10 Vector Functions

10.1 Vector Functions and Space Curves

Suggested Time and Emphasis

1 class   Essential Material

Transparencies Available

- Transparency 36 (Exercises 5–10, graphs I–VI, page 710)

Points to Stress

1. The connection between space curves and ranges of vector functions.
2. Matching vector equations with their curves.
3. Parametrizations of curves in space are not unique.
4. Visualization of curves in three dimensions.

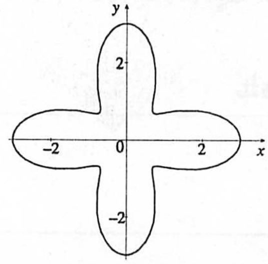
Text Discussion

- Give an example of two vector functions whose space curves lie in the plane $y + z = 2$.

Materials for Lecture

- Explain the differences and similarities between vector functions and parametric equations for a space curve.
- Point out that in going from a vector function $\mathbf{r}(t)$ to a space curve described by a variable point $P(t)$, we are picking the origin as the base of the vectors (position vectors) and the tip of $\mathbf{r}(t)$ traces out the curve: $\mathbf{r}(t) = \overrightarrow{OP}(t)$. For example, writing $\mathbf{r}(t) = \langle t, t^2 \rangle$ really means $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$.
- Consider a circle with radius 2. Parametrize this circle several ways: using the angle θ made with the positive x axis, using the angle $-\theta$, using the arc length starting from the point $(2, 0)$, and using the parametric equations $x(t) = -2\cos t$, $y(t) = -2\sin t$.
- Look at the vector function $\mathbf{v}(t) = \langle 3\sin t\cos t, 2\cos^2 t, \sin t \rangle$. Point out that since $x = 3\sin t\cos t$, $y = 2\cos^2 t$, and $z = \sin t$, we have $\frac{1}{9}x^2 + \frac{1}{4}y^2 = \cos^2 t$ and so $\frac{1}{9}x^2 + \frac{1}{4}y^2 + z^2 = 1$. Thus, the space curve lies on the ellipsoid $\frac{1}{9}x^2 + \frac{1}{4}y^2 + z^2 = 1$.

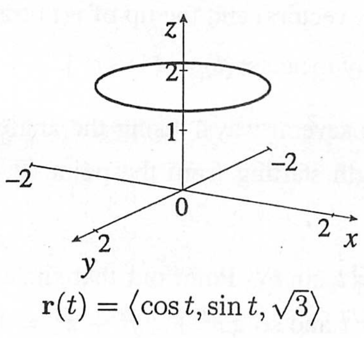
- Find the intersections of the sphere $x^2 + y^2 + z^2 = 1$ and the plane $y = z$: Solving these equations simultaneously gives $x^2 + 2y^2 = 1$. This describes an ellipse in the xy-plane with parametric equations $x = \cos t$, $y = \frac{1}{\sqrt{2}} \sin t$, $0 \le t \le 2\pi$. Thus, the space curve $x = \cos t$, $y = z = \frac{1}{\sqrt{2}} \sin t$, $0 \le t \le 2\pi$, describes the space curve which is the intersection of the two surfaces.
- Sketch the parametric curve described in polar coordinates (r, θ) by the equations $r = 2 + \cos 4t$, $\theta = t$, $0 \le t \le 2\pi$.



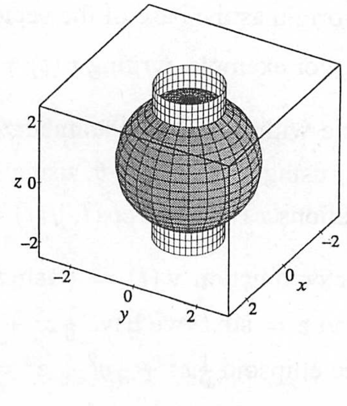
Indicate on the curve the points closest to and farthest from the origin, and determine the values of t that give these points. Then determine the Cartesian coordinates $(x(\theta), y(\theta))$ of a point on this curve as a function of θ.

Workshop/Discussion

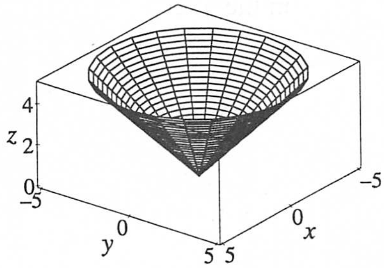
- Have the students explain why the vector functions $\langle \sin t, \cos t \rangle$ and $\langle \cos t, \sin t \rangle$ describe the same set of points. Then have them think about the three-dimensional functions $\langle \sin t, \cos t, t \rangle$ and $\langle \cos t, \sin t, t \rangle$, and notice that this time the sets of points described are different.
- This is a good opportunity to solidify the students' knowledge of $\mathbb{R}^3$. For example, they should be able to describe the curve $\langle \cos t, \sin t, t \rangle$ at this point in the course without too much difficulty.
- Describe the difference between the parametrizations $\langle \sin t, \cos t \rangle$ and $\langle \sin t^2, \cos t^2 \rangle$ graphically, in terms of both the domain of the parameter needed for one full revolution ($0 \le t \le 2\pi$ in the first case, $0 \le t \le \sqrt{2\pi}$ in the second case) and the "speed" with which the curve is traced out.
- Ask students to show that the vector function $\mathbf{v}(t) = \langle 2t + 1, 3t + 2, -5t \rangle$ describes a line, and that this line lies in the plane $x + y - z = 5$.
- Show that the vector function $\mathbf{r}(t) = \langle \cos t, \sin t, \sqrt{3} \rangle$ lies on the intersection of the sphere $x^2+y^2+z^2 = 4$ and the cylinder $x^2+y^2 = 1$. Describe this intersection, and then have the students attempt to give another vector function which parametrizes the other piece of the intersection.



$\mathbf{r}(t) = \langle \cos t, \sin t, \sqrt{3} \rangle$



- Consider the helix $x = \cos t, y = \sin t, z = t^2$. Sketch all three coordinate planar projections of this curve (that is, the projections onto the xy-plane, the yz-plane, and the xz-plane). Help the students visualize the entire curve by looking at the projections.
- Consider the cone $z = \sqrt{x^2 + y^2}$. Show that $\langle t\cos t, t\sin t, t\rangle$ parametrizes a curve that spirals around the cone, and that there are several other choices for other spiral curves.



Group Work 1: Many Paths

Ask the students to parametrize three curves on the unit sphere which connect the North Pole $(0, 0, 1)$ to the South Pole $(0, 0, -1)$. The first one should be an arc of a great circle that lies in one of the coordinate planes, the second should be an arc of a great circle that does *not* lie in one of the coordinate planes, and the third should be a curve that spirals once around the sphere. After some time, give the hint that they should use spherical coordinates.

If a group finishes early, have the students replace $(0, 0, -1)$ with $(1, 0, 0)$ and parametrize the path of shortest length between these two points.

A good way to illustrate these curves is to draw them on the surface of a ball or balloon with a felt-tip pen.

Group Work 2: Intersections and Curves

Make sure that students doing the first problem obtain both curves: $x^2 + y^2 = 1$ and $z = \pm\frac{3\sqrt{3}}{2}$.

Students doing the second problem should check their answers with you, or present their answers to the class.

Group Work 3: Visualizing Curves from their Projections

This group work attempts to give students a better understanding of the relationships between a space curve and its various two-dimensional projections.

Homework Problems

Core Exercises: 1, 5, 6, 7, 8, 9, 10, 15, 19, 29

Sample Assignment: 1, 4, 5, 6, 7, 8, 9, 10, 12, 15, 19, 27, 29, 34

Note: If three-dimensional graphics are available, add Exercises 21–24 and 25.

Exercise	C	A	N	G	V
1		×			
4			×		
5–10				×	×
12					×
15					×
19		×			×

Exercise	C	A	N	G	V
19		×			×
21–24				×	
25		×		×	
27		×			
29		×			
34		×		×	×

Group Work 2, Section 10.1
Intersections and Curves

1. In this group work, you are given two surfaces whose intersection is two distinct curves. We want to describe these curves.

(a) Describe, intuitively, the two curves in the intersection of the ellipsoid $\frac{9}{4}x^2 + \frac{9}{4}y^2 + z^2 = 9$ and the cylinder $x^2 + y^2 = 1$.

(b) Parametrize the two curves that make up this intersection.

2. (a) Show that the curve $\mathbf{r} = \langle \sin t, \sin t, \cos t \rangle$ lies on the intersection of the two cylinders $x^2 + z^2 = 1$, $y^2 + z^2 = 1$.

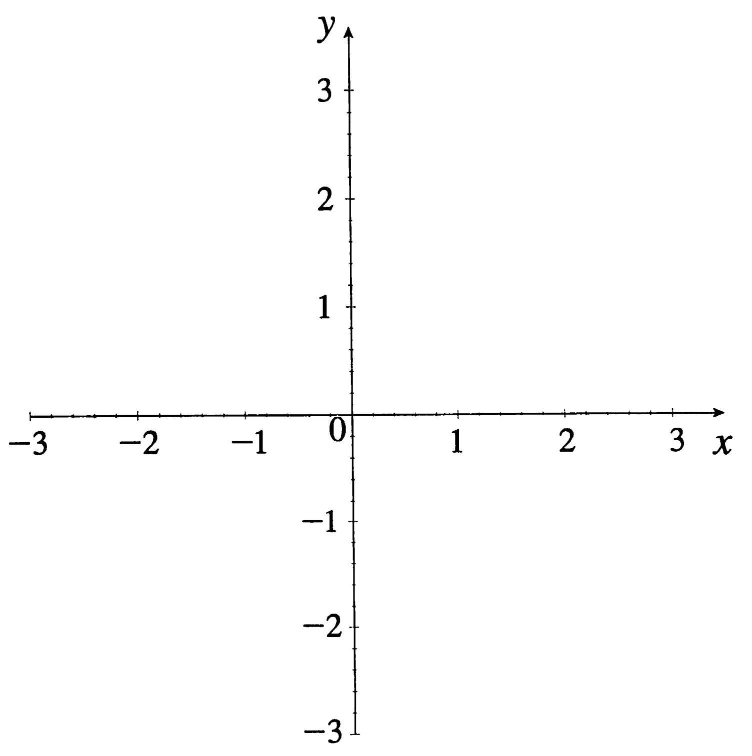
(b) Show the same for $\mathbf{s} = \langle \cos t, \cos t, \sin t \rangle$. Do they describe the same set of points? If not, what is the difference?

Group Work 3, Section 10.1

Visualizing Curves from their Projections

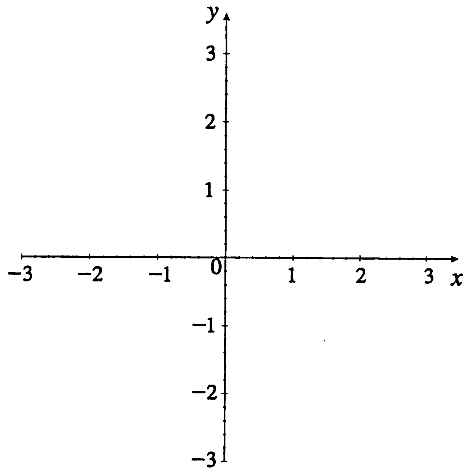
1. Consider the conical spiral $x = t\cos t$, $y = t\sin t$, $z = t$. Sketch all three planar projections. Can you visualize the entire curve by looking at the projections? Describe this curve in words.

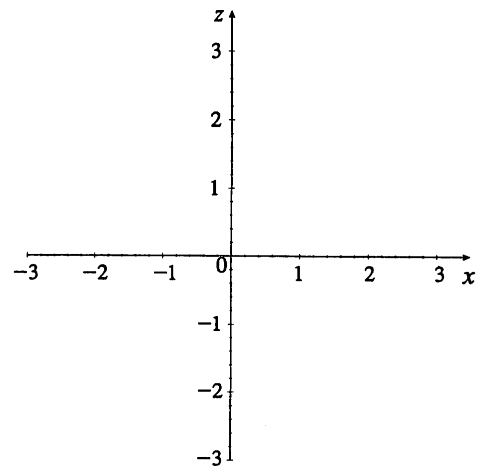
2. Consider the trefoil knot as parametrized in Exercise 34. Sketch the projection of the trefoil knot onto the xy-plane.

