# 12 Multiple Integrals

**Note:** This material is very difficult for students who do not remember their single variable integration principles and techniques. One possible strategy is to give the students a review session or gateway exam on integration techniques before starting Chapter 12.

## 12.1 Double Integrals over Rectangles

### Suggested Time and Emphasis

$\frac{1}{2}$–$\frac{3}{4}$ class Essential Material

### Transparencies Available

- Transparency 50 (Figures 4 and 5, page 841)
- Transparency 51 (Figures 7 and 8, pages 842–43)
- Transparency 52 (Figure 11, page 845)
- Transparency 53 (Figures 12 and 13, pages 845–46)

### Points to Stress

1. The definition of the double integral.
2. The analogy between single and double integration.
3. Volume interpretations of double integrals.
4. Average value in two dimensions.

### Text Discussion

- Compute $\sum_{i=1}^{2}\sum_{j=1}^{3} 2^i 3^j$.
- If we partition $[a, b]$ into $m$ subintervals of equal length and $[c, d]$ into $n$ subintervals of equal length, what is the value of $\Delta A$ for any subrectangle $R_{ij}$?
- For a positive function $f(x, y)$, what is a physical interpretation of the average value of $f$ over a region $R$?

### Materials for Lecture

- Briefly review a few basic properties of double sums, such as $\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}\right)$ and $\sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j = \left(\sum_{i=1}^{m} a_i\right)\left(\sum_{j=1}^{n} b_j\right)$, if needed.

- Show how double integration extends single-variable integration, including Riemann sums.
- Do a problem on numerical estimation, such as estimating $\iint (x^2 + 2y^2)\,dA$ over $0 \le x \le 2$, $0 \le y \le 2$, using both the lower left corners and midpoints as sample points. Also approximate $f_{\text{ave}}$ over $R$.
- Use a geometric argument to directly compute $\iint_R (3 + 4x)\,dA$ over $R = [0,1] \times [0,1]$.
- Discuss what happens when $f(x,y)$ takes negative values over the region of integration. Start with an odd function such as $x^3 + y^5$ being integrated over $R = [-1,1] \times [-1,1]$, and then discuss what happens when the function goes negative, but is not odd (for example $z = 2 - 3x$ on $R = [0,1] \times [0,1]$). Illustrate numerically. (This can also be done using the group work "An Odd Function".)
- Briefly discuss average value. Point out that if $f$ is continuous, then for some point $(x_0, y_0)$ in the region of integration $R$, $f_{\text{ave}} = f(x_0, y_0)$, as in the single-variable case. So $\iint_R f(x,y)\,dA = A(R) \cdot f(x_0, y_0)$, as shown in Figure 11 (page 845).

## Workshop/Discussion

- The following example can be used to help solidify the idea of approximating an area. Consider a square pyramid with vertices at $(1,1,0)$, $(1,-1,0)$, $(-1,1,0)$, $(-1,-1,0)$ and $(0,0,1)$. Derive an equation for the surface of the pyramid, using functions such as $z = 1 - \max(|x|, |y|)$, and then the equations of the planes containing each of the five faces ($z = 0$, $y + z = 1$, and so on). Approximate the volume using the Midpoint Rule for the following equal subdivisions. Note that $m$ is the number of equal subdivisions in the $x$-direction and $n$ is the number of equal subdivisions in the $y$-direction.

  **1.** $m = n = 2$ (Approximation is 2)

  **2.** $m = n = 3$ (Approximation is $\frac{44}{27}$)

  **3.** $m = n = 4$ (Approximation is $\frac{3}{2}$)

  **4.** $m = n = 5$ (Approximation is $\frac{36}{25}$)

  Compare your answers with the actual volume $\frac{4}{3}$ computed using the formula $V = \frac{1}{3}Bh$.

- Consider $\iint \sqrt{4 - y^2}\,dA$, with $R = [0,3] \times [-2,2]$. Use a geometric argument to compute the actual volume after approximating as above with $m = n = 3$ and $m = n = 4$. Show that the average value is $\frac{6\pi}{12} = \frac{\pi}{2}$, and that the point $\left(0, \sqrt{4 - \frac{1}{4}\pi^2}\right) \approx (0, 1.24)$ in $R$ satisfies $f\left(0, \sqrt{4 - \frac{1}{4}\pi^2}\right) = \frac{\pi}{2}$.

## Group Work 1: Back to the Park

This group work is similar to Example 4 (page 845) and uses the Midpoint Rule.

## Group Work 2: An Odd Function

The goal of the exercise is to estimate $\iint_{[-2,2] \times [-2,2]} (x^3 + y^5)\,dA$ numerically. There are three different problem sheets, each one suggesting a different strategy to obtain sample points for the estimation. After all the students are finished, have them compare their results. The exact value of the integral is zero, by symmetry.

## Group Work 3: Justifying Properties of Double Integrals

Put the students into groups. Have them read page 847 of the text carefully, and then have some groups try to justify Equation 7, some Equation 8, and some Equation 9 for nonnegative functions $f$ and $g$. They don't have to do a formal proof, but they should be able to justify these equations convincingly, either using sums or geometrical reasoning.

