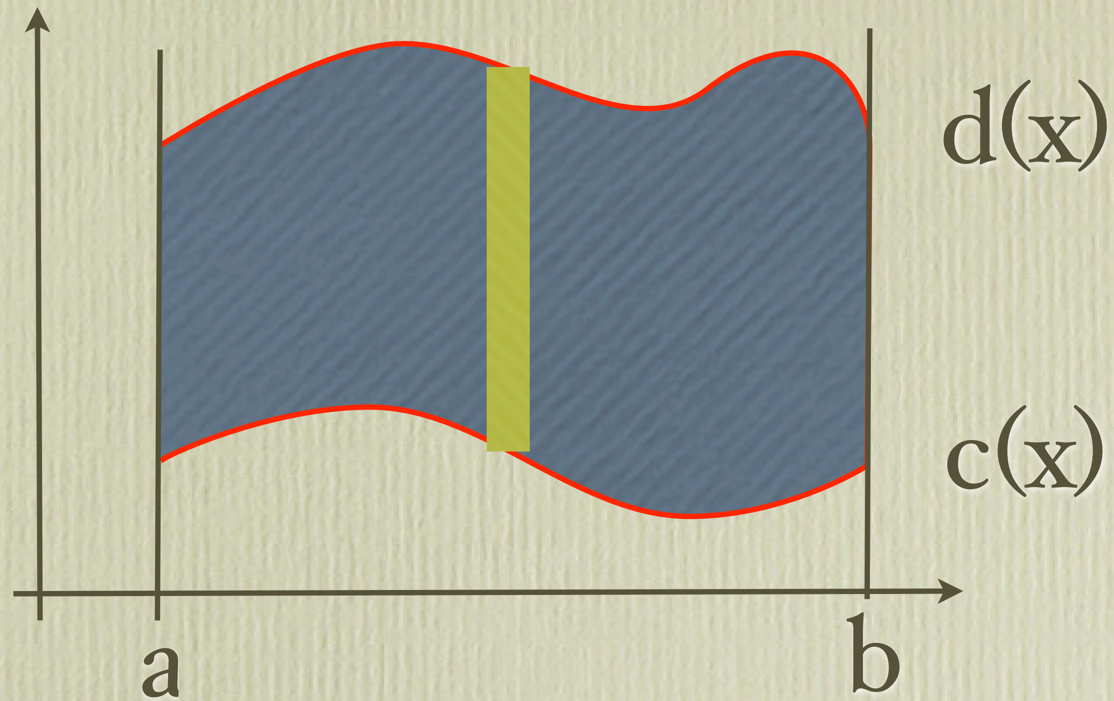




$$\int_a^b \int_c^d f(x,y)\, dy\, dx$$

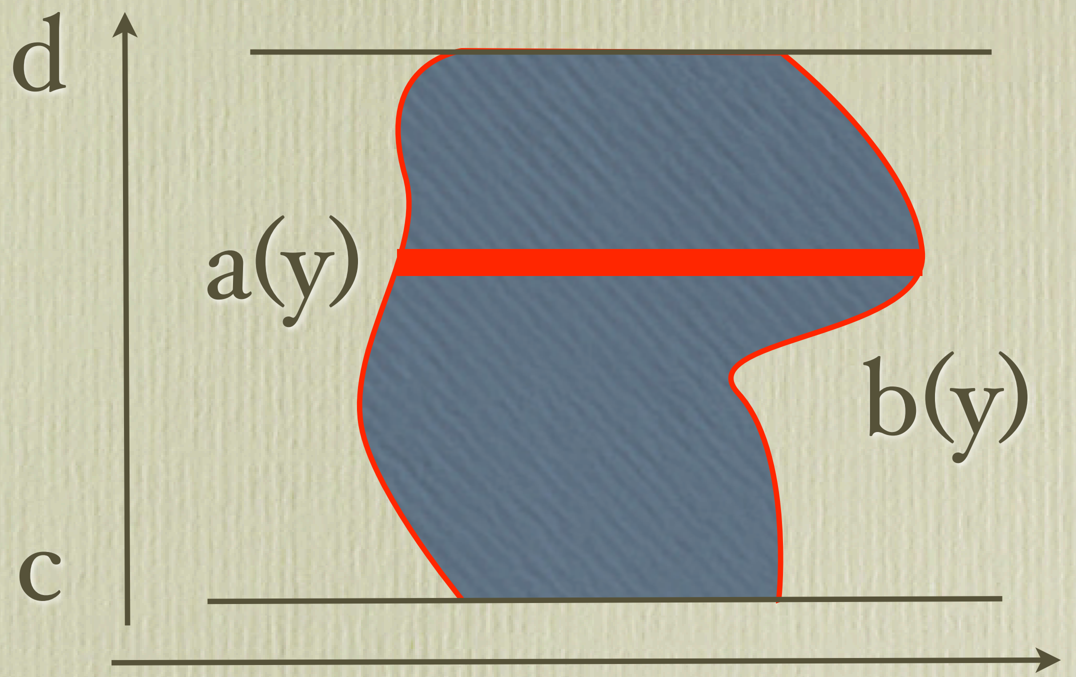
$$\int_c^d \int_a^b f(x,y)\, dx\, dy$$

Type I integrals