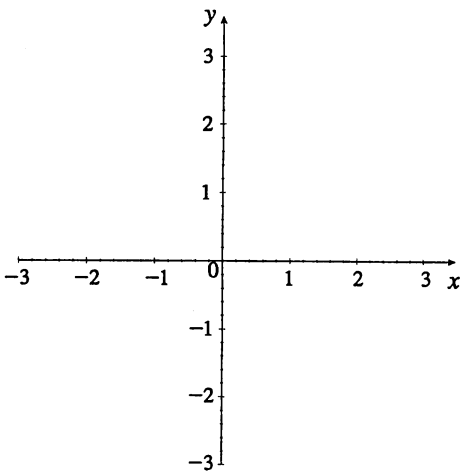


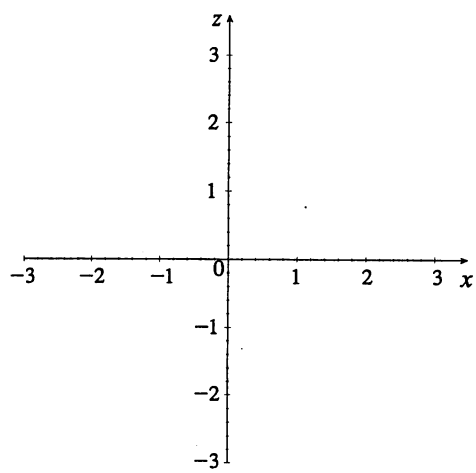
3. Consider the vector functions $\mathbf{r}(t) = \langle t, t^2, \sin t \rangle$ and $\mathbf{s}(t) = \langle t, \sin t, t^2 \rangle$.

(a) Sketch the projections of both functions in the xy- and xz-planes.



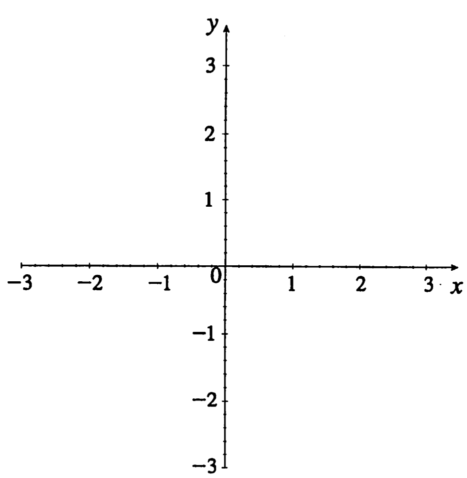


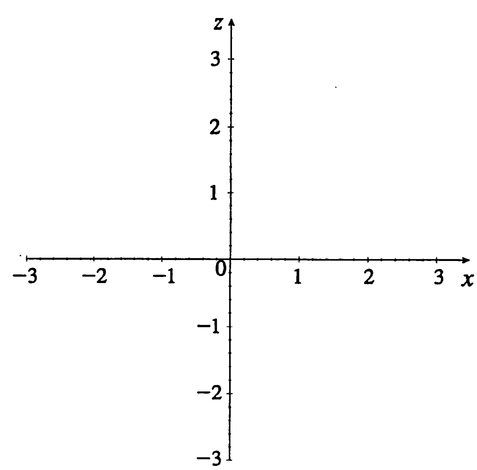




(b) Describe in words the differences between these two functions.

(c) Sketch the xy- and xz-projections for the function $\mathbf{w}(t) = \langle \sin t, t^2, t \rangle$ and describe the differences in this case.





10.2 Derivatives and Integrals of Vector Functions

Suggested Time and Emphasis

$\frac{3}{4}$ class Essential Material: Vector derivatives and unit tangent vectors.
Optional Material (recommended if time permits): Integrals of vector functions.

Points to Stress

1. The vector derivative and the unit tangent vector.

2. The definition of the tangent line to a space curve.

3. The geometric interpretation of the tangent vector and smooth curves.

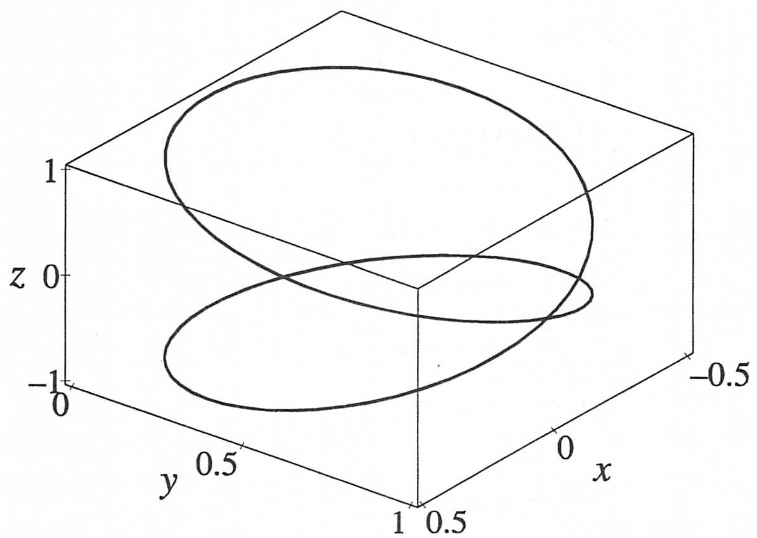
4. Integrals of vector functions.

Text Discussion

- Give an example of a space curve that has a cusp. Sketch one if you can.

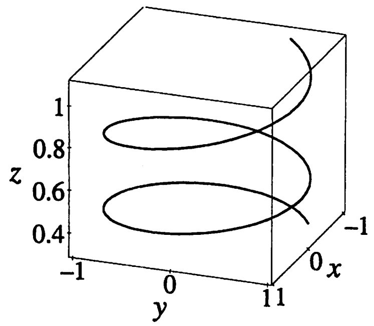
Materials for Lecture

- Elaborate on Example 5 as follows: Consider the curve $\mathbf{r}(t) = \langle \sin t \cos t, \cos^2 t, \sin t \rangle$. Show that $\mathbf{r}'$ is perpendicular to $\mathbf{r}$ by computing $|\mathbf{r}|$ and showing that it is constant. Explain why this guarantees that $\mathbf{r}'$ is perpendicular to $\mathbf{r}$. Point out that this problem could also be done analytically by computing $\mathbf{r}' \cdot \mathbf{r}$, but that the technique of showing that $|\mathbf{r}|$ is constant is often simpler.

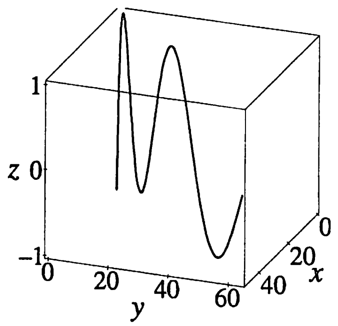


$\mathbf{r}(t) = \langle \sin t \cos t, \cos^2 t, \sin t \rangle$

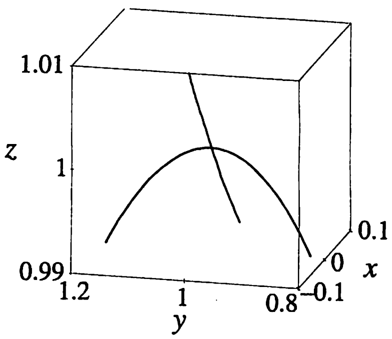
- Consider the "decaying spiral" $\mathbf{r}(t) = \langle \sin t, \cos t, e^{-0.1t} \rangle$, $t \geq -1$. Examine the curve traced out by this function, and explain why its name is appropriate. Show that the unit tangent vector at time t is $\frac{1}{\sqrt{1 + 0.01e^{-0.2t}}} \langle \cos t, -\sin t, -0.1e^{-0.1t} \rangle$. Next, let $\mathbf{q}(s) = \left\langle (s-1)^2, s^2, \sin \frac{\pi}{2} s \right\rangle$, and show that $\mathbf{r} \perp \mathbf{q}$ at their point of intersection $(0, 1, 1)$. If graphing software is available, have the students graph the two curves as shown below to verify the conclusion.



$\mathbf{r}(t) = \langle \sin t, \cos t, e^{-0.1t} \rangle,\ t \geq -1$



$\mathbf{q}(s) = \left\langle (s-1)^2, s^2, \sin \frac{\pi}{2} s \right\rangle$



$\mathbf{r} \perp \mathbf{q}$ at $\langle 0, 1, 1 \rangle$

- Since the unit tangent $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ has constant length 1, the only quantity that changes over time is its direction. Illustrate this fact using $\mathbf{r}(t) = t^3\,\mathbf{i} + t^6\,\mathbf{j},\ t > 0$. Hence, $|\mathbf{T}'(t)|$ measures the rate of change of direction of a unit tangent vector. Also, since $|\mathbf{T}(t)| = 1$, we know that $\mathbf{T}'(t) \perp \mathbf{T}(t)$. Check this fact analytically for $\mathbf{r}(t) = t^3\,\mathbf{i} + t^6\,\mathbf{j}$.
- Point out that $\mathbf{r}(t) = \langle t^3, t^6 \rangle$ and $\mathbf{s}(t) = \langle t^6, t^3 \rangle$ satisfy $\mathbf{r}'(0) = \mathbf{s}'(0) = \mathbf{0}$, and hence neither is smooth for $-1 \leq t \leq 1$. Note that $\mathbf{s}(t)$ has a cusp when $t = 0$ at $(0,0)$, while $\mathbf{r}(t)$ does not have a cusp.
- The main idea to convey about definite vector integrals is that the students should not think of them as areas under curves, but rather as vectors.

Workshop/Discussion

- Find the tangent line to $\mathbf{r}(t) = \langle \sin(e^t \pi), \cos(e^t \pi)\, e^t \rangle$ at $t = 0$. (Answer: $\langle 0,1,1 \rangle + t \langle -\pi, 0, 1 \rangle$)
- Compute the tangent vectors *and* the unit tangent vectors to the two curves $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \frac{\sqrt{5}}{3}\,\mathbf{k}$ and $\mathbf{s}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} - \frac{\sqrt{5}}{3}\,\mathbf{k}$ which form the intersections of the ellipsoid $\frac{4}{9}x^2 + \frac{4}{9}y^2 + z^2 = 1$ and the cylinder $x^2 + y^2 = 1$. Ask the following questions:
  1. Why do these vectors have zero $\mathbf{k}$-components?
  2. Why are these vectors the same on each curve?
  3. What angular change in direction is measured by $\mathbf{T}'(t)$ for both of these curves?
- Point out that different parametrizations of the same curve C can lead to quite different vector integrals of C. As an example, consider the line segment $y = x$ from $(0,0)$ to $(1,1)$. Compute and interpret vector integrals of different parametrizations of the same segment, to show that the integrals of vector functions do *not* generally have an easy geometric representation or any relationships among themselves. Possible parametrizations to try are $\int_0^1 \langle t,t \rangle\, dt$, $\int_0^{\pi/2} \langle \cos t, \cos t \rangle\, dt$, $\int_0^{\pi} \langle \sin \frac{1}{2}t, \sin \frac{1}{2}t \rangle\, dt$, $\int_0^1 \langle t^2, t^2 \rangle\, dt$.

Group Work 1: Velocity Vectors

See if the students recognize that Problem 3 is identical to Example 5 (page 715) from the text.

Group Work 2: Many Parametrizations

This is an open-ended group work. The word "explore" is deliberately left undefined. Closure is very important in an exercise of this nature; the students should have a chance to share their discoveries with others.

