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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-2, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) T F (answer: F)

Given a unit vector v, define $g(x) = D_v f(x)$. If at a critical point, for all vectors v we have $D_v g(x) > 0$, then f is a local maximum.

Solution:
On every line through the critical point, we have a local minumum. So, it is a local minimum, not a local max.

2) T F (answer: T)

Assume f satisfies the PDE $f_x = f_y$. If $g = f_x$, then $g_x = g_y$.

Solution:
Use Clairot's theorem tells $g_x = f_{yx} = f_{xy} = g_y$.

3) T F (answer: F)

The equation $\phi = \pi/4$ in spherical coordinates ($\rho \geq 0, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$ as usual) and the surface $x^2 + y^2 = z^2$ (with no further restrictions on x, y, z) are the same surface.

Solution:
The first equation is a single cone, the second equation is a double cone.

4) T F (answer: F)

Even with $f_x(a, b) = 0$ and $f_y(a, b) = 0$, it is possible that some directional derivative $D_{\vec{v}}(f)$ of $f(x, y)$ at (a, b) is non-zero.

Solution:
We have $D_v f(x, y) = f_x v_1 + f_y v_2$.

5) T F (answer: T)

There exists a pair of different points on a sphere, for which the tangent planes are parallel.

Solution:
Take the antipodes.

6) T F (answer: T)

If $\vec{u}$ is a unit vector tangent at (x, y, z) to the level surface of $f(x, y, z)$ then $D_u f(x, y, z) = 0$.

Solution:
The directional derivative measures the rate of change of f in the direction of u. On a level surface, in the direction of the surface, the function does not change (because f is constant by definition on the surface).

7) | T | **F** | Assume we have a smooth function $f(x,y)$ for which the lines $x = 0, y = 0$ and $x = y$ are level curves $f(x,y) = 0$. Then $(0,0)$ is a critical point with $D < 0$.

Solution:
It can not be a saddle point, nor can it be a local maximum or local minimum. If it would be a saddle point, the level curves through the point consist of two lines. A concrete example, where three level curves pass through the same point is a monkey saddle. This is a case, where $D = 0$.

8) | T | **F** | The gradient of $f(x,y)$ is perpendicular to the graph of f.

Solution:
This is a misconception. The gradient of f is a two dimensional vector. It is perpendicular to level curves.

9) | **T** | F | The level curves of a linearization $L(x,y)$ of a function $f(x,y) = \sin(x+y)$ at $(0,0)$ consist of lines.

Solution:
The function $L(x,y)$ is a linear function of the form $ax + by + c$.

10) | T | **F** | If $x^4 y + \sin(y) = 0$ then $y' = 4x^3 y/(x^4 + \cos(y))$.

Solution:
The sign is wrong.

11) | **T** | F | The linearization $L(x,y)$ at a critical point (x_0, y_0) of a function $f(x,y)$ is a constant function.

Solution:
If the gradient is $(a, b) = \nabla f$, then the linearization has the form $L(x, y) = ax + by + c$.

12) T **F** The surface $x^2 + y^2 - z^2 = 1$ has a parametrization of the form $\langle x(s,t), y(s,t), z(s,t)\rangle = \langle s, t, f(s,t)\rangle$ for some function $f(s,t)$ for which the parametrization covers the entire surface.

Solution:
The surface is not a graph.

13) **T** F The tangent plane to the graph of $f(x, y)$ at a point $(x_0, y_0, f(x_0, y_0))$ is a level surface of the linearization $L(x, y, z)$ of $z - f(x, y)$.

Solution:
It is the level surface $L(x, y, z) = f(x_0, y_0, f(x_0, z_0)$.

14) **T** F The critical points of $F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$ are solutions to the Lagrange equations when extremizing the function $f(x, y)$ under the constraint $g(x, y) = 0$.

Solution:
The critical points of F are points where $f_x = \lambda g_x, f_y = \lambda g_y, g = 0$ which is exactly the Lagrange equations.

15) T **F** The curve defined by $z = 1, \theta = \frac{\pi}{4}$ in cylindrical coordinates is a circle.

Solution:
It is a half line, the intersection of a plane with a half plane.