$$\int_a^b \int_{c(x)}^{d(x)} f(x,y)\, dy\, dx$$

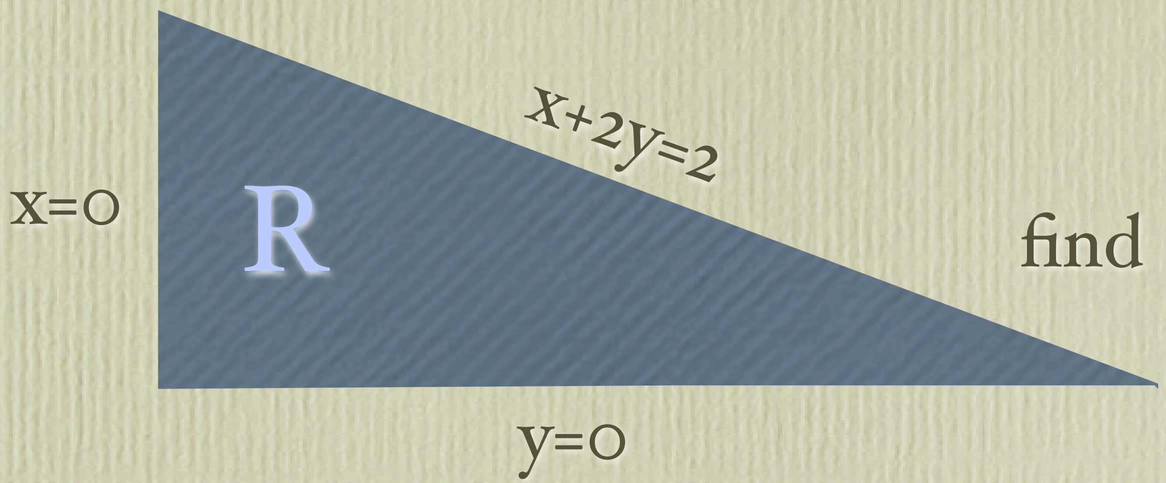
Type II integrals



$$\int_c^d \int_{a(y)}^{b(y)} f(x,y)\, dx\, dy$$

Order of integration!

Problem:

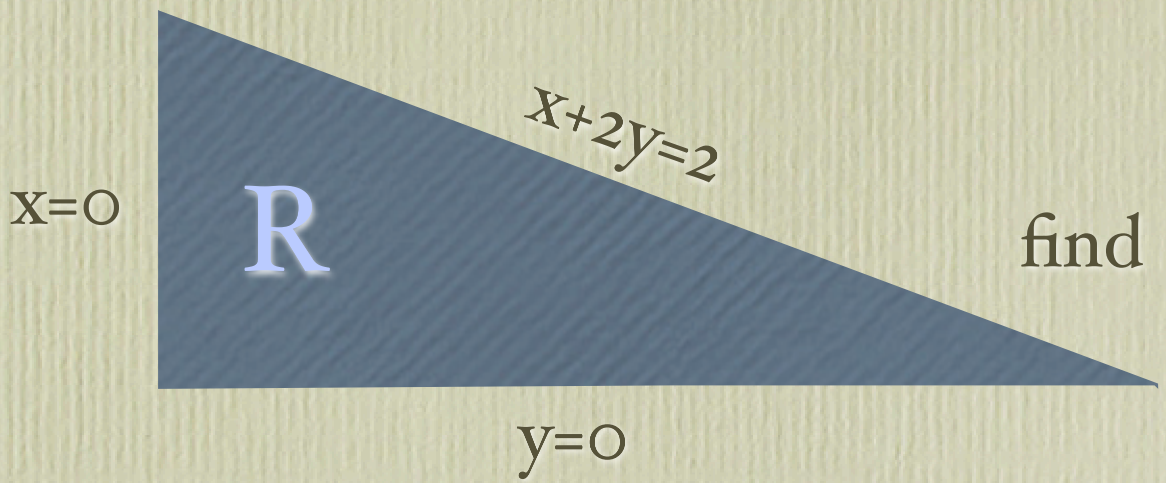


find $\iint_R e^y/(y+1)\, dA$

As a type I region

As a type II region

Problem:

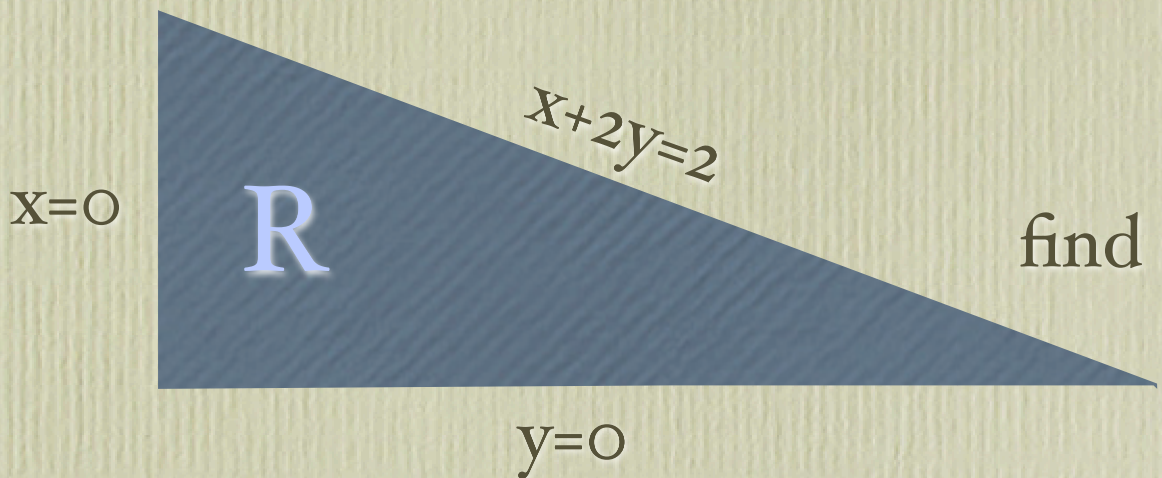


find $\displaystyle\iint_R e^y/(y+1)\, dA$

As a type I region $\displaystyle\int_0^2 \int_0^{x/2-1} e^y/(y+1)\, dy\, dx$

As a type II region

Problem:



find $$\iint_R e^y/(y+1)\, dA$$

As a type I region $$\int_0^2\int_0^{x/2-1} e^y/(y+1)\, dy\, dx$$

As a type II region $$\int_0^1\int_0^{2+2y} e^y/(y+1)\, dx\, dy$$

Problem on setting up integrals:

Let R be the region bounded by

$$y = x^2 - 1$$

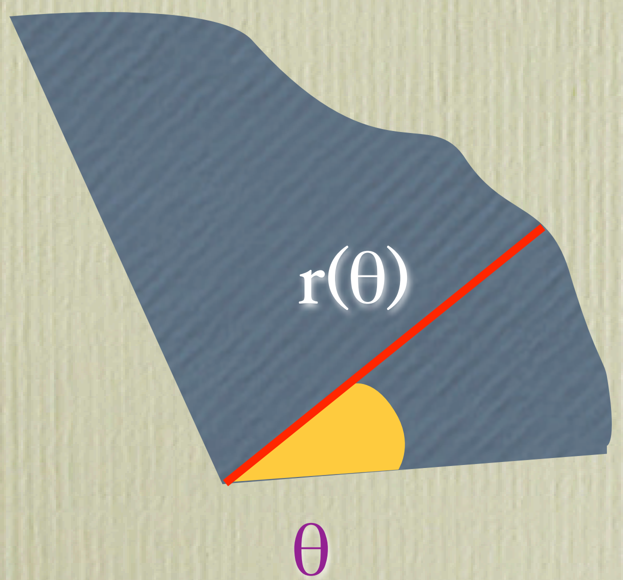
and the unit circle $x^2 + y^2 = 1$ Find

$$\iint_R f(x,y)\, dA$$

Polar integration

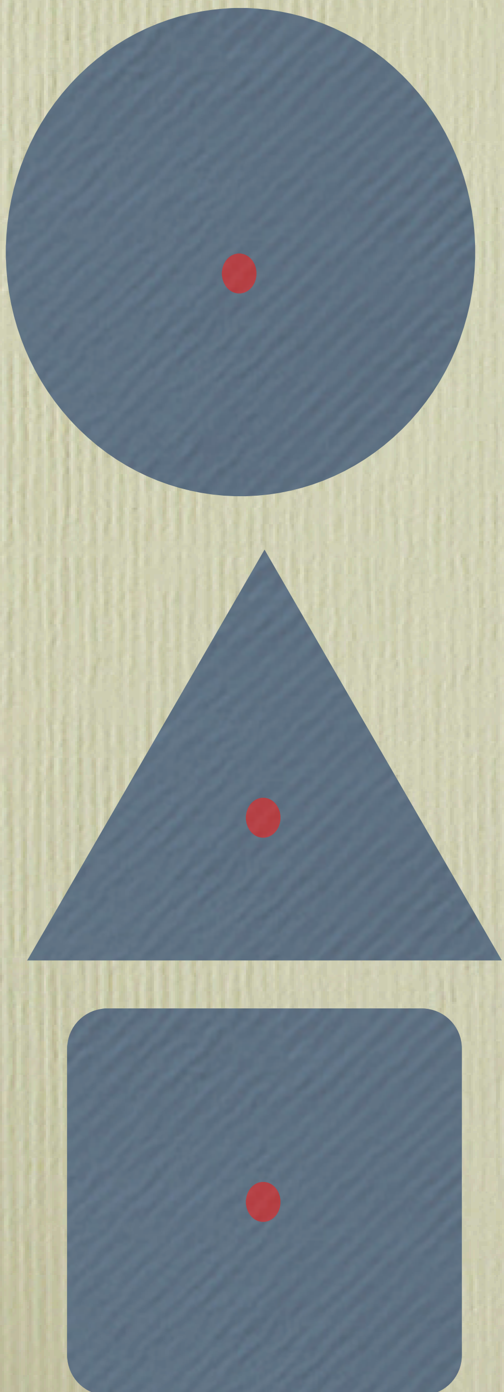
$$\iint f(x,y)\,dx\,dy \quad = \iint f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta$$