Group Work 3: Whom to Believe?

Group Work 4: The Grim Reaper

This exercise allows students to practice working with tangent vectors, using a function that has some surprising properties. For example, the parameter turns out to be the angle that the unit tangent makes with the x-axis.

Homework Problems

Core Exercises: 1, 4, 9, 17, 22, 25, 28, 32

Sample Assignment: 1, 2, 3, 4, 7, 8, 9, 11, 17, 22, 24, 25, 26, 28, 32, 43

Note: Exercises 24 and 26 require a CAS or calculator with three-dimensional graphing capability.

Exercise	C	A	N	G	V
1		×		×	
2		×		×	
3–8		×		×	
9		×			
11		×			
17		×			
22		×			

Exercise	C	A	N	G	V
24		×		×	
25		×			
26		×		×	
28		×			
32		×			
43		×			

Group Work 1, Section 10.2
Velocity Vectors

1. Consider a vector function $\mathbf{v} = \langle f(t), g(t), h(t) \rangle$. Is it true that $|\mathbf{v}'(t)| = |\mathbf{v}(t)|'$? Why or why not?

2. Show that the vector derivative for the helix $\mathbf{h}(t) = \langle a\cos t, a\sin t, bt \rangle$ makes a constant angle with the z-axis. What is this angle if $\mathbf{h}(t) = \left\langle \cos t, \sin t, \sqrt{3}t \right\rangle$?

3. Suppose a point moves along the surface of a sphere. Show that its velocity vector is always perpendicular to its position vector.

Group Work 2, Section 10.2
Many Parametrizations

1. Which of the following vector functions parametrize the helix $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$, $t \geq 0$, presented in Example 4 of Section 10.1?

(a) $\langle \cos 2t, \sin 2t, 2t \rangle$, $t \geq 0$

(b) $\langle \cos t^3, \sin t^3, t^3 \rangle$, $t \geq 0$

(c) $-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \left(t + \frac{\pi}{2}\right)\mathbf{k}$, $t \geq -\frac{\pi}{2}$

2. Check that the following curves parametrize the unit circle. For each curve, compute the unit tangent vector $\mathbf{T}(t)$.

(a) $\langle \sin t, \cos t \rangle$, $0 \leq t \leq 2\pi$

(b) $\cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $\pi \leq t \leq 3\pi$

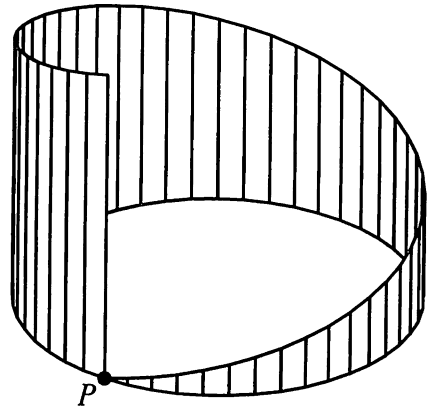
(c) $\langle \sin 2t, \cos 2t \rangle$, $0 \leq t \leq \pi$

(d) $\sin t^2\,\mathbf{i} + \cos t^2\,\mathbf{j}$, $0 \leq t \leq \sqrt{2\pi}$

(e) $\left\langle \dfrac{2t}{1+t^2}, \dfrac{t^2-1}{1+t^2} \right\rangle$, $-\infty < t < \infty$

Group Work 3, Section 10.2
Whom to Believe?

Sally Dart, the great sculptor, had just finished designing her newest masterstroke of minimalism, "The Spiral to the Stars." It was to be a rising, curving thin wall, the top of which would follow the curve $\mathbf{r}(t) = \langle \cos \pi t, \sin \pi t, t \rangle$ from $t = 0$ to $t = 2$ as shown.



In order to help her calculate her costs, she wanted to know the area of this shape, in order to determine how much stainless Italian sheet steel she would need. To answer this question, she visited noted textbook author James Stewart.

Suddenly there is a crashing sound in the hallway! The door slams open and in walks Professor Stewart's evil twin sister Onad! She looks at the problems, laughs, and then faces Sally Dart. "My brother is noted for making things complicated for no good reason. What you really want is the area under a curve, right? And just like in single-variable calculus, the area under a parametric curve is given simply by the definite integral $\left| \int_0^2 \mathbf{r}(t)\, dt \right| = \left| \int_0^2 \langle \cos \pi t, \sin \pi t, t \rangle\, dt \right|$."

1. Following Onad's advice, derive the area of the spiral surface.

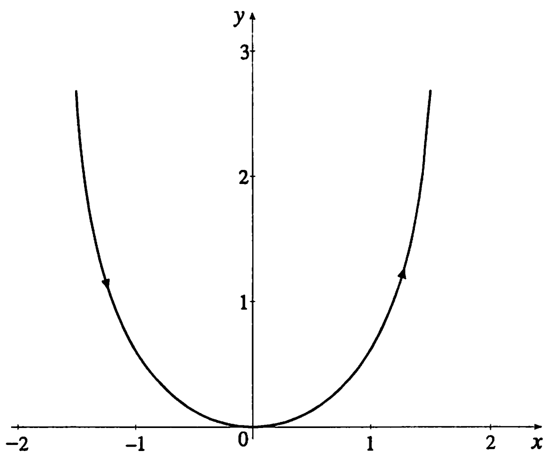
Professor Stewart smiles and says, "There is a very simple solution. Take a pair of bolt cutters and snip your piece of steel at point P. Then flatten it out and *voila*, we have a right triangle. So the area is now really easy to compute."

2. Following Professor Stewart's advice, find the area of the spiral surface.

3. Sally looks from brother to sister and back again. They have given her two different answers! Which of the two answers is correct, and why?

Group Work 4, Section 10.2
The Grim Reaper

Consider the curve $\mathbf{r}(w) = w\,\mathbf{i} - \ln(\cos w)\,\mathbf{j}$ on the interval $-\frac{\pi}{2} < w < \frac{\pi}{2}$.



1. Compute the tangent vector $\mathbf{r}'(w)$. Sketch tangent vectors corresponding to $w = -\frac{\pi}{3}$, $w = -\frac{\pi}{4}$, $w = -\frac{\pi}{6}$, $w = 0$, $w = \frac{\pi}{6}$, $w = \frac{\pi}{4}$, and $w = \frac{\pi}{3}$.

2. For each value of w, what is the length of the tangent vector $\mathbf{r}'(w)$? Find an equation for the unit tangent vector $\mathbf{T}(w)$.

3. For each value of w, what angle does the unit tangent vector $\mathbf{T}(w)$ make with the x-axis?

4. Find a vector $\mathbf{N}(w)$ perpendicular to $\mathbf{T}(w)$ and pointing away from the curve $\mathbf{r}(w)$.

10.3 Arc Length and Curvature

Suggested Time and Emphasis

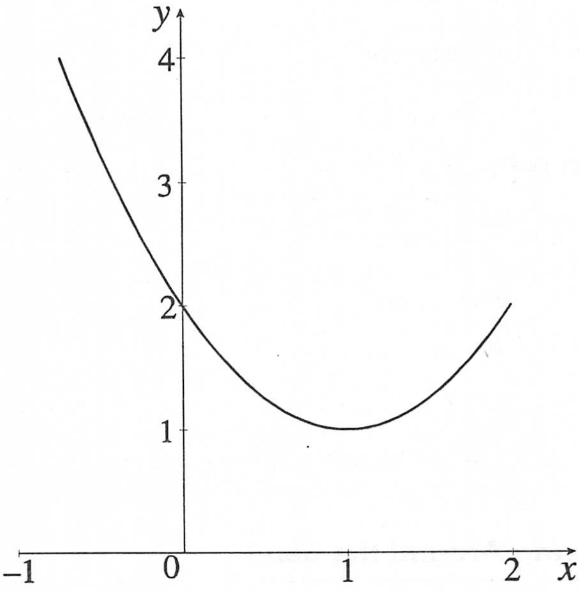
$\frac{3}{4}$–1 class Essential Material: Arc length and the basic ideas of curvature.
Optional Material: The TNB frame.

Points to Stress

1. The arc length and curvature formulas.
2. The independence of arc length and parametrization.
3. The geometric definition of curvature.
4. The TNB frame.

Text Discussion

- Why do we need to assume that the curve C is traversed exactly once as t increases by the vector function $\mathbf{r} = \mathbf{r}(t)$ in order to define the arc-length function $s = s(t)$?
- For the following curve, sketch in an approximation of the osculating circle at the point $(1, 1)$.



Materials for Lecture

- Find a condition on $\mathbf{d}$ so that the straight line segment $\mathbf{r}(t) = t\,\mathbf{d} + \mathbf{b}$, $0 \le t \le 2$ is parametrized by arc length. (Answer: $|\mathbf{r}'(t)| = |\mathbf{d}|$, $s = \int_0^t |\mathbf{d}|\,du = |\mathbf{d}|\,t$, $t = \frac{s}{|\mathbf{d}|}$, and $\mathbf{r}(t(s)) = \frac{s}{|\mathbf{d}|}\mathbf{d} + \mathbf{b} = s\frac{\mathbf{d}}{|\mathbf{d}|} + \mathbf{b}$. So $\mathbf{d}$ needs to be a unit vector.) Show why this answer makes sense geometrically.
- Compute the length of the curve traced out by $\mathbf{r}(t) = \frac{1}{2}e^t\,\mathbf{i} + \cos e^t\,\mathbf{j} + \sin e^t\,\mathbf{k}$ as t goes from 0 to 1. [Answer: $L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 \sqrt{e^{2t} + \frac{1}{4}e^{2t}}\,dt = \frac{\sqrt{5}}{2}(e - 1)$]

- Point out that since the unit tangent vector satisfies $|\mathbf{T}(t)| = 1$, we have $\mathbf{T}'(t) \perp \mathbf{T}(t)$ and so $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$ is perpendicular to $\mathbf{T}(t)$. Hence $\mathbf{N}(t)$ is a unit normal.
- Use the following method to intuitively describe the curvature $\kappa = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$: Since the length of $\mathbf{T}$ is constant, $|\mathbf{T}'(t)|$ is the rate of change of direction, or turning, of the unit tangent, and $|\mathbf{r}'(t)|$ measures the speed along the curve. So κ is essentially the rate of turning of the unit tangent $\mathbf{T}$ divided by the speed along the curve. For a circle parametrized in the standard way, $\mathbf{r}(t) = \langle a\cos t, a\sin t\rangle$, the angle of the unit tangent will change by 2π as we go around the circle, and so the rate of turning is 1, and the speed a is the radius of the circle. Therefore $\kappa = \dfrac{1}{\text{radius}}$. For a straight line, since the unit tangent never turns, κ is zero. One can also conclude that κ is zero by noting that in the case of a straight line, the radius is "infinite".

