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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-2, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) T **F** The directional derivative $D_{\vec{v}}f$ is a vector perpendicular to $\vec{v}$.

Solution:
The directional derivative is a scalar, not a vector.

2) **T** F Using linearization of $f(x, y) = xy$ we can estimate $f(0.9, 1.2) \sim 1 - 0.1 + 0.2 = 1.1$.

Solution:
$L(x, y) = 1 - 1 \cdot 0.1 + 1 \cdot 0.2$.

3) **T** F Given a curve $\vec{r}(t)$ on a surface $g(x, y, z) = 1$, then $\frac{d}{dt}g(\vec{r}(t)) = 0$.

Solution:
This fact is used in the proof that level surfaces are perpendicular to gradients.

4) T **F** Given a function $f(x, y)$ such that $\nabla f(0, 0) = \langle 2, -1 \rangle$. Then $D_{\langle 0,-1 \rangle} f(0, 0) = 0$.

Solution:
The directional derivative in the direction $\langle 0, -1 \rangle$ is equal to 1 which is nonzero.

5) T **F** $\vec{r}(u, v) = \langle u \cos(v), u \sin(v), v \rangle$ is a surface of revolution.

Solution:
The parametrization given is a helicoid and not rotationally symmetric. It ressembles the parametrization $\vec{r}(u, v) = \langle u \cos(v), u \sin(v), u \rangle$ of the cone, which is rotationally symmetric.

6) **T** F If $(1, 1)$ is a critical point for the function $f(x, y)$ then $(1, 1)$ is also a critical point for the function $g(x, y) = f(x^2, y^2)$.

Solution:
If $\nabla f(1,1) = (f_x(1,1), f_y(1,1)) = (0,0)$ then also $\nabla g(1,1) = (f_x(1,1)2x, f_y(1,1)2y) = (0,0)$.

7) T **F** If $f(x,y)$ has a local maximum at $(0,0)$ then it is possible that $f_{xx}(0,0) > 0$ and $f_{yy}(0,0) < 0$.

Solution:
The conditions imply that $D = f_{xx}f_{yy} - f_{xy}^2 < 0$.

8) T **F** The integral $\int_0^x \int_0^y 1 \, dxdy$ computes the area of a region in the plane.

Solution:
This is not a valid double integral. The outer integral should not contain variables.

9) **T** F The function $f(x,y) = x^2 + y^4$ has a local minimum at $(0,0)$.

Solution:
One can not use the second derivative test but the function is zero at $(0,0)$ but positive everywhere else.

10) **T** F The integral $\int_0^1 \int_0^1 x^2 + y^2 \, dxdy$ is the volume of the solid bounded by the 5 planes $x = 0, x = 1, y = 0, y = 1, z = 0$ and the paraboloid $z = x^2 + y^2$.

Solution:
In general $\int \int_R f(x,y) \, dydx$ is the volume under the graph of f

11) **T** F There exists a region in the plane, which is neither a type I integral, nor a type II integral.

Solution:
Take for example an S shaped region.

12) T **F** Fubini's theorem assures that $\int_0^1 \int_0^x f(x,y)\, dydx = \int_0^1 \int_0^y f(x,y)\, dxdy$.

Solution:
Fubini only applies to rectangular regions.

13) T **F** The function $f(x,y) = \sin(x)\cos(y)$ satisfies the partial differential equation $f_{xx} + f_{yy} = 0$.

Solution:
Just differentiate. It is a solution to the wave wequation, not the Laplace equation.

14) **T** F Let $L(x,y)$ be the linearization of $f(x,y) = \sin(x(y+1))$ at $(0,0)$. Then, the level curves of $L(x,y)$ consist of lines.

Solution:
The function $L(x,y)$ is a linear function of the form $ax + by + c$ it has lines.

15) T **F** For any smooth function $f(x,y)$, the inequality $|\nabla f| \geq |f_x + f_y|$ is true.

Solution:
If $\nabla f = \langle a, b \rangle$, we square the claim, we get $a^2 + b^2 \geq (a+b)^2$. This is wrong for $(a,b) = (1,1)$.

16) **T** F Any differentiable function $f(x,y)$ which satisfies the partial differential equation $||\nabla f||^2 = 0$ is constant.

Solution:
The condition $||\nabla f||^2 = 0$ implies that that the gradient is zero and so that all directional derivatives are zero.

