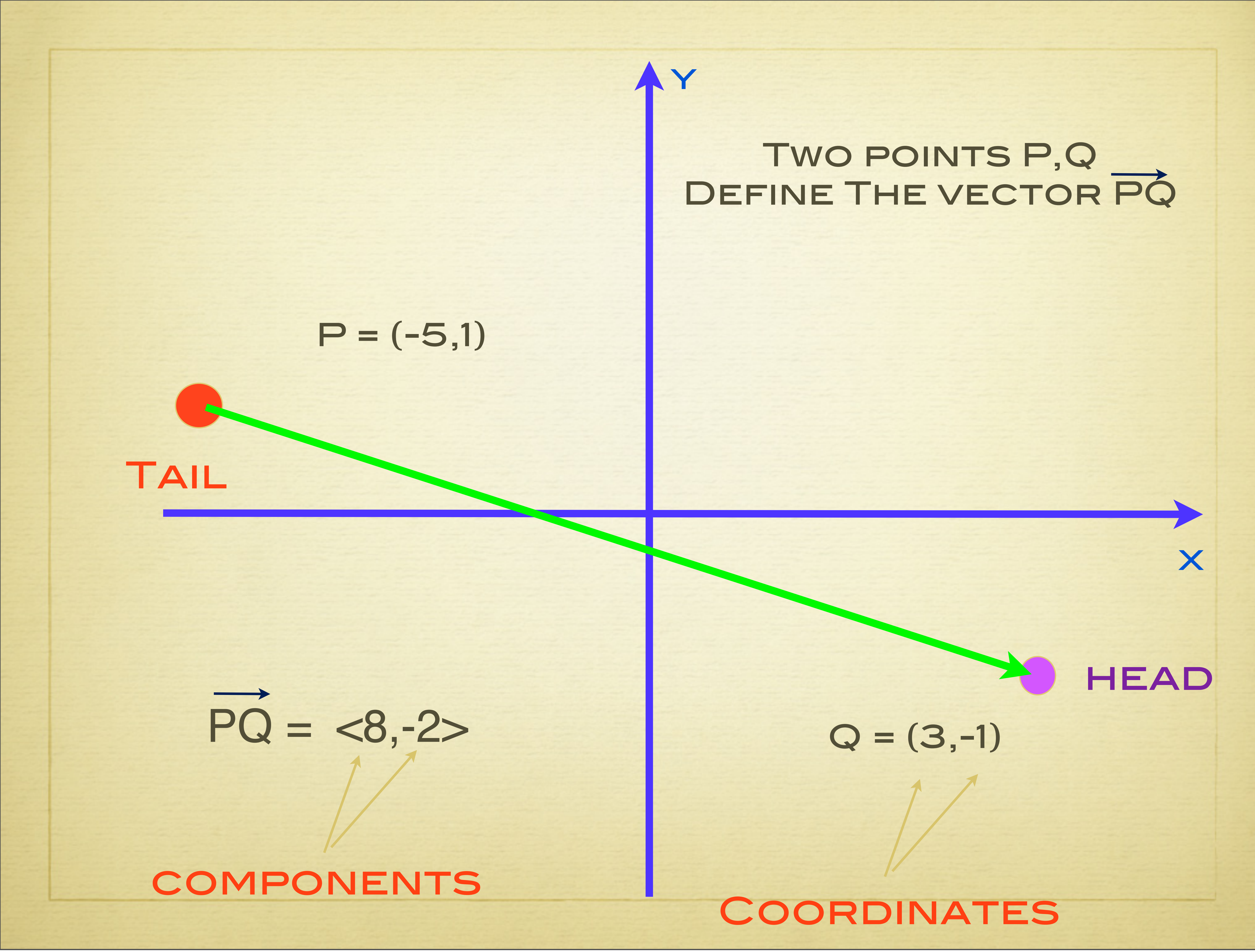
Vectors and Dot Product

- 1. Vectors
- 2. Vector Operations
- 3. Dot Product
- 4. Angle Formula
- 5. Projections
- 6. Theorems

Oliver Knill, Math21a, Fall 2009

1. Vectors



Tuesday, September 8, 2009

One often sees the definition: A Vector is defined as an object with length and direction.

DON'T!

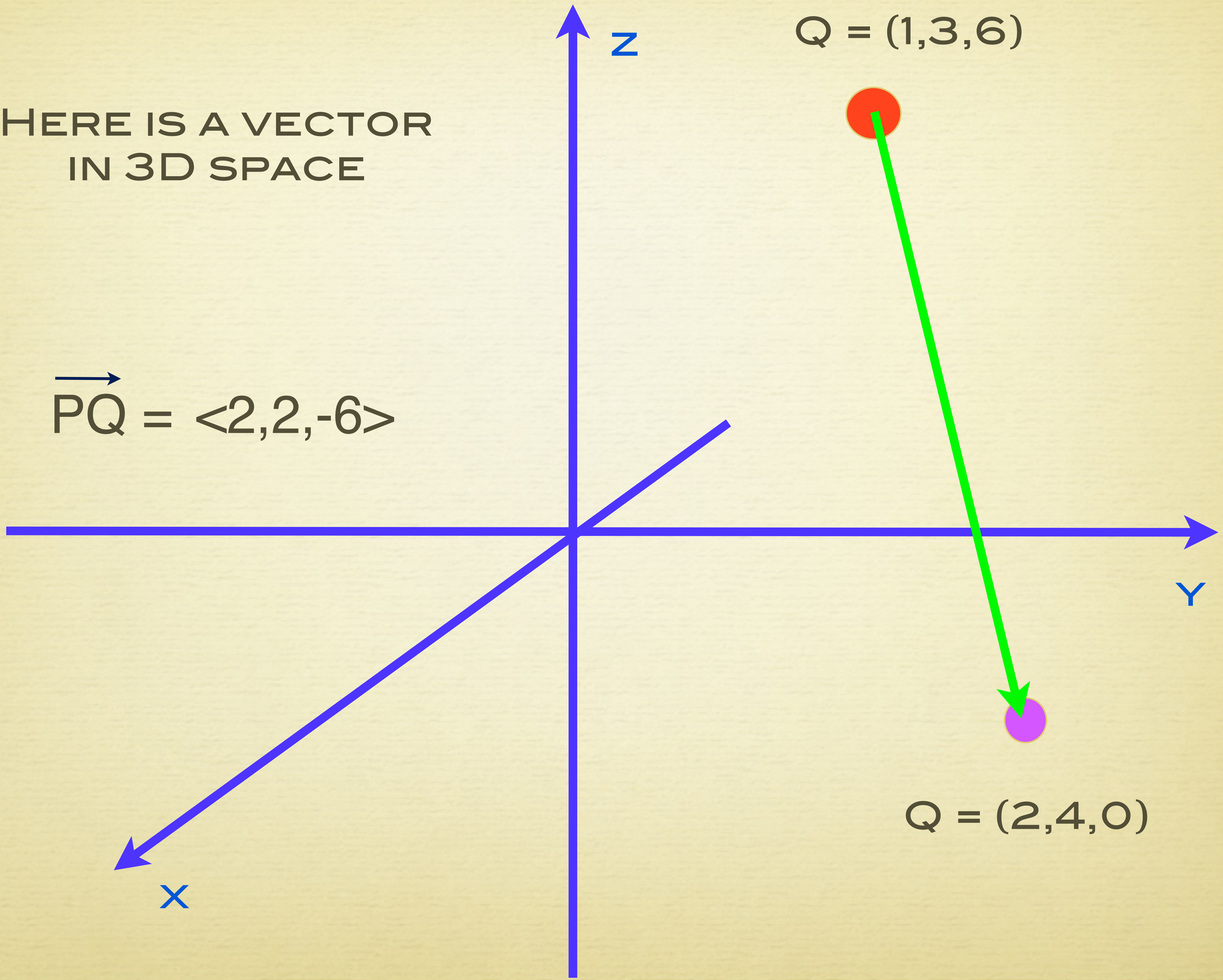
The zero vector 0 = <0,0,0> has no direction but is still a vector.