Workshop/Discussion

- Consider the curve $\mathbf{r}(t) = \sin t\,\mathbf{i} + t\,\mathbf{j} + \cos t\,\mathbf{k}$, $0 \le t \le 2\pi$. Use this example to show that arc length is independent of parametrization by computing the arc length for various parametrizations:

$$
\begin{aligned}
L &= \textstyle\int_0^{2\pi} |\mathbf{r}'(t)|\,dt \\
L_1 &= \textstyle\int_0^{\sqrt{2\pi}} |\mathbf{s}'(t)|\,dt, \text{ where } \mathbf{s}(t) = \left\langle \sin t^2, t^2, \cos t^2 \right\rangle, 0 \le t \le \sqrt{2\pi}, \text{ and} \\
L_2 &= \textstyle\int_{\pi/2}^{5\pi/2} |\mathbf{v}'(s)|\,ds, \text{ where } \mathbf{v}(s) = \left\langle \cos s, s + \frac{\pi}{2}, -\sin s \right\rangle, \ -\frac{\pi}{2} \le s \le \frac{3\pi}{2}
\end{aligned}
$$

- Talk about the curvature of the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$. Mention that it is possible to first find the arc length parameterization $\mathbf{r}(s) = \cos\left(\frac{1}{\sqrt{2}}s\right)\mathbf{i} + \sin\left(\frac{1}{\sqrt{2}}s\right)\mathbf{j} + \frac{1}{\sqrt{2}}s\,\mathbf{k}$, then compute $\mathbf{T}(s) = \mathbf{r}'(s)$, and finally, $\left|\dfrac{d\mathbf{T}}{ds}\right|$. But a direct computation using one of the formulas for curvature with the given parametrization gives $\kappa = \frac{1}{2}$, with less work.
- Compute $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ for a simple curve such as $\left\langle t^2, -t, t\right\rangle$ at $t = 1$.
- Consider the curve $\mathbf{r}(t) = \left\langle e^{-t}\cos t, e^{-t}\sin t, e^{-t}\right\rangle$, $t \ge 0$. Explain why this curve spirals down to the origin along the surface $z^2 = x^2 + y^2$. Show that the curvature is $\kappa(t) = \frac{\sqrt{2}}{3}e^t$, and explain what this means about the curve as $t \to \infty$.

Group Work 1: Going Around the Bend

There are two ways to do Problem 1, either intuitively by thinking about osculating circles, or computationally, as a max/min problem. We recommend instructing the students to first make an intuitive guess about the correct answer, and then use computations to verify their guess. Problem 2 answers a question raised in the workshop/discussion material.

Group Work 2: The Length of the Reaper

Parts 1–4 are straightforward. Parts 5–7 further develop the arc-length parametrization.

Group Work 3: T, N, B

Extended Group Work 4: The Evolute of a Plane Curve

iis project can be assigned to a group of students who wish to explore the material more deeply in an teresting context. It should take between one and three hours to complete.

Lab Project: Osculating Circles

se a CAS to plot the plane curve and its osculating circle at various points:

. $y = x^3$

. $y = \sin x$

. $y = e^x$

Extended Lab Project: Estimating Arc Length

Homework Problems

re Exercises: 1, 9, 14, 17, 27, 33, 37

mple Assignment: 1, 5, 9, 10, 14, 17, 24, 26, 27, 28, 33, 35, 37, 39, 42, 44

te: • Exercise 42 requires a CAS.

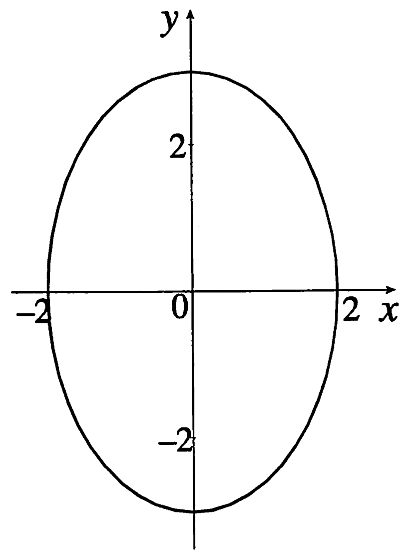
• Problems 8 from Focus on Problem Solving (page 747) is very challenging and would make a good project.

Exercise	C	A	N	G	V
1		×			
5				×	
9	×	×			
10		×			
14		×			
17		×			
24		×			
26		×			

Exercise	C	A	N	G	V
27					×
28		×		×	
33		×			
35		×			
37		×			
39		×		×	
42		×			
44	×	×			

Group Work 1, Section 10.3
Going Around the Bend

1. Consider the ellipse $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$.



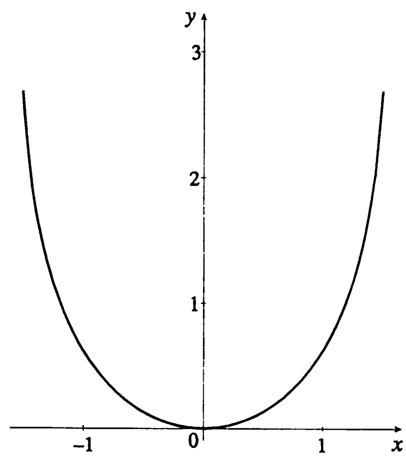
(a) Where is the curvature maximal? Give reasons for your answer.

(b) Where is the curvature minimal? Give reasons for your answer.

2. Consider the curve $\mathbf{r}(t) = \langle e^{-t}\cos t, e^{-t}\sin t, e^{-t}\rangle$, $t \geq 0$. Show that the distance traveled by a particle along this curve over *all* time is $\sqrt{3}$. (**Hint:** What is the appropriate improper integral to evaluate?)

Group Work 2, Section 10.3
The Length of the Reaper

Consider the graph of $y = -\ln(\cos x)$.



1. Fill in the blank: This graph can be written as a parametrized curve by $\mathbf{r}(t) = \langle t, \_\_\_\_\_\_\_\_\_\_ \rangle$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

2. Compute $\mathbf{r}'(t)$ and $\kappa(t)$.

3. Find the unit tangent vector $\mathbf{T}(t)$ and the unit normal vector $\mathbf{N}(t)$.

4. Find the arc length function $s(t)$ by solving $s(t) = \int_0^t |\mathbf{r}'(u)|\, du$

5. Solve for $\sec t$ in terms of s. (*Hint:* You'll have to solve the arc length function you found in the previous part for $\tan t$, and then use a trigonometric identity.)

6. Solve for $\cos t$ in terms of s.

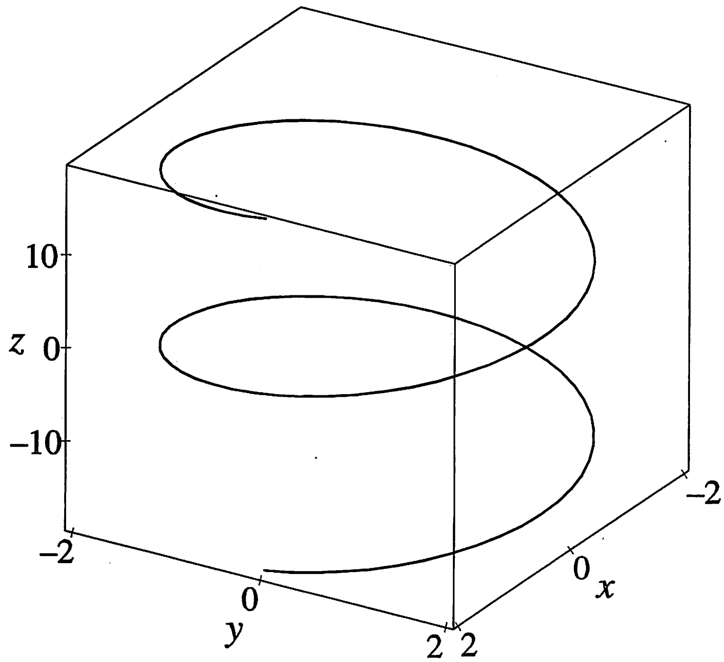
7. Parametrize $\mathbf{r}(t)$ with respect to arc length. That is, find $\mathbf{r}(s)$.

Notice that even though we now have parametrized the Grim Reaper with respect to arc length, these computations in terms of s are very awkward. Would you like to, say, find r when $s = 1$ using the equation you derived above? We wouldn't either. In theory, we can compute $|\mathbf{r}'(s)| = 1$ for all s and $\kappa(s) = |\mathbf{r}''(s)|$, but these computations are often messy.

Group Work 3, Section 10.3

T, N, B

If $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 3t\,\mathbf{k}$, find $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$. Sketch $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ when $t = \frac{3\pi}{2}$.



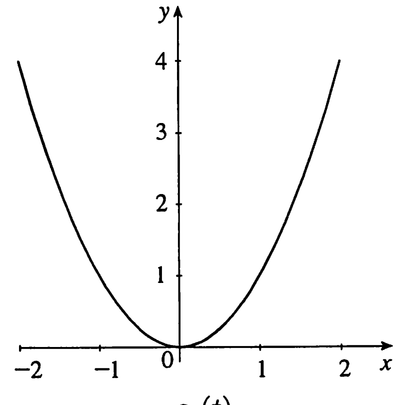
Extended Group Work 4, Section 10.3
The Evolute of a Plane Curve

Suppose the vector function $\boldsymbol{\alpha}(t)$ describes a plane curve, and $\mathbf{N}(t)$ is the unit normal vector as definined in the text. Now define a different vector function $\boldsymbol{\beta}(t) = \boldsymbol{\alpha}(t) + \frac{1}{\kappa(t)}\mathbf{N}(t)$, where $\kappa(t)$ is the curvature to $\boldsymbol{\alpha}$. This function $\boldsymbol{\beta}$ is called the **evolute** of $\boldsymbol{\alpha}$. In this exercise, we will figure out exactly what makes the evolute so special.

1. Let $\boldsymbol{\alpha}(t) = \langle t, t^2 \rangle$. Compute $\mathbf{N}(t)$ and show that $\kappa(t) = \frac{2}{(1+4t^2)^{3/2}}$.

2. Use Problem 1 to compute the evolute $\boldsymbol{\beta}(t)$ of $\boldsymbol{\alpha}(t)$.

3. Sketch the curve given by $\boldsymbol{\beta}(t)$ on the axes below.



$\boldsymbol{\alpha}(t)$

4. Show that $|\alpha(t) - \beta(t)|$ is the radius of the osculating circle for the above curve.

5. Show that the endpoint of the position vector $\beta(t)$ is the center of the osculating circle at $\alpha(t)$ for the above curve.

6. Verify your solutions to the problems above by specifically computing the curvature of $\alpha(t)$ at $t = 0$ and $t = 1$.

10.4 Motion in Space

Suggested Time and Emphasis

$\frac{3}{4}$–1 class Essential Material: Velocity and acceleration
Optional Material: Kepler's Laws

Points to Stress

1. Definitions of velocity and acceleration as vector functions.
2. How to derive velocity from acceleration and position from velocity.
3. Tangential and normal components of acceleration.

Text Discussion

- What is the relationship between the velocity vector and the tangent vector?
- How is ds/dt, the rate of change of distance along the direction of motion with respect to time, related to the velocity vector $\mathbf{v}(t)$?
- Why must the acceleration vector $\mathbf{a}$ always lie in the osculating plane at a point on a curve?

Materials for Lecture

- Emphasize that the velocity vector $\mathbf{v}$ gives the direction and speed with which a particle would travel if it flew off the curve at that instant. Also, the speed $|\mathbf{v}(t)| = ds/dt$ is the rate of change of distance along the direction of motion with respect to time.
- Reinforce Example 4 by reminding students that $\theta(t)$ is the angle that $\mathbf{r}(t)$ makes with the positive x-axis, and that the angular speed $d\theta/dt$ is the rate of change of θ with respect to time.
- Point out that the model described in Figure 5 is not a good description of the way the Earth moves about the Sun. The Earth moves around the Sun in an elliptical orbit with the Sun at a focus. Note that the acceleration of the Earth points toward the Sun, not toward the center of our elliptical orbit.
- Emphasize that the acceleration vector $\mathbf{a}(t)$ is not generally perpendicular to $\mathbf{v}(t)$. (Use $\mathbf{r}(t) = \langle t, e^t, e^{-t} \rangle$ as an example.) However, if $|\mathbf{v}(t)|$ is constant, then $\mathbf{a}(t) \perp \mathbf{v}(t)$.
- Remind students that since the unit tangent vector $\mathbf{T}$ satisfies $|\mathbf{T}| = 1$, then $\mathbf{T}$ is orthogonal to $\mathbf{T}'$ and hence $\mathbf{N} = \dfrac{1}{|\mathbf{T}'|}\mathbf{T}'$. This is important in deriving the expressions for the components of acceleration $a_\mathbf{N}$ and $a_\mathbf{T}$.