17) **T** F If $x + \sin(xy) = 1$, $dy/dx = \frac{-(1+y\cos(yx))}{(x\cos(xy))}$.

Solution:
This is implicit differentiation.

18) T F — The directional derivative $D_v f(1,1)$ is zero if v is a unit vector tangent to the level curve of f which goes through $(1,1)$.

Solution:
The level curve is perpendicular to the gradient.

19) T F — If (a,b) is a maximum of $f(x,y)$ under the constraint $g(x,y) = 0$, then the Lagrange multiplier λ there has the same sign as the discriminant $D = f_{xx}f_{yy} - f_{xy}^2$ at (a,b).

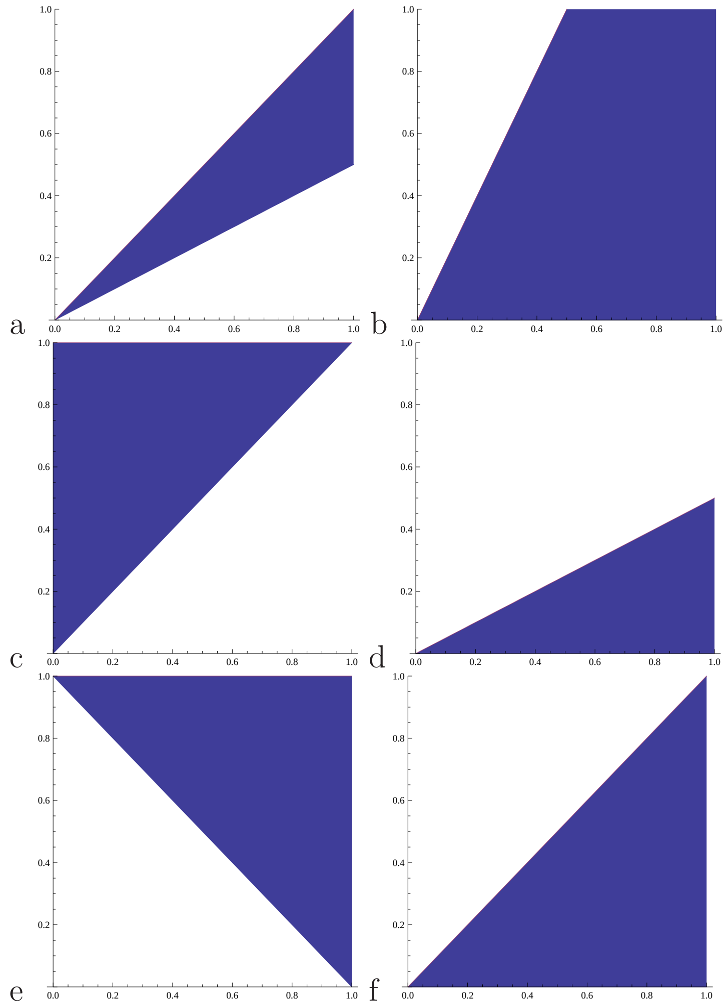
Solution:
False, by changing g to $-g$, we can change the Lagrange multiplier, but the discriminant stays the same.

20) T F — If $D_{\langle 1/\sqrt{2}, 1/\sqrt{2}\rangle} f(1,2) = 0$ and $D_{\langle -1/\sqrt{2}, 1/\sqrt{2}\rangle} f(1,2) = 0$, then $(1,2)$ is a critical point.

Solution:
Indeed, if $\nabla f = \langle a,b \rangle$, then $\langle a,b \rangle \cdot \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle = 0$ and $\langle a,b \rangle \cdot \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle = 0$ which imples $a = b = 0$.

Problem 2) (10 points)

Match the regions with the corresponding double integrals



Enter a,b,c,d,e or f	Integral of Function $f(x,y)$
	$\int_0^1 \int_{x/2}^{x} f(x,y)\; dydx$
	$\int_0^1 \int_0^{y} f(x,y)\; dxdy$
	$\int_0^1 \int_0^{x/2} f(x,y)\; dydx$
	$\int_0^1 \int_{y/2}^{1} f(x,y)\; dxdy$
	$\int_0^1 \int_0^{x} f(x,y)\; dydx$
	$\int_0^1 \int_{1-x}^{1} f(x,y)\; dydx$

Solution:
a,c,d,b,f,e.