Typical region

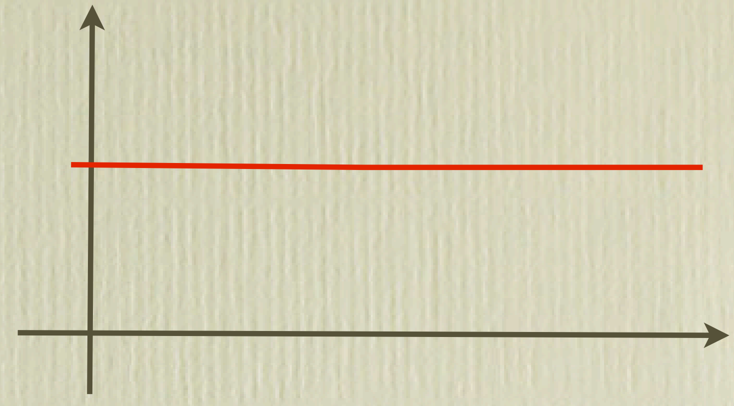


$$\int_a^b \int_0^{r(\theta)}$$

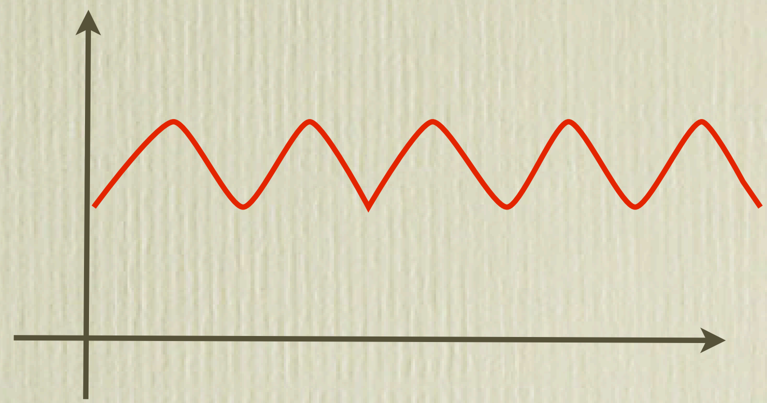
Matching regions with polar curves



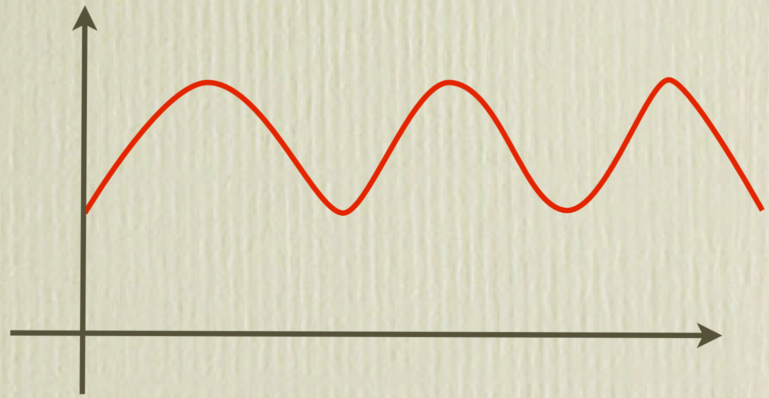
A $\displaystyle\int_0^{2\pi}\int_0^{f(\theta)}$



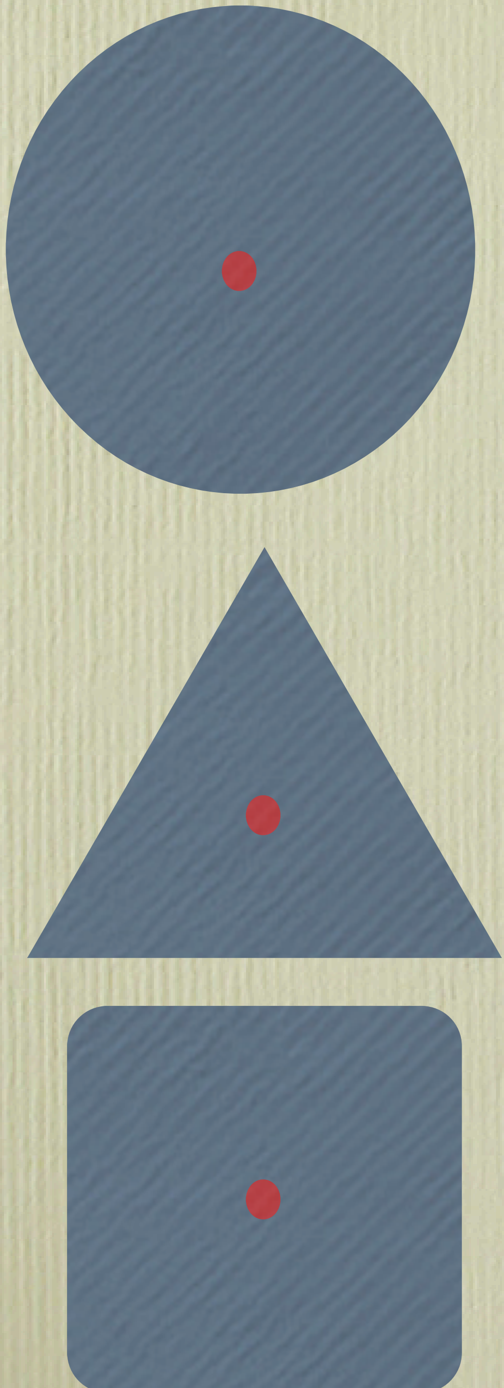
B $\displaystyle\int_0^{2\pi}\int_0^{g(\theta)}$