Workshop/Discussion

- Compute velocity and acceleration vectors for $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ and $\mathbf{s}(t) = \langle \cos t, \sin t, t^2 \rangle$. Point out that $\mathbf{r}$ has no z-component of acceleration, while $\mathbf{s}$ has a constant z-component of acceleration. Then note that the length of the acceleration vector is still constant for both curves.
- Compute the tangential and normal accelerations in Example 5.

- Give an example to show how different vector functions whose ranges are the same can have the same speed but different velocities (for example, $\mathbf{q} = \langle \sin t, \cos t \rangle$ and $\mathbf{r} = \langle \cos t, \sin t \rangle$).
- Graph and analyze an example of three-dimensional centripetal force. (*Centripetal* means "directed from the outside towards the center.") For example, the function $\mathbf{r}(t) = \langle 5\sin 2t, 4\cos 2t, 3\cos 2t \rangle$ gives $\mathbf{r}'' = -4\mathbf{r}$, while $\mathbf{r} = \langle \cos t, \sin t, \cos t + \sin t \rangle$ gives $\mathbf{r}'' = -\mathbf{r}$.
- Describe the motion of $\mathbf{w}(t) = \left\langle 10e^{-2t}, 0.2e^{2t}, 3e^{-2t} \right\rangle$. Show that $\mathbf{w}'' = 4\mathbf{w}$ and explain why this force is called centrifugal (directed from the center to the outside.)

Group Work 1: Checking Out the Action

The equation of motion in Problem 2 is $\mathbf{r}(t) = \frac{1}{k}\left(e^{kt} - 1\right)\mathbf{v}_0 + \mathbf{r}_0$, and this curve lies along the line $(s) = s\mathbf{v}_0 + \mathbf{r}_0$.

Group Work 2: Back to Start

The second question is impossible to solve as stated, for it would violate the Mean Value Theorem. The idea s to illustrate a fundamental difference between graphs of ordinary functions and curves in $\mathbb{R}^2$ or $\mathbb{R}^3$. After llowing the students to try the second question for awhile, announce that an acceptable answer would be a eason why it cannot be solved.

Group Work 3: Find the Error

This group work has a subtle solution, and the students may require some guidance. The main flaw in the easoning presented is that the argument doesn't take into account that the base points of the vectors are not he same. Hence, the vector labelled $\mathbf{v}(t + \Delta t) - \mathbf{v}(t)$ is really $\mathbf{v}(t + \Delta t) + \mathbf{r}(t + \Delta t) - (\mathbf{v}(t) + \mathbf{r}(t))$ and he limit vector is $\mathbf{a}(t) + \mathbf{v}(t)$. Notice that our *formula* is correct, but the *picture* gives the wrong acceleration ector.

Homework Problems

ore Exercises: 1, 2, 5, 7, 14, 17, 32

ample Assignment: 1, 2, 5, 7, 12, 14, 16, 17, 23, 26, 27, 32

lote: For a physical extension of this material, Problems 3 and 4 from Focus on Problem Solving (page 746) can be assigned. The computations for these problems are quite challenging.

Exercise	C	A	N	G	V
1			×		
2				×	
5		×		×	
7		×		×	
12		×			
14		×			

Exercise	C	A	N	G	V
16		×		×	
17		×			
23		×			
26		×			
27		×			
32		×		×	×

Group Work 1, Section 10.4
Checking Out the Action

1. Using the tangential and normal components of acceleration, describe in words those curves $\mathbf{r}$ for which

(a) $\mathbf{a} \parallel \mathbf{v}$

(b) $\mathbf{a} \perp \mathbf{v}$

2. We now look at the case $\mathbf{a} \parallel \mathbf{v}$ in a little more detail. Suppose $\mathbf{a} \parallel \mathbf{v}$; for example, say $\mathbf{a} = k\mathbf{v}$, k a non-zero constant. Assuming initial position $\mathbf{r}_0$ and initial velocity $\mathbf{v}_0$,

(a) Find an equation for the velocity function $\mathbf{v}(t)$ in terms of $\mathbf{v}_0$.

Hint: Remember that $\mathbf{a} = \dfrac{d\mathbf{v}}{dt}$.

(b) Find an equation for the position function $\mathbf{r}(t)$ in terms of $\mathbf{v}_0$ and $\mathbf{r}_0$.

(c) Describe the line which contains the range of the position function $\mathbf{r}(t)$.

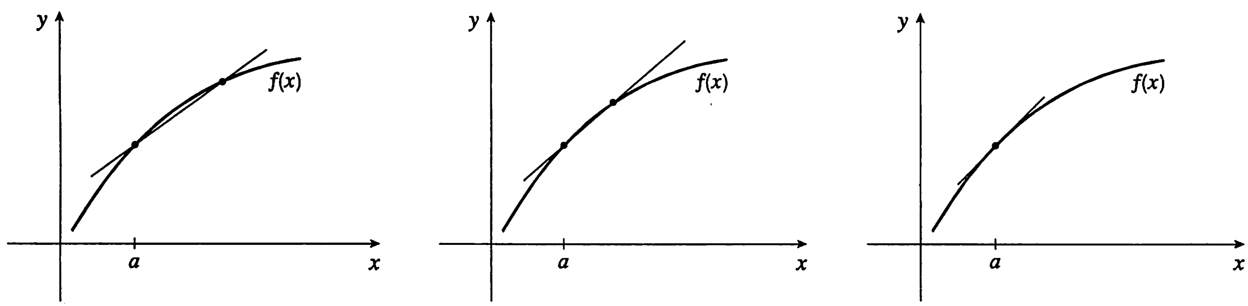
Group Work 2, Section 10.4
Back to Start

1. Give an example of a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ for which the position at time $t = 10$ is the same as the position at time $t = 0$, yet the speed is never zero.

2. Is it possible to find a differentiable function of a single variable with this property, that is, a function $y = f(x)$ for which $f(0) = f(10)$ and $f'(x)$ is never zero for $x \in [0, 10]$? Why or why not?
Hint: Think of some important theorems about derivatives.

Group Work 3, Section 10.4
Find the Error (Part 1)

As you recall, one way to figure out the derivative of a function $f(x)$ at a point was to compute the slope of the line tangent to f at the point x by taking the limit of secant lines as follows:

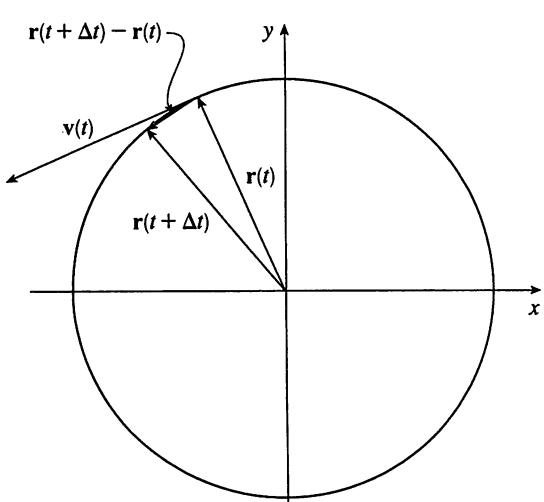


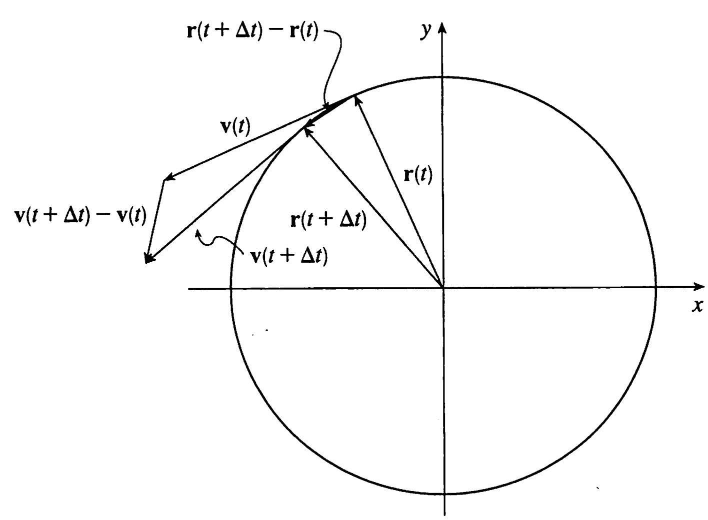
We found that $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

At first glance, it may seem reasonable to try to compute the vector acceleration of a function using that same technique. In this case, however, something goes wrong. We present the false argument below:

Consider the circle parametrized by $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$. The direction of $\mathbf{r}(t + \Delta t) - \mathbf{r}(t)$ should be a good approximation for the direction of $\mathbf{v}(t)$ since $\mathbf{v}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$ (see the diagram on the left).

Now consider the same circle. The direction of $\mathbf{v}(t + \Delta t) - \mathbf{v}(t)$ should be a good approximation to the direction of $\mathbf{a}(t)$ since $\mathbf{a}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t}$ (see the diagram on the right).





Since $\mathbf{a}(t) = \langle -\cos t, -\sin t \rangle$, we see that $\mathbf{a}(t)$ should point toward the origin, however the direction of $\mathbf{v}(t + \Delta t) - \mathbf{v}(t)$ is not the same as that of $\mathbf{a}(t)$.

Find the Error (Part 1)

1. Why should $\mathbf{a}(t)$ point towards the origin?

2. What, then, is wrong with our analysis above?

Applied Project: Kepler's Laws

This is a project that is best assigned in its entirety, although Problem 3 or Problem 4 can be omitted if there is a time crunch. (We recommend keeping Problem 4.) Make sure to impart to the students how amazing the results of this project really are: Given *only* that

1. $\mathbf{F} = m\mathbf{a}$ and
2. the gravitational force between two objects is proportional to the product of their masses and inversely proportional to the square of the distance between them,

it is possible to use calculus to deduce Kepler's three laws without even looking out of the window to make a measurement!

10.5 Parametric Surfaces

Suggested Time and Emphasis

$\frac{3}{4}$–1 classes   Essential Material, but it can be delayed until just before Section 13.6.

Transparencies Available

- Transparency 37 (Figure 5, page 737)
- Transparency 38 (Exercises 11–16, graphs 1–VI, page 741)

Points to Stress

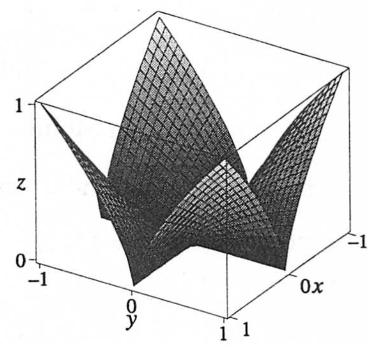
1. Parametric surfaces and the role of grid curves in studying these surfaces.
2. How the form and/or symmetry of a surface helps one in choosing a parametrization.
3. Different parametrizations for surfaces.

Text Discussion

- Why parametrize a surface?

Materials for Lecture

- Revisit the discussion from Section 9.5 on parametrizing a plane.
- Present an example of how to choose a parametrization for a surface using form or symmetry. A good example is the top half of the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{4} = 1$, $y \geq 0$. Notice that $x^2 + z^2 = 4\left(1 - y^2\right)$, so if we let $u = 2\sqrt{1 - y^2}$, then $x^2 + z^2 = u^2$. So we let $x = u\cos v$, $z = u\sin v$, and we have the parametrization $\mathbf{r}(u, v) = \left\langle u\cos v, \sqrt{1 - \frac{1}{4}u^2}, u\sin v\right\rangle$ for $0 \leq u \leq 2$, $0 \leq v \leq 2\pi$. Note that now the surface is the graph of a parametric function, and that for the same values of u and v, $\mathbf{s}(u, v) = \left\langle u\cos v, -\sqrt{1 - \frac{1}{4}u^2}, u\sin v\right\rangle$ parametrizes the bottom half of the ellipsoid.
- Present examples of how to determine what a surface looks like from its parametrization. Perhaps start with the example $q(s, t) = \langle s, t, st^2\rangle$, which parametrizes the surface $z = xy^2$. Then look at the parametrization $\mathbf{w}(s, t) = \langle st^2, s^2t, s^2t^2\rangle$, $x = st^2$, $y = s^2t$, and $z = s^2t^2$. We have $xy = s^3t^3$ and $(xy)^2 = s^6t^6 = \left(s^2t^2\right)^3 = z^3$, so an equation of the surface is $z^3 = (xy)^2$. This surface can be visualized by setting $y = 1$ and thinking of the plane curve $z = x^{2/3}$ or setting $x = 1$ and thinking of the plane curve $z = y^{2/3}$.