Problem 3) (10 points)

Let $g(x, y, z) = x^2 + 2y^2 - z - 3$.
a) (5 points) Find the equation of the tangent plane to the level surface $g(x, y, z) = 0$ at the point $(x_0, y_0, z_0) = (2, 0, 1)$.
b) (5 points) The surface in a) is the graph $z = f(x, y)$ of a function of two variables. Find the tangent line to the level curve $f(x, y) = 1$ at the point $(x_0, y_0) = (2, 0)$.

Solution:
a) $\nabla g(x, y, z) = \langle 2x, 4y, -1 \rangle$.
$\nabla g(2, 0, 1) = \langle 4, 0, -1 \rangle$ The equation of the tangent plane is

$$4x + 0y - z = 7 \ .$$

where the value of 7 has been obtained by plugging in the point $(2, 0, 1)$. The final answer is $\boxed{4x - z = 7}$.
b) $z = x^2 + 2y^2 - 3 = f(x, y)$.

$$\nabla f(x, y) = \langle 2x, 4y \rangle \ .$$

and

$$\nabla f(2, 0) = \langle 4, 0 \rangle \ .$$

The equation of the tangent line is

$$4x = 8$$

or $\boxed{x = 2}$.

Problem 4) (10 points)

a) (5 points) Use the technique of linear approximation to estimate $f(\pi/2 + 0.1, 2.9)$ for

$$f(x, y) = (10 \sin(x) - 5y^2 + 8)^{1/3} \ .$$

b) (5 points) Find the unit vector at $(\pi/2, 3)$, in the direction where the function increases fastest.

Solution:
a) $f(\pi/2, 3) = -3$ and

$$\nabla f(x,y) = \frac{1}{3}(10\sin(x) - 5y^2 + 8)^{-2/3}\langle 10\cos(x), -10y\rangle \ .$$

so that

$$\nabla f(\pi/2, 3) = \frac{1}{27}\langle 0, -30\rangle = \langle 0, -10/9\rangle \ .$$

The estimation is $-3 + 0.1 \cdot 0 - 0.1 \cdot (-10/9) = -3 + 1/9 = -26/9$.
b) It is $(0, -1)$.

Problem 5) (10 points)

The pressure in the space at the position (x, y, z) is $p(x, y, z) = x^2 + y^2 - z^3$ and the trajectory of an observer is the curve $\vec{r}(t) = \langle t, t, 1/t\rangle$.

a) (2 points) State the chain rule which applies in this situation.
b) (4 points) Using the chain rule in a) compute the rate of change of the pressure the observer measures at time $t = 2$.
c) (4 points) At which time t does the observer go in the direction, in which the pressure decreases most?

Solution:
a) The multivariable chain rule is

$$\frac{d}{dt}p(\vec{r}(t)) = \nabla p(\vec{r}(t)) \cdot \vec{r}\,'(t) \ .$$

b) $\nabla p(x, y, z) = \langle 2x, 2y, -3z^2\rangle$, $\vec{r}\,'(t) = \langle 1, 1, -1/t^2\rangle$. We have $\vec{r}(2) = \langle 2, 2, 1/2\rangle$ and $\vec{r}\,'(2) = \langle 1, 1, -1/4\rangle$. By the chain rule in a), we have

$$\nabla p(2, 2, 1/2) \cdot \vec{r}\,'(2) = \langle 4, 4, -3/4\rangle \cdot \langle 1, 1, -1/4\rangle = 8 + 3/16 \ .$$

c) The direction in which the pressure decreases most at the observers position $\vec{r}(t)$ is $-\nabla p(\vec{r}(t)) = \langle -2t, -2t, 3/t^2\rangle$. The question is, when this vector is parallel to the velocity vector $\langle 1, 1, -1/t^2\rangle$. If we set

$$\langle -2t, -2t, 3/t^2\rangle = c\langle 1, 1, -1/t^2\rangle \ ,$$

we get by comparing the first coordinate $c = -2t$. The third component equation reads $3/t^2 = 2t/t^2$ and gives $3 = 2t$ leading to $\boxed{t = 3/2}$.