## Group Work 4: Several Ways to Compute Double Integrals

## Homework Problems

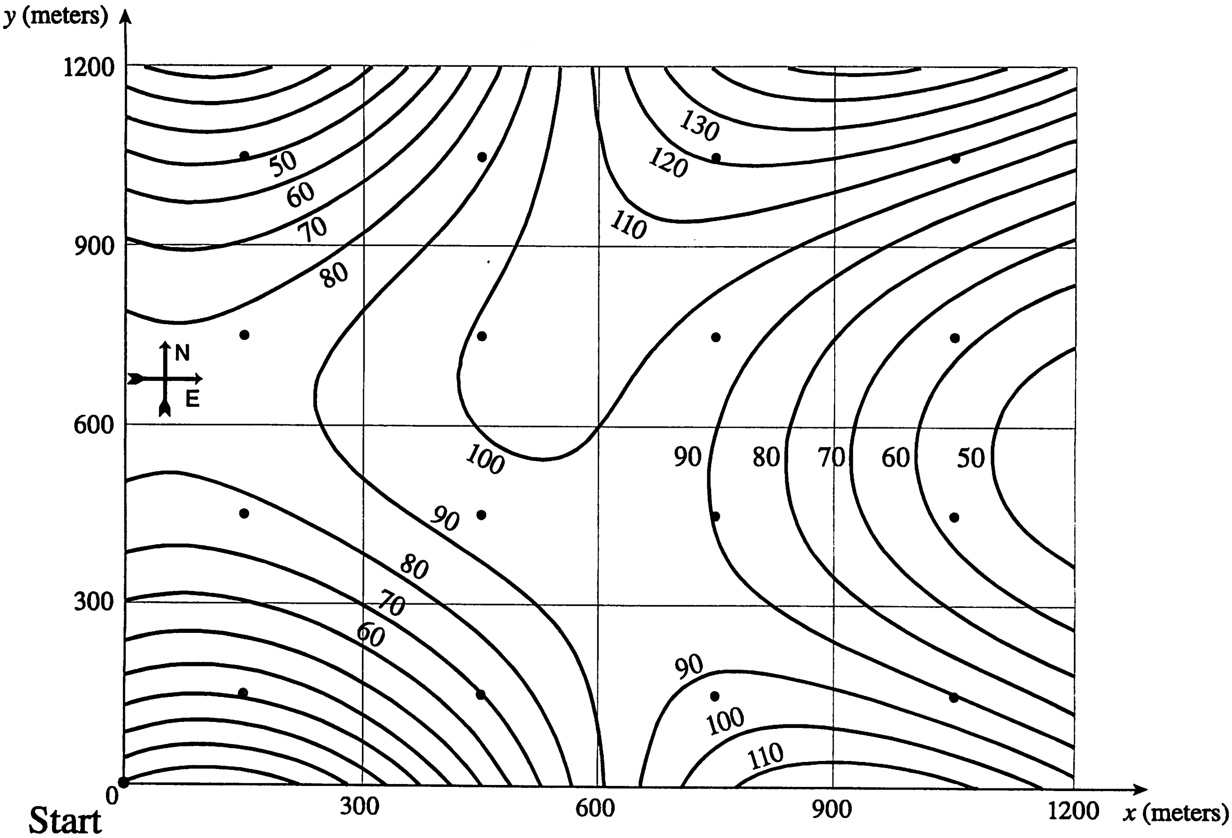
**Core Exercises:** 1, 3, 5, 6, 7, 10

**Sample Assignment:** 1, 3, 5, 6, 7, 10, 12, 17

| Exercise | C | A | N | G | V |
|---|---|---|---|---|---|
| 1 | | | × | | |
| 3 | | | × | | |
| 5 | | | × | | |
| 6 | | | × | | |
| 7 | | | × | | |
| 10 | | | × | × | |
| 12 | | | × | | × |
| 17 | | × | | | |

## Group Work 1, Section 12.1
## Back to the Park

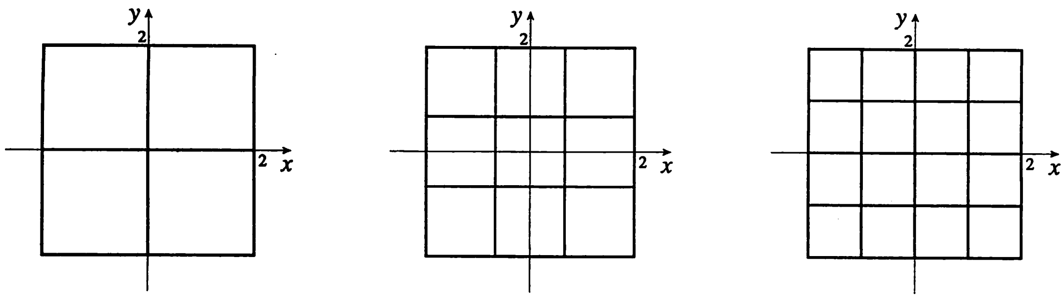
The following is a map with curves of the same elevation of a region in Orangerock National Park:



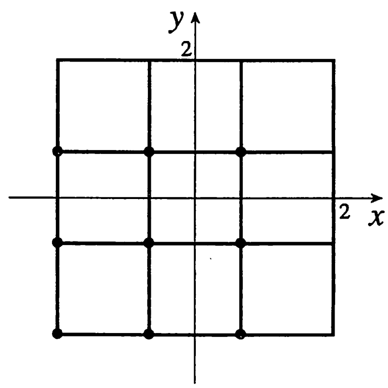

Estimate (numerically) the average elevation over this region using the Midpoint Rule.

## Group Work 2, Section 12.1
## An Odd Function (Version 1)

In this exercise, we are going to try to approximate the double integral $\iint_{[-2,2]\times[-2,2]} (x^3 + y^5)\, dA$. We start by partitioning the region $[-2, 2] \times [-2, 2]$ into four smaller regions, then nine, then sixteen, like this:




Now we approximate the double integral as discussed in the text, picking one point in every smaller region. To make things simple, just choose the lower left corner of every small region, like so:




Approximation for four regions: ______________________

Approximation for nine regions: ______________________

Approximation for sixteen regions: ______________________

When you are finished, try again using a point of your choice in each region.