$z^3 = (xy)^2$

- Point out that surface of revolution S formed by rotating $y = f(x)$ about the x-axis satisfies $y^2 + z^2 = [f(x)]^2\left(\cos^2\theta + \sin^2\theta\right) = [f(x)]^2$, which describes the surface directly as an equation in x, y, and z. Similarly, the surface T formed by rotating $z = g(y)$ about the y-axis has equation $x^2 + z^2 = [g(y)]^2$ or parametric equations $x = g(y)\cos\theta$, $y = y$, $z = g(y)\sin\theta$.

Workshop/Discussion

- Parametrize the elliptic paraboloid $x^2+y^2-z^2=1$. (Answer: $\mathbf{r}(u,v) = \left\langle u\cos v, u\sin v, \pm\sqrt{u^2-1}\right\rangle$)
- Identify the surfaces parametrized by $\mathbf{r}(s,t) = \langle s\sin t, s\cos t, s\rangle$ if

  1. $0 \le s \le 1, 0 \le t \le \pi$ **2.** $0 \le s \le 4, 0 \le t \le 2\pi$ **3.** $0 \le s, 0 \le t \le 2\pi$
- Give two different parametrizations of a cone, one in which the grid curves meet at right angles and one in which one set of grid curves spirals up the cone.
  [Examples: $\mathbf{A}(s,t) = \langle s\cos t, s\sin t, s\rangle$ and $\mathbf{B}(s,t) = \langle st\cos t, st\sin t, st\rangle$]
- Find two parametrizations for the parabolic surface $z = 16 - x^2 - y^2$ that lies above the xy-plane, first by using the Cartesian parametrization ($x = x$, $y = y$, $z = 16 - x^2 - y^2$, $z \ge 0$) and then identifying z as a surface of revolution for the curve $y = \sqrt{16-z}$, $z \ge 0$ (which gives the parametrization $x = \sqrt{16-z}\,\cos\theta$, $y = \sqrt{16-z}\,\sin\theta$, $z = z$, $z \ge 0$). Note that the grid curves are parabolas in the first case and parabolas/circles in the second. A third approach is to use cylindrical coordinates.

Group Work 1: The Propeller Problem

Be sure that the students understand that $\mathbf{r}(\theta, z)$ is being given to them in cylindrical coordinates (r, θ, z). For instance, $2 + \cos(4(\theta - z))$ is the value of r at angle θ and height z above the xy-plane.
If the parameter z is imagined to represent time, this surface represents a spinning propeller. Note that the $z = 0$ grid curve of this surface appeared in Group Work 2: Intersections and Curves, from Section 10.1.

Group Work 2: Bagels, Bagels, Bagels!

This is Exercise 32 from the text. There are two versions of this group work included. The second requires more independent thought on the part of the students.

Group Work 3: More With Möbius Strips

This group work extends Exercise 30 and requires technology.

The surface with parametric equation $\mathbf{r}(\theta,t) = \left\langle 2\cos\theta + t\cos\frac{1}{2}\theta, 2\sin\theta + t\cos\frac{1}{2}\theta, t\sin\frac{\theta}{2}\right\rangle$, $0 \le \theta \le 2\pi$, $-\frac{1}{2} \le t \le \frac{1}{2}$ is called a Möbius strip.

1. Graph this surface and consider several viewpoints. What is unusual about it?

2. Find the coordinates (x,y,z) corresponding to

(a) $\theta = 0, t = \frac{1}{2}$ $\theta = \pi, t = \frac{1}{2}$ $\theta = 2\pi, t = \frac{1}{2}$ $\theta = 3\pi, t = \frac{1}{2}$ $\theta = 4\pi, t = \frac{1}{2}$

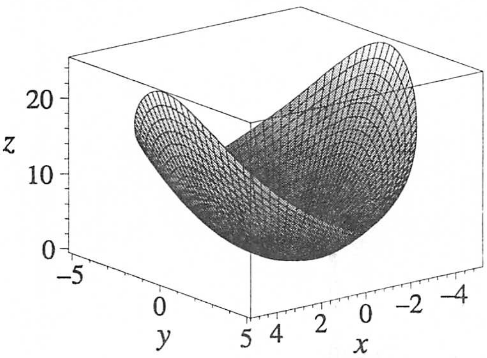
(b) $\theta = 0, t = -\frac{1}{2}$ $\theta = \pi, t = -\frac{1}{2}$ $\theta = 2\pi, t = -\frac{1}{2}\theta = 3\pi, t = -\frac{1}{2}$ $\theta = 4\pi, t = -\frac{1}{2}$

3. Graph the grid curves corresponding to $t = \frac{1}{2}$ and $t = -\frac{1}{2}$ and note that they are the same set of points. Also note that each curve makes two circuits about the z-axis and then closes up at $\theta = 4\pi$. However, the complete Möbius strip is created when $0 \le \theta \le 2\pi$. How can this be?

Lab Project: A Difficult Plot

1. Have the students plot the surface described by $x^2 + y^2 = z\left[1 + \left(\tan^{-1}\frac{x}{y}\right)^2\right]$. Allow them to notice that this is very hard to do, even with a sophisticated graphing package like Maple or Mathematica. They may fail to get a picture of this surface that makes sense. This is okay; let them fail this time.

2. Now have them draw the parametric surface $\mathbf{r}(s,t) = \left\langle s\cos t, s\sin t, \frac{s^2}{1+t^2}\right\rangle$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$. This one should be straightforward. They should get a picture like this:



3. Now have them discover, using algebra, that the two surfaces are the same. When they do this, discuss how important it is to be able to parametrize surfaces.

4. Let them investigate the surface for $t \in \mathbb{R}$. Ask what the relationship is between the two intersecting surfaces which result.

Homework Problems

Core Exercises: 1, 4, 11–16, 27, 32(a)

Sample Assignment: 1, 4, 6, 9, 11–16, 19, 21, 27, 32

Note:
- Problems 6, 9, 27, and 32 require a CAS.
- Problem 32 should be done in class, as group work, or as part of the assignment.

Exercise	C	A	N	G	V
1		×			×
4		×			×
6				×	
9				×	
11–16					×

Exercise	C	A	N	G	V
19		×			×
21		×			×
27		×		×	
32		×		×	

Group Work 1, Section 10.5
The Propeller Problem

Consider the parametrized surface $\mathbf{r}(\theta, z) = \langle 2 + \cos(4(\theta - z)), \theta, z \rangle$, where $\langle r, \theta, z \rangle$ is the cylindrical representation of the position vector $\mathbf{r}(\theta, z)$.

1. Sketch grid curves for $(\theta, 0)$, $(\theta, 1)$, and $(\theta, 2)$.

2. Describe this surface and sketch it if you can.

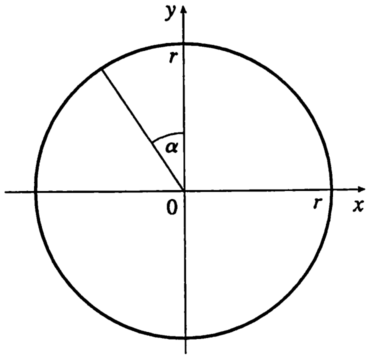
3. If the parameter z is imagined to represent time, that is, increasing time is represented by increasing the z-coordinate, what can this surface represent?

Group Work 2, Section 10.5
Bagels! Bagels! Bagels! (Version 1)

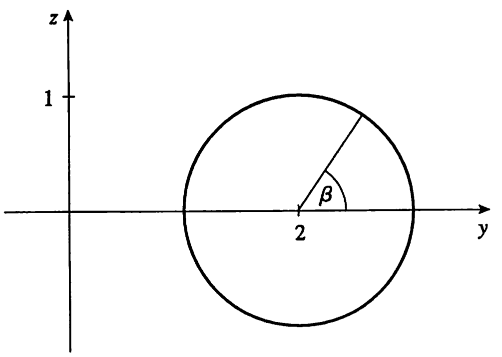
There are many ways to create bagels. Some people prefer boiling them, some prefer baking them, and some prefer defrosting then toasting them. Today we shall be creating bagels by the method of parametrizing them.

In single-variable calculus, it was shown that one can compute the volume of a torus, or doughnut shape, by thinking of it as a circle rotated about a horizontal or vertical line.

1. Parametrize a circle of radius r centered at the origin in the xy-plane starting at $(0, r)$. Let α be the angle between the position vector and the y-axis.



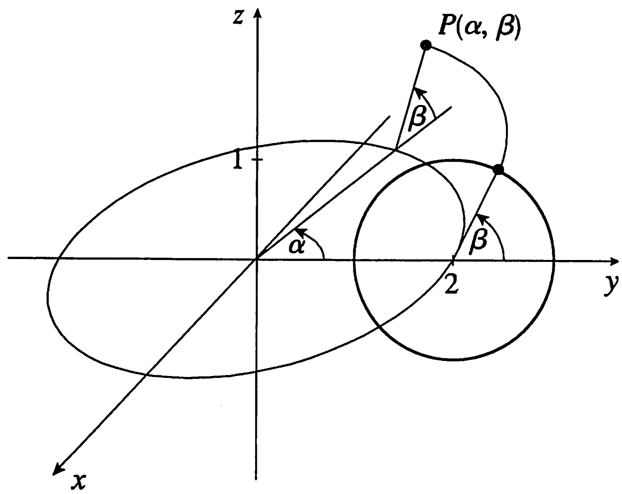
2. Parametrize a circle of radius 1 in the yz-plane with center $(2, 0)$ starting at $(0, 3)$. Let β be the angle between the position vector and the positive y-axis.



3. Now we want to characterize a typical point on our bagel, so we can write a vector function $s(\alpha, \beta)$ whose range is the entire breakfast treat. To find any specific point we

(a) Move α radians along the horizontal curve, then

(b) Rotate β radians along the vertical curve.



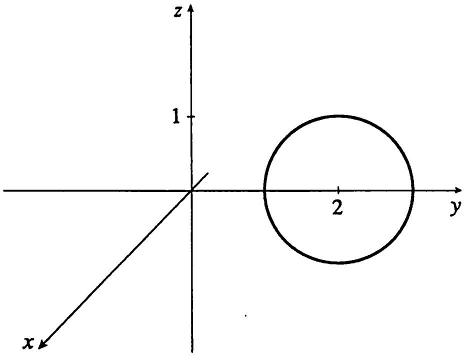
Now express the bagel as the range of a vector function $s(\alpha, \beta)$.

Group Work 2, Section 10.5
Bagels! Bagels! Bagels! (Version 2)

There are many ways to create bagels. Some people prefer boiling them, some prefer baking them, and some prefer defrosting and then toasting them. Today we shall be creating bagels by the method of parametrizing them.

We are going to parametrize a bagel obtained by rotating the circle $(y-2)^2 + z^2 = 1$ about the z-axis.

1. Find a parametrization for the circle below.



2. Write a parametrization for the circle when it has been rotated about the z-axis through angles of

(a) 0

(b) $\frac{\pi}{2}$

(c) π

(d) $\frac{3\pi}{2}$

(e) 2π

3. When the above circle has been rotated through an angle of $\frac{\pi}{4}$ about the z-axis, in what plane does the circle lie? What is a parametrization of the circle that lies in the plane?

