11 Partial Derivatives

11.1 Functions of Several Variables

Suggested Time and Emphasis

1 class Essential material

Transparencies Available

- Transparency 39 (Figure 6, page 752)
- Transparency 40 (Figure 10, page 754)
- Transparency 41 (Figure 12, page 755)
- Transparency 42 (Exercises 31–36, Graphs A–F and I–VI, page 759)

Points to Stress

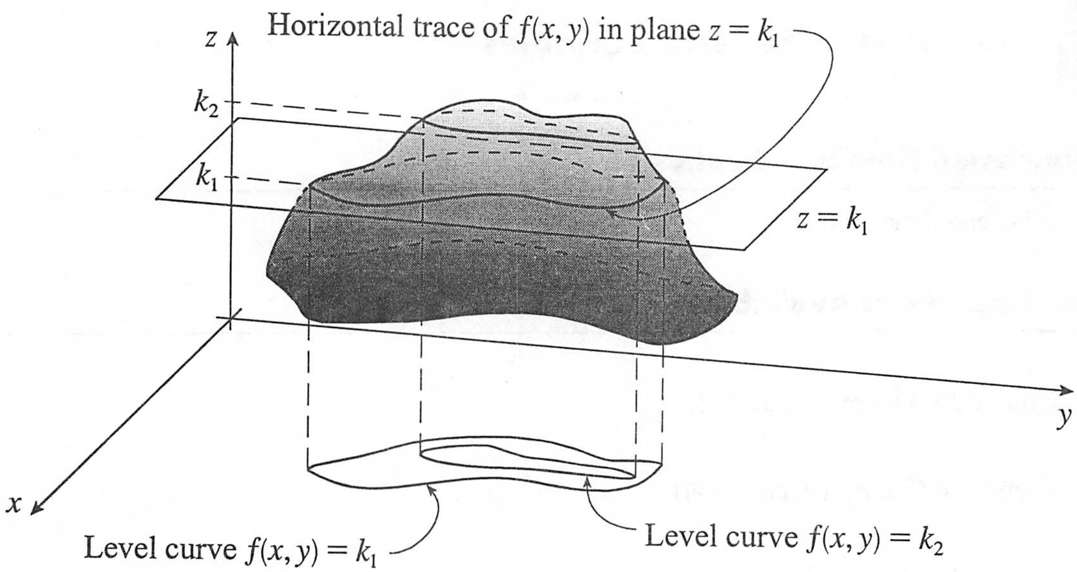
1. The notions of level curves (contour lines) and level surfaces.
2. Functions of two variables can be represented as surfaces, and can be described in two dimensions by contour maps and horizontal traces.
3. Functions of three variables can be described by level surfaces, and are generally more diffi visualize than functions of two variables.

Text Discussion

- Why is the domain of the function in Example 3 (page 750), $g(x, y) = \sqrt{9 - x^2 - y^2}$, sh
- Why are the level curves for $f(x, y) = 6 - 3x - 2y$ straight lines with the same slope y-intercept of these lines change as a function of k, if $f(x, y) = k$?

Materials for Lecture

- Discuss contour or level curves carefully, invoking the image of a moving plane slicing a three-dimensional surface. A picture like the following should help the students visualize the situation:



- Discuss the level curves for $f(x, y) = e^{x-y}$. Point out that they reduce to the simple equations $y = x - \ln k$ for $z = k > 0$.

- Describe the domain and range of $f(x, y, z) = \ln\left(z - \sqrt{xy}\right)$. Point out why $f(1, 2, 1)$ is not defined, and why points (x, y, z) in the domain must have (x, y) in the first or third quadrants.

- One way to describe functions of two variables $f(x, y)$ is to have the students think of the contour curve $f(x, y) = t$ as existing at time t. With this way of looking at things, a sphere is a point that becomes a circle, grows, and then shrinks back to a point. This approach then makes it easier to describe functions of three variables. A function of three variables can be thought of a level surface that changes with time. Example 12 (page 756) can be revisited in this context. $f(x, y, z) = x^2 + y^2 + z^2$ can be pictured as a point at time $t = 0$, a sphere of radius 1 at time $t = 1$, a sphere of radius $\sqrt{2}$ at $t = 2$, and so on. In other words, $f(x, y, z) = x^2 + y^2 + z^2$ can be pictured as a growing sphere, and the "level surfaces" of the function as snapshots of the process. Another good function to describe with this method is $f(x, y, z) = x^2 + y^2 - z$.

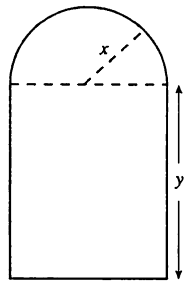
- An alternate way to approach the subject is to think of one dimension as "color". A surface such as $f(x, y) = \sin x + \cos y$ could be then drawn on a sheet of graph paper, with red representing the contour curve $f(x, y) = -2$, violet representing the contour curve $f(x, y) = 2$, and any number in between represented by the appropriate color. Some software packages represent functions of three variables using a method similar to this one.

Workshop/Discussion

- Calculate the domain and range for each of the following functions:

$$f(x,y) = \sin\left(\sqrt{1-(x^2+y^2)}\right) \qquad f(x,y) = \exp\left(\frac{x+y}{xy}\right) \qquad f(x,y,z) = \frac{\sqrt{z^2-3}}{\sqrt{2x^2+3y^2-4}}$$

- Let A be the area of the Norman window shown below. Revisit the formula for A as a function of two variables x and y. Have them use level curves to determine what the graph of A looks like.



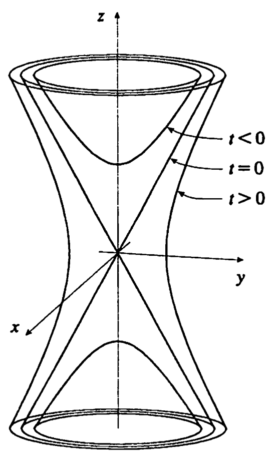
- Pass around some interesting solid figures, and have the students attempt to sketch the appropriate level curves for the solids.
- Sketch the ellipsoidal level surfaces for $f(x,y,z) = 2x^2 + y^2 + z^2$.
- Give the students a start on Group Work 5: The M. R. Project, doing either the two-dimensional problems or one of the three-dimensional problems.

Group Work 1: Dali's Target

This activity is designed for students who are having some difficulty with the concept of level curves.

Group Work 2: Level Surfaces

Part 2 of Problem 4 requires that the students have some familiarity with conic sections, or have some type of graphing software. The solution to Part 4 (and hence Parts 1–3) is given by the picture below:



Group Work 3: The M. R. Project

This exercise involves looking at some interesting functions given a particular domain in the xy-plane or in space. Each group should get the same worksheet, but a different domain. (There is a blank space on the sheet in which to write the assigned domain.) Possible domains to give the students are:

$$\text{(Two-dimensional)} \quad 0 \le x, y \le 1,\ z = 0$$
$$\text{(Two-dimensional)} \quad x^2 + y^2 \le 1,\ z = 0$$
$$0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$$
$$x^2 + y^2 \le 1, 0 \le z \le 1$$
$$x^2 + y^2 + z^2 = 1$$
$$x^2 + y^2 = 1,\ |z| \le 1$$

If a group finishes early, they could be given another domain to do, or instructed to prepare a presentation about their solution to give to the class. Ideally, each group should solve the problem themselves for at least one domain, and see a discussion of at least two other domains.

In some classes, it may be most appropriate to go through the domain $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$ with the students as an example.

If the students are able to do the last problem well, make sure to point out that what they are really doing is trying to comprehend a four-dimensional object, with the fourth dimension being time.

Group Work 4: Applied Contour Maps

Give the students a list of today's temperature in various cities, along with a map of the country. The temperatures are available in many newspapers. Have the students draw a contour map showing curves of constant temperature. Then give them a copy of the temperature contour map from a newspaper to compare with their map to see how they did. Discuss what a three-dimensional representation of today's weather would be like. You can have the students cut the contours out of corrugated cardboard and make actual three-dimensional weather maps.

Homework Problems

Core Exercises: 8, 9, 10, 11, 12, 16, 31–36, 37, 40

Sample Assignment: 1, 2, 4, 8, 9, 10, 11, 12, 16, 20, 28, 29, 31–36, 37, 40

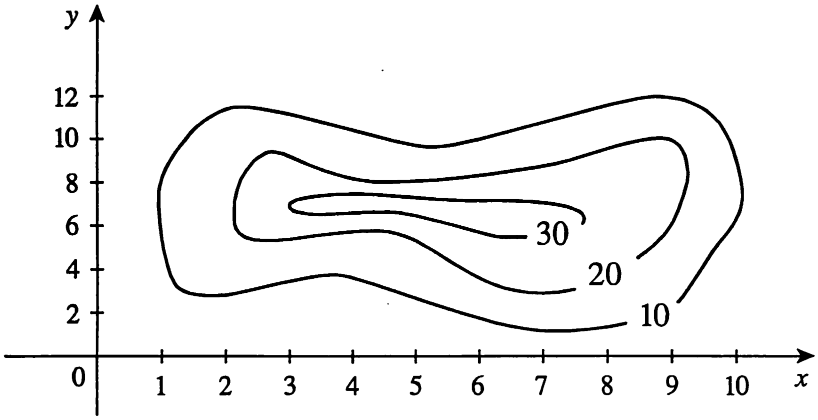
Note: Exercises 28 and 29 require a CAS.

Exercise	C	A	N	G	V
1	×				
2	×		×		
4			×		
8		×		×	
9				×	×
10					×
11					×
12					×

Exercise	C	A	N	G	V
16				×	
20				×	
28				×	
29				×	
31–36					×
37					×
40					×

Group Work 1, Section 11.1
Dali's Target

Consider the following contour map of a continuous function $f(x, y)$:



1. For approximately what values of y is it true that $10 \leq f(5, y) \leq 30$?

2. What can you estimate $f(2, 4)$ to be, and why?

3. Do we have any good estimates for $f(5, 8)$? Explain.

4. How many values y satisfy $f(7, y) = 20$?

5. How many values of x satisfy $f(x, 9) = 20$?

Group Work 2, Section 11.1
Level Surfaces

It can be difficult to visualize functions of three variables. One way to do it is by thinking of each level surface as representing a different point in time. As we let t vary in the equation for the level surface $f(x, y, z) = t$ we can think of the function $f(x, y, z)$ as a surface whose shape and size vary as time changes.

Consider the function $f(x, y, z) = x^2 + y^2 - z^2$.

1. What is the level surface $f(x, y, z) = 0$?

2. What is the level surface $f(x, y, z) = 1$?

3. For $t > 0$, what do the level surfaces $f(x, y, z) = t$ look like?

4. What is the level surface $f(x, y, z) = -1$?

5. Describe all the level surfaces $f(x, y, z) = t$.

Group Work 3, Section 11.1
The M.R. Project

Consider the region ______________________.

1. Sketch or describe this region.

We are now going to describe some functions of three variables for which the region in Part 1 is the domain. In other words, every point in your domain will have a function value for the functions below. The functions are:

$M(x,y,z) = \max(x,y,z)$ $\quad$ $m(x,y,z) = \min(x,y,z)$ $\quad$ $R(x,y,z) = x + y + z$

2. Evaluate M, m, and R at several different points in your domain. The first line in the following table is an example for you to look at.

Point	$M(x,y,z)$	$m(x,y,z)$	$R(x,y,z)$
$\left(\frac{1}{5}, \frac{1}{3}, \frac{1}{2}\right)$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{31}{30}$
(, ,)			
(, ,)			
(, ,)			

3. Find the maximum values of M, m, and R on your domain.

4. Sketch the level surfaces $M = \frac{1}{2}$, $R = \frac{1}{2}$, $R = 0$, and $m = \frac{1}{2}$ for your domain.

5. For an extra challenge, try to describe the level surfaces $M = t$, $R = t$, and $m = t$, for $0 \leq t \leq 2$. If we let t stand for time, and make a movie of the level surface changing as t goes from 0 to 2, what would the movie look like?

11.2 Limits and Continuity

Suggested Time and Emphasis

1 class Recommended material

Note: This material can be covered from a variety of perspectives, and at a variety of depths. (For example, the nonexistence of certain limits can be de-emphasized.) The instructor should feel especially free to pick and choose from the suggestions below.

Points to Stress

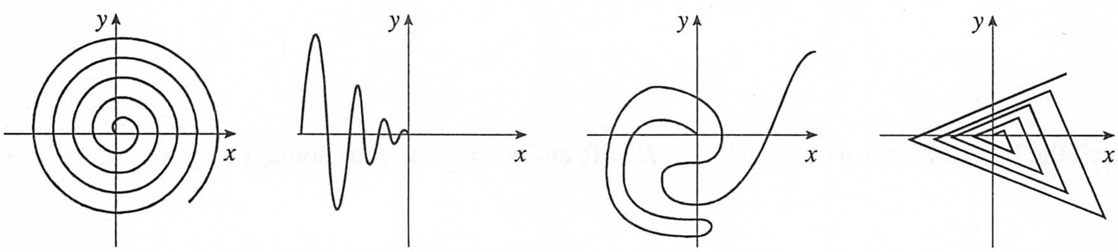
1. While the definitions of limits and continuity for multivariable functions are nearly identical to those of their single variable counterparts, very different behavior can take place in the multivariable case.
2. The idea of points being "close" in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Text Discussion

- When talking about limits for functions of several variables, why isn't it sufficient to say, "$\lim_{(x,y)\to(0,0)} f(x,y) = L$ if $f(x,y)$ gets close to L as we approach $(0,0)$ along the x-axis ($y=0$) and along the y-axis ($x=0$)"?