Using Arbitrarily Chosen Points

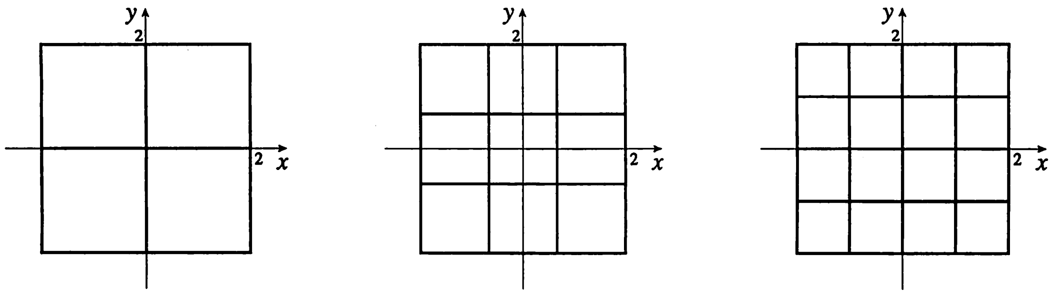
Approximation for four regions: ______________________

Approximation for nine regions: ______________________

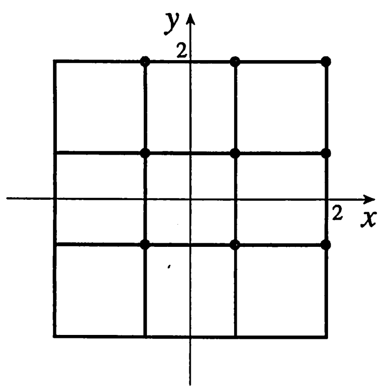
Approximation for sixteen regions: ______________________

## Group Work 2, Section 12.1
## An Odd Function (Version 2)

In this exercise, we are going to try to approximate the double integral $\iint_{[-2,2]\times[-2,2]} (x^3 + y^5)\, dA$. We start by partitioning the region $[-2, 2] \times [-2, 2]$ into four smaller regions, then nine, then sixteen, like this:




Now we approximate the double integral as discussed in the text, picking one point in every smaller region. To make things simple, just choose the upper right corner of every small region, like so:




Approximation for four regions: ____________________

Approximation for nine regions: ____________________

Approximation for sixteen regions: ____________________

When you are finished, try again using a point of your choice in each region.

Using Arbitrarily Chosen Points

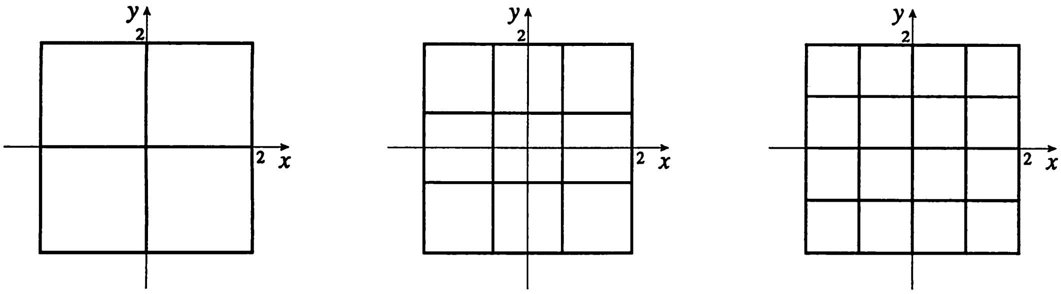
Approximation for four regions: ____________________

Approximation for nine regions: ____________________

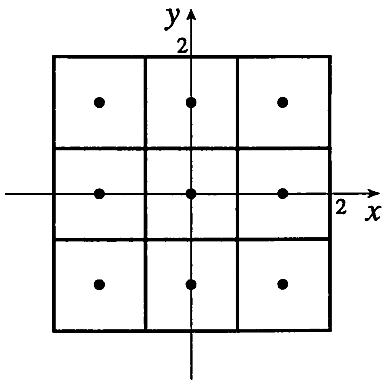
Approximation for sixteen regions: ____________________

## Group Work 2, Section 12.1
## An Odd Function (Version 3)

In this exercise, we are going to try to approximate the double integral $\iint_{[-2,2]\times[-2,2]} (x^3 + y^5)\, dA$. We start by partitioning the region $[-2,2] \times [-2,2]$ into four smaller regions, then nine, then sixteen, like this:




Now we approximate the double integral as discussed in the text, picking one point in every smaller region. To make things simple, just choose the midpoint of every small region, like so:




Approximation for four regions: ____________________

Approximation for nine regions: ____________________

Approximation for sixteen regions: ____________________

When you are finished, try again using a point of your choice in each region.

Using Arbitrarily Chosen Points

Approximation for four regions: ____________________

Approximation for nine regions: ____________________

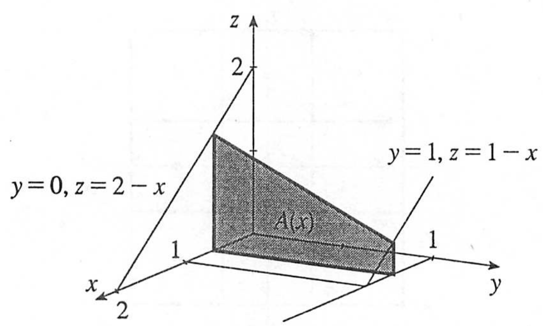
Approximation for sixteen regions: ____________________

## Group Work 4, Section 12.1
## Several Ways to Compute Double Integrals

Consider the double integral $\iint (2 - x - y)\, dA$, where $R = [0, 1] \times [0, 1]$.

**1.** Estimate the value of the double integral, first using two equal subdivisions in each direction, then three.

**2.** Fix $x$ such that $0 \le x \le 1$. What is the area $A(x)$ of the slice shown below?




**3.** Find the exact volume of the solid with cross-sectional area $A(x)$ using single variable calculus.

# 12.2 Iterated Integrals

## Suggested Time and Emphasis

$\frac{3}{4}$–1 class  Essential Material

## Points to Stress

1. The meaning of $\int_a^b \int_c^d f(x,y)\,dy\,dx$ for a positive function $f(x,y)$ over a rectangle $[a,b] \times [c,d]$.
2. The geometric meaning of Fubini's Theorem: slicing the area in two different ways.
3. The statement of Fubini's Theorem and how it makes computations easier.

## Text Discussion

- Consider Figures 1 and 2 in the text. Why is $\int_a^b A(x)\,dx = \int_c^d A(y)\,dy$?