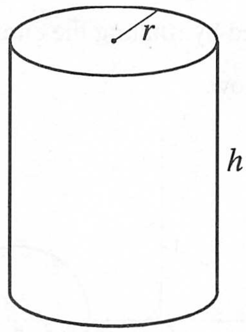
4. What is a parametrization of the circle when it has been rotated through an angle of α about the z-axis?

5. If you have a graphics program, graph the surface described by your parametrization in Problem 4.

10 Sample Exam

Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

1. The volume of a can of base radius r and height h is a function of the variables r and h.



(a) Write a formula for the volume V of a can of radius r and height h.

(b) What is the domain of $V(r, h)$?

(c) Graph the traces $r = 2$ and $h = 3$.

(d) Graph the level curve $V = 4\pi$.

2. Describe a non-zero vector function $\mathbf{r}(t)$ in $\mathbb{R}^3$ whose acceleration vector $\mathbf{a}(t)$ satisfies $\mathbf{a}(t) = -\mathbf{r}(t)$ for all t.

3. Let $\mathbf{r}(s)$ be the arc-length parametrization of a circle of radius R.

(a) What is the domain of $\mathbf{r}$?

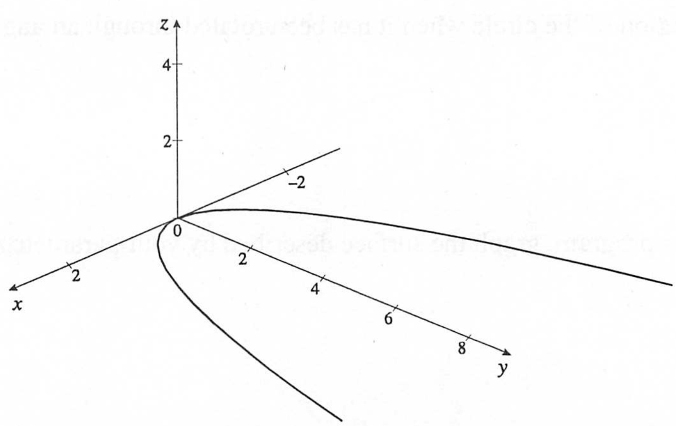
(b) Find a value of $s \neq 0$ such that $\mathbf{r}(s) = \mathbf{r}(0)$, or explain why no such value exists.

4. Let C be the circle of radius 3 centered at the point $(2, 5)$ in the xy-plane.

(a) Give the arc-length parametrization $\mathbf{r}(s)$ of this curve, starting at the point $(2, 8)$.

(b) Verify that the curvature of C is constant.

5. Let $y = x^2$ be a parabola in the xy-plane parametrized by $\mathbf{r}(t) = \langle t, t^2, 0 \rangle$. What are the vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ at the origin?



6. Match the equations with their graphs. Give reasons for your choices.

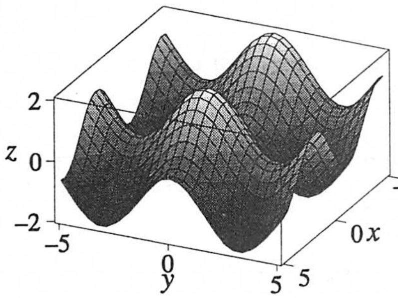
(a) $8x + 2y + 3z = 0$

(b) $z = \sin x + \cos y$

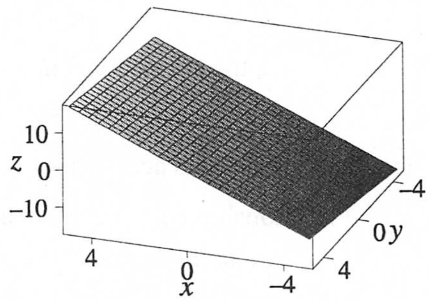
(c) $z = \sin\left(\dfrac{\pi}{2 + x^2 + y^2}\right)$

(d) $z = e^y$

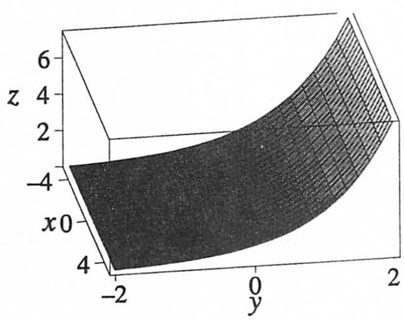
I



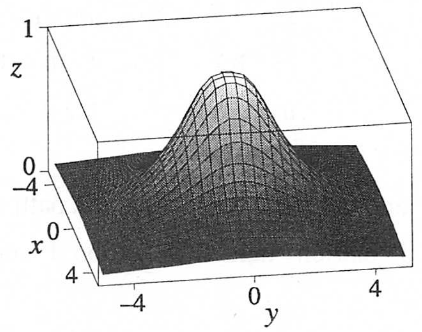
II



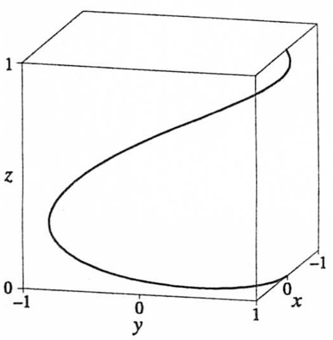
III



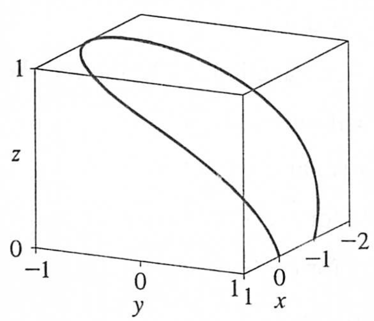
IV



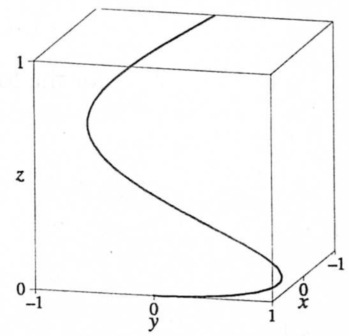
7. Which curve below is the path traced out by $\mathbf{r}(t) = \left\langle \sin \pi t, \cos \pi t, \frac{1}{4}t^2 \right\rangle$, $0 \le t \le 2$? Justify your answer.



Graph 1

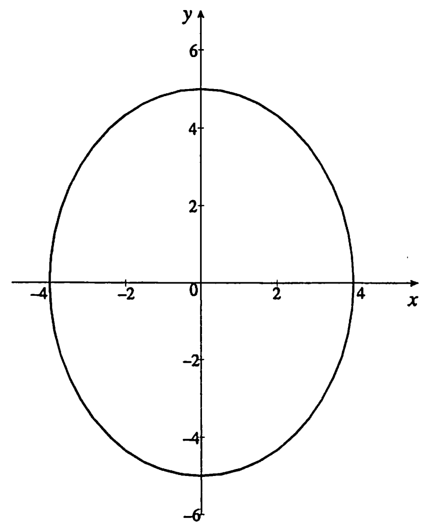


Graph 2



Graph 3

8. Consider the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$.



(a) Find two different parametrizations of this ellipse.

(b) Find a point where the curvature is minimal. Give a reason for your answer.

9. Consider the curve $\mathbf{w}(t) = \langle 3e^{t/2}, 4e^{-t/2} \rangle$.

(a) Compute the velocity and acceleration vectors $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

(b) What is the relationship between $\mathbf{w}(t)$ and $\mathbf{a}(t)$?

(c) Is the unit tangent vector at 0 orthogonal to the acceleration vector at 0?

10. Consider the intersection of the paraboloid $z = x^2 + y^2$ with the plane $x - 2y = 0$. Find a parametrization of the curve of intersection and verify that it lies in each surface.

11. (a) Find a parametrization of the circular path along a circle of radius 2 going counterclockwise from $(0, 2)$ to $(0, -2)$.

(b) Give another parametrization of a path from $(0, 2)$ to $(0, -2)$.

12. A crazed ostrich named Rhomboid runs along a mountain path with coordinates given by $\mathbf{r}(t) = \langle e^t, e^{-t}, \sqrt{2}t \rangle$.

(a) What is the change in Rhomboid's altitude from $t = 0$ to $t = 10$?

(b) What is Rhomboid's speed in the x-direction when $t = 4$?

(c) What is Rhomboid's speed in the y-direction when $t = 4$?

(d) Find a formula for the total distance travelled by Rhomboid the Crazed Ostrich from $t = 0$ to $t = 4$.

13. A point moves along a circle with position vector given by $\mathbf{r}(t) = \langle \cos t^2, \sin t^2 \rangle$.

(a) Write $\mathbf{a}(t)$ as a combination of $\mathbf{r}(t)$ and $\mathbf{v}(t)$, for $t \neq 0$.

(b) Find the unit tangent vector $\mathbf{T}(t)$ for $t \neq 0$.

(c) Find the unit normal vector $\mathbf{N}(t)$ for $t \neq 0$.

(d) Find the tangential and normal components of the acceleration vector $\mathbf{a}(t)$.

14. A point moves with position given by $\mathbf{p}(t) = \langle 1, \cos 2t, t^2 + 4 \rangle$. Find the osculating plane at $t = 4$.

15. Suppose that the function given by $z = f(x, y)$ below represents height above sea level in kilometers.

$y \setminus x$	-3	-2	-1	0	1	2	3
-3	0.1	0.2	0.5	1	0.5	0.2	0.1
-2	0.2	1	4	5	4	1	0.2
-1	0.5	4	8	9	8	4	0.5
0	1	5	9	10	9	5	1
1	0.5	4	8	9	8	4	0.5
2	0.2	1	4	5	4	1	0.2
3	0.1	0.2	0.5	1	0.5	0.2	0.1

(a) What is the change in height from

(i) $(1, 0)$ to $(1, 1)$?

(ii) $(-1, -1)$ to $(0, 0)$?

(b) What is the average change in height if you walk from $(1, 0)$ to $(1, 1)$? From $(-1, -1)$ to $(0, 0)$?

(c) Approximately where are the points of steepest descent?

16. The formula for curvature for a function $y = f(x)$ is given by

$$\kappa(x) = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}}.$$

(a) Using this formula, find the points in $[0, 2\pi]$ where the curvature of $y = \sin x$ is maximal.

(b) Using this formula, find the points in $[0, 2\pi]$ where the curvature of $y = \sin x$ is minimal.

(c) Give a geometric interpretation of your results from (a) and (b).

17. Find the length of the curve $\left\langle \sqrt{1+t^3}, \sqrt{1+t^3} \right\rangle$ between $t = 0$ and $t = 2$.

18. Give a parametric representation for the intersection of the cylinder $x^2 + y^2 = 2$ and

(a) the plane $z = 2$.

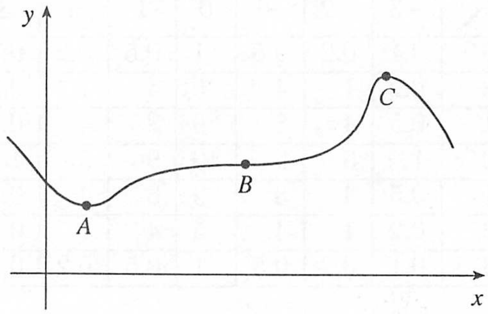
(b) the plane $x - 2z = 2$.

19. Let $\mathbf{r}(t) = \langle \sin 2t, 3t, \cos 2t \rangle$, $-\pi \le t \le \pi$ be the position vector of a particle at time t.

(a) Show that the velocity and acceleration vectors are always perpendicular.

(b) Is there any time t for which $\mathbf{r}(t)$ and the velocity vector are perpendicular? If so, find all s of t.

20. Consider the vector function $\mathbf{r}(t)$ describing the curve shown below. Put the curvatures of $\mathbf{r}$ at A, B, and C in order from smallest to largest.