Also, it is not Tasteless to define something New with 2 undefined concepts (length and direction).

Also a movie has Length and direction, but it is not a Vector

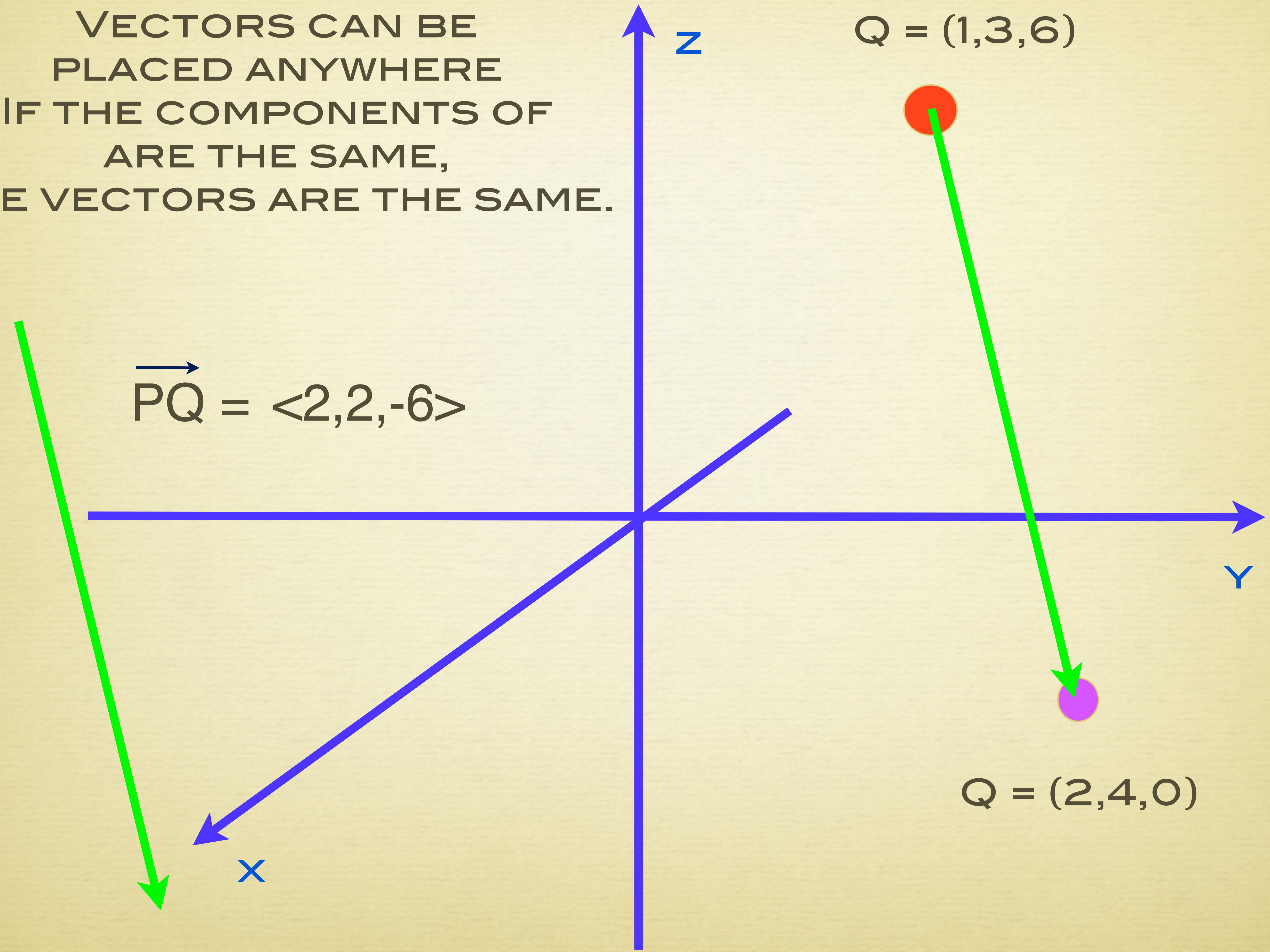
Here is a vector in 3D space

$\overrightarrow{PQ} = \langle 2,2,-6 \rangle$

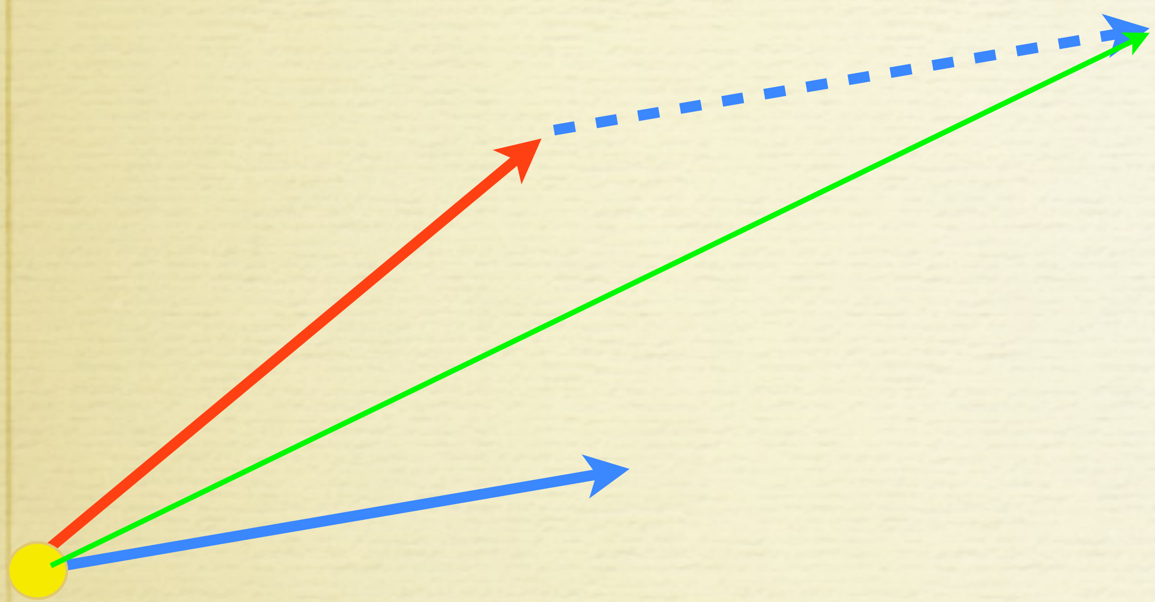


Vectors can be placed anywhere
If the components of are the same,
the vectors are the same.