## Materials for Lecture

- Use an alternate approach to give an intuitive idea of why Fubini's Theorem is true. Using equal intervals of length $\Delta x$ and $\Delta y$ in each direction and choosing the lower left corner in each rectangle, we can write the double sum in Definition 12.1.5 (page 841) as the iterated sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta x\,\Delta y = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta x\right)\Delta y$$

  which in the limit gives an iterated integral. Go through some examples such as $\iint_{[-1,3]\times[-1,3]} xy\,dA$ and $\iint_{[0,1]\times[0,1]} (2-x-y)\,dA$ to demonstrate this approach.
- Revisit the example $\iint_{[-2,2]\times[-2,2]} \left(x^3+y^5\right)dA$ using integrated integrals, to illustrate the power of the technique of iteration.
- Illustrate that for rectangles, $\iint_R f(x)\,g(y)\,dA = \int_a^b f(x)\,dx \int_c^d g(y)\,dy$ using $\dfrac{x}{y}e^{x^2}$ over $[0,1]\times[1,2]$. Stress that this does *not* mean that $\iint_R f(x,y)\,g(x,y)\,dA = \iint_R f(x,y)\,dA \iint_R g(x,y)\,dA$.

## Workshop/Discussion

- Remind the students how, in single-variable calculus, volumes were found by adding up cross-sectional areas. Take the half-cylinder $f(x,y) = \sqrt{1-y^2}$, $0 \le x \le 2$, and find its volume, first by the "old" method, then by expressing it as a double integral. Show how the two techniques are, in essence, the same.
- Have the students work several examples, such as $\iint_{[0,1]\times[0,1]} y\sqrt{1-x^2}\sin\left(2\pi y^2\right)dA$, which can be computed as $\int_0^1 f(x)\,dx \int_0^1 g(y)\,dy$.
- Find the volume of the solids described by $\iint_{[-\sqrt{2},\sqrt{2}]\times[-2,3]} \left(2-x^2\right)dA$ and $\iint_{[-1,1]\times[-1,1]} \left(1+x^2+y^2\right)dA$.

## Group Work 1: Regional Differences

If the students get stuck on this one, give them the hint that Problem 1(b) can be done by finding the double integral over the square, and then using the symmetry of the function to compute the area over $R$. The regions in the remaining problems can be broken into rectangles.

## Group Work 2: Practice with Double Integrals

It is a good idea to give the students some practice with straightforward computations of the type included in Problem 1. It is advised to put the students in pairs or have them work individually, as opposed to putting them in larger groups. The students are not to actually compute the integral in Problem 2; rather, they should recognize that each slice integrates to zero. Think about what happens when integrating with respect to $x$ first.

## Group Work 3: The Shape of the Solid

In Problem 3, the students could first change the order of integration to more easily recognize the "pup tent" shape of the resulting solid.

## Homework Problems

**Core Exercises:** 2, 5, 10, 16, 17, 25

**Sample Assignment:** 2, 5, 7, 10, 15, 16, 17, 20, 25, 31

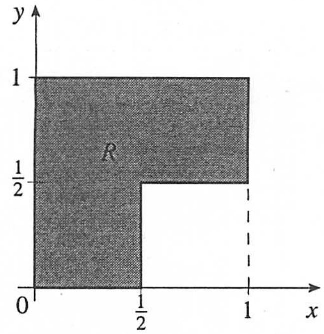
**Note:**
- Exercise 31 requires a CAS.
- Problem 5 from Focus on Problem Solving (page 914) can be assigned as an optional project.

| Exercise | C | A | N | G | V |
|---|---|---|---|---|---|
| 2 | | × | | | |
| 3–10 | | × | | | |
| 15 | | × | | | |
| 16 | | × | | | |
| 17 | | | | | × |
| 20 | | × | | | × |
| 25 | | × | | | × |
| 31 | × | | | | |

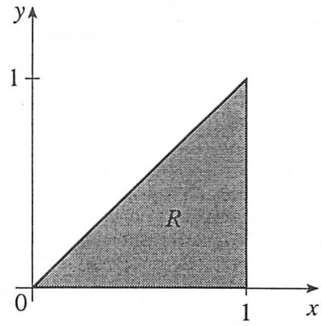
## Group Work 1, Section 12.2
## Regional Differences

1. Calculate the double integral $\iint_R (x+y)\,dA$ for the following regions $R$:

(a)

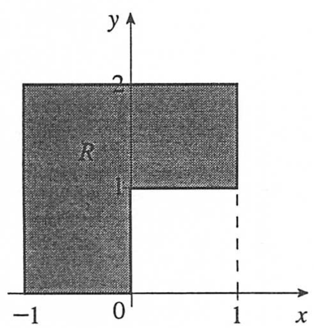



(b)

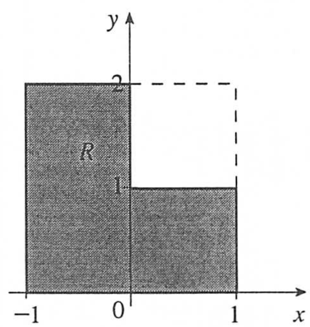



2. Calculate the double integral $\iint_R (xy - y^3)\,dA$ for the following regions $R$.

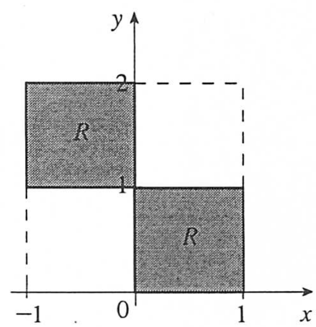
(a)




(b)




(c)



## Group Work 2, Section 12.2
## Practice with Double Integrals

Compute the following double integrals:

**1.** (a) $\displaystyle\iint_{[1,2]\times[0,1]} x\sqrt{1+y+x^2}\,dA$

(b) $\displaystyle\iint_{[0,1]\times[1,2]} \frac{x}{x+y}\,dx\,dy$

(c) $\displaystyle\iint_{[0,1]\times[1,2]} ye^{xy}\,dx\,dy$

**2.** Is the statement $\iint_{[0,1]\times[0,1]} \cos\left(2\pi\left(y^2+x\right)\right)dA = 0$ true or false?

# Group Work 3, Section 12.2
## The Shape of the Solid

For each of the following integrals, describe the shape of the solid whose volume is given by the integral, then compute the volume.

