Name:

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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-2, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) T **F** Every function $f(x,y)$ of two variables has either a global minimum or a global maximum.

Solution:
Take for example $f(x,y) = x + y$. This function has a constant nonzero gradient and so no critical point. It is unbounded above and below.

2) **T** F The linearization of the function $f(x,y) = e^{x+3y}$ at $(0,0)$ is $L(x,y) = 1 + x + 3y$.

Solution:
Use the definition of linearization. The gradient of f is $\nabla f = \langle e^{x+3y}, 3e^{x+3y} \rangle$. At $(0,0)$this is $\langle 1, 3 \rangle$. We have $f(0,0) = 1$ so that $L(x,y) = 1 + x + 3y$.

3) **T** F The function $f(x,y,z) = x^2 \cos(z) + x^3 y^2 z + (y-2)^3 y^5$ satisfies the partial differential equation $f_{xyxzxy} = 12$.

Solution:
Use Clairot.

4) **T** F If $xe^z = y^2 z$, then $\partial z/\partial x = e^z/(y^2 - xe^z)$.

Solution:
This is a direct application of implicit differentiation $z_x = -f_x/f_z$.

5) **T** F The function $\cos(x^2)\cos(y^2)$ has a local maximum at $(0,0)$.

Solution:
The value at $(0,0)$ is equal to 1. The functions and so the product take values between -1 and 1.

6) **T** F The value of the double integral $\int_0^{\pi/4} \int_0^2 x^3 \cos(y) \;\, dxdy$ is the same as $(\int_0^2 x^3 \; dx)(\int_0^{\pi/4} \cos(y) \; dy)$.

Solution:
The function $\cos(y)$ is a constant for the inner integral so that we can pull it out of the inner integral.

7) T **F** The gradient of $f(x, y)$ is always tangent to the level curves of f.

Solution:
It is perpendicular

8) **T** F If $f(x, y, z) = x - 2y + z$, then the largest possible directional derivative $D_{\vec{u}}f$ at any point in space is $\sqrt{6}$.

Solution:
The gradient has length $\sqrt{6}$. The directional derivative into the direction of the gradient is the length of the gradient.

9) T **F** $\int_0^1 \int_0^1 (x^2 + y^2)\ dxdy = \int_0^1 \int_0^1 r^3\ dr d\theta$.

Solution:
While the substitution of the function and the r factor have been done correctly, the region changes. The right integral defines a sector, while the left integral is an integral over the unit square.

10) T **F** It is possible that the directional derivative $D_{\vec{v}}f$ is positive for all unit vectors $\vec{v}$.

Solution:
The directional derivative changes sign if $\vec{v}$ is replaced by $-\vec{v}$.

11) **T** F Using linearization of $f(x, y) = xy$ we can estimate $f(0.999, 1.01) \sim 1 - 0.001 + 0.01 = 1.009$.

Solution:
$L(x, y) = 1 - 1 \cdot 0.001 + 1 \cdot 0.01$.

12) T **F** Given a curve $\vec{r}(t)$ on a surface $g(x, y, z) = -1$, then $\frac{d}{dt}g(\vec{r}(t)) < 0$.

Solution:
It is zero.

13) **T** F If $f(x, y)$ has a local minimum at $(0, 0)$ then it is possible that $f_{xy}(0, 0) > 0$.

Solution:
$D = f_{xx}f_{yy} - f_{xy}^2 > 0$ is still possible, if f_{xx} and f_{yy} are large. For example $x^2 + y^2 + xy/10$ has a local minimum at $(0, 0)$ even so $f_{xy} > 0$.

14) T **F** The function $f(x, y) = -x^8 - 2x^6 - y^8$ has a local minimum at $(0, 0)$.

Solution:
One can not use the second derivative test because the discriminant is zero. But the function is zero at $(0, 0)$ and strictly negative everywhere else. Therefore, $(0, 0)$ is a global maximum. It is definitly not a minimum.

15) T **F** If $\vec{r}(t)$ is a curve in space and f is a function of three variables, then $\frac{d}{dt}f(\vec{r}(t)) = 0$ for $t = 0$ implies that $\vec{r}(0)$ is a critical point of $f(x, y, z)$.

Solution:
We can have $r(t) = (t, 0, 0)$ and $f(x, y, z) = x^2 + (y - 1)^2$.

16) T **F** Let a, b, c be the number of saddle points, maxima and minima of a function $f(x, y)$. Then $a \leq b + c$.

Solution:
Already $x^2 - y^2$ is a counter example.