16) **T** F If $(0, 0)$ is a critical point of $f(x, y)$ and the discriminant D is zero but $f_{xx}(0, 0) > 0$ then $(0, 0)$ can not be a local maximum.

Solution:
If $f_{xx}(0, 0) > 0$ then on the x-axes the function $g(x) = f(x, 0)$ has a local minimum. This means that there are points close to $(0, 0)$ where the value of f is larger.

17) T F If $f(x,y,z) = x^2 + y^2 + z^2$, then $\nabla f = 2x + 2y + 2z$.

Solution:
The gradient is a vector, not a scalar.

18) T F A function $f(x,y)$ in the plane always has a local minimum or a local maximum.

Solution:
Take the example $f(x,y) = xy$.

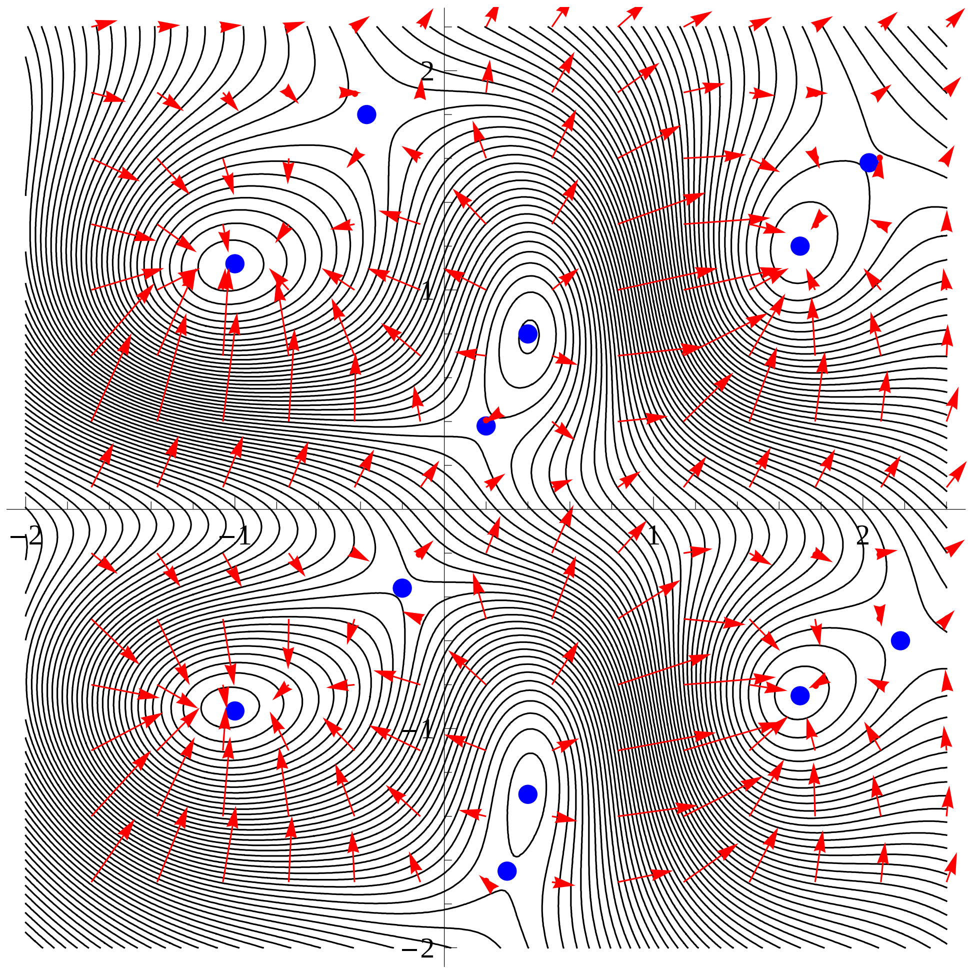
19) T F For any smooth function $f(x,y)$, the inequality $\|\nabla f\| \geq |f_x + f_y|$ is true.

Solution:
If $\nabla f = \langle a, b \rangle$, we square the claim, we get $a^2 + b^2 \geq (a+b)^2$. This is wrong for $(a,b) = (1,1)$.