Materials for Lecture

- Stress that $f(x,y) \to L$ as $(x,y) \to (a,b)$ means that we can make $f(x,y)$ as close to L as we like by taking (x,y) close to (a,b) in distance, *regardless of path*. Give examples of exotic paths to $(0,0)$ such as the following:



- This is a rich example of a limit that exists:

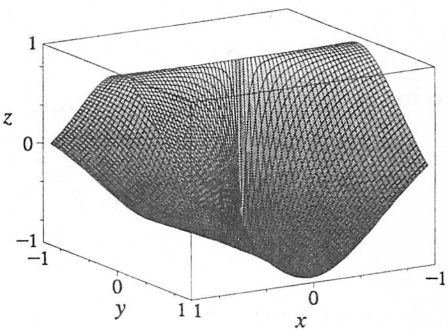
$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1$$

Since $\frac{\sin(x^2+y^2)}{x^2+y^2}$ is constant on circles centered at the origin, we want to look at the distance w between (x,y) and $(0,0)$: $w = \sqrt{x^2+y^2}$. Computationally, it is best to look at what happens when $w^2 \to 0$. In this case, $\frac{\sin(x^2+y^2)}{x^2+y^2} = \frac{\sin w^2}{w^2}$, and single-variable calculus gives us that $\lim_{w\to 0} \frac{\sin w^2}{w^2} = 1$. Stress that in general it does *not* suffice to just let $x=0$ or $y=0$ and then compute the limit.

- This is a good example of a limit that does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

The text shows this fact in an interesting way: If we let $x = 0$, then we get $\lim_{y\to 0} \frac{-y^2}{y^2} = -1$, but if we let $y = 0$ then we get $\lim_{x\to 0} \frac{x^2}{x^2} = 1$. So if we approach the origin by one radial path, we get a different limit than we do if we go by a different radial path. In fact, assume we go to the origin by a straight line $y = mx$. Then $\frac{x^2 - y^2}{x^2 + y^2} = \frac{x^2(1 - m^2)}{x^2(1 + m^2)}$ and $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \frac{1 - m^2}{1 + m^2}$. So this limit can take any value from -1 to 1 if we approach the origin by a straight line. For example, if we use the line $y = \frac{1}{\sqrt{3}}x$, we get $\frac{1}{2}$ as a limit.

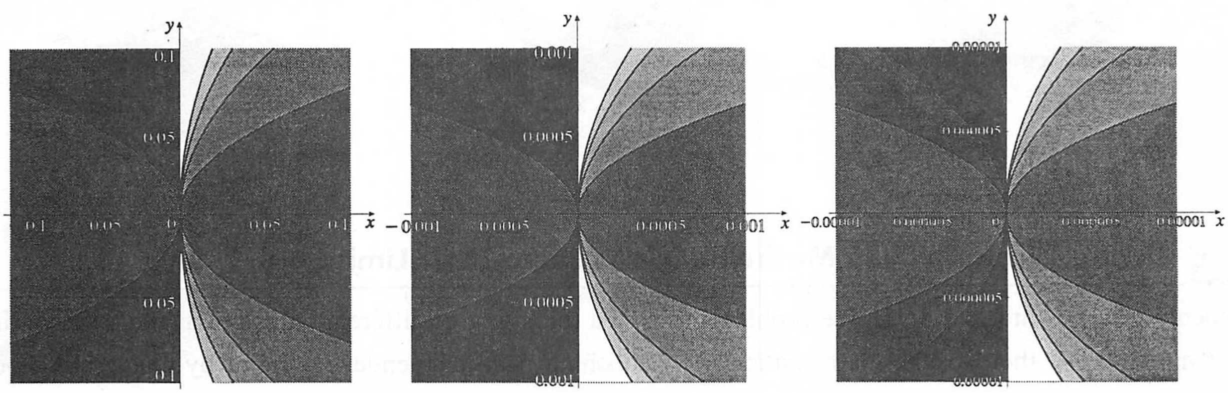


- Note that just because a function has a limit at a point doesn't imply that it is continuous there. For example,

$$f(x,y) = \begin{cases} \dfrac{\sin(x^2 + y^2)}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is discontinuous at the origin even though $\lim_{(x,y)\to(0,0)} f(x,y)$ exists.

- Expand the explanation of Example 3 (page 762), where $f(x,y) = \frac{xy^2}{x^2 + y^4}$, using a visual approach, perhaps using figures like the following or using algebra to compute the limit along the general parabola $x = my^2$. The figures show progressively smaller viewing rectangles centered at the origin. The black regions correspond to larger negative values of $f(x,y) = \frac{xy^2}{x^2 + y^4}$, and the white regions correspond to larger positive values. Notice that when travelling along any straight line $y = mx$, the color of the points on the path eventually becomes gray [at points where $f(x,y) = 0$] as the origin is approached. This effect is best observed (even if m is large) using the later pictures.



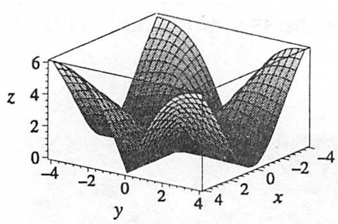
$-0.1 \le x \le 0.1$ $\quad$ $-0.001 \le x \le 0.001$ $\quad$ $-0.00001 \le x \le 0.00001$

However, when approaching the origin on a parabolic path, $x = my^2$, the color of the points on the path always stays the same! This phenomenon is best illustrated by the earlier pictures. Therefore, this set of plots illustrates how the limit as $(x, y) \to (0, 0)$ of $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$ does not exist, although one would erroneously believe it to be 0 if one looked only at the "obvious" linear paths.

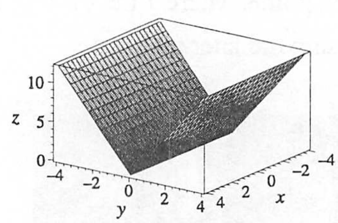
- Point out that while $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist, $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}$ does exist. To see this, use cylindrical coordinates to get $\dfrac{xyz}{x^2+y^2+z^2} = \dfrac{r^2 z \sin\theta\cos\theta}{r^2+z^2}$, and compare to $\dfrac{r^2 z}{r^2+z^2}$, which approaches 0 as in Example 8.

Workshop/Discussion

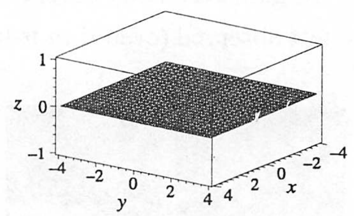
- Try to compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y^2}$. Show that the function is unbounded when restricted to the y-axis ($x = 0$), so there is no limit.
- Introduce the use of polar coordinates by trying to compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{\sqrt{x^2+y^2}}$. Point out how intractable the problem looks at first glance. But notice that in polar coordinates, the statement "$(x, y) \to (0, 0)$" translates to the much simpler "$r \to 0$", so the limit can be rewritten as $\displaystyle\lim_{r\to 0} \frac{3(r\cos\theta)^2 r\sin\theta}{r}$, which simplifies to $\displaystyle\lim_{r\to 0} 3r^2\cos^2\theta\sin\theta = 0$.
- Check that $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq -y}} \frac{x+y-\sin(x+y)}{(x+y)^3} = \frac{1}{6}$, by noticing that this function is constant on $x + y = k$. This suggests using the substitution $u = x+y$ and applying the single-variable version of l'Hospital's Rule. This is also a good limit to first investigate numerically, plugging in small values of x and y.
- Go over Exercises 33 and 34 (page 766) using polar coordinates.
- Discuss the squeeze principle by looking at the three graphs below, and computing $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$ by squeezing f between g and h.



$f(x, y) = \dfrac{3x^2|y|}{x^2+y^2}$



$g(x, y) = 3|y|$



$h(x, y) = 0$

Group Work 1: Even Mathematicians Have Their Limits

When a group is finished doing the problems as stated (plugging in different values of x and y) have them attempt to prove their results mathematically, establishing path-independence, either by changing to polar coordinates, or by using a substitution.

Group Work 2: Limits in $\mathbb{R}^3$

Group Work 3: There Is No One True Path

This is a difficult project, but might be of interest for strong students.

Have the students use a calculator or CAS to do an analysis of $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^4+y^8}$. Guide them through the following steps:

1. Setting $x = 0$ and letting $y \to 0$ gives a limit of 0.
2. More generally, for any m, approaching the origin along the path $y = mx$ gives a limit of 0.
3. Following the path $y = x^2$ gives a limit of 0 as well.
4. However, the actual limit is not zero! For example, following the path $y = \sqrt{x}$ gives a limit of 1.

Homework Problems

Core Exercises: 1, 2, 4, 5, 8, 11, 25, 30

Sample Assignment: 1, 2, 4, 5, 8, 11, 13, 19, 25, 28, 30, 33, 35, 36

Note: Exercise 19 requires a CAS.

Exercise	C	A	N	G	V
1	×				
2	×				
4			×		
5–18		×			
19				×	×
25–32		×			
35		×			
36		×		×	

Group Work 1, Section 11.2
Even Mathematicians Have Their Limits

Try to estimate the following limits by plugging small values of x and y into the appropriate function. Remember that path independence is important: Try some values where $x = y$ and some where $x \neq y$.

1. $\displaystyle\lim_{(x,y)\to(0,0)} 5$

2. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{xy}}{x^2+y^2}$

3. $\displaystyle\lim_{(x,y)\to(0,0)} \ln\left(x^2+y^2\right)$

4. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq -y}} \frac{\sin\left(2\left(x+y\right)\right)}{x+y}$

Group Work 2, Section 11.2
Limits in $\mathbb{R}^3$

Determine whether or not the following limits exist. If a limit exists, compute its value.

1. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x+y+z}{x^2+y^2+z^2}$

2. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{\sin\left(x^2+y^2+z^2\right)}{x^2+y^2+z^2}$

3. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{(x+y+z)^2}{x^2+y^2+z^2}$

4. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x+y+z}{e^{x^2+y^2+z^2}}$

11.3 Partial Derivatives

Suggested Time and Emphasis

$\frac{3}{4}$–1 class   Essential material

Transparencies Available

- Transparency 43 (Figures 2–5, page 770)
- Transparency 44 (Exercise 7, page 777)

Points to Stress

1. The meaning of f_x and f_y, both analytically and geometrically.
2. The various notations for f_x and f_y. [Be sure to point out that in the notation $f_x(x, y)$, x is playing two different roles: $f_x(x, y)$ can be written as $f_1(x, y)$, where the 1 indicates that the derivative is taken with respect to the first variable.]
3. Higher-order partial derivatives.

Text Discussion

- In computing $f_x(1, 1)$ for $f(x, y) = 4 - x^2 - 2y^2$, what function of x are we differentiating at $x = 1$?
- Describe the line which has slope $f_y(1, 1)$ for $f(x, y) = 4 - x^2 - 2y^2$.
- Find a function $f(x, y)$ for which $\frac{\partial f}{\partial x} = x + y$ and $\frac{\partial f}{\partial y} = x$.

Materials for Lecture

- Provide an alternate geometric interpretation for the partial derivative in terms of vector functions. The graph C_1 of the function $g(x) = f(x, b)$ is the curve traced out by the *vector* function $\mathbf{g}(x) = \langle x, b, f(x, b)\rangle$ whose *vector* derivative $\mathbf{g}'(a) = \langle 1, 0, f_x(a, b)\rangle$ is determined by $f_x(a, b)$. [That is, its "slope" is $f_x(a, b)$.] Similarly, the graph C_2 of $h(y) = f(a, y)$ is the curve traced out by the vector function $\mathbf{h}(y) = \langle a, y, f(a, x)\rangle$ whose vector derivative $\mathbf{h}'(b) = \langle 0, 1, f_y(a, b)\rangle$ is determined by $f_y(a, b)$.
- Compute $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$ for $f(x, y) = \sin(\pi x e^{xy})$.
- Let $f(x, y, z) = xy^4z^3$ or some other easy-to-differentiate function. Verify that $f_{xyz} = f_{xzy} = \cdots = f_{zyx}$. Perhaps then show that $f_{xzz} = f_{zxz}$.
- The idea of partial derivatives being continuous is going to be very important in this chapter, so make sure to stress both the hypothesis and the conclusion of Clairaut's Theorem.

- Define the function $f(x,y) = \begin{cases} x^{4/3}\sin(y/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ and compute $\frac{\partial f}{\partial x}(0,1)$ and $\frac{\partial f}{\partial y}(0,1)$ using the definitions of f_x and f_y.

- Demonstrate that the functions $f(x,y) = 5xy$, $f(x,y) = e^x \sin y$, and $f(x,y) = \arctan(y/x)$ all solve the Laplace equation $f_{xx} + f_{yy} = 0$.

Workshop/Discussion

We believe that it is crucial that the students do not leave their workshop/discussion session without knowing how to compute partial derivatives. There should be some opportunity for the students to practice in class, even by just trying one or two easy problems, so they can get instant feedback.

- Compute f_x and f_y for $f(x,y) = (x+y^2)^3 + \sin(x+y) + e^{x^2 y}$ and g_x and g_y for $g(x,y) = \dfrac{xy}{x^2+y^2}$.

- Compute some second- or even third-order partial derivatives for $f(x,y,z) = \ln(xy^2 + z^3)$.

- Expand on the text discussion of Clairaut's Theorem by showing that $f_{xyy} = f_{yxy} = f_{yyx}$ (provided that all the first-, second-, and third-order partial derivatives are continuous).