**1.** $\iint_{[0,1] \times [0,1]} (3 - 2x - y)\, dA$

**2.** $\int_{-3}^{3} \int_{-2}^{2} \sqrt{4 - y^2}\, dy\, dx$

**3.** $\int_{-1}^{1} \int_{-2}^{3} (1 - |x|)\, dy\, dx$

# 12.3 Double Integrals over General Regions

## Suggested Time and Emphasis

1 class    Essential Material

## Points to Stress

**1.** The geometric interpretation of $\int_a^b \int_{f(y)}^{g(y)} dx\,dy$ and $\int_c^d \int_{h(x)}^{k(x)} dy\,dx$.

**2.** Setting up the limits of double integrals, given a region over which to integrate.

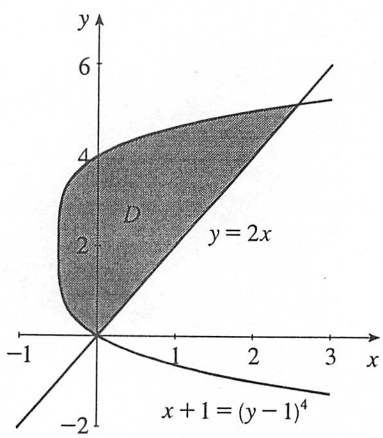
**3.** Changing the order of integration.

## Text Discussion

- Why is the type I region illustrated in Figure 8 (page 856) not considered type II?
- Sketch a region that is type II and not type I, and then sketch one that is both type II and type I.
- Is it true that $\int_0^1 \int_x^1 f(x,y)\,dy\,dx = \int_0^1 \int_y^1 f(x,y)\,dx\,dy$?

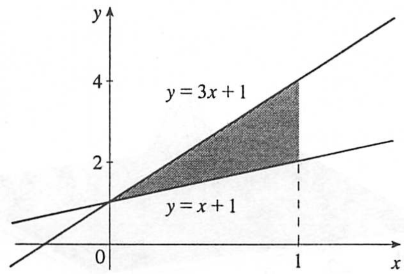
## Materials for Lecture

- Clarify why we bother with the fuss of defining $F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D \end{cases}$ instead of "just integrating over $D$".
- Show how the area under a curve in the $xy$-plane can be thought of as the double integral $\int_a^b \int_0^{f(x)} dy\,dx$.
- Let $D$ is the region shown below. Set up $\iint_D f(x,y)\,dA$ both as a type I integral $\left[\iint_D f(x,y)\,dx\,dy\right]$ and a type II integral $\left[\iint_D f(x,y)\,dy\,dx\right]$.




- Discuss Example 4, emphasizing that $z = 2 - x - 2y$ is the *height* of the solid being described. Show how to integrate with respect to $x$ first, noting that the answer is still $\frac{1}{3}$.

- Show that the order of integration matters when computing $\iint e^{x^2}\,dA$, where $D$ is the region shown below.




- Show how to change order of integration in $\int_0^1 \int_{x^4}^{x^{1/3}} f(x, y)\,dy\,dx$.

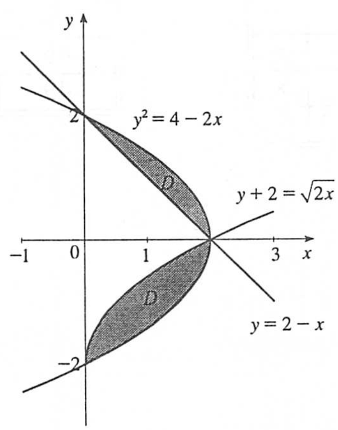
## Workshop/Discussion

- Point out that $\iint_D 1\,dA$ gives the area of $D$, and that $\int_a^b \int_{f(x)}^{g(x)} 1\,dy\,dx$ gives the usual formula for the area between curves.

- Evaluate $\iint_D yx^2\,dA$, where $D$ is the unit circle, both as a type I integral and as a type II integral.

- Evaluate $\iint_D h\,dA$, where $D$ is a circle of radius $r$ and $h$ is constant. Show how this gives the general formula for the volume of a cylinder. Ask the students to evaluate $\iint_D h\,dA$, where $D$ is the parallelogram with vertices $(1, 1)$, $(2, 3)$, $(5, 1)$, and $(6, 3)$. Have them interpret their answer in terms of volume.

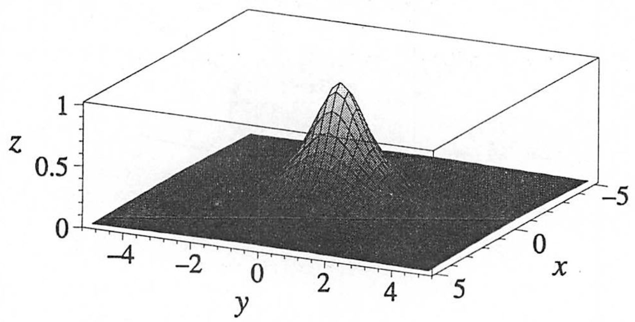
- Evaluate $\iint_D x\,dA$, where $D$ is the region shown below.




- Change order of integration for $\int_0^1 \int_{(2/\pi)\arcsin y}^{y^{1/3}} f(x, y)\,dx\,dy$.

- Find a good upper bound for $\displaystyle\iint_D \frac{1}{x^2 + y^2 + 1}\,dA$ where $D$ is $[-5, 5] \times [-5, 5]$. Perhaps show that $\dfrac{100}{51}$

is a lower bound, since $\dfrac{1}{x^2+y^2+1} \geq \dfrac{1}{51}$ on $D$. The graph of $\dfrac{1}{x^2+y^2+1}$ is given below.