20) T F If a function $f(x,y)$ satisfies $||\nabla f(x,y)|| = 1$ everywhere in the plane, then f is constant.

Solution:
Counter examples are $f(x,y) = \sqrt{x^2 + y^2}$, or $f(x,y) = x$.

Problem 2) (10 points)



a) The picture above shows a contour map of a function $f(x,y)$ of two variables. This function has 12 critical points and all of them are marked. Each of them is either a local max, a local min or a saddle point. The picture shows also some gradient vectors. Count the number of critical points in the following table. No justifications are necessary.

The function $f(x,y)$ has		local maxima
The function $f(x,y)$ has		local minima
The function $f(x,y)$ has		saddle points

b) (4 points) Match the following partial differential equations with the names. No justifications are needed.

Enter A,B,C,D here	PDE
	$u_{xx} + u_{yy} = 0$
	$u_{xx} - u_{yy} = 0$

Enter A,B,C,D here	PDE
	$u_x - u_{yy} = 0$
	$u_x - u_y = 0$

A) Wave equation	B) Heat equation	C) Transport equation	D) Laplace equation

Solution:
a) 4 maxima, 2 minima and 6 saddle points (look at the direction of the arrows.
b) Left: Laplace and wave, Right: heat and transport.

Problem 3) (10 points)

Find the cos of the angle between the sphere

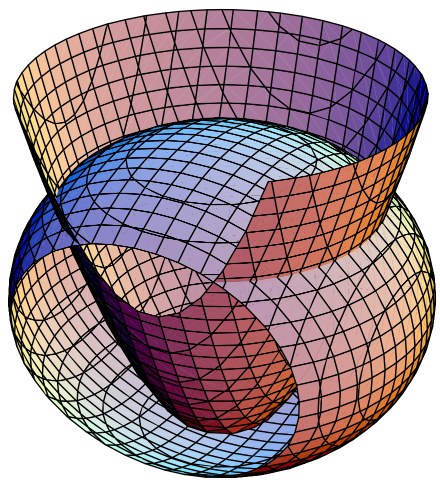
$$x^2 + y^2 + z^2 - 9 = 0$$

and the paraboloid

$$z - x^2 - y^2 + 3 = 0$$

at the point $(2, -1, 2)$.

Note: The angle between two general surfaces at a point P is defined as the angle between the tangent planes at the point P.



Solution:
The angle between the surfaces is the angle between the normal vectors. The gradients are $(4, -2, 4)$ and $(-4, 2, 1)$. The angle beween them satisfies $\cos(\theta) = -16/\sqrt{36 \cdot 21} = -8/3\sqrt{21}$. Both signs are ok.

Problem 4) (10 points)

a) You know that

$$-2x + 5y + 10z = 2$$

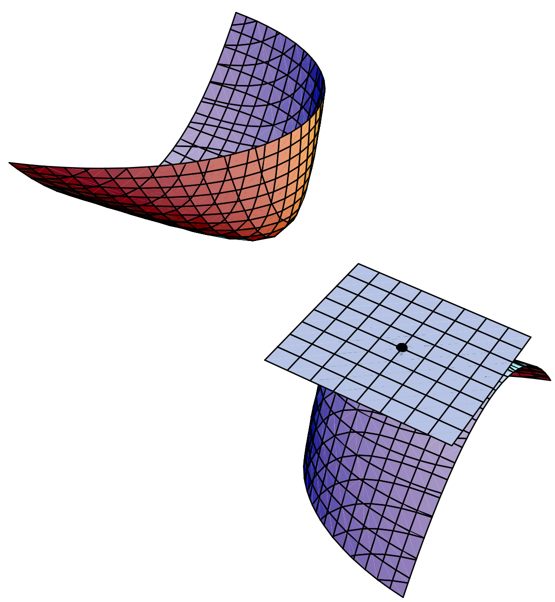
is the equation of the tangent plane to the graph of $f(x, y)$ at the point $(-1, 2, -1)$. Find the gradient $\nabla f(-1, 2)$ at the point $(-1, 2)$ and Estimate $f(-0.998, 2.0001)$ using linear approximation.

b) Let $f(x, y, z) = x^2 + 2y^2 + 3xz + 2$. Find the equation of the tangent plane to the surface $f(x, y, z) = 0$ at the point $(2, 0, -1)$ and estimate $f(2.001, 0.01, -1.0001)$.