17) T **F** If $f(x, y)$ is a nonzero function of two variables and R is a region, then $\int\int_R f(x, y)\, dxdy$ is the volume under the graph of f and therefore a positive value.

Solution:
if f is replaced by $-f$, then the sign of the integral changes too.

18) T F

We extremize $f(x,y)$ under the constraint $g(x,y) = c$ and obtain a solution (x_0, y_0). If the Lagrange multiplier λ is positive, then the solution is a minimum.

Solution:
There is no relation between the sign of λ and minima and maxima. Change $g = c$ to $-g = -c$ and the sign of λ changes.

19) T F

The tangent plane to a surface $f(x,y,z) = 1$ intersects the surface in exactly one point.

Solution:
Take a one sheeted hyperboloid.

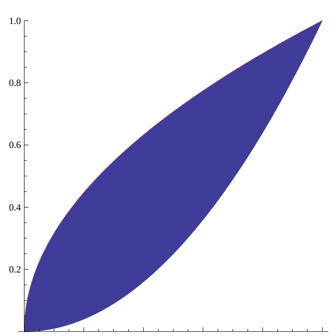
20) T F

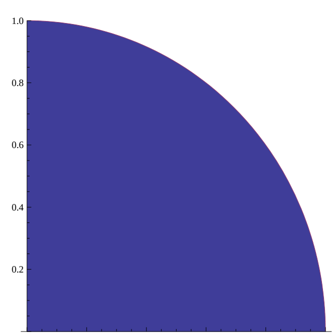
Let $\vec{v}$ be a vector of length 1 in space. Given a function $f(x,y,z)$ of three variables. If (x_0, y_0, z_0) is a critical point of f, then it is a critical point of $g(x,y,z) = D_v f(x,y,z)$.

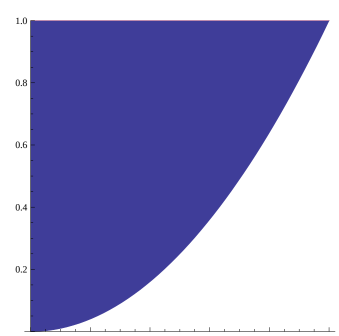
Solution:
Let $\vec{v} = \langle 1, 0, 0 \rangle$. Now $g(x,y,z) = f_x(x,y,z)$ and $\nabla g = \langle f_{xx}, f_{xy}, f_{xz} \rangle$.

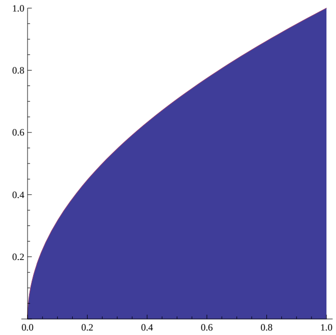
Problem 2) (10 points)

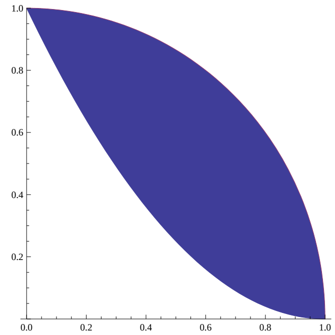
a) (6 points) Match the regions with the corresponding double integrals

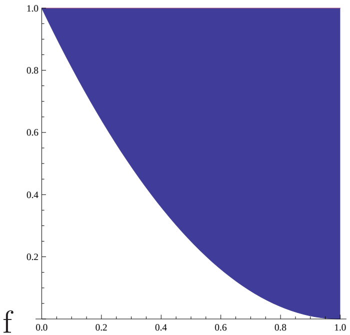
a 

b 

c 

d 

e 

f 

Enter a,b,c,d,e or f	Integral of $f(x,y)$
	$\int_0^1 \int_{x^2}^{\sqrt{x}} f(x,y)\ dydx$
	$\int_0^1 \int_0^{\sqrt{y}} f(x,y)\ dxdy$
	$\int_0^1 \int_{y^2}^1 f(x,y)\ dxdy$

Enter a,b,c,d,e or f	Integral of $f(x,y)$
	$\int_0^1 \int_0^{\sqrt{1-x^2}} f(x,y)\ dydx$
	$\int_0^1 \int_{(1-x)^2}^1 f(x,y)\ dydx$
	$\int_0^1 \int_{(1-x)^2}^{\sqrt{1-x^2}} f(x,y)\ dydx$

b) (4 points) Match the PDE's with the names. No justifications are needed.