## Group Work 1: Type I or Type II?

Before handing out this activity, remind the students of the definitions of type I and type II regions given in the text.

## Group Work 2: Fun with Double Integrals

## Group Work 3: Writing a Quarter-Annulus as One or More Iterated Integrals

Problem 2 can be done using Problem 1 and symmetry.

## Group Work 4: Bounding on a Disk

## Homework Problems

**Core Exercises:** 2, 8, 19, 22, 30, 35

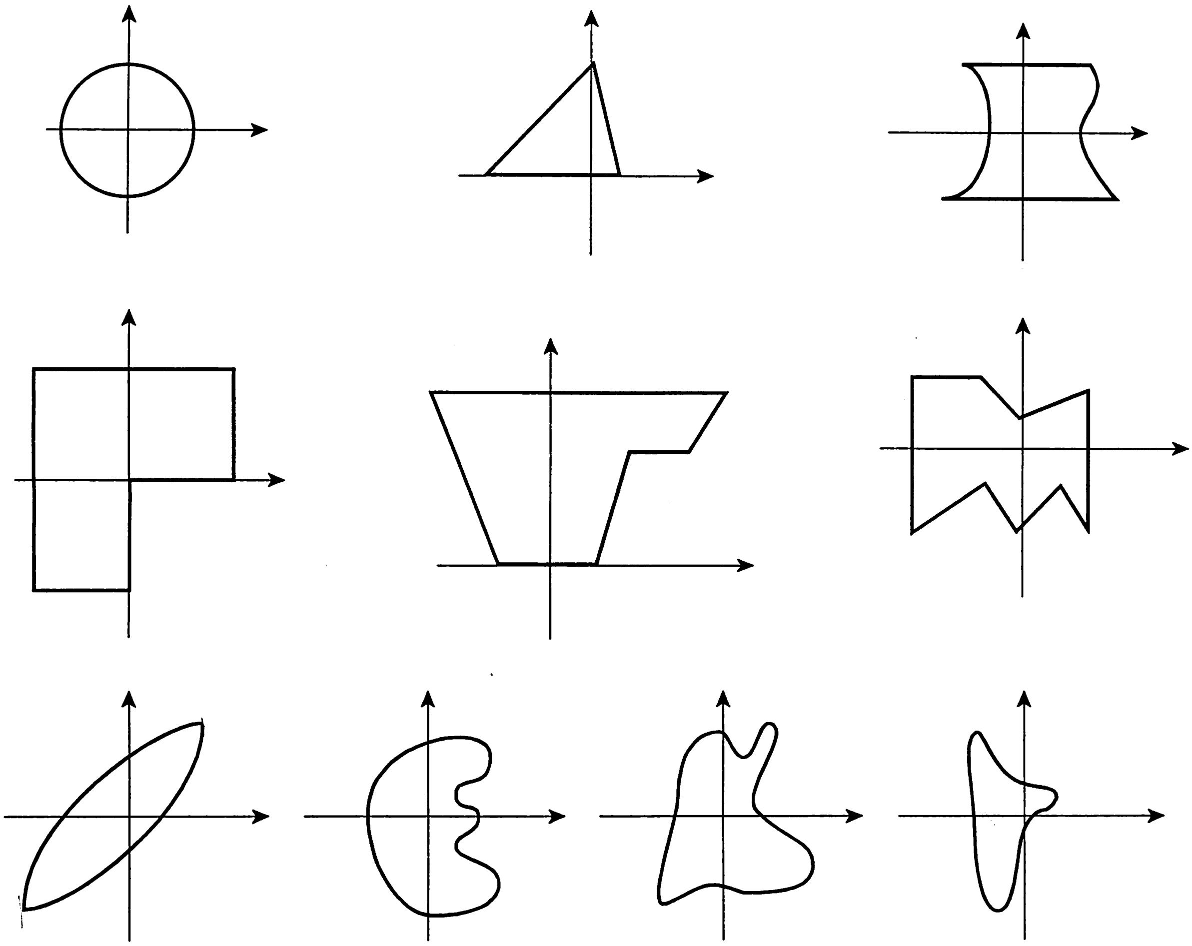
**Sample Assignment:** 2, 5, 8, 11, 15, 19, 22, 24, 30, 33, 35, 38, 44, 46, 49

| Exercise | C | A | N | G | V |
|---|---|---|---|---|---|
| 2 | | × | | | |
| 5 | | × | | | |
| 7–16 | | × | | | |
| 17–24 | | × | | | × |
| 30 | | × | | | × |
| 33 | | × | | | × |

| Exercise | C | A | N | G | V |
|---|---|---|---|---|---|
| 35 | | × | | | |
| 38 | | × | | | |
| 44 | | | × | | |
| 46 | | × | | | × |
| 49 | | × | | | × |

## Group Work 1, Section 12.3
## Type I or Type II?

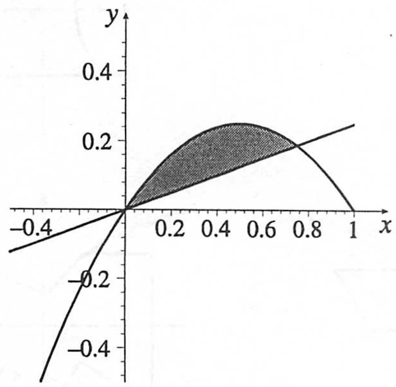
Classify each of the following regions as type I, type II, both, or neither.