Solution:
a) The gradient is parallel to $\langle -2, 5, 10 \rangle$. Because the gradient of the graph f is the gradient to the level surface $z - f(x, y) = 0$ which is $\nabla g(x, y) = \langle -f_x, -f_y, 1 \rangle$, the gradient is $\langle f_x, f_y \rangle = \langle 2/10, -5/10 \rangle = \langle 1/5, -1/2 \rangle$.
The linearization is

$$L(x, y) = f(-1, 2) + (1/5)(x + 1) - (1/2)(y - 2) = -1 + (x + 1)/5 - (y - 2)/2$$

We get $L(-0.998, 2.0001) = -1 + 0.0004 - 0.0005 = -0.99965$.
b) The gradient is $\nabla f(x, y, z) = \langle 2x + 3z, 4y, 3x \rangle$. At the point $(2, 0, -1)$, we get $\nabla f(2, 0, -1) = \langle 1, 0, 6 \rangle$. The equation of the tangent plane is $x + 6z = d$, where d is obtained by plugging in the point $(2, 0, -1)$. We have $x + 6z = -4$.
The linearization is $L(x, y, z) = 0 + (x - 2) + 6(z + 1)$. Plugging in the point $(x, y, z) = (2.001, 0.01, -1.0001)$ gives $L(2.001, 0.01, -1.0001) = 0.001 + 6(-0.0001) = 0.0004$.



Problem 5) (10 points)

a) (4 points) Find all the critical points of the function $f(x, y) = xy$ in the interior of the elliptic domain

$$x^2 + \frac{1}{4}y^2 < 1 \ .$$

and decide for each point whether it is a maximum, a minimum or a saddle point.

b) (4 points) Find the extrema of f on the boundary

$$x^2 + \frac{1}{4}y^2 = 1 \ .$$

of the same domain.

c) (2 points) What is the global maximum and minimum of f on $x^2 + \frac{1}{4}y^2 \leq 1$.

Solution:
a) $\nabla f(x,y) = \langle y, x \rangle = \langle 0, 0 \rangle$ implies $x = y = 0$. The only critical point in the interior is $\boxed{(0,0)}$. The discriminant is $D = -1^2 = -1$. The point is a $\boxed{\text{saddle point}}$.
b) With $g(x,y) = x^2 + y^2/4$, we have the Lagrange equations

$$\begin{aligned} y &= \lambda 2x \\ x &= \lambda 2y/4 \\ x^2 + y^2/4 &= 1 \end{aligned}$$

Dividing the first equation by the second to get $y/x = 4x/y$ which means $y^2 = 4x^2$ or $y = \pm 2x$. The third equation gives $2x^2 = 1$ or $x = \pm 1/\sqrt{2}$. The third equation gives $y = 2\sqrt{1-x^2} = \pm\sqrt{2}$. The critical points are

$$\{(1/\sqrt{2}), \sqrt{2}\}, \{(-1/\sqrt{2}), \sqrt{2}\}, \{(1/\sqrt{2}), -\sqrt{2}\}, \{-(1/\sqrt{2}), -\sqrt{2}\} \ .$$

The value of the function at these points are $1, -1, -1, 1$. The first and last are maxima, the second and third are minima.
c) There is no global maximum, nor a global minimum in the interior of the disc because there is no local maximum in the interior of the disc. The global maxima as well as the minima are on the boundary.

Problem 6) (10 points)

a) Assume $f(x,y) = e^{2x-y-2} + y + \sin(x-1)$ and $x(t) = \cos(5t), y(t) = \sin(5t)$. What is

$$\frac{d}{dt} f(x(t), y(t))$$

at time $t = 0$.

b) The relation

$$xyz + z^3 + xy + yz^2 = 4$$

defines z as a function of x and y near $(x, y, z) = (1, 1, 1)$. Find the gradient

$$\langle \frac{\partial z}{\partial x}(1,1), \frac{\partial z}{\partial y}(1,1) \rangle$$

of $z(x, y)$ at the point $(1, 1)$.