- Set up the tangent plane determined by the partials f_x and f_y by using the vector functions $\mathbf{g}(x) = \langle x, b, f(x,b)\rangle$, $\mathbf{h}(y) = \langle a, y, f(a,y)\rangle$ and the vector derivatives $\mathbf{a} = \langle 1, 0, f_x(a,b)\rangle$, $\mathbf{b} = \langle 0, 1, f_y(a,b)\rangle$ (as discussed in Materials for Lecture above). Form the plane through $(a, b, f(a,b))$ with normal vector $\mathbf{N} = \mathbf{a} \times \mathbf{b}$. Find the plane tangent to $f(x,y) = e^{xy}$ at the point $(1, 2, e^2)$.

- Have the students do Exercise 61 or Exercise 63 (page 778), either as a class or in groups.

Group Work 1: Partial Derivatives on the Sphere

Group Work 2: Partial Derivatives on Hyperboloids

Group Work 3: Clarifying Clairaut's Theorem

Problem 3 foreshadows the process of solving exact differential equations by finding $f(x,y)$ given that $f_{xy} = f_{yx}$. Students should be led carefully through this component.

Group Work 4: Back to the Park

Note that Problem 5 foreshadows the technique of linear approximation.

Group Work 5: The Geometry of Partial Derivatives

This group work gives a concrete example to illustrate the geometry behind the usual way of computing partial derivatives. It might help the students if a graph of $z = xy$ is given to them when they get to Problem 3.

Homework Problems

Core Exercises: 1, 4, 21, 30, 36, 46, 53, 60, 63

Sample Assignment: 1, 2, 4, 8, 18, 21, 24, 30, 33, 36, 42, 46, 53, 60, 61, 62(b), 62(c), 63, 65, 66, 76, 79

Note:
- Exercise 76 is a particularly good modeling problem.
- Exercises 76 and 79 require a CAS.

Exercise	C	A	N	G	V
1	×				
2	×		×		
4	×		×		
8			×	×	×
13–34		×			
36		×			
42		×			
46		×			
53		×			

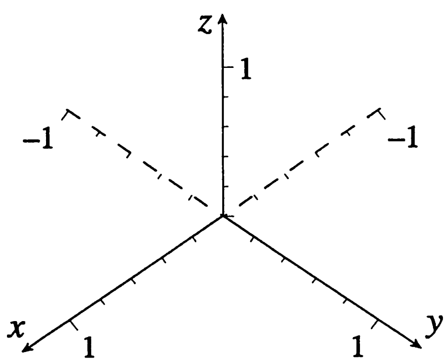
Exercise	C	A	N	G	V
60				×	×
61		×			
62		×			
63		×			
65		×			
66		×			
76	×	×		×	
79	×	×	×	×	×

Group Work 1, Section 11.3
Partial Derivatives on the Sphere

Consider the surface formed by the top half of the unit sphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$.

1. Write the equation for the top half of the sphere in the form $z = f(x, y)$.

2. Draw the surface.



3. Compute $\frac{\partial z}{\partial x}(0,0)$ and $\frac{\partial z}{\partial y}(0,0)$, and justify your answer by looking at your drawing.

4. Compute $\frac{\partial z}{\partial x}\left(\frac{1}{\sqrt{2}},0\right)$ and $\frac{\partial z}{\partial y}\left(\frac{1}{\sqrt{2}},0\right)$ and similarly justify your answer.

Group Work 2, Section 11.3
Partial Derivatives on Hyperboloids

1. Consider the hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$.

(a) For what values of (x, y) is $z = f(x, y)$, $z \geq 0$ defined?

(b) Sketch a graph of the surface.

(c) Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, both in general and at the point $(2, 2)$.

2. Next consider the hyperboloid of two sheets $x^2 + y^2 - z^2 = -1$.

(a) For what values of (x, y) is $z = g(x, y)$, $z \geq 0$ defined?

(b) Sketch a graph of the surface.

(c) Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, both in general and at the point $(2, 2)$.

3. Compare your answers for part (c) of Problems 1 and 2. Can you give a geometric explanation why the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in Problem 1 are larger than the corresponding values in Problem 2?

Group Work 3, Section 11.3
Clarifying Clairaut's Theorem

Consider $f(x,y) = x^2 \cos(y^3 + 2y)$.

1. Why do we know that $f_{yxyyxx} = 0$ without doing any computation?

2. Do we also know, without doing any computation, that $f_{yxyyxy} = 0$? Why or why not?

3. Suppose that $f_x = 3x + ay^2$, $f_y = bxy + 2y$, $f_y(1,1) = 3$, and f has continuous mixed second partial derivatives f_{xy} and f_{yx}.

(a) Find values for a and b and thus equations for f_x and f_y. *Hint:* What does Clairaut's Theorem say about the mixed partial derivatives of a function? When does the theorem apply?

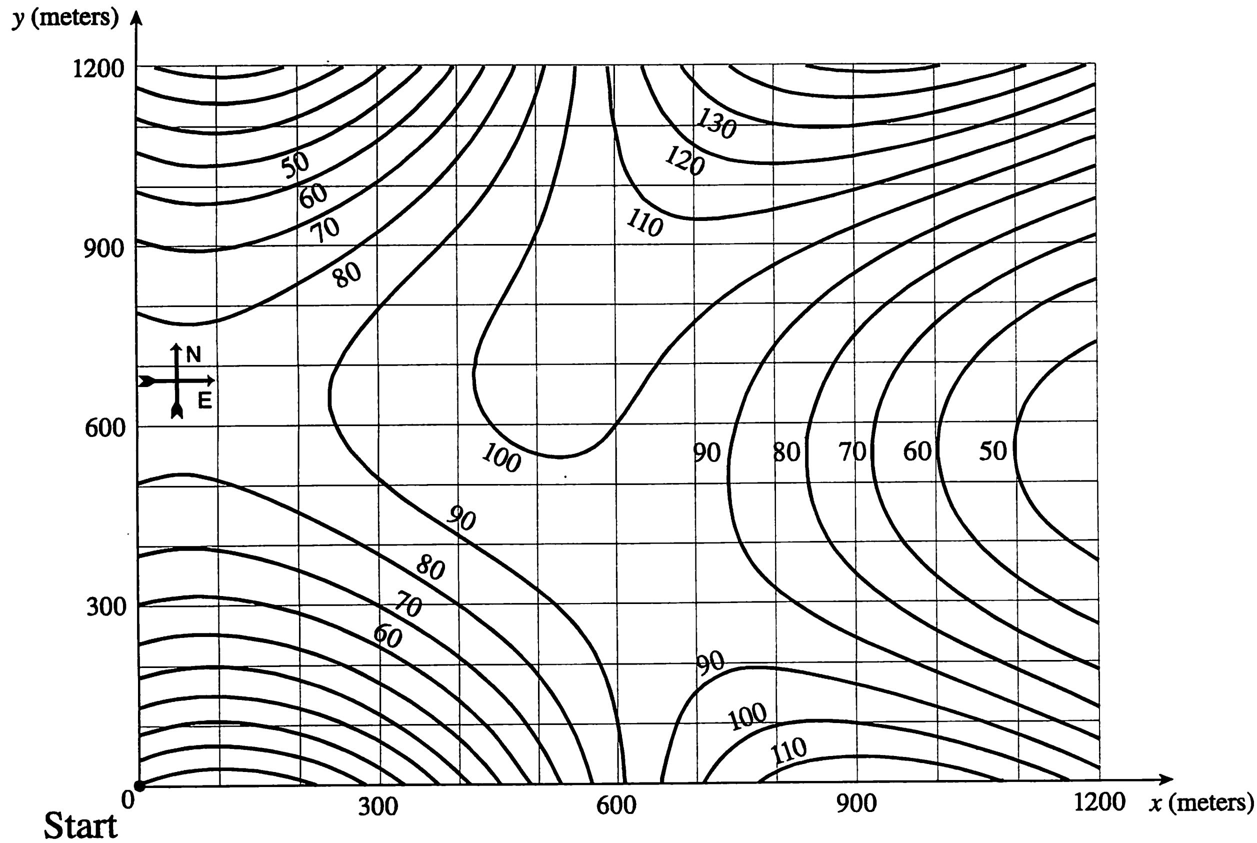
(b) Can you find a function $F(x,y)$ such that $\dfrac{\partial F}{\partial x} = f_x$ in part (a)?

(c) Can you find a function $G(x,y) = F(x,y) + k(y)$ such that $\dfrac{\partial G}{\partial y} = f_y$ in part (a)? What is $k(y)$?

(d) What is $\dfrac{\partial G}{\partial x}$? Can you now find $f(x,y)$?

Group Work 4, Section 11.3
Back to the Park

The following is a map with curves of the same elevation of a region in Orangerock National Park:



We define the altitude function, $A(x, y)$, as the altitude at a point x meters east and y meters north of the origin ("Start").

1. Estimate $A(300, 300)$ and $A(500, 500)$.

2. Estimate $A_x(300, 300)$ and $A_y(300, 300)$.

3. What do A_x and A_y represent in physical terms?

4. In which direction does the altitude increase most rapidly at the point $(300, 300)$?

5. Use your estimates of $A_x(300, 300)$ and $A_y(300, 300)$ to approximate the altitude at $(320, 310)$.

Group Work 5, Section 11.3
The Geometry of Partial Derivatives

Consider the function $f(x, y) = xy$.

1. Write an equation for the points in the intersection of the graph $z = xy$ with the plane $x = 1.5$. Since $x = 1.5$ is constant, we can consider this curve to be the graph of a function $g_{1.5}(y) = f(1.5, y)$. Compute the slope of the curve $g_{1.5}(y)$ and explain why $g_{1.5}'(y) = \frac{\partial f}{\partial y}(1.5, y)$.

2. Do the same thing for the intersection of $z = xy$ and the plane $y = 2$. As before, we can think of this curve as the graph of a function $h_2(x) = f(x, 2)$. Compute the slope of the curve $h_2(x)$ and explain why $h_2'(x) = \frac{\partial f}{\partial x}(x, 2)$.

3. Graph the function $z = xy$. Why are the curves on the graph that are parallel to either the xz-plane or the yz-plane always straight lines? What variables are being held constant in these cases? Imagine yourself walking on the surface. If you are at the point $(1.5, 2, 3)$ on the surface, the slope in the x-direction is 2 and the slope in the y-direction is 1.5. Verify this by using the results of parts (a) and (b) and by looking at the graph.

11.4 Tangent Planes and Linear Approximations

Suggested Time and Emphasis

1–$1\frac{1}{4}$ classes   Recommended material

Transparencies Available

- Transparency 45 (Figure 2, page 781)
- Transparency 46 (Figure 7, page 785)

Points to Stress

1. The tangent plane and its analogy with the tangent line.
2. Approximation along the tangent plane and its analogy with approximation along the tangent line.
3. The meaning of differentiability in $\mathbb{R}^2$ and $\mathbb{R}^3$.
4. The difference between f being differentiable and the existence of f_x and f_y.

Text Discussion

- Is it possible for a function f to be differentiable at (a, b) even though f_x and f_y do not exist at (a, b)?
- Is it possible for a function f to be *not* differentiable at (a, b) even though f_x and f_y exist at (a, b)?

Materials for Lecture

- Discuss tangent planes using both the development on pages 779–80, and the geometric and vector approaches. This latter approach uses the "vector" derivative with $\mathbf{a} = \langle 1, 0, f_x(a, b)\rangle$, $\mathbf{b} = \langle 0, 1, f_y(a, b)\rangle$, and normal $\mathbf{N} = \mathbf{a} \times \mathbf{b}$ to define the tangent plane at $(a, b, f(a, b))$.
- Find an equation for the tangent plane to the top half of the unit sphere $x^2 + y^2 + z^2 = 1$ at the point $(0, 0, 1)$ and then at $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ using both algebraic and geometric reasoning.
- Discuss differentiability using both the definition on page 782, and this alternate definition involving the tangent plane approximation:
  f is differentiable at (a, b) if both $f_x(a, b)$ and $f_y(a, b)$ exist and no matter how we choose (x, y) sufficiently close to (a, b), the linearization of f at (a, b) closely approximates $f(x, y)$.
- One good example of a function that is not differentiable at the origin is $f(x, y) = \begin{cases} \dfrac{(x-y)^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$ Here $f(x, 0) \equiv 1$ and $f(0, y) \equiv 1$, so f_x and f_y are both zero, but the tangent plane $z = 1$ fails to be a good approximation, no matter how close to the origin we look. For example, $f(x, x) \equiv 0$ for all $x \neq 0$.
- Note that if the partial derivatives exist and are continuous, then the tangent plane exists.

- Present examples of functions which are not differentiable, such as $f(x,y) = \begin{cases} \dfrac{x-y}{x+y} & \text{if } x+y \neq 0 \\ 0 & \text{if } x+y=0 \end{cases}$ Include an example, such as $g(x,y) = \sqrt{x^2+y^2}$, where the function is continuous.