$\overrightarrow{PQ} = \langle 2,2,-6 \rangle$



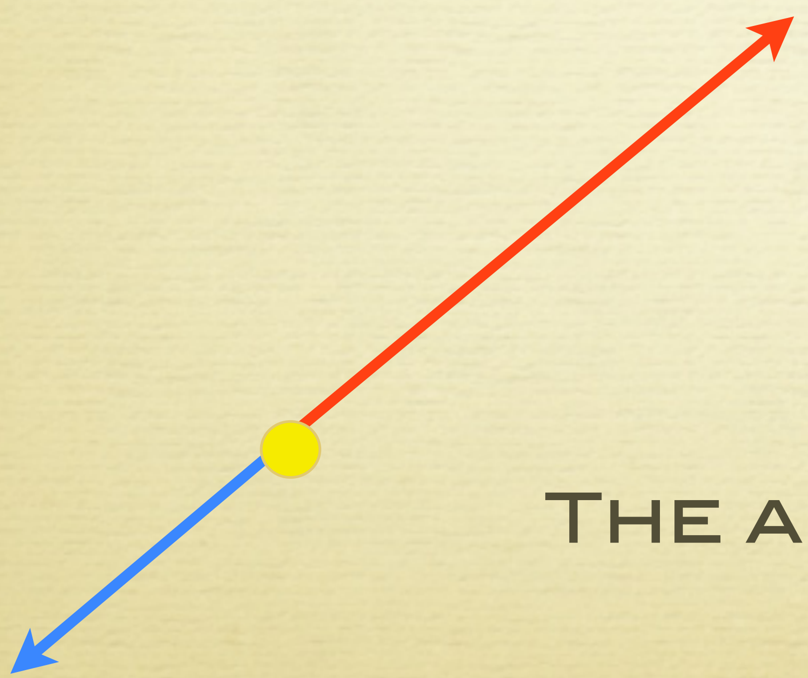
1. Vector Operations



$\vec{v} = \langle 2,-2,4 \rangle$

$\vec{w} = \langle 1,1,3 \rangle$

$\vec{v}+\vec{w} = \langle 3,-1,7 \rangle$



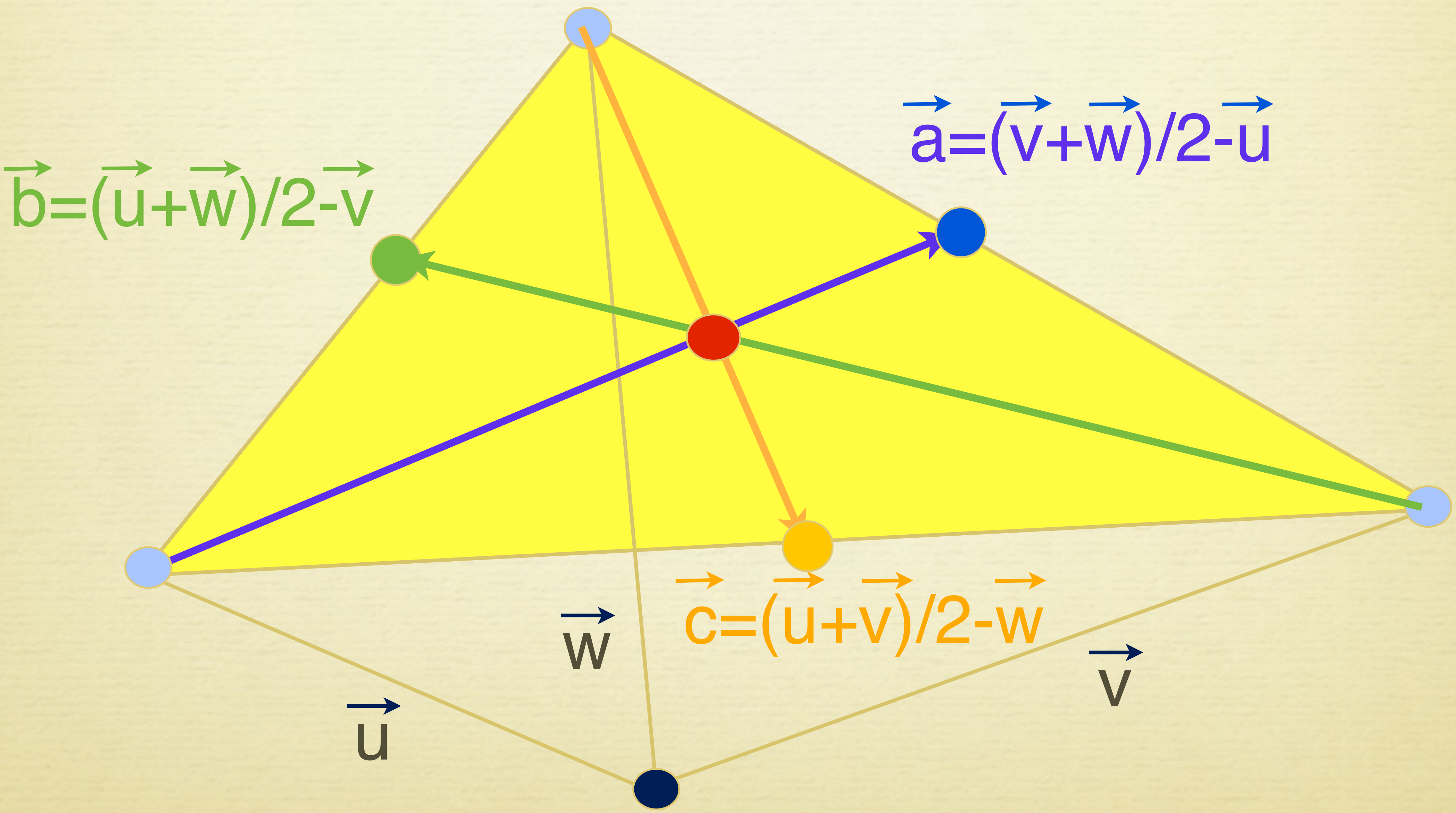
$\vec{v} = \langle 2,-2,4 \rangle \qquad \lambda = -0.5$

$\lambda \vec{v} = \langle -1,1,-2 \rangle$

THE ADDITION AND SCALING OPERATIONS DEFINE A VECTOR SPACE.

WE CAN WORK WITH VECTORS SIMILAR AS WITH NUMBERS.

Problem: show that all the medians in a triangle intersect in one point



$$\vec{u}+(2/3)\,\vec{a} = \vec{v}+(2/3)\,\vec{b} = \vec{w} + (2/3)\,\vec{c}$$

● Centroid

check it!

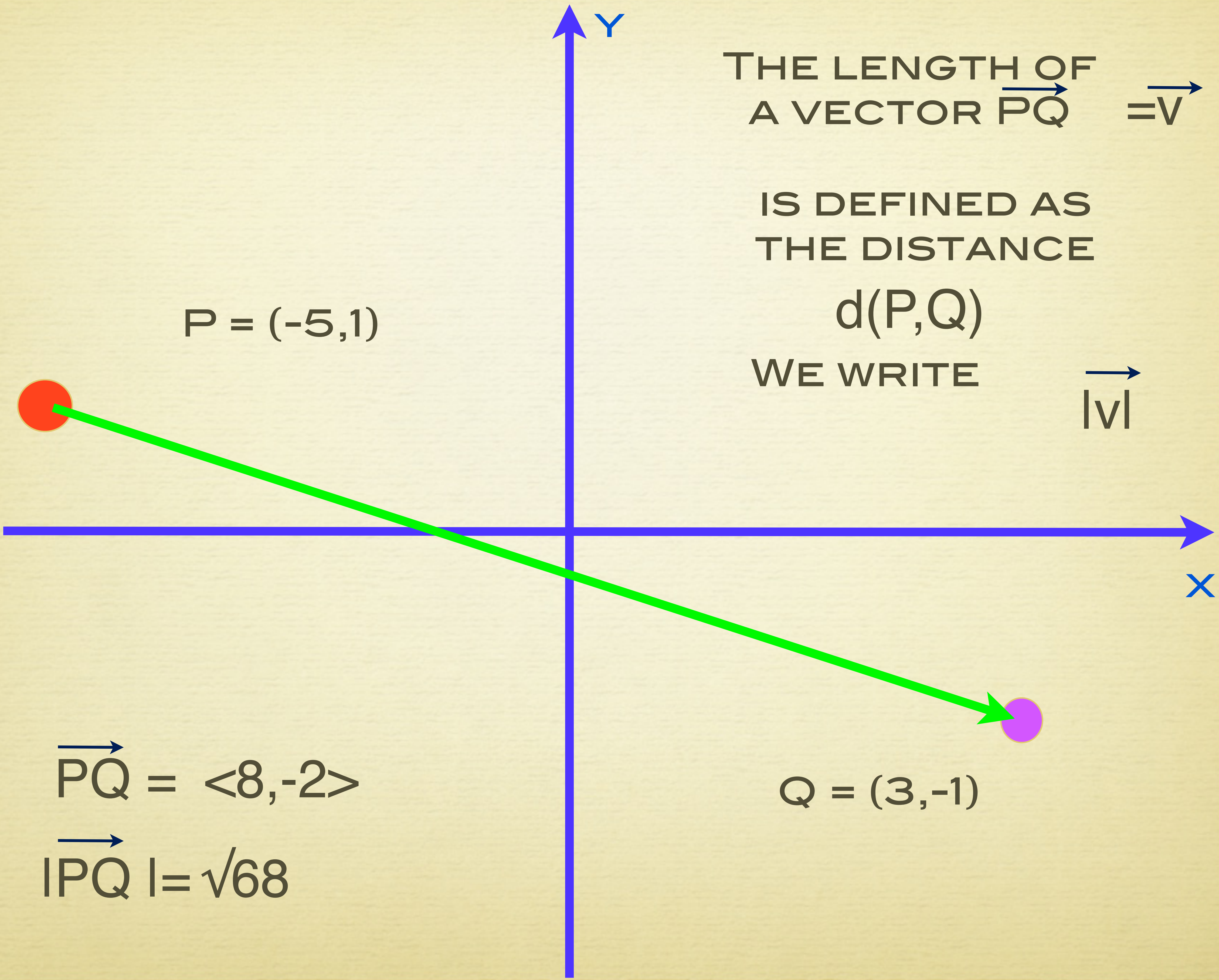
THE LENGTH OF A VECTOR $\overrightarrow{PQ} = \overrightarrow{v}$