Solution:
a) At time $t = 0$, the curve passes through the point $(1, 0)$. The gradient of f is $\nabla f(x, y) = \langle 2e^{2x+y-2} + \cos(x-1), -e^{2x-y-2} + 1 \rangle$. At the point $(1, 0)$, we have $\nabla f(1, 0) = \langle 3, 0 \rangle$. The velocity vector at time $t = 0$ is $\vec{r}\,'(t) = \langle -5\sin(t), 5\cos(t) \rangle$. At time $t = 0$, the velocity vector is $\langle 0, 5 \rangle$. By the chain rule, the rate of change

$$\frac{d}{dt} f(x(t), y(t)) = \nabla f(x(t), y(t)) \cdot \vec{r}\,'(t) = \langle 2, 0 \rangle \langle 0, 5 \rangle = 0 \ .$$

b) We have $z_x = -f_x/f_z, z_y = -f_y/f_z$. Because $\nabla f = \langle yz+y, xz+x+z^2, xy+3z^2+2yz \rangle = \langle f_x, f_y, f_z \rangle$. and $\nabla f(1,1,1) = \langle 2, 3, 6 \rangle$, we have $z_x = -1/3, z_y = -1/2$. In summary $\boxed{\nabla z(1,1) = \langle -1/3, -1/2 \rangle}$.

Problem 7) (10 points)

The temperature in a room is given by $T(x, y, z) = x^2 + 2y^2 - 3z + 1$.

a) Barry B. Benson is hovering at the point $(1, 0, 0)$ and feels cold. Which direction should he go to heat up most quickly? Make sure that your answer is a unit vector.

b) At some later time, Barry arrives at the point $(3, 2, 1)$ and decides that this is a nice temperature. Find a direction (a unit vector) in which he can go, to stay at the same temperature and the same altitude.

Solution:
$\nabla T(x,y,z) = \langle 2x, 4y, -3 \rangle$. $\nabla T(1,0,0) = \langle 2,0,-3 \rangle$. To increase the heat, we have to go into the direction $\boxed{\langle 2,0,-3 \rangle/\sqrt{13}}$.
b) We have to find a direction $\vec{v} = \langle x, y, 0 \rangle = \langle \cos(\theta), \sin(\theta), 0 \rangle$ such that $D_{\vec{v}} f(3,2,1) = 0$. We have $\nabla T(3,2,1) = \langle 6,8,-3 \rangle$ and so $D_{\vec{v}} f(3,2,1) = 6 \cdot 3 + 8 \cdot 2 = 0$. This means $x = -4/3y$. Normalized, the vector is $\vec{v} = \langle -4/5, 3/5, 0 \rangle$.

Problem 8) (10 points)

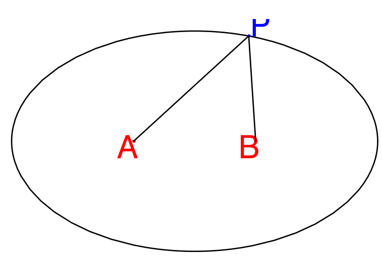
Let $g(x,y)$ denote the distance of a point $P = (x,y)$ to a point A and $h(x,y)$ the distance from P to a point B. The set of points (x,y) for which $f(x,y) = g(x,y) + h(x,y)$ is constant, forms an ellipse. In other words, the level curves of f are ellipses.

a) (4 points) Why is $\nabla g + \nabla h$ perpendicular to the ellipse?

b) (3 points) Show that if $\vec{r}(t)$ parametrizes the ellipse, then $(\nabla g + \nabla h) \cdot \vec{r}' = 0$ or $\nabla g \cdot r' = -\nabla h \cdot r'$.

c) (3 points) Conclude from this that the lines AP and BP make equal angles with the tangent to the ellipse at P.

You have now shown that light rays originating at focus A will be reflected from the ellipse to focus at the point B.