Workshop/Discussion

- Visit or revisit examples of functions such as $f(x,y) = \begin{cases} \dfrac{x-y}{x+y} & \text{if } x+y \neq 0 \\ 0 & \text{if } x+y=0 \end{cases}$ and $g(x,y) = \sqrt{x^2+y^2}$, providing details where necessary.
- Compute some approximations to values of differentiable functions. For example, if $f(x,y) = \sin\left(\pi\left(x^2+xy\right)\right)$, then $f\left(\frac{1}{2},0\right) = \frac{1}{\sqrt{2}}$. Show the students how to use this fact and the partial derivatives of f to estimate $f(0.55,-0.01)$.
- Go over Example 5 in detail. Point out that although the errors are not small numbers in absolute terms, they are small numbers relative to the total volume (approximately 2618 cm^3) at the specified values.
- Use the approach on pages 779–80 to find the plane tangent to a surface described by a function. For example, given $f(x,y) = x^2y^3$, find the equation for the tangent plane at $(1,1,1)$, using the formula $z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$ (Answer: $z = 2x+3y-4$). Then have the students find the tangent plane at the point $(3,1,9)$ (Answer: $z = 6x+27y-36$).
  This example can be extended by asking how well the tangent plane approximates the function at the given point, perhaps by comparing $f(1.1,1.1) = 1.61$ to the approximating function $2x+3y-4$ at $(1.1,1.1)$, which has the value 1.5, and then comparing $f(1.01,1.01) \approx 1.051$ to $2x+3y-4$ at $(1.01,1,01)$, which has the value 1.05. Point out that $f(2,2) = 32$, while the value of $2x+3y-4$ at $(2,2)$ is 6.
- Review the geometric and vector approaches to planes using a vector normal to the plane. Consider the graph of f and the curves C_1 and C_2 as described in Section 11.3 on pages 769–70. Given that the vectors $\langle 1,0,f_x\rangle$ and $\langle 0,1,f_y\rangle$ lie in the tangent plane, taking the cross product gives a normal vector $\mathbf{N} = \langle -f_x,-f_y,1\rangle$. Now we can compute the equation of the tangent plane to be $z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$, exactly as derived by a different approach in the text on pages 779–80.
- Describe the tangent plane for a surface using either geometric reasoning or formal computations. A good example is the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{4} = 1$ and its tangent planes at $(0,0,4)$, $\left(\sqrt{2},0,\sqrt{2}\right)$, $\left(\sqrt{2},0,-\sqrt{2}\right)$, and $(0,0,-4)$.

Group Work 1: Trying it All Out

Problem 1 of this exercise requires the student to recognize where a complicated function is continuous. Problem 2 can be done without a picture, but students should be encouraged to draw a picture to verify their conclusions.

Group Work 2: Voluminous Approximations

Group Work 3: Self-Intersection of Surfaces

This one is a challenge for the students. For Problem 2, it will probably be necessary to give them a hint. If $\mathbf{r}(u_1, v_1) = \mathbf{r}(u_2, v_2)$, then we know that $u_1 = \pm u_2$ and $v_1 = \pm v_2$ in order that the first two coordinates be the same. Given this fact, we have to find the relationship between the pairs to make $u_1 + 2v_1 = u_2 + 2v_2$. There are three nontrivial cases to check:

$$u_1 = u_2, v_1 = -v_2 \qquad u_1 = -u_2, v_1 = v_2 \qquad u_1 = u_2, v_1 = -v_2$$

When all is said and done, the only nontrivial result occurs when $u_1 = -u_2$ and $v_1 = -v_2$, and that result is $u = -2v$. So the self-intersecting set occurs when $u = -2v$, and the self-intersecting points are of the form $(4v^2, v^2, 0)$, that is, the half-line $x = 4y$, $y \geq 0$.

Homework Problems

Core Exercises: 1, 4, 10, 18, 20, 26, 33

Sample Assignment: 1, 4, 5, 8, 10, 11, 13, 18, 20, 25, 26, 31, 33, 36, 40

Note: Exercises 5, 8, 33, and 36 require a CAS.

Exercise	C	A	N	G	V
1		×			
4		×			
5		×		×	
8		×		×	
10		×			
11		×			
13		×	×		
18			×		

Exercise	C	A	N	G	V
20		×			
25		×	×		
26		×	×		
31		×	×		
33		×			
36		×		×	
40		×			

Group Work 1, Section 11.4
Trying it All Out

1. Determine for what points the following functions are differentiable, and describe (qualitatively) why your answer is correct.

(a) $f(x,y) = e^{xy}\cos(\pi(xy+1))$

(b) $g(x,y) = \dfrac{x^4 - y^4}{x+y}$

(c) $h(x,y) = x - 2y\ln|x+y|$

2. Consider the surface $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{16} = 1.$

(a) Find the equation of the tangent plane to this surface at the point $(0,3,0)$.

(b) Can you find a point at which the tangent plane to this surface is horizontal? Is there any other such point?

(c) Can you find a point at which the tangent plane to this surface is vertical? Is there any other such point?

3. Consider the surface $\dfrac{x^2}{4} + \dfrac{y^2}{9} - \dfrac{z^2}{16} = 1.$

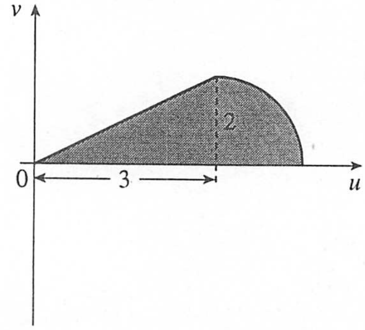
(b) Find the equation of the tangent plane to this surface at the point $(0,3,0)$.

(b) Can you find a point at which the tangent plane to this surface is horizontal? Is there any other s point?

(c) Can you find a point at which the tangent plane to this surface is vertical? Is there any c point?

Group Work 2, Section 11.4
Voluminous Approximations

Consider the solid obtained when rotating the following region about the u-axis.



Note that the region is composed of a right triangle and a quarter-circle.

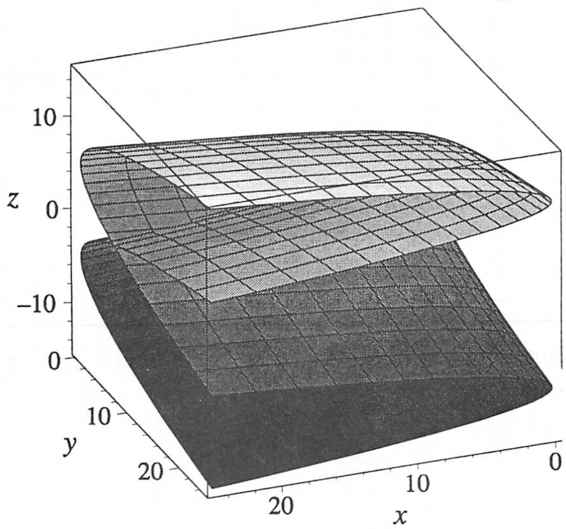
1. Compute the volume of this solid.

2. Find the volume $V(x, y)$ of a similar solid created by rotating a region with horizontal dimension x and vertical dimension y instead of 3 and 2.

3. Oh yeah — we forgot to tell you that in Problem 1, the "2" and the "3" were really just rounded-off numbers. The actual quantities can be off by up to 0.5 in either direction. Use linear approximation to estimate the maximum possible error in your answer to part (a).

Group Work 3, Section 11.4
Self-Intersection of Surfaces

Consider the parametric surface given by $\mathbf{r}(u,v) = \left(u^2, v^2, u+2v\right)$, shown below.



1. There is a set of points at which this surface intersects itself. If we know that $\mathbf{r}(u_1, v_1) = \mathbf{r}(u_2, v_2)$, what conditions does this place on u_1 and u_2, and on v_1 and v_2?

Hint: Look at each of the three coordinates separately.

2. Using your answer to Problem 1, find a relationship between u and v which describes the set of self-intersecting points.

3. Give a geometric description of the set of points on the surface that are self-intersecting points.

11.5 The Chain Rule

Suggested Time and Emphasis

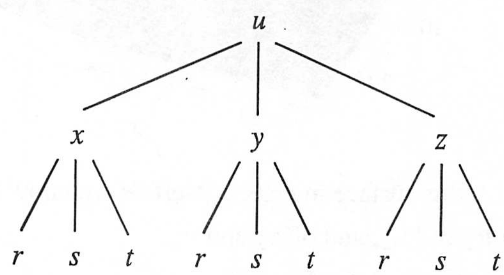
1 class Essential material

Points to Stress

1. The extension of the Chain Rule for functions of several variables.
2. Tree diagrams.
3. Implicit differentiation.

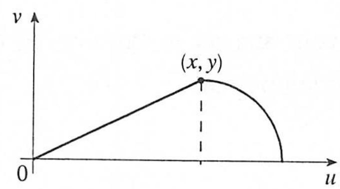
Text Discussion

- What was the following figure illustrating in the text? Specifically, how was it used?



Materials for Lecture

- Review the single-variable Chain Rule $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$ using the same language and symbols that will be used in presenting the multivariable Chain Rule.
- Develop the formulas and derivatives for chain rules involving two and three independent variables.
- Find the volume $V(x, y)$ generated by rotating the region at right about the u-axis. (Note that the region is composed of a right triangle and a quarter-circle.) Now suppose that x and y vary with time: $x = t + \sin t$, $y = 2^t - t\cos t$. Compute $\frac{dV}{dt}$.

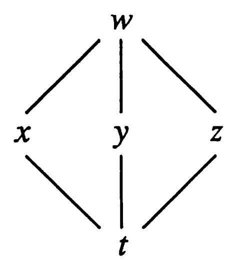


- Carefully state the Implicit Function Theorem. Illustrate it for the "fat circle" $x^4 + y^4 = 1$, and show why it fails when $\frac{\partial F}{\partial y} = 0$ [that is, at the points $(-1, 0)$ and $(1, 0)$.]
- Give an example of implicit differentiation on the ellipsoid $x^2 + \frac{y^2}{2} + \frac{z^2}{3} = 1$. Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, both in general and at the point $\left(0, 1, \sqrt{\frac{3}{2}}\right)$.
- Consider a cylindrical can of radius r and height h. Let V and S be the volume and surface area of the can. Find $\frac{\partial V}{\partial r}$ and $\frac{\partial S}{\partial h}$ and discuss what these quantities mean in practical terms. Then find $\frac{\partial V}{\partial h}$ when $r = 5$.

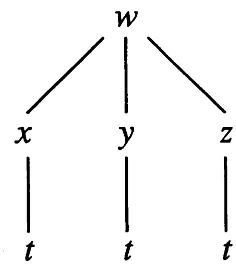
Workshop/Discussion

- Set up tree diagrams in two ways for the set of functions

$$w = f(x, y, z) \qquad x = g(t) \qquad y = h(t) \qquad z = k(t)$$



or



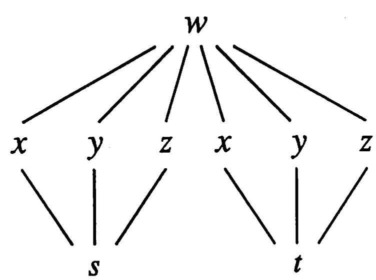
Then write out the Chain Rule for this case:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

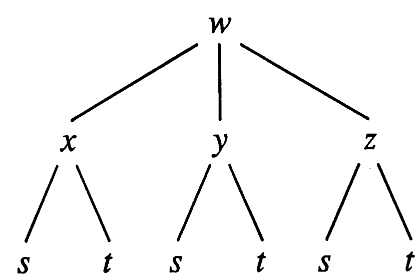
Extend the demonstration by considering

$$w = f(x, y, z) \qquad x = g(s, t) \qquad y = h(s, t) \qquad z = k(s, t)$$

Set out the relevant tree diagrams as shown below and write out the Chain Rule for $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$.



or



- Consider the function $f(x, y) = x^2 + y^2$, where $x = 3t$ and $y = e^{2t}$. Compute $\frac{df}{dt}$, first by using the Chain Rule, and then by actually performing the substitution to get $f(t) = 9t^2 + e^{4t}$ and taking the derivative. Show how this process of first substituting can become more complicated when using functions of two variables by discussing the function $g(x, y) = (x^2 + xy + y^2)^2$ with $x = 3(t + s)$ and $y = e^{2st}$, and computing $\frac{\partial g}{\partial t}$.

Group Work 1: Chain Rule Examples

This exercise contains three problems from very different disciplines. It would probably be overkill to have every group do every problem. Assign the one that most closely matches the interest of the class, and then groups that finish early can work on a different one.

Group Work 2: The Elephant Podium

Notice that Problems 1 and 2 can be done with calculus, by rotating the relevant line segment around the y-axis, or without calculus, by subtracting the volume of one cone from the volume of another cone. There may be even more methods, but they are surely irrelephant.

Group Work 3: Partial to Polar

Consider a surface $z = f(x, y)$ in polar coordinates and compute $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$. Do an explicit example, such as $z^2 = x^2 - y^2$, and evaluate $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ when $r = 1$ and $\theta = \frac{\pi}{2}$.

Homework Problems

Core Exercises: 2, 7, 11, 16, 22, 28, 34, 36

Sample Assignment: 1, 2, 7, 10, 11, 16, 22, 25, 28, 29, 34, 35, 36, 40

Note: Many students remember the Chain Rule by telling themselves that $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$ "because the dx's cancel". In single variable calculus, it is hard to persuade them that this is not what is really happening. Problem 45 shows the students an example of where that kind of thinking can get them into trouble.