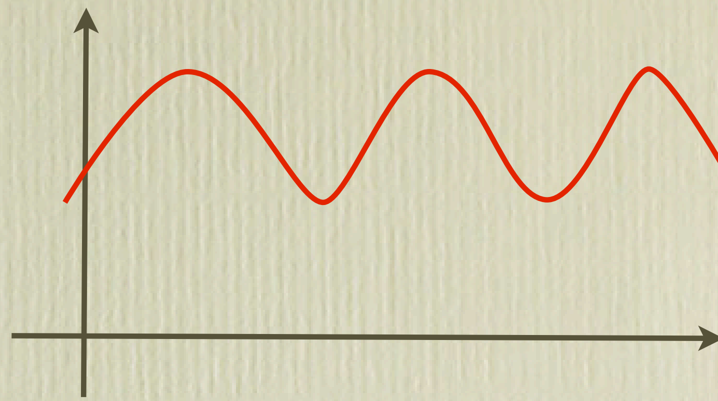
C $\displaystyle\int_0^{2\pi}\int_0^{h(\theta)}$



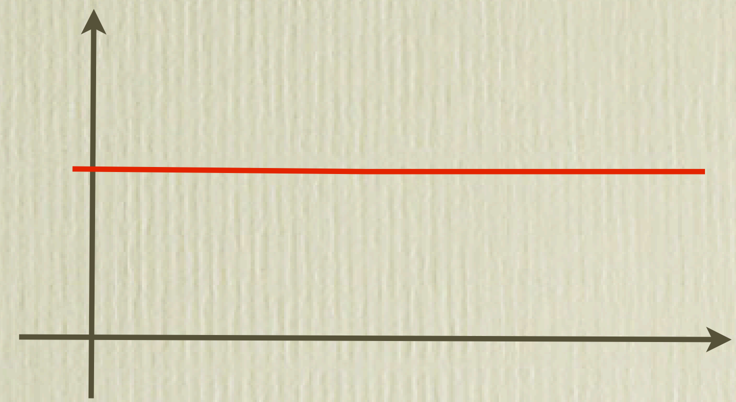
Matching regions with polar curves



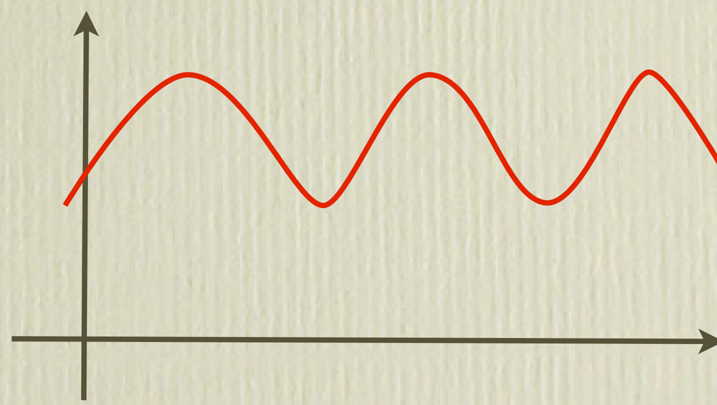
A $\displaystyle\int_0^{2\pi}\int_0^{g(\theta)}$