Enter A,B,C,D here	PDE
	$f_{xx} = -f_{yy}$
	$f_x = f_y$

Enter A,B,C,D here	PDE
	$f_{xx} = f_{yy}$
	$f_x = f_{yy}$

A) Wave equation	B) Heat equation	C) Transport equation	D) Laplace equation

Solution:

a)

a	b
c	f
d	e

b)

D	A
C	B

Problem 3) (10 points)

a) (3 points) Find and classify all the critical points of $f(x,y) = xy - x$ on the plane.

b) (2 points) Decide whether an absolute maximum or an absolute minimum of f exists on the plane R^2.

c) (3 points) Use the method of Lagrange multipliers to find the maximum and minimum of f on the boundary $x^2 + 4y^2 = 12$ of the elliptical region $G : x^2 + 4y^2 \leq 12$.

d) (2 points) Find the absolute maximum and absolute minimum of f on the region G given in c).

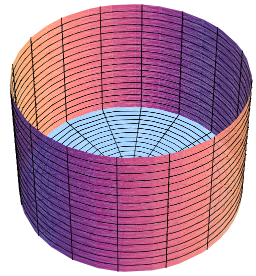
Solution:
a) $\nabla f = \langle y - 1, x \rangle = \vec{0}$ for $(x,y) = (0,1)$. Since $f_{xx} = f_{yy} = 0$ and $f_{xy} = 1$, the discriminant is $D = 0^2 - 2^2 < 0$ and $\boxed{(0,1) \text{ is a saddle point}}$.
b) There is $\boxed{\text{no global maximum, nor any global minimum}}$ on the plane. On the x-axes $y = 0$ for example, we have $f(x,0) = -x$ which is unbounded both from above and from below.
c) The Lagrange equations are

$$\begin{aligned} y - 1 &= \lambda\, 2x \\ x &= \lambda\, 8y \\ x^2 + 4y^2 &= 12\,. \end{aligned}$$

$y \neq 0$, because otherwise the second equation would give $x = 0$, contradicting the constraint. Also $x \neq 0$, because otherwise, the first equation would give $y = 1$, again contradicting the constraint. Dividing the first by the second gives $(y-1)/x = (1/4)x/y$ or $4y(y-1) = x^2$. Plugging this into the constraint gives $4y(y-1) + 4y^2 = 12$. The solutions of this quadratic equation are $y = 3/2$ or $y = -1$. The extrema are $\boxed{(\pm 2\sqrt{2}, -1) \text{ and } (\pm\sqrt{3}, 1.5)}$. Since $f(2\sqrt{2}, -1) = -4\sqrt{2}, f(-2\sqrt{2}, -1) = 4\sqrt{2}, f(\sqrt{3}, 1.5) = \frac{\sqrt{3}}{2}$ and $f(-\sqrt{3}, 1.5) = -\frac{\sqrt{3}}{2}$, $\boxed{\text{the maximum is } (x,y) = (-2\sqrt{2}, -1)}$ and $\boxed{\text{the minimum is } (x,y) = (2\sqrt{2}, -1)}$.
d) From parts (a) and (c) we have a list of all candidates for global extrema. The global maximum value of f on G is $f(-2\sqrt{2}, -1) = \boxed{4\sqrt{2}}$, the global minimal value on G is $f(2\sqrt{2}, -1) = \boxed{-4\sqrt{2}}$.

Problem 4) (10 points)

Find the cylindrical basket which is open on the top has has the largest volume for fixed area π. If x is the radius and y is the height, we have to extremize $f(x,y) = \pi x^2 y$ under the constraint $g(x,y) = 2\pi xy + \pi x^2 = \pi$. Use the method of Lagrange multipliers.



Solution:
The Lagrange equations are

$$\begin{aligned} 2xy\pi &= (2x\pi + 2y\pi)\lambda \\ \pi x^2 &= 2\pi x\lambda \\ \pi x^2 + 2\pi xy &= \pi \end{aligned}$$

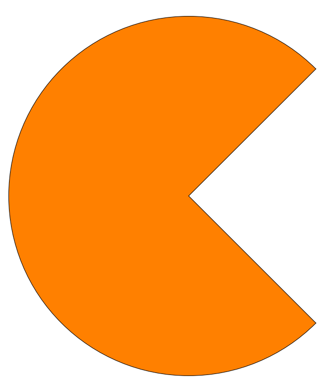
Since $x = 0$ is not possible (it would violate the constraint), we can divide the second equations by x and divide the first by the second equation. This gives $\boxed{x = y = 1/\sqrt{3}}$. The maximum value is $\boxed{\pi\sqrt{3}/9}$.