Exercise	C	A	N	G	V
1		×			
2		×			
7		×			
10		×			
11		×			
16		×			
22		×			

Exercise	C	A	N	G	V
25		×			
28	×	×	×		
29		×	×	×	
34		×			
35		×			
36		×			
40		×			

Group Work 1, Section 11.5
Chain Rule Examples

Example A: Chemistry 101

Given n moles of gas, the relationship between pressure P, temperature T, and volume V can be approximated by the formula

$$PV = nRT$$

where P is in atmospheres, V is in Liters, T is in degrees Kelvin (degrees Kelvin = degrees Celsius $+273.15$), and R is the ideal gas constant $[0.08206 \text{ L} \cdot \text{atm}/(\text{mol} \cdot \text{K})]$

Assume we have 10 moles of gas in a balloon-type bladder. Initially we have a volume of 1 liter at "STP" ($T = 273.15$, $P = 1$). As time goes on, the gas is heated. The following expresses the temperature T of the gas as a function of the time elapsed t since the beginning of the experiment:

$$T = 323.15 - \frac{50}{t+1}$$

The bladder begins to expand over time as a function also of the strength s of its material, with the following formula describing how the volume V of the gas which can occupy the bladder changes as a function of the number of minutes t and the material strength s (where s is measured in millikents.)

$$V = 2\left(2 - e^{-3ts}\right)$$

1. Describe how the pressure of the gas in the bladder changes as a function of time.

2. Describe how the pressure of the gas in the bladder changes as a function of the strength of the bladder.

3. If the experiment takes place in a one-millikent bladder, what is the pressure of the gas in the box after 4 minutes?

4. If the experiment is allowed to run for a very long time, what value will P approach? Wh $\frac{dP}{dt}$ approach?

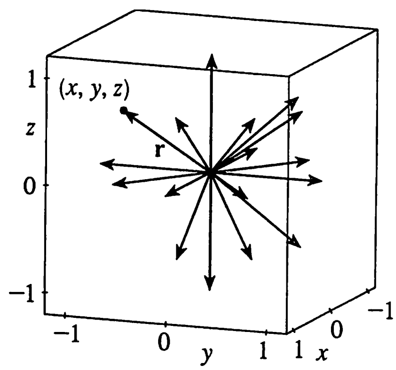
Group Work 1, Section 11.5
Chain Rule Examples

Example B: Electric Fields

Here we introduce the idea of a vector field or function defined for position vectors in space. In a vacuum, the electric field r units from a charge q at the origin is given by the vector field

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0 r^2}\mathbf{r} = \langle E_1, E_2, E_3 \rangle$$

where ε_0 is a constant, q is the charge on an electron, and $\mathbf{r}$ is a unit vector that points radially outward from the charge as shown in the figure.



1. What is the field at $\left(\frac{1}{2}, 0, 0\right)$? At $(0, 1, 0)$? At $(0, 0, 2)$?

2. Assume that we are concerned with the electric field at a point $p = (x, y, z)$. Why can we write $\mathbf{r} = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}} \langle x, y, z \rangle$?

3. Find the total x-change $\dfrac{\partial \mathbf{E}}{\partial x} = \left\langle \dfrac{\partial E_1}{\partial x}, \dfrac{\partial E_2}{\partial x}, \dfrac{\partial E_3}{\partial x} \right\rangle$ of the electric field if we start at $(5, 5, 5)$ and move in a direction parallel to the x-axis. Is this derivative a vector or a scalar?

4. Find the rate of change $\dfrac{d\mathbf{E}}{dt}$ of the electric field as we move along the line $z = 0$, $y = 2x$, starting at the point $(1, 2, 0)$, with parametrization $\mathbf{L}(t) = \langle t + 1, 2(t + 1), 0 \rangle$, $t \geq 0$. Is this derivative a vector or a scalar?

Group Work 1, Section 11.5
Chain Rule Examples

Example C: The Mutual Fund

One of the hottest investments on Wall Street today is the Share-All Mutual Fund. The Share-All Fund has issued 500,000 shares for eager investors to buy. Each share, therefore, represents $\frac{1}{500,000}$ of the fund's total net asset value. The fund owns 100,000 shares of stock in four companies, as described below, on a given day.

Company Name	Current price/share	Number of Shares Owned by Share-All	Total $ value
Allied Oil	30	30,000	900,000
Beck Keyboard Manufacturing	15	20,000	300,000
Jasmine Tea	23	28,000	644,000
Lapland Importing-Exporting	22	22,000	484,000
Total asset value			2,328,000

So, on this day, the total asset value is $2,328,000, and the price of one share of Share-All is $\frac{2,328,000}{500,000} = \$4.656$.

Many factors affect the price of a stock. For example, the worldwide exchange rate* w affects the Lapland Importing-Exporting company much more than it does the primarily domestic Beck Keyboard Manufacturing company. Similarly, the United States prime lending rate p affects the highly indebted Allied Oil company more than it affects the relatively debt-free Jasmine Tea company. Thus, we can develop models for the price of these stocks as functions of the world-wide exchange rate, the prime lending rate, and other economic factors q, r, s, and v, which are independent of these variables.

Let w be the world-wide exchange rate, and p be the U.S. prime lending rate. Let $a(w,p,q)$, $b(w,p,r)$, $j(w,p,s)$, and $l(w,p,v)$ be the current price per share of Allied, Beck, Jasmine and Lapland respectively. We have

$$\begin{aligned} a(w,p,q) &= 0.1w^2 - 100\sqrt{p} + q \\ b(w,p,r) &= 0.01w - p + r \\ j(w,p,s) &= 2w + s \\ l(w,p,v) &= -(1+w)^3 - p + v \end{aligned}$$

where q, r, s and v are composite variables representing the myriad other factors that affect the price of these stocks, and are independent of w and p. (*Note:* If we knew q, r, s, and v exactly, then we would be able to predict the price of the stock with unrealistic accuracy.)

* The worldwide exchange rate measures the value of the dollar measured versus a weighted average of other relevant currencies. In other words, it is a statistic that we made up, but which could probably fool some people.

1. If the current worldwide exchange rate is 0.85, and the current U.S. prime lending rate is 0.06, what are the current values of q, r, s, and v?

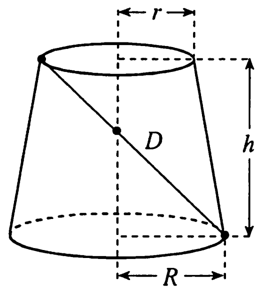
2. Assume that the values of q, r, s, and v are fixed at the quantities computed in Problem 1. Let S be the price of a share of Share-All. Write S as a function of w and p.

3. If S is the price of a share of Share-All, what is $\frac{\partial S}{\partial w}$?

4. What is $\frac{\partial S}{\partial p}$?

Group Work 2, Section 11.5
The Elephant Podium

The solid shown below has top radius r, bottom radius R, and height h.



1. If $r = 2$, $R = 4$, and $h = 8$, then what is the volume V of the solid?

2. Find a general expression for V in terms of r, R, and h.

3. Compute $\dfrac{\partial V}{\partial R}$.

4. Compute $\dfrac{\partial V}{\partial h}$.

5. If $r = 2$, $R = 4$, and $h = 8$, then what is the length of the diagonal D of the solid?

6. Find a general expression for D in terms of r, R, and h

7. Compute $\frac{\partial D}{\partial R}$.

8. Compute $\frac{\partial D}{\partial h}$.

9. Compute $\frac{\partial R}{\partial r}$ if we know that $V = 10\pi$. What does this quantity represent in practical terms?

11.6 Directional Derivatives and the Gradient Vector

Suggested Time and Emphasis

$1\text{–}1\frac{1}{4}$ classes Essential Material

Transparencies Available

- Transparency 47 (Figure 3, page 799 and Figure 5, page 801)

Points to Stress

1. The geometric meaning of a directional derivative.
2. The geometric meanings of the gradient vector, as defining the direction of greatest change of the directional derivative, and as a normal vector to a surface.
3. The relationships between directional derivatives, gradient vectors, and tangent planes.

Text Discussion

- The text shows that $D_{\mathbf{u}}f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$, where $\mathbf{u}$ is a unit vector Why does this show that the directional derivative in the direction of $\mathbf{u}$ is the scalar projection of the gradient vector onto $\mathbf{u}$?

Materials for Lecture

- Give geometric interpretations of the gradient, particularly of how the gradient gives the direction of maximal increase of a function $\left[\text{using the unit vector } \mathbf{u} = \dfrac{\nabla f(x,y)}{|\nabla f(x,y)|}\right]$ and its negative the direction of maximal decrease.
- Consider the surface $f(x,y) = xy$ at the point $(0,0)$. Note that although the maximum rate of change is zero at that point, it is not the case that the function is identically zero near the origin. Thus, if $\nabla f(a,b) = \mathbf{0}$, we cannot talk about the direction of maximal change at (a,b).
- Take any function $z = f(x,y)$ and write out its linearization at a point (a,b):

$$z = g(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

Show that the two functions f and g have the same gradient at the point (a,b), namely

$$\nabla f = \nabla g = \langle f_x(a,b), f_y(a,b)\rangle$$

Hence their directional derivatives are the same at (a,b). However, the graph of the function g is a plane, so it is visually obvious that there is a unique direction of maximal increase, *unless* the plane is horizontal. Note that the plane is horizontal precisely when

$$f_x(a,b) = f_y(a,b) = 0$$

that is, when (a,b) is a critical point of f. This reasoning gives an informal explanation as to why f has only one direction of maximal increase.

- Let $f(x,y) = x^2y^2$. Compute $D_{\mathbf{u}}f(x,y)$ for unit vectors $\mathbf{u}$ making angles of 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$ with the positive x-axis, and fill in the following table. Point out that the coefficient of $2xy^2$ decreases from

1 to 0 while the coefficient of $2x^2y$ increases from 0 to 1. Have the students reason intuitively why this should be the case, just using the concept of "directional derivative". Have them figure out (intuitively) $D_{\mathbf{u}}f(x,y)$ for angles π and $\frac{3\pi}{2}$.

Angle	$D_u f(x,y)$
0	$(1)\,2xy^2 + (0)\,2x^2y$
$\frac{\pi}{6}$	$(0.866\ldots)\,2xy^2 + (0.5)\,2x^2y$
$\frac{\pi}{4}$	$(0.707\ldots)\,2xy^2 + (0.707\ldots)\,2x^2y$
$\frac{\pi}{3}$	$(0.5)\,2xy^2 + (0.866...)\,2x^2y$
$\frac{\pi}{2}$	$(0)\,2xy^2 + (1)\,2x^2y$

Note: At a given point (x,y), one cannot maximize D_u merely by looking at the chart.

- Review that the direction of any vector $\mathbf{v} \neq \mathbf{0}$ is determined by the unit vector $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \langle \cos\theta, \sin\theta \rangle$ where θ is the angle that $\mathbf{v}$ makes with the positive x-axis. Using this interpretation, the directional derivative formula can be rewritten as

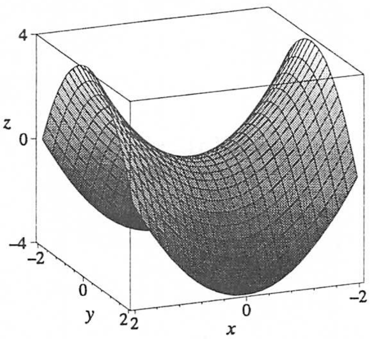
$$D_u f = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

- Define S to be a level surface of the function $f(x,y,z)$. Explain why $\nabla f(x_0,y_0,z_0)$ is orthogonal to the surface at any point $P(x_0,y_0,z_0)$ on S. Define the tangent plane to S at P to be the plane with normal vector $\nabla f(x_0,y_0,z_0)$.
- Show how Equation 2 in Section 11.4 (page 780) is really just a special case of Equation 19 in this section (page 806).

Workshop/Discussion

- Have the students practice finding the directional derivatives of $f = x^2y$ and $f = e^{xy}$ in the directions $-\mathbf{i}$, $\mathbf{i}+\mathbf{j}$, $-\mathbf{i}-\mathbf{j}$, and $\mathbf{i}-\mathbf{j}$. Also have them find the directional derivatives of $f(x,y,z) = z^2e^{xy^2}$ in the directions of $\langle -1,-1,-1\rangle$ and $\langle 0,0,-1\rangle$.
- Show that the gradient vector $\nabla f(x_0,y_0)$ is normal to the line tangent to the level curve $k = f(x,y)$ at the point (x_0,y_0). Look at the example $f(x,y) = 5x^4 + 4xy + 3y^2$ and show that $\nabla f(-1,-1) = \langle -24,-10\rangle$. Conclude that we now know that f is decreasing in both the x- and y-directions and that the direction of maximal increase is $\langle -24,-10\rangle$. Ask the students to resolve these seemingly contradictory observations: That the gradient is supposed to point in the direction of maximal *increase*, yet the components $f_x(-1,-1) = -24$ and $f_y(-1,-1) = -10$ of the gradient are pointing in the direction of decreasing x and y. (Although no real paradox exists, students are often confused by this type of situation.)
- Note that $-\nabla f$ points in the direction of maximal decrease of f, and that the rate of change is $-|\nabla f|$.
- Do Exercise 32 (page 810), adding some additional points on the outer contour lines and drawing the curves of steepest ascent, and then choosing a point on the innermost contour line and drawing the curve of steepest descent.

- Solve the problem of finding the direction u for which the rate of increase of $f(x, y) = x^2y^3 + x + y$ at $(1, 1)$ is 2. Pose the same problem to the students for a rate of increase of 4 and for a rate of decrease of 5.
- Analyze Figure 13. Ask why the gradients near the y-axis point toward the vicinity of the origin and downhill (have negative z-coordinate) while those near the x-axis are pointing uphill, as the text claims. Show how the shape of $z = x^2 - y^2$ reflects this behavior.



$z = x^2 - y^2$

Group Work 1: Two Ways

Group Work 2: Computation Practice

It is a good idea to give the students a chance for guided practice using the types of computations that will be required on the homework. We recommend having them do either Problem 1 or Problem 2 in groups, and then handing the remaining problem out as a worksheet.

Group Work 3: Bowling Balls and Russian Weebles

This exercise may seem trivial, but it is a good setup for discussions of Lagrange multipliers. If the students do this exercise, ask them to remember the result, and make sure to remind them of the bowling balls and Russian weebles when discussing Lagrange multipliers.