Problem 6) (10 points)

The coffee chain **Astrbucks** [1] has branches at $(0,0),(0,3)$ and $(3,3)$ (JFK street, Church street, and Broadway) near Harvard square. A caffeine addicted [politically correct: loving] mathematician wants to rent an apartment at a location, where the sum of the squared distances $f(x,y)$ to all those shops is a local minimum. The function is

$$f(x,y)=(x-0)^2+(y-0)^2+(x-0)^2+(y-3)^2+(x-3)^2+(y-3)^2=27-6x+3x^2-12y+3y^2\ .$$

a) (5 points) Where does the mathematician have to live to locally minimize $f(x,y)$?
b) (3 points) For every local minimum answer: Is this local minimum a **global** minimum?
c) (2 points) Is there a global maximum to this problem? If yes, give it. If no, why not?

Solution:
a) $\nabla f(x,y)=\langle 6x-6,6y-12\rangle$ is the zero vector for $(x,y)=(1,2)$. By the second derivative test, this is a local minimum.
b) Yes, this is a global minimum. The function can be written with a completion of squares as

$$27-3+(3-6x+3x^2)-12+(12-12y+3y^2)=12+3(1-x)^2+3(2-y)^2$$

which has as a graph an elliptic paraboloid with global minimum at $(1,2)$. Many different attempts have been used here to justify the fact that we have a global minimum. Note it is not true that if a function $f(x,y)$ has one local minimum and no other critical point, then this local minimum has to be a global minimum. An example (provided by Chen-Yu Chi) is $f(x,y)=x^3+e^{(3y)}-3xe^y$. It has a local minimum at $(1,0)$ but not other critical point and no global minimum nor maximum.
c) It is possible to argue that the function increases monotonically if x^2+y^2 is large enough and goes to ∞. This prevents the existence of a global maximum.

Problem 7) (10 points)

[1] This problem was sponsored by *Astrbucks*©.

Find all the critical points of $f(x, y) = 3xy + x^2y + xy^2$ and classify them as saddle points, local maxima or local minima.

Solution:
The gradient is

$$\nabla f(x, y) = \langle 3y + 2xy + y^2, 3x + x^2 + 2xy \rangle \ .$$

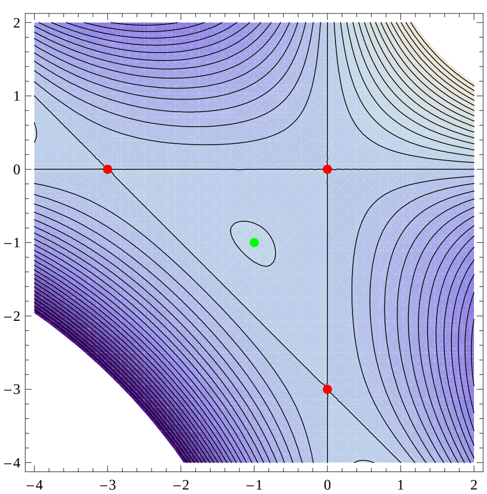
Factor out y in the first component and x in the second component.

$$\nabla f(x, y) = \langle y(3 + 2x + y), x(3 + x + 2y) \rangle = \langle 0, 0 \rangle \ .$$

If $y = 0$, then the first equation holds and the second equation needs either $x = 0$ or $x = -3$. If $x = 0$ then the second equation holds and the first equation needs either $y = 0$ or $y = -3$. If both x and y are not zero, then $3 + 2x + y = 0, 3 + x + 2y = 0$ which has the solution $x = y = -1$. We have therefore 4 solutions. We evaluate the discriminant $D = 4xy - (3 + 2x + 2y)^2$ and the second derivative $f_{xx} = 2y$ at each point:

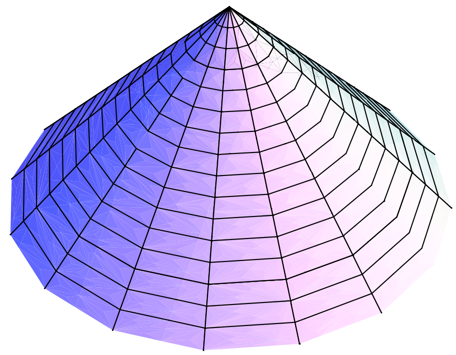
point	D	f_{xx}	nature of the critical point
(-3,0)	-9	0	saddle point
(-1,-1)	3	-2	local max
(0,-3)	-9	-6	saddle point
(0,0)	-9	0	saddle point

The picture shows some level curves of the function $f(x, y)$:



Problem 8) (10 points)

A solid cone of height h and with base radius r has the volume $f(h, r) = h\pi r^2/3$ and the surface area $g(h, r) = \pi r\sqrt{r^2 + h^2} + \pi r^2$. Among all cones with fixed surface area $g(h, r) = \pi$ use the Lagrange method to find the cone with maximal volume.