Problem 5) (10 points)

The Pac-Man region R is bounded by the lines $y = x, y = -x$ and the unit circle. The number

$$a = \frac{\int\int_R x \; dxdy}{\int\int_R 1 \; dxdy}$$

defines the point $C = (a, 0)$ called center of mass of the region. Find it.



Solution:

$$\int_{\pi/4}^{7\pi/4} \int_0^1 r\cos(\theta)\ r dr d\theta = (1/3)\sin(\theta)|_{\pi/4}^{7\pi/4} = -\sqrt{2}/3 \ .$$

$$\int_{\pi/4}^{7\pi/4} \int_0^1 r dr d\theta = (1/2)(7\pi/4 - \pi/4) = 6\pi/8 = 3\pi/4 \ .$$

The second integral is the area of the Pac-Man, which is 3/4 of the area of the full disc. Dividing the first by the second integral gives the result $\boxed{a = -4\sqrt{2}/(9\pi)}$. The center of mass is $\boxed{(-4\sqrt{2}/(9\pi), 0)}$.

Problem 6) (10 points)

a) (5 points) Find the tangent plane to the surface $\sqrt{xyz} = 60$ at $(x, y, z) = (100, 36, 1)$.

b) (5 points) Estimate $\sqrt{100.1 * 36.1 * 0.999}$ using linear approximation. Here, for clarity reasons, we use $*$ for the usual multiplication for numbers.

Solution:
a) We have

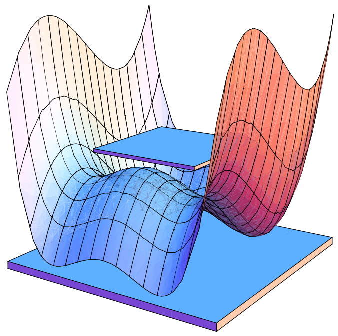
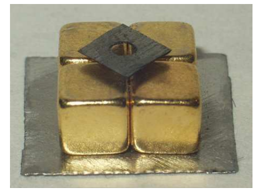
$$\nabla f(x, y, z) = \langle \sqrt{\frac{yz}{x}}, \sqrt{\frac{xz}{y}}, \sqrt{\frac{xy}{z}} \rangle / 2$$

$$\nabla f(100, 36, 1) = \langle \frac{6}{10}, \frac{10}{6}, 60 \rangle / 2$$

The tangent plane is $\boxed{(3/10)x + (5/6)y + 30z = 90}$. We have obtained the constant on the right by plugging in the point $(x, y, z) = (100, 36, 1)$.
b) Since $f(100, 36, 1) = 60$, we have $L(x, y, z) = 60 + (3/10)(x-100) + (5/6)(y-36) + 30(z-1)$. We have $L(100.1, 36.1, 0.999) = 60 + 0.03 + 0.08333.. - 0.03 = \boxed{60.08333... = 60 + 1/12}$. This is very close to the actual value 60.0832455.... You have in this problem computed the square root of a real number by hand with an accuracy of 4 digits after the comma.

Problem 7) (10 points)

Oliver got a diagmagnetic kit, where strong magnets produce a force field in which pyrolytic graphic flots. The gravitational field produces a well of the form $f(x, y) = x^4 + y^3 - 2x^2 - 3y$. Find all critical points of this function and classify them. Is there a global minimum?

Right picture credit: Wikipedia.

Solution:
To find the critical points, we have to solve the system of equations $f_x = 4x^3 - 4x = 0, f_y = 3y^2 - 3 = 0$. The first equation gives $x = 0$ or $x = \pm 1$. The second equation $f_y = 3y^2 - 3 = 0$ gives $y = \pm 1$. There are $3 \cdot 2 = 6$ critical points. We compute the discriminant $D = 6y(12x^2 - 4)$ and $f_{xx} = 12x^2 - 4$ at each of the 6 points and use the second derivative test to determine the nature of the critical point.

point	D	$f_x x$	nature	value
(-1, -1)	-48	8	saddle	1
(-1, 1)	48	8	min	-3
(0, -1)	24	-4	max	2
(0, 1)	-24	-4	saddle	-2
(1, -1)	-48	8	saddle	1
(1, 1)	48	8	min	-3

There is no global minimum, nor any global maximum since for $x = 0$, the function is $f(0, y) = y^3 - 3y$ which is unbounded from above and from below (it goes to $\pm\infty$ for $y \to \pm\infty$).