Homework Problems

Core Exercises: 1, 5, 8, 14, 19, 23, 30, 32, 41

Sample Assignment: 1, 5, 8, 10, 12, 14, 19, 20, 23, 28, 30, 31, 32, 34, 35, 39, 41, 44, 53

Note: Exercise 39 requires a CAS.

Exercise	C	A	N	G	V
1			×	×	×
5		×			
8		×			
10		×			
12		×			
14		×			
19		×			
20		×			
23		×			
28		×			

Exercise	C	A	N	G	V
30		×			
31		×			
32				×	
34	×			×	
35		×			
39		×		×	
41		×		×	
44		×			
53	×	×			

Group Work 1, Section 11.6
Two Ways

Consider the function $f(x,y) = x^2 + 2xy^2$.

1. What is $f(1,1)$?

2. What is the directional derivative $D_u f(1,1)$ if $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$?

3. What is the directional derivative $D_u f(1,1)$ if $\mathbf{u}$ is the unit vector that makes an angle θ with the positive x-axis?

4. In Problem 3, you expressed $D_u f(1,1)$ as a function of the angle θ. Let's say we want to find the maximum value of the directional derivative. This is now a single-variable calculus problem! Use your single-variable calculus techniques, coupled with your answer to Problem 3, to find the find the maximum value of the directional derivative.

5. What is the angle θ for which f increases the fastest? (You should be able to use your computations for Problem 4 to answer this one quickly.)

6. What is the unit vector that makes that angle θ with the positive x-axis?

7. Now compute $\dfrac{\nabla f(1,1)}{|\nabla f(1,1)|}$, but before you do so, discuss with your group members what the answer should be. You should be able to anticipate the correct answer.

Group Work 2, Section 11.6
Computation Practice

1. Find the directional derivative of the function at the given point in the direction of the given vector $\mathbf{v}$.

(a) $f(x,y) = e^{xy} - x^2$, $(1,1)$, $\mathbf{v} = \langle 1,0 \rangle$

(b) $f(x,y) = e^{xy} - x^2$, $(1,1)$, $\mathbf{v} = \langle 0,1 \rangle$

(c) $f(x,y) = e^{xy} - x^2$, $(1,1)$, $\mathbf{v} = \langle 1,1 \rangle$

(d) $f(x,y,z) = z \ln\left(x^2 + y^2\right)$, $(-1,1,0)$, $\mathbf{v} = \langle 1,1,-1 \rangle$

(e) $f(x,y,z) = z \ln\left(x^2 + y^2\right)$, $(-1,1,0)$, $\mathbf{v} = \langle 2,1,1 \rangle$

2. Find the maximum and minimum rates of change of f at the given point and the directions in which they occur.

(a) $f(x,y) = e^{xy} - x^2$, $(1,1)$

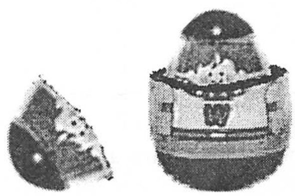
(b) $f(x,y,z) = z\ln\left(x^2+y^2\right)$, $(-1,1,0)$

Group Work 3, Section 11.6
Bowling Balls and Russian Weebles

1. Assume that there are two bowling balls in a ball-return machine. They are touching each other. At the point at which they touch, what can you say about their respective normal vectors? What about their tangent planes? What would happen if the bowling balls were of different sizes?

2. A *weeble* is a doll that is roughly egg-shaped. It is an ideal toy for little children, because weebles wobble but they don't fall down.

A Russian Weeble is a hollow weeble, with one or more weebles inside it. Picture two nested hollow eggs as shown.



At the point at which two nested Russian weebles touch each other, what can you say about their respective normal vectors? What about their tangent planes?

3. Now picture your two favorite differentiable surfaces that touch at exactly one point. What can you say about their normal vectors at the point where they touch? What about their tangent planes?

11.7 Maximum and Minimum Values

Suggested Time and Emphasis

1 class Essential Material

Transparencies Available

- Transparency 48 (Figures 7–9, page 815)

Points to Stress

1. The contrast between optimization problems in single-variable calculus (relatively few cases) and in multivariable calculus (many possible solutions).
2. Critical points and local maxima and minima.
3. The Second Derivative Test.
4. Absolute maxima and minima.

Text Discussion

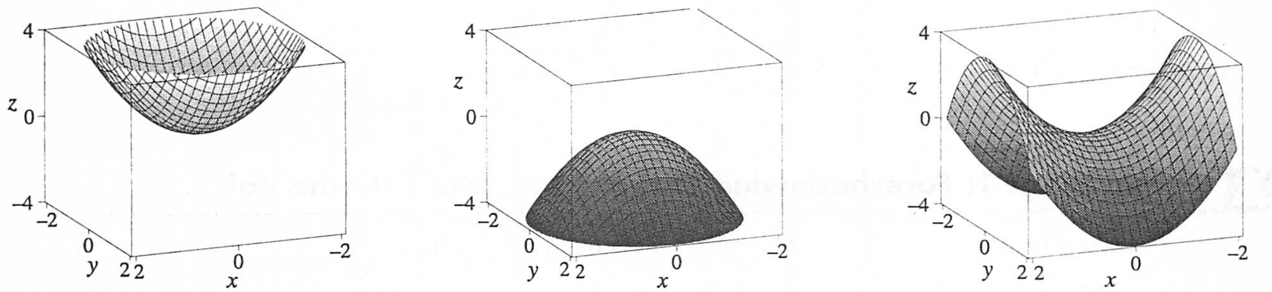
- Can a differentiable function f have a local maximum at a point (a, b) with $f_x(a, b) = 3$?
- Can you give an example of a function f with the property that $f_x(a, b) = 0$, $f_y(a, b) = 0$, and f does *not* have a local maximum or minimum at (a, b)?
- Can you always apply the Second Derivative Test at any critical point? If you can, does it always give information?

Materials for Lecture

Note: One way to introduce this topic is to have the students initially do Group Work 1: Foreshadowing Critical Points and Extreme Values.

- Stress geometric interpretations: If f is differentiable at a local maximum or minimum, then the tangent plane must be horizontal.
- Note that there are critical points at which there is no local maximum or minimum. For example, examine the saddle points at the origin for $f(x, y) = xy$ and $g(x, y) = x^2 - y^2$.

- Illustrate the idea behind the Second Derivative Test, using $f(x,y) = \frac{1}{2}(ax^2 + by^2)$. Note that $D = ab$ means that
  - $D > 0$, $a > 0$ gives $b > 0$ and hence $f(x,y)$ has a local minimum at $(0,0)$ [See Picture (a)]
  - $D > 0$, $a < 0$ gives $b < 0$ and hence $f(x,y)$ has a local maximum at $(0,0)$ [See Picture (b)]
  - $D < 0$ means that a and b have opposite signs, and hence $f(x,y)$ has a saddle point at $(0,0)$ [See Picture (c)]



(a) Minimum (b) Maximum (c) Saddle Point ($a > 0$, $b < 0$)

- Describe some of the ways saddle points can occur for functions of two variables. (For examples, see Example 3 and Figure 4 on page 813, and Example 4 and Figures 7 and 8 on pages 813–15.) Contrast with the single-variable case, where there are fewer possibilities.

- Use $f(x,y) = x^4 + y^4$, $g(x,y) = x^4 - y^4$, $h(x,y) = -(x^4 + y^4)$ to show that no information is given about local extrema when $D = 0$.

- Review Example 5 (pages 815–16), which finds the distance between a point and a plane. Contrast this approach to the method used in Example 8 in Section 9.5 (page 682).

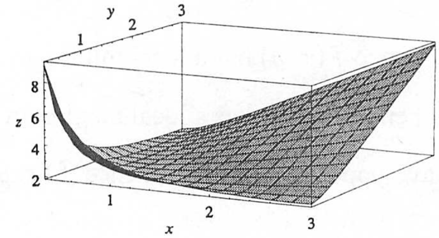
- Briefly discuss absolute maximum and minimum values of continuous functions on closed bounded sets. Point out that these values can occur either at boundaries or "inside" the region. One good example is $f(x,y) = x^2 + y^2$ on the rectangle $D = \{(x,y) \mid -1 \le x \le 1, -1 \le y \le 2\}$, where the absolute minimum value is $f(0,0) = 0$ and the absolute maximum value is $f(\pm 1, 2) = 5$.

Workshop/Discussion

- Pose the problem of finding the absolute maximum of $f(x,y) = ax + by + c$ on the set of points $x^2 + y^2 \le 4$. Note that the gradient of f is never zero, so the maximum and minimum values must occur on the boundary. One way to find these maximum and minimum values is by parametrizing the boundary $x^2 + y^2 = 4$ by $\mathbf{r}(\theta) = \langle 2\cos\theta, 2\sin\theta \rangle$, where θ is the angle made by the position vector with the x-axis, and then optimizing the function $g(\theta) = f(\mathbf{r}(\theta))$.

- Expand on Example 5 (pages 815–16) by indicating that it is sometimes easier to optimize $f^n(x,y)$ instead of $f(x,y)$ for a function f that is always positive. Point out that f^n has the same maxima as f for any n. One good example to use is $f(x,y) = \left[(x-1)^2 + (y+1)^2 + 1\right]^{1/3}$. Another example is the problem of finding the points on the surface $z^2 = xy + x^2 + 1$ that are closest to the origin. [Answer: $\left(\frac{1}{7}, -\frac{4}{7}, \frac{3\sqrt{2}}{7}\right)$]

- Discuss local and absolute maxima and minima for $f(x,y) = xy + 1/(xy)$, $x > 0$, $y > 0$.



- Consider $f(x,y) = x^2 - y^2 + 2xy$. Show that f has no absolute maximum or minimum and also that f has no local maximum or minimum.

Group Work 1: Foreshadowing Critical Points and Extreme Values

This group work is best done just before Section 11.7 is covered. First present the single-variable definitions of local and global maximum and minimum. (This was done in single-variable calculus, but the students have probably forgotten the technical definitions by this point.) Then put the students into groups and ask them to come up with good multivariable definitions of the same concepts. They should present their definitions and discuss them. At the end of the activity, look up the definition presented in the text, and compare it with the student definitions.

If there is time, do a similar activity for the various types of critical points. Graph $y = x^2$, $y = -x^2$, $y = x^3$, $y = -x^3$, $y = |x|$, and $y = -|x|$ on the board to show different types of critical values at $x = 0$. Then have the students try to come up with the variety of types that can occur for functions of two variables.

Group Work 2: The Squares Conjecture

If a group finishes this problem early, have them try to solve it without using any calculus at all.

Group Work 3: Strange Critical Points

In this case, f_x and f_y do not exist at the critical point $(1, -1)$ and so the students cannot use the Second Derivative Test. Acceptable answers include graphing the surface or recognizing that it is an elliptic cone.

Extended Lab Project: The Genetic Algorithm

The use of "genetic algorithms" for finding maxima and minima for functions of several variables has become popular in recent years. Usually this technique is used to optimize functions of hundreds of variables, but we'll look at the simpler case of functions of two variables.

Although we don't intend to give a complete description of how genetic algorithms work, an outline is as follows:

Suppose you want to maximize a function of several variables. Start by selecting several arbitrary points (at random or otherwise) from your domain. Select two points among these which give the two largest values of your function. Now choose several more arbitrary points close to these selected points. Continue to repeat this process until you have what seems to be a maximum value.

We will study this process for the complicated function $100e^{-(|x|+1)(|y+1|+1)}\dfrac{\sin(y\sin x)}{1+x^2y^2}$. Let D be the square $[-3,3]\times[-3,3]$.

(i) Use your computer program to select 5 points at random in this square and then evaluate the function at these 5 points.

(ii) Select the points which give the two largest values for $f(x,y)$ and then select 4 points at random close to each of these points. Again, selecting the points at random near these points isn't so trivial. Evaluate the function at the 10 points you now have. Select the two points among these which give the largest value for $f(x,y)$. Repeat (b) until it appears that you have a maximum.

(iii) Is the value you found in (ii) likely to be an absolute maximum?

Homework Problems

Core Exercises: 2, 3, 7, 8, 12, 16, 23, 25

Sample Assignment: 1, 2, 3, 4, 7, 8, 12, 13, 15, 16, 23, 25, 29, 32, 35, 40, 45

Note:
- Exercise 29 requires a CAS.
- Problem 8 in Focus on Problem Solving (page 837) makes a good group project, particularly because it incorporates material from several sections of the text.

Exercise	C	A	N	G	V
1					
2	×				
3		×		×	×
4		×		×	×
5–14		×		×	
15		×		×	
16		×		×	

Exercise	C	A	N	G	V
23		×			
25		×			
29		×		×	
32		×			
35		×			
40		×			
45		×			

Group Work 2, Section 11.7
The Squares Conjecture

You are given a government grant to prove or disprove the Squares Conjecture:

> There exist three positive numbers, r, s, t whose product is 100, yet have the property that the sum of their squares is less than 65.

Either find three such numbers, or show that none exist.

Group Work 3, Section 11.7

Strange Critical Points

Let $f(x, y) = 2 + \sqrt{3(x-1)^2 + 4(y+1)^2}$.

1. Find the critical points of f.

2. Find the local and absolute minimum values of f. Where do these values occur?

Applied Project: Designing a Dumpster

This project requires the students to solve an extended real-world problem that involves them going out and measuring a nearby dumpster. They will have to make approximations, and figure out how best to get an accurate answer. A good sample answer is given in the *Complete Solutions Manual.*

Discovery Project: Quadratic Approximations and Critical Points

Problems 1–3 serve as a good introduction to Taylor's Theorem for two variables, and to quadratic polynomial approximation in two variables. Problems 4 and 5 justify the Second Derivative Test, the proof of which is given in Appendix E.