IS DEFINED AS THE DISTANCE

d(P,Q)

WE WRITE

$|\overrightarrow{v}|$



$\overrightarrow{PQ} = \langle 8,-2 \rangle$

$|\overrightarrow{PQ}| = \sqrt{68}$

2. Dot Product

Definition:

$\vec{v} = <3,4,5>$

$\vec{w} = <-2,1,3>$

$\vec{v} \cdot \vec{w} = -6+4+15$

We define the dot product algebraically as the sum of the product s of all components.

$$\vec{v} \cdot \vec{w} = <v_1, v_2, v_3> \cdot <w_1, w_2, w_3> = v_1 w_1 + v_2 w_2 + v_3 w_3$$

We will use this Product to introduce more geometry.

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$$

$$\vec{v} \cdot \vec{v} = |\vec{w}|^2$$

WE CAN COMPUTE LENGTH

MANY RULES FROM USUAL MULTIPLICATION HOLD

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

NOTE THAT WE CAN NOT "DIVIDE VECTORS"

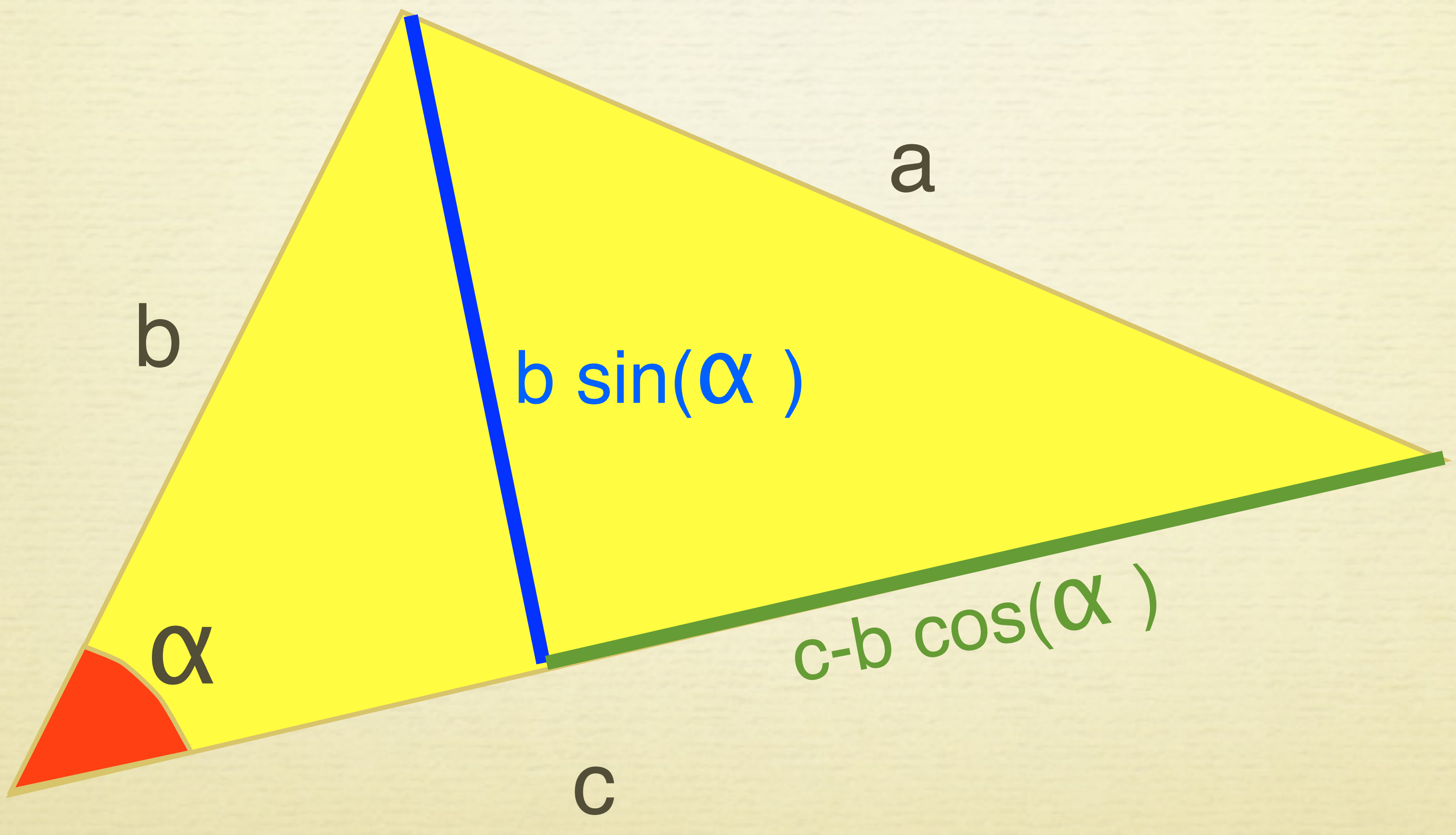
$$\vec{v} / |\vec{v}|$$

UNIT VECTOR

Cos-Theorem

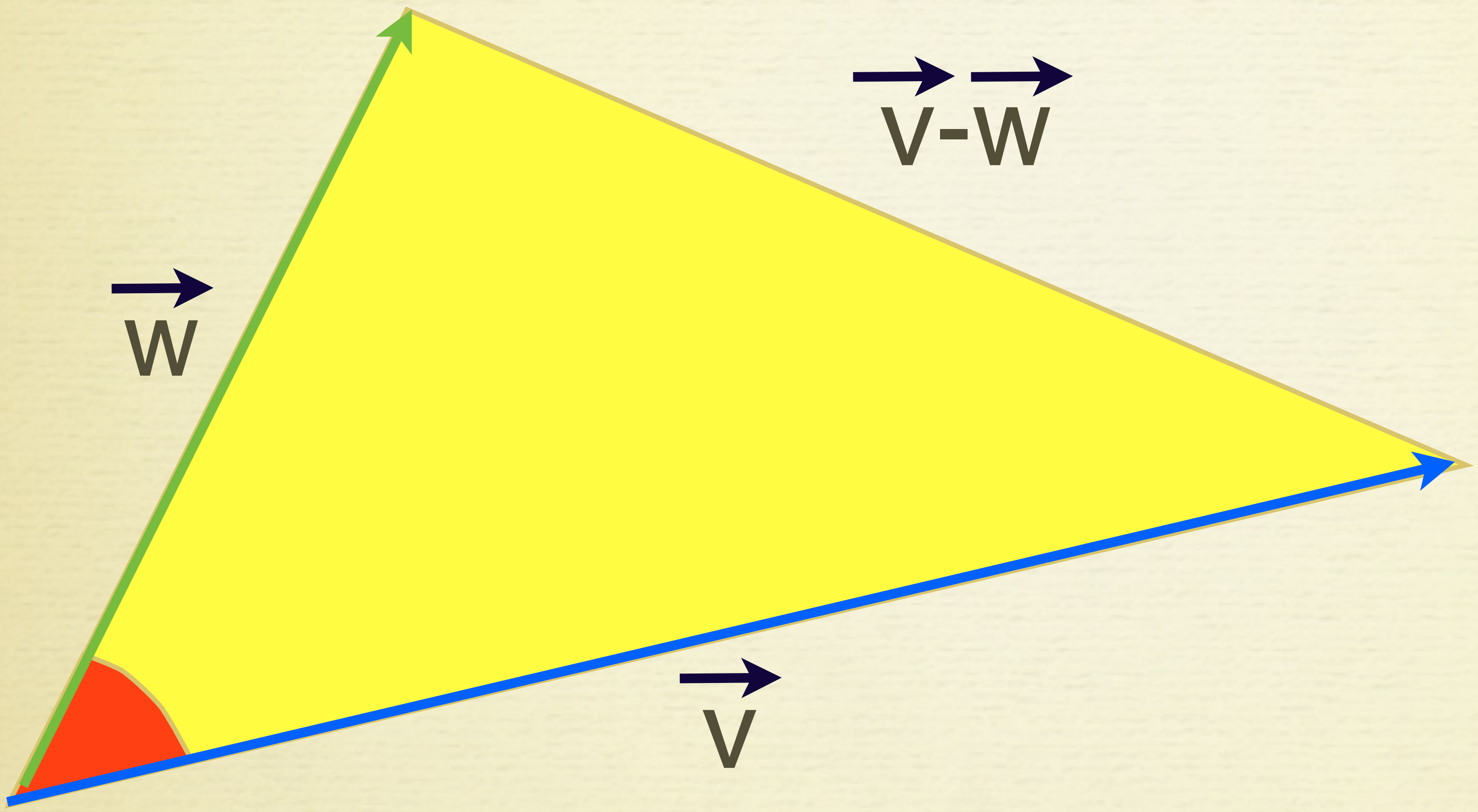
$$a^2 = (b \sin(\alpha))^2 + (c - b\cos(\alpha))^2$$

$$= c^2 + b^2 - 2\,b\,c\cos(\alpha)$$



WE USED PYTHAGORAS AND GOT A GENEARLIZATION OF THE PYTHAGOREAN THEOREM

4. Angle Formula



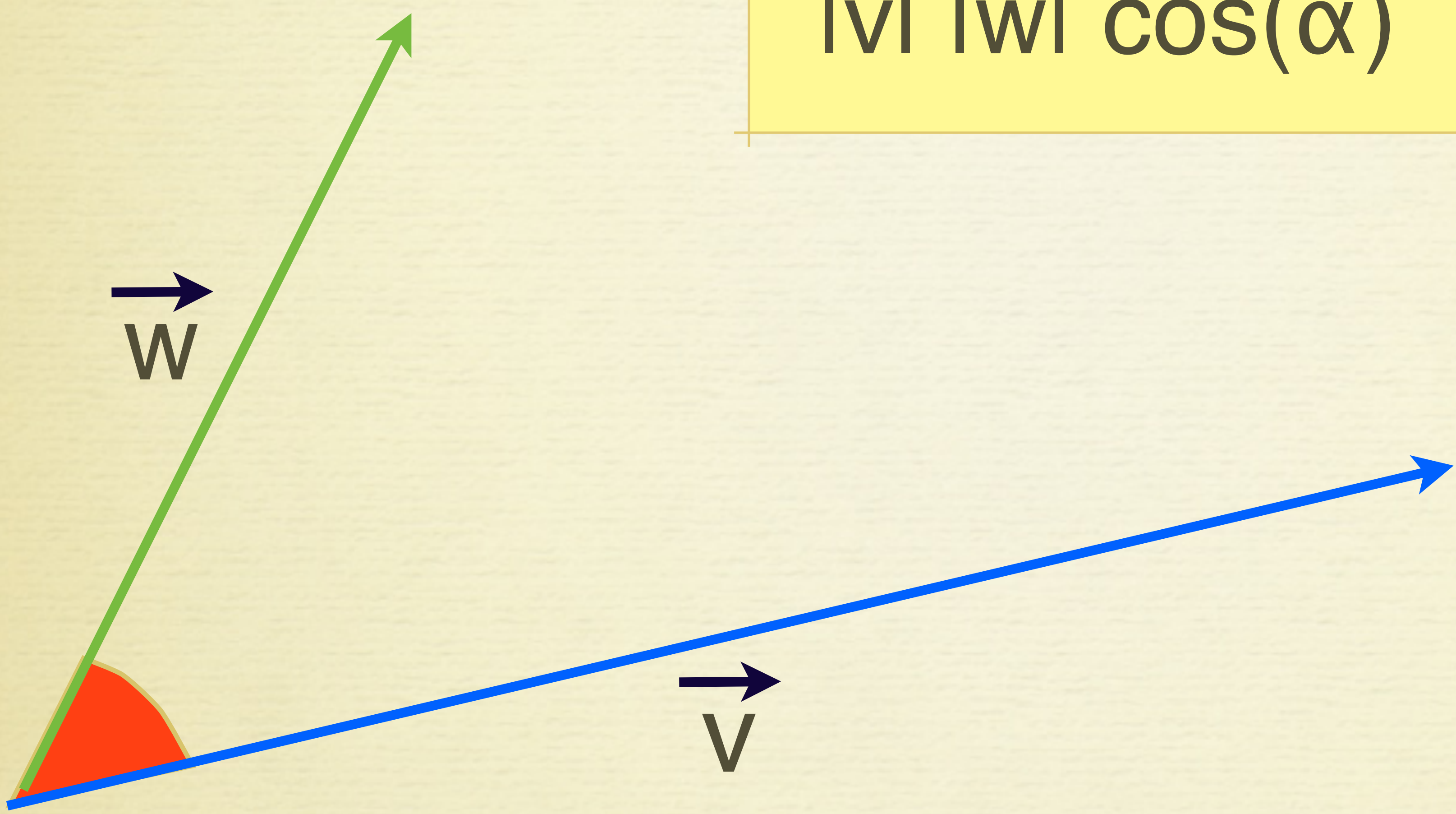
Compare $a^2 = (\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w}) = \vec{v}\cdot\vec{v} - 2\,\vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{w}$

$= c^2 + b^2 - 2\,\vec{v}\cdot\vec{w}$

with $a^2 = c^2 + b^2 - 2\,b\,c\cos(\alpha)$

We get:

$$|\vec{v}|\,|\vec{w}|\cos(\alpha) = \vec{v}\cdot\vec{w}$$



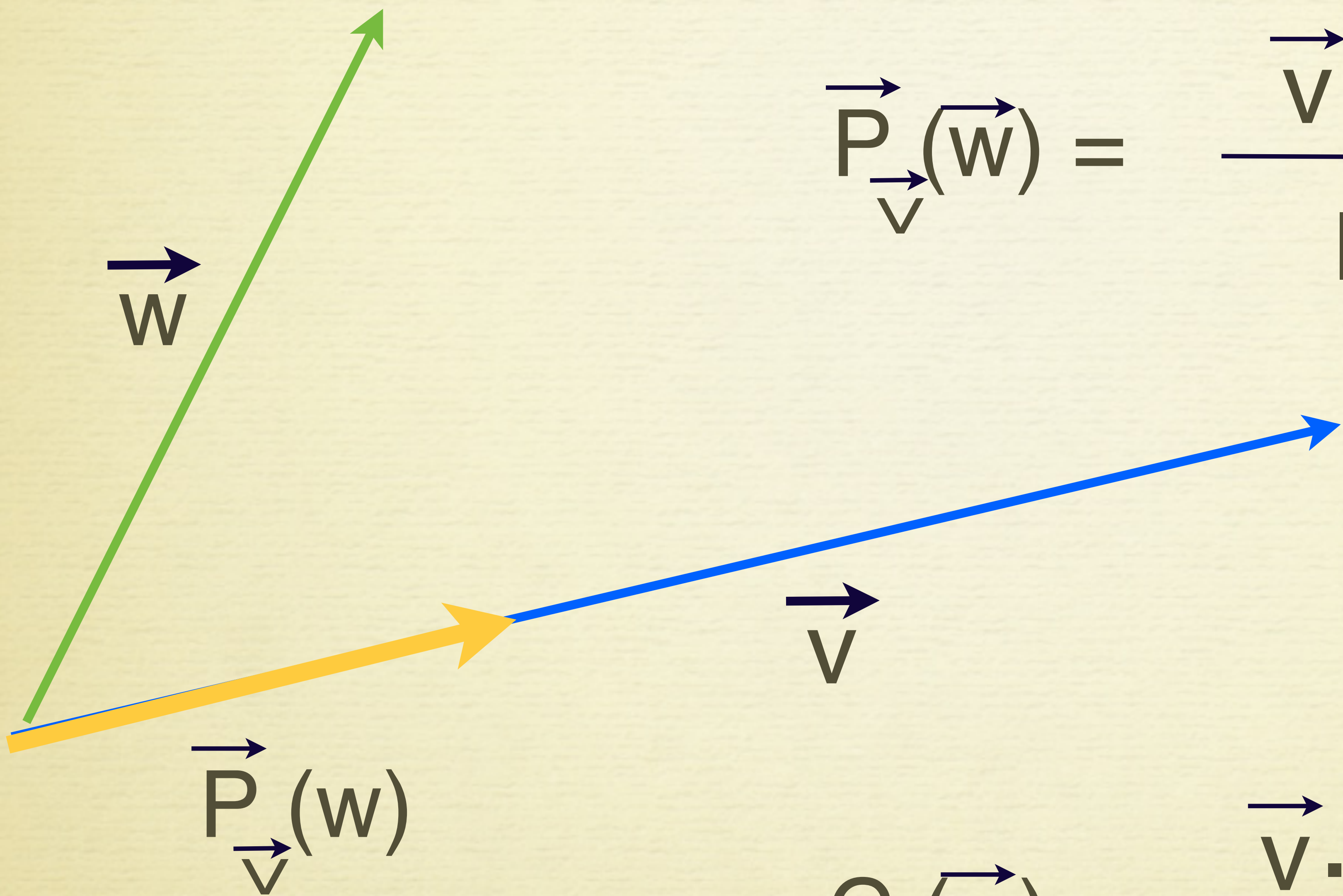
This motivates to DEFINE the angle between two vectors as

$$\cos(\alpha) = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|\,|\vec{w}|}$$

This defines the geometric concept "angle" from an algebraic definition.

5. Projections

Definition: Projection

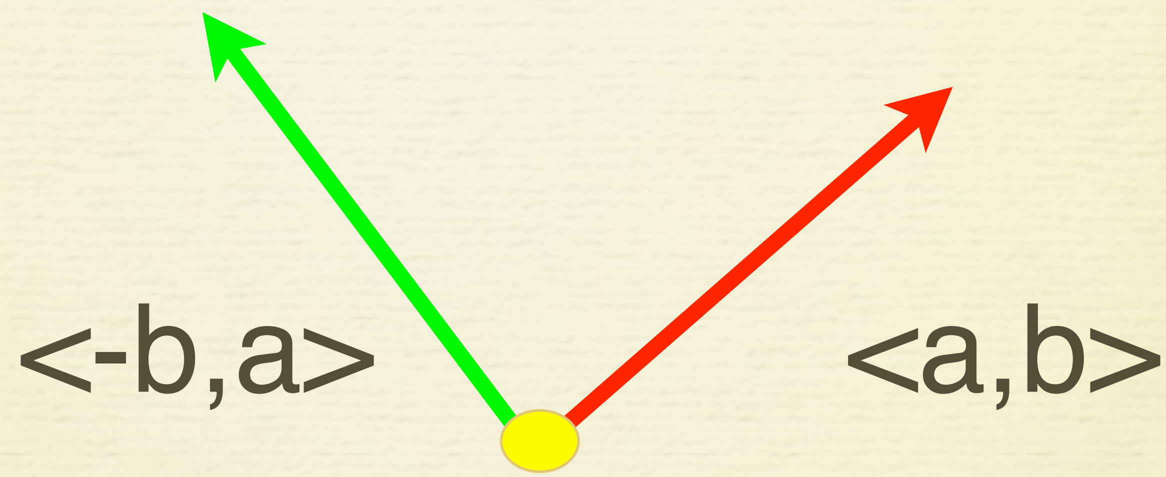


$$\vec{P}_{\vec{v}}(\vec{w}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}|^2} \vec{v}$$

$$C_{\vec{v}}(\vec{w}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}|}$$

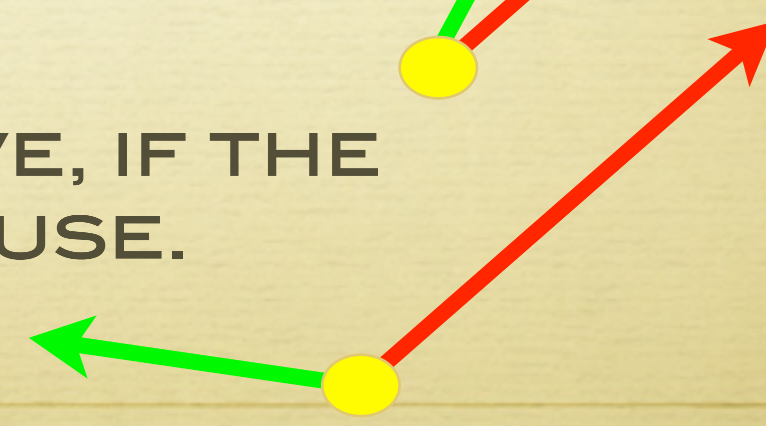
COMPONENT

- Two vectors are Called perpendicular (orthogonal), if their dot product is zero.