Solution:
The Lagrange equations are after dividing out π factors in all 3 equations

$$\begin{aligned} f_r = 2rh/3 &= \lambda(2r + \sqrt{r^2+h^2} + r^2/\sqrt{r^2+h^2} = g_r \\ f_h = r^2/3 &= \lambda rh/\sqrt{r^2+h^2} = g_h \\ r\sqrt{r^2+h^2} + r^2 &= 1 \end{aligned}$$

Because $r = 0$ is incompatible with the third equation, we can divide by r in the second and third equation equation:

$$\begin{aligned} 2rh/3 &= \lambda(2r + \sqrt{r^2+h^2} + r^2/\sqrt{r^2+h^2} \\ r/3 &= \lambda h/\sqrt{r^2+h^2} \\ \sqrt{r^2+h^2} &= \frac{1-r^2}{r} \end{aligned}$$

Now plug the third equation into the first two to have a simpler system and also square the third equation

$$\begin{aligned} 2rh/3 &= \lambda(2r + (1-r^2)/r + r^3/(1-r^2)) \\ r/3 &= \lambda hr/(1-r^2) \\ r^2+h^2 &= \frac{(1-r^2)^2}{r^2} \end{aligned}$$

We get rid of λ by dividing the first by the second equation

$$\begin{aligned} 2h &= 1/(hr^2) \\ r^2+h^2 &= \frac{(1-r^2)^2}{r^2} \end{aligned}$$

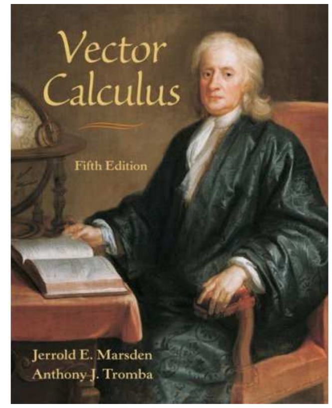
The first equation gives $h^2 = 1/(2r^2)$. Plugging this into the third gives $r = 1/2$. The solution is $\boxed{r = 1/2, h = \sqrt{2}}$.

Problem 9) (10 points)

Marsden and Tromba pose in their textbook the following riddle: Suppose $w = f(x, y)$ and $y = x^2$. By the chain rule

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + 2x\frac{\partial w}{\partial y}$$

so that $0 = 2x\frac{\partial w}{\partial y}$ and so $\frac{\partial w}{\partial y} = 0$.



a) Find an explicit example of a function $f(x, y)$, where you see the argument is false.

b) What is flawed in the above application of the chain rule?

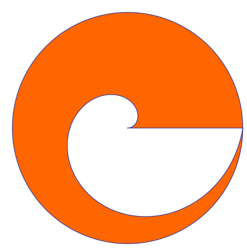
Solution:
a) Take $w = f(x, y) = x^2 + y^2$ for example. Then $w_x = 2x, w_y = 2y$. b) On the left hand side of the chain rule, we have not a partial derivative any more. Lets write this more clearly. We have a function of two variables x, y and both variables x, y are functions of a third variable t. In our case $x(t) = t, y(t) = t^2$. The chain rule gives the derivative $\frac{d}{dt}f(x(t), y(t))$ as $f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) = f_x(t, t^2) + f_y(t, t^2)2t$. The false argument had written the left hand side of the chain rule as f_x again. Because the variable t was the variable x itself, this confusion was possible.

Problem 10) (10 points)

Evaluate the double integral

$$\int\int_R \sqrt{x^2 + y^2}\; dxdy$$

where R is the region bounded by the positive x-axes, the spiral curve $\vec{r}(t) = \langle t\cos(t), t\sin(t)\rangle, 0 \leq t \leq 2\pi$ and the circle with radius 2π.



Solution:
Use polar coordinates, where $\sqrt{x^2 + y^2} = r$ and where $dxdy$ is replaced by $rdrd\theta$.

$$\int_0^{2\pi}\int_\theta^{2\pi} r^2\; drd\theta = \int_0^{2\pi} (8\pi^3 - \theta^3)/3\; d\theta\ .$$

This integral is $\boxed{4\pi^4}$.