Problem 8) (10 points)

Let $f(x, y) = xy$.

a) (2 points) Find the direction of maximal increase at the point $(1, 1)$.

b) (3 points) Find the directional derivative at $(1, 1)$ in the direction $\langle 3/5, 4/5 \rangle$.

c) (2 points) The curve $\vec{r}(t) = \langle \sqrt{2}\sin(t), \sqrt{2}\cos(t) \rangle$ passes through the point $(1, 1)$ at some time t_0. Find $\frac{d}{dt} f(\vec{r}(t))$ at time t_0 directly.

d) (3 points) Find $\frac{d}{dt} f(\vec{r}(t))$ at time t_0 using the multivariable chain rule.

Solution:
a) $\nabla f(x,y) = \langle y, x \rangle$, $\nabla f(1,1) = \langle 1, 1 \rangle$. The direction of maximal increase is $\boxed{\langle 1,1 \rangle/\sqrt{2}}$.
b) $D_v f(1,1) = \langle 1,1 \rangle \cdot \langle 3/5, 4/5 \rangle = \boxed{7/5}$.
c) It is at the time $t_0 = \pi/4$, where the curve passes through thte point $(1,1)$. We have $f(\vec{r}(t)) = 2\cos(t)\sin(t) = \sin(2t)$ and $d/dt f(\vec{r}(t)) = 2\cos(2t)$ which is $\boxed{0}$ at time $t = \pi/4$.
d) By the multi variable chain rule, $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(1,1) \cdot \langle -\sin(\pi/4), \cos(\pi/4) \rangle = \boxed{0}$.

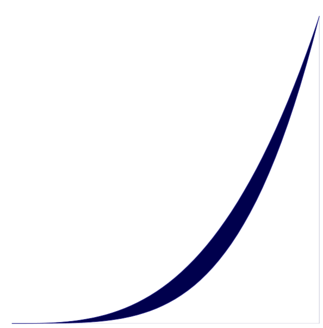
Problem 9) (10 points)

Integrate the function

$$f(x,y) = \frac{y^5 - 1}{y^{1/3} - y^{1/4}}$$

on the finite region bounded by the curves $y = x^3$ and $y = x^4$.

Solution:
Make a picture! The two graphs intersect at 0 and 1 forming a grass shaped region.



The type I integral

$$\int_0^1 \int_{x^4}^{x^3} \frac{y^5 - 1}{y^{1/3} - y^{1/4}} \, dydx$$

can not be evaluated (at least not without going through difficult substitution/partial fraction procedures which can fill pages).
We decide therefore, to change the order or integration and write a type II integral:

$$\int_0^1 \int_{y^{1/3}}^{y^{1/4}} \frac{y^5 - 1}{y^{1/3} - y^{1/4}} \, dxdy$$

Now the inner integral can be solved and give $(1 - y^5)$. We end up with $\int_0^1 (1 - y^5) \, dy = \boxed{5/6}$.

Problem 10) (10 points)

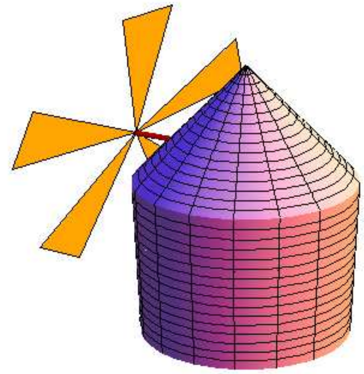
The main building of a mill has a cone shaped roof and cylindrical walls. If the cylinder has radius r, the height of the side wall is h and the height of the roof is h, then the volume is

$$V(h,r) = \pi r^2 h + h\pi r^2/3 = (4\pi/3)hr^2$$

and assume the cost of the building is

$$A(h,r) = \pi r^2 + 2\pi rh + \pi 2r^2 = \pi(3r^2 + 2rh)$$

which is the area of the ground plus the area of the wall plus $2\pi rh$, the cost for the roof. For fixed volume $V(h,r) = 4\pi/3$, minimize the cost $A(h,r)$ using the Lagrange multiplier method.



Solution:
After dividing out some constants and taking $g = hr^2 = 1$, the Lagrange equations become

$$\begin{aligned} 6r + 2h &= \lambda 2hr \\ 2r &= \lambda r^2 \\ r^2 h &= 1 \end{aligned}$$

The second equation can be divided by r since $r = 0$ is incompatible with the third equation. The first can be divided by 2. We get

$$\begin{aligned} 3 * r + h &= \lambda hr \\ 2 &= \lambda r \\ r^2 h &= 1 \end{aligned}$$

You can plug in λr from the second equation into the first to get

$$\begin{aligned} 3r + h &= 2h \\ r^2 h &= 1 \end{aligned}$$

The first equation shows $h = 3r$ and plugging this into the third equation gives $\boxed{r = 1/3^{1/3}}$ and $\boxed{h = 3r = 3^{2/3}}$.