B $\displaystyle\int_0^{2\pi}\int_0^{f(\theta)}$



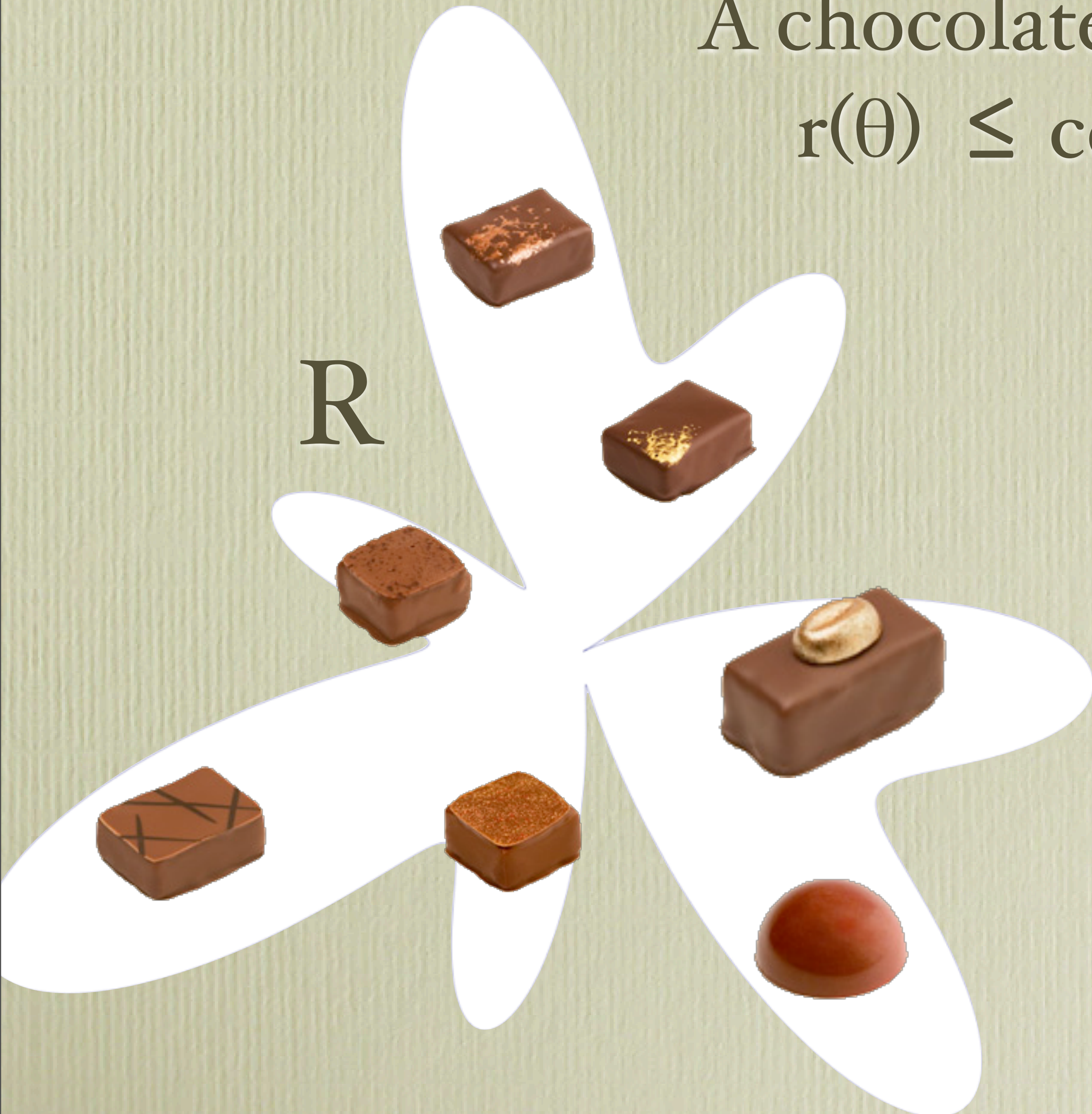
C $\displaystyle\int_0^{2\pi}\int_0^{g(\theta)}$



Chocolate cookie problem

A chocolate cookie is satisfies

$$r(\theta) \le \cos(7\,\theta) - \sin(3\,\theta) + 2$$



Find:

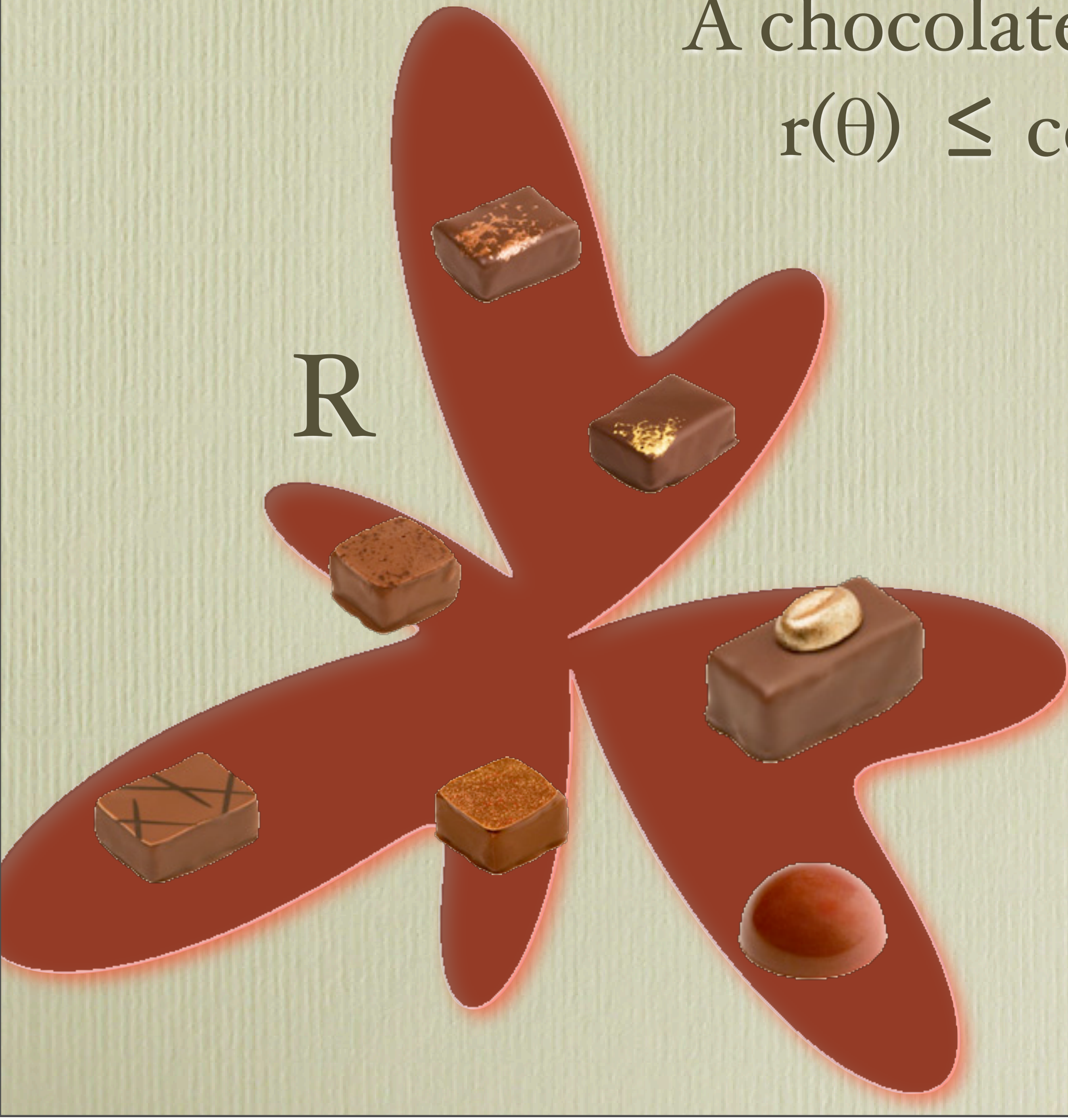
$$\iint_R 1 \, dx \, dy$$

Chocolate cookie problem

A chocolate cookie is satisfies

$$r(\theta) \leq \cos(7\,\theta) - \sin(3\,\theta) + 2$$

R



Find:

$$\iint_R 1 \, dx\, dy$$

Advise

- make a picture - in any case!
- consider change order of integration
- are Polar coordinates better?