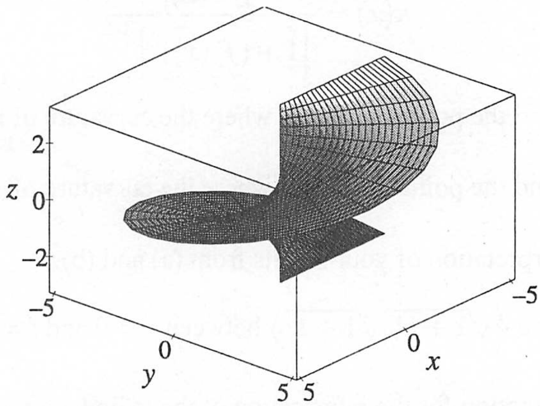


21. Show that the surface parametrization given by $\mathbf{r}(s,t) = \left\langle 2\cos t \sin s, \sin t \sin s, \frac{1}{\sqrt{2}}\cos s \right\rangle$, where $0 \le t \le 2\pi$, $0 \le s \le \pi$, describes the ellipsoid $\frac{1}{4}x^2 + y^2 + 2z^2 = 1$.

22. Find the equations for the following parametrized surfaces in rectangular coordinates, and describe them in words.

(a) $\left\langle t, \sqrt{1-t^2}\sin s, \sqrt{1-t^2}\cos s \right\rangle$ (b) $\langle t^2, s^2, s^2 + t^2 \rangle$

23. Find a parametric representation for the surface $z = \theta$ described in cylindrical coordinates.

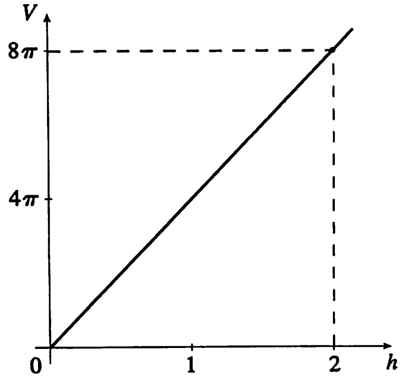


10 Sample Exam Solutions

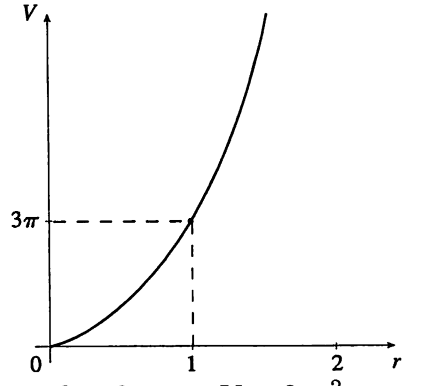
1. (a) $V = \pi r^2 h$

(b) $r > 0$, $h > 0$

(c)

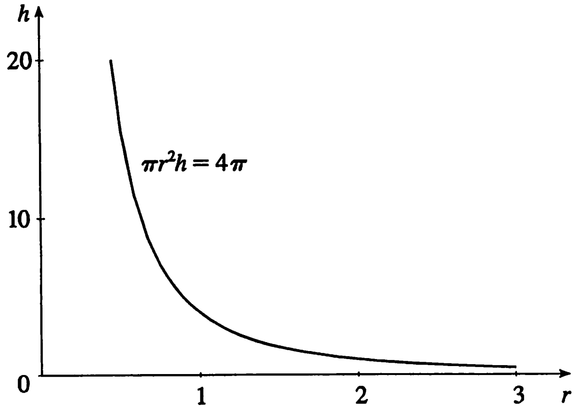


$r = 2 \Rightarrow V = 4\pi h$



$h = 3 \Rightarrow V = 3\pi r^2$

(d)



2. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos t\,\mathbf{k}$, $\mathbf{a}(t) = \mathbf{r}''(t) = -\mathbf{r}(t)$

3. (a) $[0, 2\pi R]$

(b) $s = 2\pi R$, the circumference of the circle

4. (a) A parametrization is $x = 3\cos t + 2$, $y = 3\sin t + 5$, $0 \le t \le 2\pi$. If s is the arc length, then $s = 3t$ or $t = s/3$. So the answer is $x = 3\cos(s/3) + 2$, $y = 3\sin(s/3) + 5$, $0 \le s \le 6\pi$.

(b) $\kappa = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{|-\cos t\,\mathbf{i} + \sin t\,\mathbf{j}|}{3} = \dfrac{1}{3}$

5. $\mathbf{r}'(t) = \langle 1, 2t, 0\rangle$, so $\mathbf{T}(t) = \dfrac{\mathbf{i}}{\sqrt{1+4t^2}} + \dfrac{2t\,\mathbf{j}}{\sqrt{1+4t^2}}$, $\mathbf{T}(0) = \mathbf{i}$, $\mathbf{N}(0) = \mathbf{j}$, and hence $\mathbf{B}(0) = \mathbf{k}$.

6. (a) II (b) I (c) IV (d) III

7. Graph I $[\mathbf{r}(0) = \langle 0,1,0\rangle$ and $\mathbf{r}(2) = \langle 0,1,1\rangle]$

8. (a) $\mathbf{r}_1(t) = 4\cos t\,\mathbf{i} + 5\sin t\,\mathbf{j}$. $\mathbf{r}_2(t) = 4\sin t\,\mathbf{i} + 5\cos t\,\mathbf{j}$

(b) The curve is most "flat" (minimal curvature) at $(\pm 4, 0)$.

9. (a) $\mathbf{v}(t) = \left\langle \frac{3}{2}e^{t/2}, -2e^{-t/2}\right\rangle$, $\mathbf{a}(t) = \left\langle \frac{3}{4}e^{t/2}, e^{-t/2}\right\rangle$

(b) $\mathbf{a}(t) = \frac{1}{4}\mathbf{w}(t) \parallel \mathbf{w}(t)$

(c) $\mathbf{v}(0) = \left\langle \frac{3}{2}, -2\right\rangle$, which is not perpendicular to $\left\langle \frac{3}{4}, 1\right\rangle = \mathbf{a}(0)$, so $\mathbf{T}(0)$ is not perpendic

10. $\mathbf{r}(t) = \langle 2t, t, 5t^2 \rangle$

11. (a) $\mathbf{r}(t) = \langle -2\sin t, 2\cos t \rangle$, $0 \le t \le \pi$

(b) $\mathbf{r}(t) = \langle 0, 2-t \rangle$, $0 \le t \le 4$

12. (a) The change is $z(10) - z(0) = 10\sqrt{2}$.

(b) $\mathbf{r}'(t) = \langle e^t, -e^{-t}, \sqrt{2} \rangle$, so the change in x is e^4.

(c) e^{-4}

(d) $L = \int_0^4 |\mathbf{r}'(t)|\, dt = \int_0^4 \sqrt{e^{2t} + e^{-2t} + 2}\, dt = \int_0^4 \sqrt{(e^t + e^{-t})^2}\, dt = \int_0^4 (e^t + e^{-t})\, dt$
$= \left[e^t - e^{-t}\right]_0^4 = e^4 - e^{-4}$

13. (a) $\mathbf{v}(t) = \langle -2t\sin t^2, 2t\cos t^2 \rangle$, $\mathbf{a}(t) = \langle -2\sin t^2 - 4t\cos t^2, 2\cos t^2 - 4t^2 \sin t^2 \rangle = \frac{1}{t}\mathbf{v} - 4t^2 \mathbf{r}(t)$ for $t \neq 0$.

(b) $\mathbf{T}(t) = \frac{\mathbf{v}}{2t}$, $t \neq 0$

(c) $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\langle -\cos t^2, -\sin t^2 \rangle}{1} = -\mathbf{r}(t)$, $t \neq 0$

(d) $\mathbf{a}(t) = \frac{1}{t}\mathbf{v} - 4t^2\mathbf{r}(t) = 2\mathbf{T} + 4t^2\mathbf{N}$, so $a_{\mathbf{N}} = 4t^2$, $a_{\mathbf{T}} = 2$.

14. Since $\mathbf{p}(t) = \langle 1, \cos 2t, t^2 + 4 \rangle$ lies in the plane $x = 1$, so does the osculating plane at $t = 4$.

15. (a) (i) -1 (ii) 2

(b) Change in height from $(1,0)$ to $(1,1)$ is -1; from $(-1,-1)$ to $(0,0)$ is $\sqrt{2}$

(c) The four points $(\pm 1, \pm 1)$

16. (a) $\kappa(x) = \frac{|\sin x|}{(1+\cos^2 x)^{3/2}}$. κ is maximal when $|\sin x| = 1$ and $\cos x = 0$, that is, when $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. The maximum value is $\kappa = 1$.

(b) κ is minimal when $|\sin x| = 0$, or $x = 0$ or π. The minimum value is $\kappa = 0$.

(c) The minimum value occurs where $\sin x$ is "flat", and the maximum value occurs at maxima and minima of $\sin x$.

17. $\mathbf{v} = \sqrt{1+t^3}\langle 1, 1 \rangle = \sqrt{1+t^3}(\mathbf{i} + \mathbf{j})$ is a line with direction vector $\mathbf{i} + \mathbf{j}$. Since $\mathbf{v}(0) = 1$, $\mathbf{v}(2) = 3$, the length is $2|\mathbf{i} + \mathbf{j}| = 2\sqrt{2}$.

18. (a) $\mathbf{v}(t) = \langle \sqrt{2}\cos t, \sqrt{2}\sin t, 2 \rangle$

(b) Since $z = \frac{1}{2}(x-2)$,
$\mathbf{v} = \langle \sqrt{2}\cos t, \sqrt{2}\sin t, \frac{1}{2}(\sqrt{2}\cos t - 2) \rangle = \left\langle \sqrt{2}\cos t, \sqrt{2}\sin t, \frac{1}{\sqrt{2}}\left(\cos t - \sqrt{2}\right) \right\rangle$.

19. (a) $\mathbf{v}(t) = \langle 2\cos 2t, 3, -2\sin 2t \rangle$, $\mathbf{a}(t) = \langle -4\sin 2t, 0, -4\cos 2t \rangle$.
$\mathbf{v}(t) \cdot \mathbf{a}(t) = -8\sin 2t\cos 2t + 8\sin 2t \cos 2t = 0$ for all t.

(b) $\mathbf{r}(t) \cdot \mathbf{v}(t) = 2\sin 2t\cos 2t + 9t - 2\sin 2t\cos 2t = 0$ when $t = 0$, the only time at which $\mathbf{r}(t) \perp \mathbf{v}(t)$.

20. B, A, C

21. If $x = 2\cos t \sin s$, $y = \sin t \sin s$, $z = \frac{1}{\sqrt{2}} \cos s$, then

$$\tfrac{1}{4}x^2 + y^2 = \tfrac{1}{4}\left(4\cos^2 t \sin^2 s\right) + \sin^2 t \sin^2 s = \sin^2 s\left(\cos^2 t + \sin^2 t\right) = \sin^2 s$$

and so

$$\tfrac{1}{4}x^2 + y^2 + 2z^2 = \sin^2 s + 2\left(\tfrac{1}{\sqrt{2}} \cos s\right)^2 = \sin^2 s + 2\left(\tfrac{1}{2}\cos^2 s\right) = \sin^2 s + \cos^2 s = 1$$

22. (a) $x = t$, $y = \sqrt{1-t^2}\sin s$, and $z = \sqrt{1-t^2}\cos s$ gives $y^2 + z^2 = 1 - t^2 = 1 - x^2$, or $x^2 + y^2 + z^2 = 1$, a sphere of radius 1.

(b) $x = t^2$, $y = x^2$, and $z = s^2 + t^2 = y + x$, $x \geq 0$, $y \geq 0$, part of a plane above the first quadrant.

23. If $z = \theta$, then $x = r\cos\theta$, $y = r\sin\theta$, $z = \theta$ and $\mathbf{R}(r,\theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + \theta\mathbf{k}$, $r \geq 0$, $0 \leq \theta \leq 2\pi$ is a parametrization.