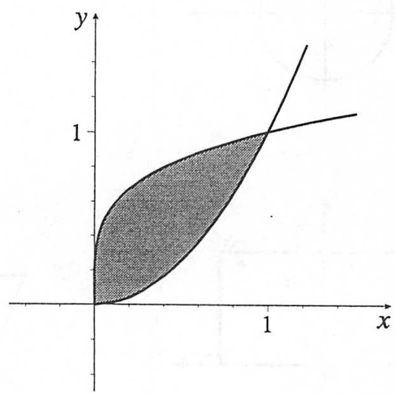
## Group Work 2, Section 12.3
## Fun with Double Integration

**1.** Write double integrals that represent the following areas.

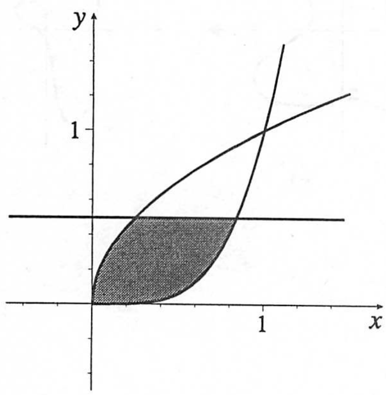
(a) The area enclosed by the curve $y = x - x^2$ and the line $y = \dfrac{x}{4}$



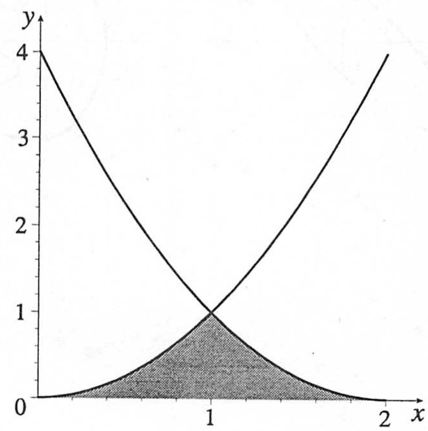

(b) The area enclosed by the curves $y = \sqrt[4]{x}$ and $\sqrt{y} = x$




(c) The area enclosed by the curves $y = \sqrt{x}$ and $\sqrt[4]{y} = x$, and the line $y = \frac{1}{2}$




(d) The area enclosed by the curves $y = x^2$ and $y = (x-2)^2$, and the line $y = 0$

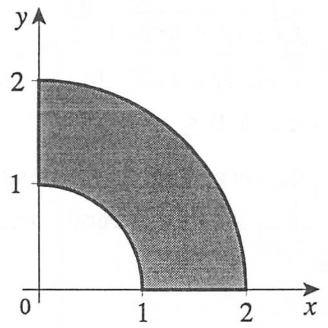



**2.** What solid region of $\mathbb{R}^3$ do you think is represented by $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dy\,dx$?

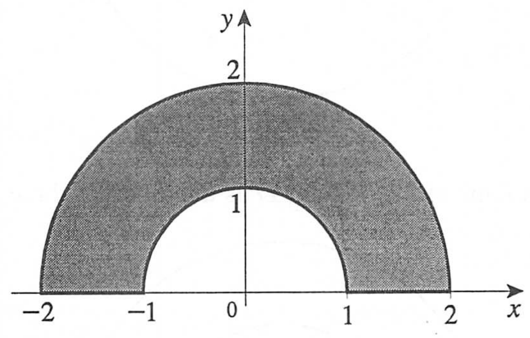
# Group Work 3, Section 12.3

## Writing a Quarter-Annulus as One or More Iterated Integrals

**1.** Express $\iint_D \sqrt{x^2 + y^2}\, dA$ using iterated integrals, where $D$ is given by the region sketched below.




**2.** Express $\iint_D \sqrt{x^2 + y^2}\, dA$ using iterated integrals, where $D$ is the region sketched below.



## Group Work 4, Section 12.3
## Bounding on a Disk

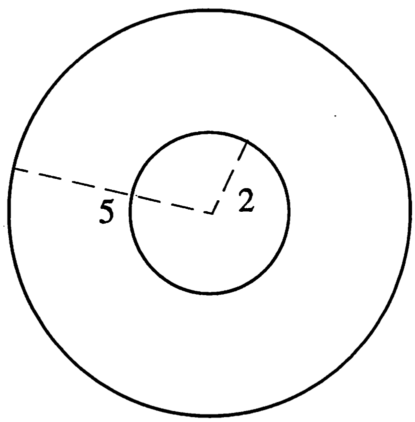
One integral that is very important in Nentebular science is the Brak integral:

$$\iint_D \frac{1}{x^2 + y^2 + 1}\, dA$$

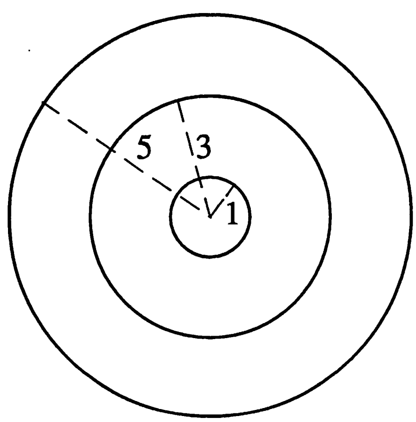
For these sorts of integrals the domain $D$ is usually a disk. In this exercise, we are going to find upper and lower bounds for this integral, where $D$ is the disk $0 \leq x^2 + y^2 \leq 25$.

**1.** It is possible to get some crude upper and lower bounds for this integral over $D$ without any significant calculations? Find upper and lower bounds for this integral (perhaps crude ones) and explain how you know for sure that they are true bounds.

**2.** One can get a better estimate by splitting up the domain as shown in the graph below, and bounding the integral over the inside disk and then over the outside ring. Using this method, what are the best bounds you can come up with?