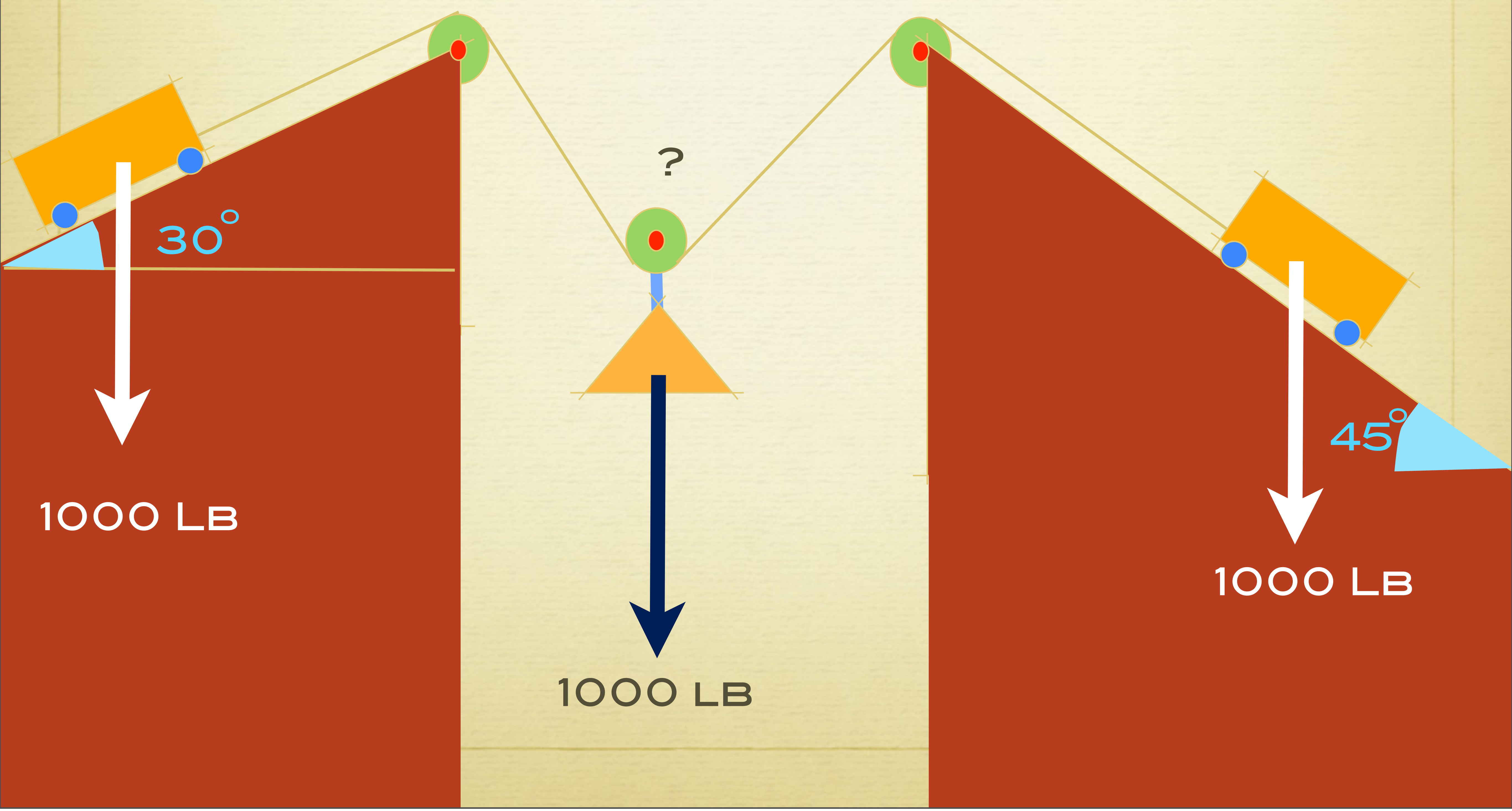
- The dot product of two vectors is positive if the angle between them is acute.

- The dot product is negative, if the angle between them is obtuse.