Solution:

a) The crucial point is that ∇f is perpendicular to the curve. That implies $\nabla g + \nabla h$ is perpendicular to the curve.

b) That is a direct consequence of a): the velocity vector $\vec{r}'(t)$ is tangent to the level curve. The vector $\nabla g(r(t)) + \nabla h(r(t))$ is perpendicular to the level curve and so to $\vec{r}\,'(t)$.

c) The cosine of the angle α between ∇g and $\vec{r}\ '$ is $|\nabla g \cdot r'|/(|\nabla g||r'|)$, the cosine of the angle β between ∇h and $\vec{r}\ '$ is $|\nabla h \cdot r'|/(|\nabla h||r'|)$. Note that $|\nabla h| = |\nabla g| = 1$. [Take $g(x,y) = (x^2 + y^2)^{1/2}$ and compute the gradient. $\nabla g(x,y) = (2x, 2y)/2(x^2 + y^2)^{1/2} = (x,y)/(x^2 + y^2)^{1/2}$. This is a special case of the Eiconal problem which appeared in other practice problems, where we take the distance to a curve.]

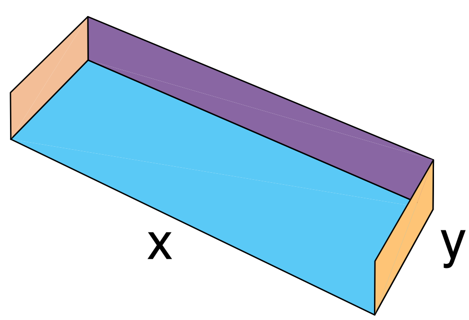
$$\cos(\alpha) = \frac{|\nabla g \cdot r'|}{|\nabla g|\ |r'|} = \frac{|\nabla g \cdot r'|}{|r'|} = \frac{|\nabla h \cdot r'|}{|r'|} = \frac{|\nabla h \cdot r'|}{|\nabla h|\ |r'|} = \cos(\beta)\ .$$

Problem 9) (10 points)

Minimize the material cost of an office tray

$$f(x,y) = xy + 2x + 2y$$

of length x, width y and height 1 under the constraint that the volume $g(x,y) = xy$ is constant and equal to 4.



Solution:
The Lagrange equations for the function given are

$$\begin{aligned} y+2 &= \lambda y \\ x+2 &= \lambda x \\ xy &= 4 \end{aligned}$$

Dividing the first by the second, gives

$$(y+2)/(x+2) = y/x$$

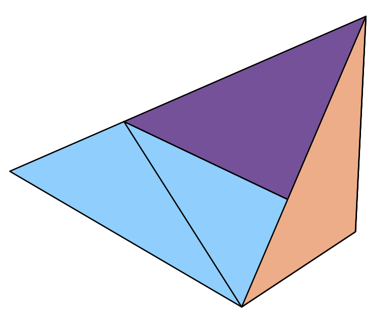
or $x(y+2) = y(x+2)$. This gives $x = y = 2$.
The picture was inconsistent with the function. We actually had intended to take

$$f(x,y) = xy + x + 2y$$

which leads to the minimum $(x,y) = (2\sqrt{2}, \sqrt{2})$. If somebody would solve this problem, then it would give full credit also.

Problem 10) (10 points)

A beach wind protection is manufactured as follows. There is a rectangular floor $ACBD$ of length a and width b. A pole of height c is located at the corner C and perpendicular to the ground surface. The top point P of the pole forms with the corners A and C one triangle and with the corners B and C an other triangle. The total material has a fixed area of $g(a,b,c) = ab + ac/2 + bc/2 = 12$ square meters. For which dimensions a, b, c is the volume $f(a,b,c) = abc/6$ of the tetrahedral protected by this configuration maximal?



Solution:
The Lagrange equations are

$$\begin{aligned} bc &= \lambda(b + c/2) \\ ac &= \lambda(a + c/2) \\ ab &= \lambda(a + b)/2 \\ ab + bc/2 + ac/2 &= 12 \ . \end{aligned}$$

Dividing the first to the second equation leads to $a = b$. Dividing the second to the third equation gives $c = 2b$. Substituting a and c gives $b^2 + b^2 + b^2 = 12$ or $b = 2$. Therefore $a = 2, b = 2, c = 4$ is the optimal configuration. The maximal volume is $f(2, 2, 4) = 8/3$.