11.8 Lagrange Multipliers

Suggested Time and Emphasis

$\frac{3}{4}$–1 class Essential Material: one-constraint problems.
Optional Material (if time permits): two-constraint problems

Transparencies Available

- Transparency 49 (Figures 2 and 3, pages 824–25)

Points to Stress

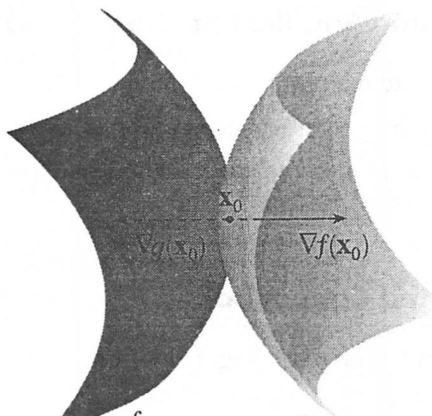
1. The geometric justification for the method of Lagrange multipliers
2. How to apply the method of Lagrange multipliers, including the extension of the method for two-constraint problems

Text Discussion

- How does the equation $\nabla f(x,y) = \lambda \nabla g(x,y)$, subject to the constraint $g(x,y) = k$, lead to three equations with three unknowns? What are the unknowns?

Materials for Lecture

- Make sure that students understand the actual "nuts and bolts" of the one-constraint method.
- Draw a picture like the one below illustrating that if two surfaces are tangent, they have parallel normals at the point of tangency.



- Give an example to show that, with functions of two variables, there are often alternate methods other than Lagrange multipliers to solve max/min problems. Perhaps redo Example 2, substituting $x^2 = 1 - y^2$ into $f(x,y) = x^2 + 2y^2$ to get the single-variable problem $g(y) = 1 + y^2$, minimize to get $y = 0$ (with $x = \pm 1$), and then get $h(x) = 2 - x^2$ with maximum at $x = 0$, $y = \pm 1$.
- Note that, for the two-variable case, $\nabla f = \lambda \nabla g$ implies that $\nabla f \times \nabla g = \mathbf{0}$. This condition can sometimes be used to replace Lagrange multipliers.
- Present some additional explanation of the use of Lagrange multipliers for functions of three variables.

- Present a complementary problem similar to Examples 2 and 3 (pages 824–25), such as finding extreme values for $f(x, y) = (x^2 + y^2)^{3/2}$ on the ellipse $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$.

- Go through the analytic solution to Example 4 in detail.

Workshop/Discussion

- Find the volume of the largest rectangular solid that can be inscribed in a sphere, that is, maximize $V(x, y, z) = (2x)(2y)(2z)$ given that $x^2 + y^2 + z^2 = a^2$.

- Discuss the geometric solution to Example 4. What basic geometric principle is being used?

Group Work 1: The Inscribed Rectangle Race

Divide the class in half. Write the following problem on the board: "What is the area of the largest rectangle that can be inscribed in a circle of radius 4?" Have one half of the class try to solve this problem using Lagrange multipliers, and the other half try to use single-variable calculus. See which side finishes first, and which side found the problem more difficult. At the end, the students should see both methods presented.
If a group finishes early, or after all groups have presented, have the students further practice the two techniques by maximizing xy^2 on the ellipse $\frac{1}{5}x^2 + \frac{1}{7}y^2 = 1$.

Group Work 2: Biggest and Smallest on Closed and Bounded Sets

This activity involves finding the absolute maximum and minimum values of a function of several variables on a closed and bounded set. Review the necessary steps outlined on page 817 in Section 11.7. Note that Problem 1 may be easier to solve by plugging in the appropriate value for x or y along a boundary curve and using single-variable methods.

Group Work 3: The Heated Cannonball

This problem appears to be quite difficult at first reading, but letting x, y, and z be the angles (in radians) and using Lagrange multipliers leads to a very easy solution.

Group Work 4: The Sum of the Sines

Group Work 5: Find the Error

This group work presents a problem where the absolute maximum value occurs on the boundary and the absolute minimum is inside the region. The idea is to reinforce the point that finding critical points is not sufficient to locate absolute extrema for functions of two variables.

Homework Problems

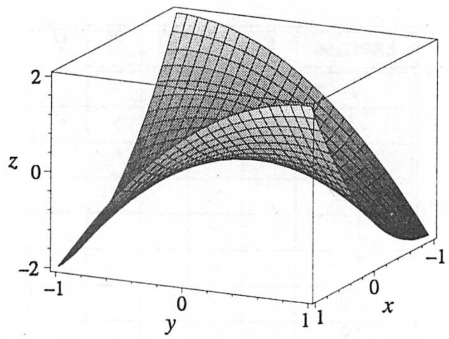
Core Exercises: 1, 2, 5, 13, 19, 23

Sample Assignment: 1, 2, 5, 8, 13, 19, 23, 26, 29, 38

Exercise	C	A	N	G	V
1	×			×	×
2		×		×	×
3–17		×			
19		×			
23		×			
26		×			
29		×			
38		×			

Group Work 2, Section 11.8
Biggest and Smallest on Closed and Bounded Sets

Let $f(x, y) = x^2 - y^2 + 2xy$.



1. What are the absolute maximum and absolute minimum values of this function on the unit square $\{0 \le x \le 1, 0 \le y \le 1\}$?

2. What are the absolute maximum and absolute minimum values of this function on the disk $x^2 + y^2 \le 1$?

3. What are the absolute maximum and absolute minimum values of this function on the elliptical disk $x^2 + \frac{y^2}{4} \le 1$?

Group Work 3, Section 11.8
The Heated Cannonball

One of the wonderful things about the British army in the eighteenth century was that they were very polite. For example, during the Revolutionary War, during the battle of Valley Forge, it was standard practice for them to gently warm their cannonballs before firing them at the colonists. Suppose that a particular cannonball with radius 1 foot has a temperature distribution $T(x, y, z) = 48 + 16(y^2 + z^2 - x^2)$ (where the center of the cannonball is at the origin).

1. What are the maximum and minimum temperatures in the cannonball, and where do they occur?

2. What is the shape of the wire frame used to apply the heat to the surface of the cannonball?

Group Work 4, Section 11.8
The Sum of the Sines

1. What is the description of the triangle for which the sum of the sines of its angles is a maximum?

2. Find the maximal value of the sums of the sines of the angles of a triangle.

3. Repeat Problems 1 and 2 if we now assume that the triangle is

(a) a right triangle.

(b) a triangle one of whose angles is 30°.

Group Work 5, Section 11.8
Find the Error

Consider the ellipsoid $\frac{1}{9}x^2 + \frac{1}{4}y^2 + z^2 = 1$. You want to locate the point(s) on this ellipse which are furthest from the origin.

1. Is it sufficient to maximize the square D of the distance instead of the distance itself?

2. Writing the ellipsoid as $z^2 = 1 - \frac{1}{9}x^2 - \frac{1}{4}y^2$, find an equation for the square D of the distance in terms of x and y.

3. Show that $(0, 0)$ is a critical point of D.

4. The points corresponding to $(x, y) = (0, 0)$ are $(0, 0, \pm 1)$ and $D = 1$ at these points. But at the point $\left(0, 1, \frac{\sqrt{3}}{2}\right)$, $D = 1 + \frac{3}{4} = \frac{7}{4}$. So the points $(0, 0, \pm 1)$ are not furthest from the origin. So calculus doesn't locate this point. What is wrong? How can you find the point(s) on the surface furthest from the origin?

Applied Project: Rocket Science

This is an excellent example of Lagrange multipliers presented in a realistic setting. If not assigned as a project, it can be given as a supplementary reading. The computations required for this problem are extensive. A CAS might help, but is not required.

Applied Project: Hydro-Turbine Optimization

The Great Northern Paper Company is a real company that has hired engineers to solve the same problem that the students are faced with. If this project is assigned, the students should be informed that they have the opportunity to solve a real engineering problem.

11 Sample Exam

Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

1. (a) Consider the function $f(x,y) = \dfrac{1}{x^2+y^2+1}$. Find equations for the following level surfaces for f, and sketch them.

(i) $f(x,y) = \frac{1}{5}$

(ii) $f(x,y) = \frac{1}{10}$

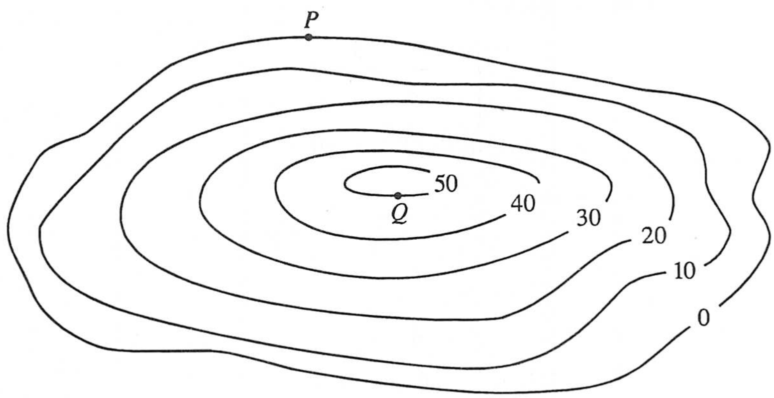
(b) Find k such that the level surface $f(x,y) = k$ consists of a single point.

(c) Why is k the global maximum of $f(x,y)$?

2. Is the function $f(x,y) = \sin^2(xy^2)$ a solution to the partial differential equation $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} = (2x+y)(2y)\cos(xy^2)\sqrt{f}$ when $\sin(xy^2) \geq 0$?

3. Is it possible to find a function for which it is true that, for all $x > 0$ and $y > 0$, $f_x > 0$ and $f_y < 0$, and $f(x,y) > 0$? If so, give an example. If not, why not?

4.



The above is a topographical map of a hill.

(a) Starting at P, sketch the path of steepest ascent to the peak elevation of 50 yards.

(b) Suppose it rains, and water runs down the hill starting at Q. At what point would you expect the water to reach the bottom? Justify your answer.

5. Find the absolute maximum and minimum of $f(x,y) = x^2+xy+y^2$ on the disk $\{(x,y) \mid x^2+y^2 \leq 9\}$.

6. Consider the ellipsoid $\dfrac{x^2}{4} + 2z^2 + \dfrac{y^2}{4} = 1$. Using geometric reasoning or otherwise, find the equation of the tangent plane at

(a) $(\sqrt{2}, \sqrt{2}, 0)$.

(b) $\left(0, 0, \frac{1}{\sqrt{2}}\right)$.

7. Describe the level surfaces $f(x,y,z) = k$ for the function $f(x,y,z) = 1 - x^2 - \dfrac{y^2}{2} - \dfrac{z^2}{3}$ and the values $k = -1$, $k = 1$, and $k = 2$.

8. Suppose that the amount of energy $F(x,y,z)$ emanating from a source at $(0,0,0)$ is inversely proportional to one more than the square of the distance from the origin measured only in the the xy-plane, and is directly proportional to the height above the xy-plane. Assume that all of the constants of proportionality are equal to 1.

(a) What is an equation for the energy as a function of x, y, and z?

(b) Where is there no energy at all?

(c) Sketch the level surface $F(x,y,z) = 1$.

9. Consider the function

$$f(x,y) = \frac{x+y}{|x|+|y|}$$

(a) Evaluate the following

(i) $f(1,1)$

(ii) $f(1,-1)$

(iii) $f(-1,1)$

(iv) $f(-1,-1)$

(b) Does this function have a limit at $(0,0)$?

10. Consider the function

$$f(x,y) = \begin{cases} \dfrac{2x^2+3y^2}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

(a) Compute $f_x(0,0)$ directly from the limit definition of a partial derivative

$$f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,y_0) - f(x_0,y_0)}{h}$$

(b) Compute $f_y(0,0)$.

11. If $f(0,0) = 0$, $f_x(0,0) = 0$, $f_y(0,0) = 0$, and $f(x,y)$ is differentiable at $(0,0)$, does this imply that $f(x,y) = 0$ for some point $(x,y) \neq (0,0)$? Justify your result, or give a counterexample.

12. Consider the sphere $x^2+y^2+z^2 = 9$. Find the equation of the plane tangent to this sphere at

(a) $(3,0,0)$.

(b) $(2,2,1)$.

13. Suppose that $f(x,y) = e^{x-y}$ and $f(\ln 2, \ln 2) = 1$. Use the technique of linear approximation to estimate $f(\ln 2 + 0.1, \ln 2 + 0.04)$.

14. Let $g(u)$ be a differentiable function and let $f(x,y) = g(x^2+y^2)$.

(a) Show that $y f_x = x f_y$.

(b) Find the direction of maximal increase of f at $(1,1)$ in terms of g'.

15. Let f be a function of two variables with the following properties:

- $\dfrac{\partial f}{\partial x}$ is defined near $(0,0)$, continuous at $(0,0)$ and $\dfrac{\partial f}{\partial x}(0,0) = 0$
- $\dfrac{\partial f}{\partial y}$ is defined near $(0,0)$, continuous at $(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0) = 0$

- $\dfrac{\partial^2 f}{\partial x \partial y}(0,0) = 1$

- $\dfrac{\partial^2 f}{\partial y \partial x}(0,0) = -1$

Answer true or false to the following, and give reasons for your answers.

(a) f is differentiable at $(0,0)$.

(b) There is a horizontal plane that is tangent to the graph of f at $(0,0)$.

(c) The functions $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ are both continuous at $(0,0)$.

(d) The linear approximation to $f(x,y)$ at $(0,0)$ is $L(x,y) = x - y$.