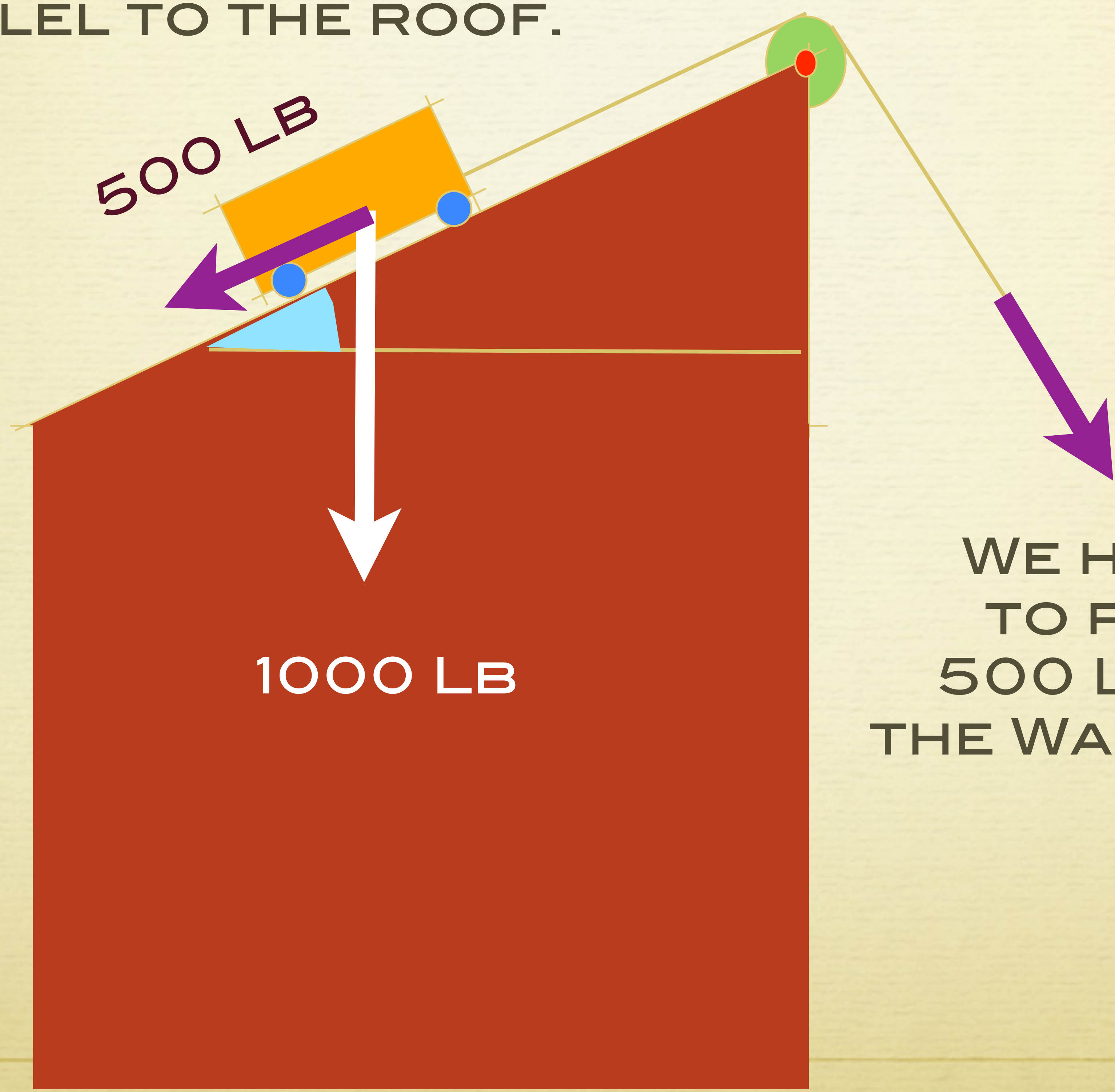


Problem

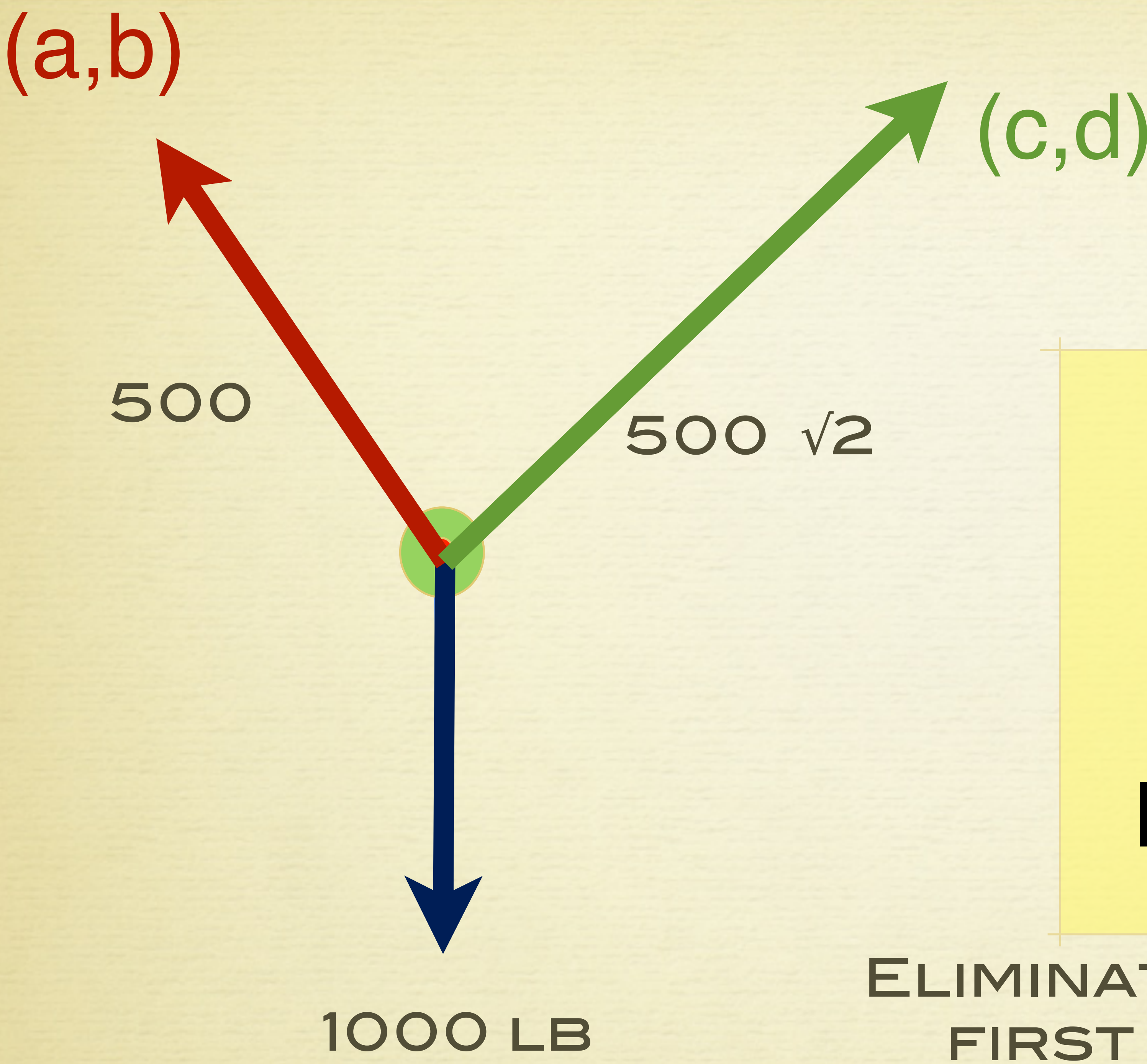
TO ILLUSTRATE THAT THESE THINGS ARE USEFUL, HERE IS A PROBLEM FOR AN ENGENIER.



The Force vector pulling the wagon down is the projection of the gravitational force vector onto a vector parallel to the roof.



We have here to pull with 500 Lb to keep the Wagon at rest



$$a^2 + b^2 = 500^2$$

$$c^2 + d^2 = 500^2\ 2$$

$$a + c = 0$$

$$b + d = 1000$$

ELIMINATE C, SUBTRACT FIRST 2 EQUATIONS

$$d^2 - b^2 = 500^2$$

$$b + d = 250$$

$b=375$, $d=625$

$a=-125\sqrt{7}$

$c=125\sqrt{7}$

5. Results

1) Pythagoras: If two vectors are perpendicular, then

$$v \cdot v + w \cdot w = (v-w) \cdot (v-w)$$

2) Cauchy-Schwarz inequality

$$|v \cdot w| \leq |v|\,|w|$$

3) Triangle inequality

$$|v-w| \leq |v| + |w|$$

PS: Only For the mathematically inclined:

Given a dot product on a vector space, we can define angle, length and Derive basic things like Pythagoras.
While we have used Pythagoras to motivate the notion of angle, this could have Been avoided:

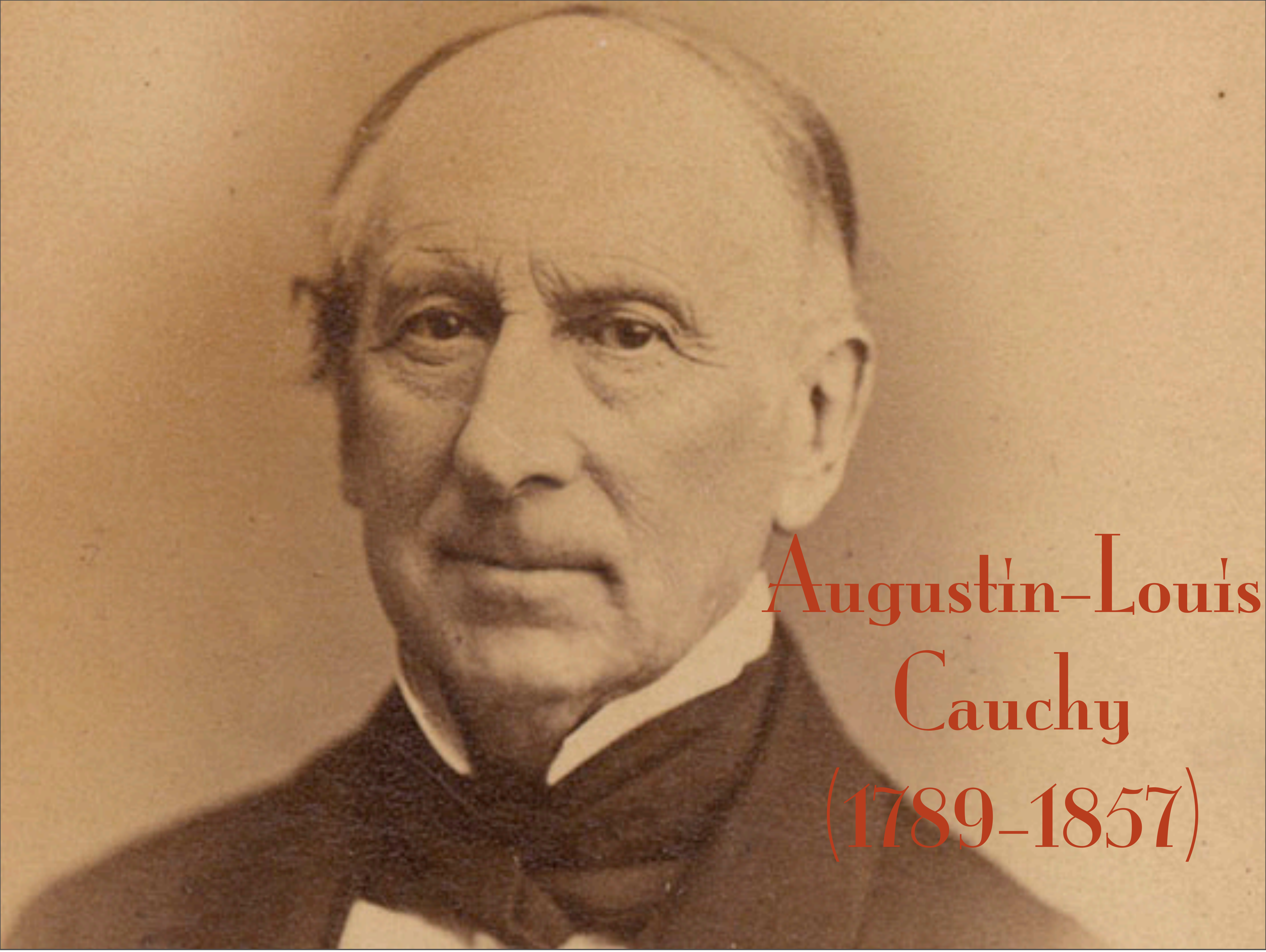
first verify the Cauchy-Schwarz inequality. This assures that the definition of angle with the angle formula makes sense. Then Pythagoras follows.

5. History

1700BC	Babilonians: Cases of Pythagorean theorem
500BC	Pythagoras
1400	Al Kashi, cos-Theorem (?)
1812	Concept of Vector
1843	Hamilton, Moebius: Dot Product
1847	Grassmann: inner product in general
1888	Peano: Abstract Vector Space

Pythagoras (580-500BC)

Al Kashi (1380–1429)



Augustin-Louis Cauchy (1789-1857)

Hermann Schwarz (1843-1921)