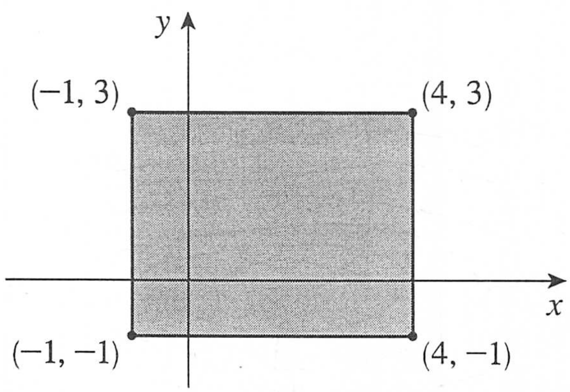
16. Suppose $\mathbf{u} = \langle 1, 0 \rangle$, $\mathbf{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, $D_{\mathbf{u}}(f(a,b)) = 3$ and $D_{\mathbf{v}}(f(a,b)) = \sqrt{2}$.

(a) Find $\nabla f(a,b)$.

(b) What is the maximum possible value of $D_{\mathbf{w}}(f(a,b))$ for any $\mathbf{w}$?

(c) Find a unit vector $\mathbf{w} = \langle w_1, w_2 \rangle$ such that $D_{\mathbf{w}}(f(a,b)) = 0$.

17. Let $f(x,y) = e^{-(x^2+y^2)}$. Find the maximum and minimum values of f on the rectangle shown below. Justify your answer.

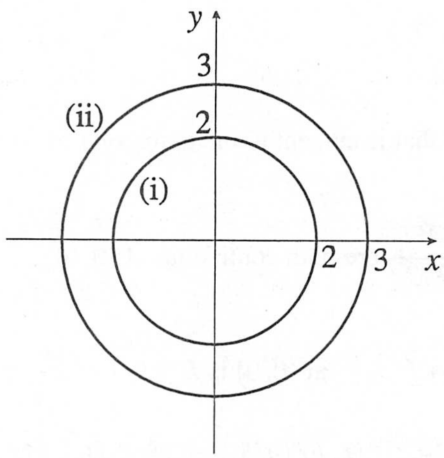


18. Which point on the surface $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$, $x, y, z > 0$ is closest to the origin?

11 Sample Exam Solutions

1. $f(x,y) = \dfrac{1}{x^2+y^2+1}$

(a) (i) $f(x,y) = \frac{1}{5} \Rightarrow 5 = x^2+y^2+1$ $\Rightarrow x^2+y^2=4$ (ii) $f(x,y) = \frac{1}{10} \Rightarrow x^2+y^2=9$



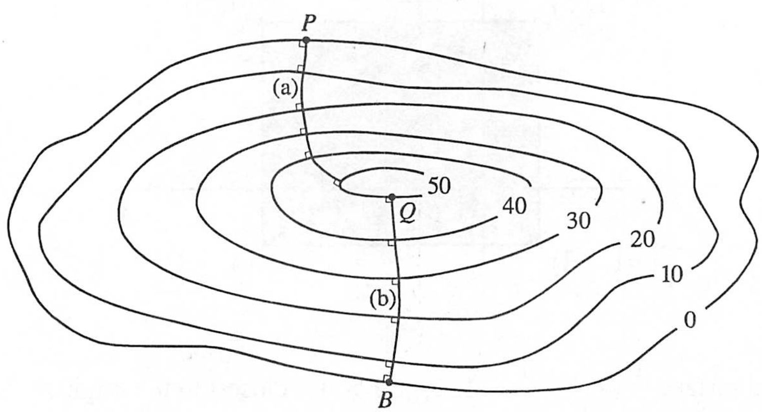
(b) $f(x,y) = 1$ consists of a single point $(0,0)$. Otherwise, $k < 1$ always gives the circle $x^2+y^2 = 1 - 1/k$.

(c) $\dfrac{1}{x^2+y^2+1} \le 1$ for any point (x,y), since $x^2+y^2+1 \ge 1$.

2. Yes. On the left-hand side we get $(2x+y)\,2y\cos(xy^2)\sin(xy^2)$ and on the right-hand side we get $(2x+y)\,2y\cos(xy^2)\,|\sin xy^2|$, so these are equal for $\sin(xy^2) \ge 0$.

3. Yes. There are many examples of such functions. One which works for all x and y is $f(x,y) = e^x + e^{-y}$, which has $f_x = e^x$ and $f_y = -e^{-y}$. A good strategy is to write $f(x,y) = g(x) + h(y)$, where $g'(x) > 0$, $h'(y) < 0$.

4.



5. $f(x,y) = x^2 + xy + y^2$ on the disk $\{(x,y) \mid x^2 + y^2 \le 9\}$.
$\nabla f(x,y) = \langle 2x + y, 2y + x \rangle = \langle 0,0 \rangle \Leftrightarrow y = -2x$ and $x = -2y \Leftrightarrow (x,y) = (0,0)$. So the minimum value on the interior of the disk is $f(0,0) = 0$.
Using Lagrange multipliers for the boundary, we solve $\nabla f = \lambda \nabla g$ where $g(x,y) = x^2 + y^2 = 9$. So $2x + y = \lambda 2x \Rightarrow \lambda = 1 + y/(2x)$ and $2y + x = \lambda 2y \Rightarrow \lambda = 1 + x/2y \Rightarrow x^2 = y^2$. But $x^2 + y^2 = 9$, so $2x^2 = 9 \Rightarrow x = \pm\frac{3}{\sqrt{2}}$ and $y = \pm\frac{3}{\sqrt{2}}$. Thus the maximum value on the boundary is $f\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = f\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = \frac{27}{2}$ and the minimum value on the boundary is $f\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = f\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = \frac{9}{2}$.
The absolute minimum value is $f(0,0) = 0$ and the absolute maximum value is $f\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = f\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = \frac{27}{2}$.

6. (a) Let $g(x,y,z) = \dfrac{x^2}{4} + \dfrac{y^2}{4} + 2z^2$, so $\nabla g = \left\langle \frac{x}{2}, \frac{y}{2}, 4z \right\rangle$ and $\nabla g\left(\sqrt{2}, \sqrt{2}, 0\right) = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$ which is normal to the surface. So the tangent plane satisfies $\dfrac{x}{\sqrt{2}} + \dfrac{y}{\sqrt{2}} = k$ and goes through $\left(\sqrt{2}, \sqrt{2}, 0\right)$. Thus $k = 1$ and the tangent plane is $\dfrac{x}{\sqrt{2}} + \dfrac{y}{\sqrt{2}} = 1$.

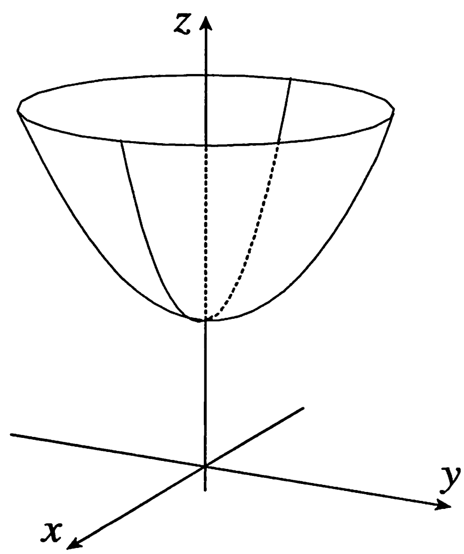
(b) Since this is a maximum value of z, the tangent plane is horizontal, that is, $z = \frac{1}{\sqrt{2}}$. Analytically, $\nabla g\left(0, 0, \frac{1}{\sqrt{2}}\right) = \langle 0, 0, 2\sqrt{2} \rangle$, so the tangent plane is $2\sqrt{2}z = 2$ or $z = \frac{1}{\sqrt{2}}$.

7. Ellipsoid for $k = -1$, single point $(0,0,0)$ for $k = 1$, no surface for $k = 2$.

8. (a) $F(x,y,z) = \dfrac{z}{1 + x^2 + y^2}$

(b) $z = 0$ is the only place where $F(x,y,z) = 0$. So there is no energy on the xy-plane.

(c) $F(x,y,z) = 1$ gives $1 = \dfrac{z}{1 + x^2 + y^2}$ or $z = 1 + x^2 + y^2$, a circular paraboloid.



9. $f(x,y) = \dfrac{x+y}{|x|+|y|}$

(a) (i) $f(1,1) = 1$

(ii) $f(1,-1) = 0$

(iii) $f(-1,1) = 0$

(iv) $f(-1,-1) = -1$

(b) No, the function does not have a limit at $(0,0)$, since if $y = -x$, then $f(x,-x) = 0$ and if $y = x$, $f(x,x) = \dfrac{x}{|x|} = \pm 1$.

10. $f(x,y) = \begin{cases} \dfrac{2x^2+3y^2}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

(a) $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{2h^2}{h} - 0}{h} = \lim\limits_{h\to 0} \dfrac{2h^2}{h^2} = \lim\limits_{h\to 0} 2 = 2$

(b) $f(0,y) = \dfrac{3y^2}{-y} = 3y = g(y)$. Then $f_y(0,0) = g'(0) = -3$.

11. A counterexample is $f(x,y) = x^2+y^2$. For this function $f_x(0,0) = f_y(0,0) = 0$; $f(0,0) = 0$ and $f(x,y) \neq 0$ for $(x,y) \neq (0,0)$.

12. $x^2+y^2+z^2 = 9$

(a) The tangent plane at $(3,0,0)$ is $x = 3$.

(b) Let $g(x,y,z) = x^2+y^2+z^2$. Then $\nabla g = \langle 2x, 2y, 2z\rangle$ and $\nabla g(2,2,1) = \langle 4,4,2\rangle$, which is normal to the surface. So the tangent plane is $4x+4y+2z = k$ and goes through $(2,2,1)$, so $k = 18$, and the tangent plane is $2x+2y+z = 9$.

13. $f(x,y) = e^{x-y}$, $f_x(x,y) = e^{x-y}$, $f_y(x,y) = -e^{x-y}$.
$L(x,y) = f(\ln 2, \ln 2) + f_x(\ln 2, \ln 2)(x - \ln 2) + f_y(\ln 2, \ln 2)(y - \ln 2)$. So the linear approximation is $f(\ln 2 + 0.1, \ln 2 + 0.04) \approx L(\ln 2 + 0.1, \ln 2 + 0.04) = 1 + 1(0.1) - 1(0.04) = 1.06$.

14. (a) $yf_x = y\left[g'(x^2+y^2)\,2x\right] = 2xyg'(x^2+y^2)$, $xf_y = x\left[g'(x^2+y^2)\,2y\right] = 2xyg'(x^2+y^2)$.

(b) The maximal increase is in the direction of $\mathbf{u} = \langle 2g'(2), 2g'(2)\rangle$, which is the same as that of $\mathbf{w} = \langle 1,1\rangle$.

15. (a) True; the partials are continuous.

(b) True (in fact the plane is $z = 0$).

(c) False; if they were continuous, then we would have $f_{xy}(0,0) = f_{yx}(0,0)$.

(d) False; the linear approximation is $L(x,y) = 0$.

16. $\mathbf{u} = \langle 1, 0 \rangle$, $\mathbf{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, $D_\mathbf{u}(f(a,b)) = 3$ and $D_\mathbf{v}(f(a,b)) = \sqrt{2}$

(a) $\nabla f(a,b) = \langle f_1, f_2 \rangle$ and $\langle f_1, f_2 \rangle \cdot \mathbf{u} = 3 \Rightarrow f_1 = 3$. $\langle f_1, f_2 \rangle \cdot \mathbf{v} = \sqrt{2} \Rightarrow \dfrac{f_1}{\sqrt{2}} + \dfrac{f_2}{\sqrt{2}} = \sqrt{2}$
$\Rightarrow \dfrac{3}{\sqrt{2}} + \dfrac{f_2}{\sqrt{2}} = \sqrt{2} \Rightarrow 3 + f_2 = 2 \Rightarrow f_2 = -1$. So $\nabla f(a,b) = \langle 3, -1 \rangle$.

(b) $D_\mathbf{w}(f(a,b))$ is maximized when $\mathbf{w}$ is in the direction of $\langle 3, -1 \rangle$. So $\mathbf{w} = \left\langle \frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}} \right\rangle$ and since $\mathbf{w} = \frac{4}{\sqrt{10}}\mathbf{u} - \frac{1}{\sqrt{5}}\mathbf{v}$, $D_\mathbf{w}(f(a,b)) = \frac{4}{\sqrt{10}} D_\mathbf{u}(f(a,b)) - \frac{1}{\sqrt{5}} D_\mathbf{v}(f(a,b)) = \frac{4}{\sqrt{10}} \cdot 3 - \frac{1}{\sqrt{5}} \cdot \sqrt{2} = \sqrt{10}$

(c) $D_\mathbf{w}(f(a,b)) = 0$ if $\mathbf{w} \cdot \langle 3, -1 \rangle = 0$, so $3w_1 - w_2 = 0$ and $w_1^2 + w_2^2 = 1$ gives $w_1^2 + 9w_1^2 = 1$, $w_1 = \frac{1}{\sqrt{10}}$ and $w_2 = \frac{3}{\sqrt{10}}$, so $\mathbf{w} = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$.

17. Since f is a function which is constant on circles $x^2 + y^2 = R$ and since f is decreasing as the radius of the circle increases, then the maximum is $f(0,0) = 1$ and the minimum is $f(4,3) = e^{-25}$.

18. Let $d^2 = x^2 + y^2 + z^2$ and minimize d^2 subject to the constraint $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$, $x, y, z > 0$. The method of Lagrange multipliers gives the point $(3, 3, 3)$.