**3.** Now refine your bounds by looking at the domain $D$ as the union of three domains as shown.



# 12.4 Double Integrals in Polar Coordinates

## Suggested Time and Emphasis

1 class    Essential Material

**Note:** If polar coordinates have not yet been covered, the students should read Appendix H.1, pages A58–A65.

## Points to Stress

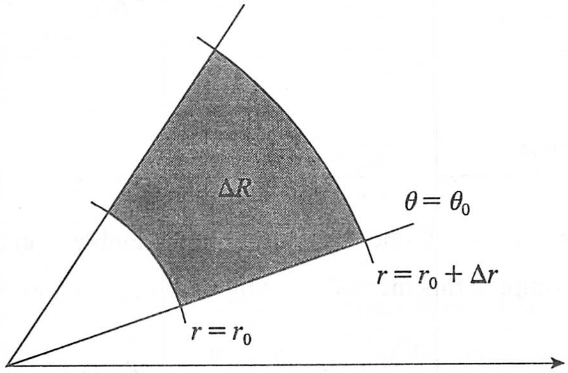
1. The definition of a polar rectangle: what it looks like, and its differential area $r\,dr\,d\theta$
2. The idea that some integrals are simpler to compute in polar coordinates
3. Integration over general polar regions

## Text Discussion

- Is the area of a polar rectangle $R = \{(r, \theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$ equal to $(b - a)(\beta - \alpha)$?
- In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. So in the formula $\iint_R f(x, y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$, where does that extra $r$ come from?

## Materials for Lecture

- Start by reminding the students of polar coordinates, and ask them what they think the polar area formula will be: what will replace $dx\,dy$?
- Draw a large picture of a polar rectangle, and emphasize that it is the region between the gridlines $r = r_0$, $r = r_0 + \Delta r$, $\theta = \theta_0$, and $\theta = \theta_0 + \Delta\theta$.




The following method can be used to show that Area$(\Delta R) \approx r\,\Delta r\,\Delta\theta$ if $\Delta r$ and $\Delta\theta$ are small:

1. Using the formula for the area of a circular sector, we obtain

$$\begin{aligned}\text{Area}(\Delta R) &= \tfrac{1}{2}(r + \Delta r)^2\,\Delta\theta - \tfrac{1}{2}(r)^2\,\Delta\theta \\ &= r\,\Delta r\,\Delta\theta + \tfrac{1}{2}(\Delta r)^2\,\Delta\theta\end{aligned}$$

**2.** We now take the limit as $\Delta r, \Delta\theta \to 0$, of $\dfrac{\text{Area}(\Delta R)}{r\,\Delta r\,\Delta\theta}$ and show that this limit is equal to 1.

$$\lim_{\Delta r\to 0,\,\Delta\theta\to 0} \frac{\text{Area}(\Delta R)}{r\,\Delta r\,\Delta\theta} = \lim_{\Delta r\to 0,\,\Delta\theta\to 0} \frac{r\,\Delta r\,\Delta\theta + \frac{1}{2}(\Delta r)^2\,\Delta\theta}{r\,\Delta r\,\Delta\theta}$$
$$= \lim_{\Delta r\to 0,\,\Delta\theta\to 0} 1 + \frac{\Delta r}{2r} = 1$$

**3.** The result follows, since when $\Delta r$ and $\Delta\theta$ are small,

$$\frac{\text{Area}(\Delta R)}{r\,\Delta r\,\Delta\theta} \approx 1$$

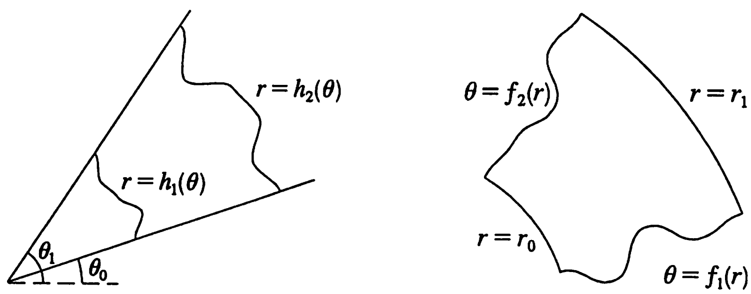
or

$$\text{Area}(\Delta R) \approx r\,\Delta r\,\Delta\theta$$

- If the quarter-annulus group activity was done in the previous section, point out that the integration problem posed can now be solved using one simple integral with the methods from this section.

- Indicate to the students that polar coordinates are most useful when one has an obvious center of symmetry for the region $R$ in the $xy$-plane.

- Show how to set up the two general types of polar regions. (The second type occurs less frequently, and may be omitted.)




## Workshop/Discussion

- Point out that polar areas can be found by setting up (double) polar integrals of the function $f(r\cos\theta, r\sin\theta) = 1$ and compare this method to using the formula $A = \int_a^b \frac{1}{2}r^2\,d\theta$. Calculate

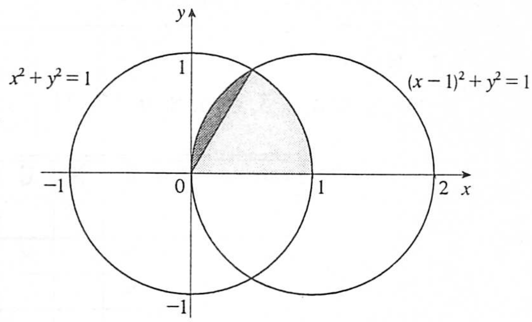
  1. The area between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$ in the second quadrant.

  2. The area inside the spiral $r = \theta$ where $0 \le \theta \le \pi$.

  3. $A = \left\{(x, y) \mid -1 \le x \le 1, 0 \le y \le \sqrt{1 - x^2}\right\}$.

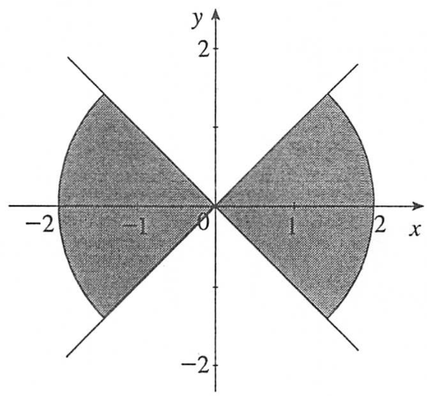
  4. The area inside the first loop of the curve $r = 2\sin\theta\cos\theta$.

- A more difficult problem is to find the area inside the circles $x^2+y^2=1$ and $(x-1)^2+y^2=1$ (or $r=2\cos\theta$). This involves finding the point of intersection [which is $\left(1,\frac{\pi}{3}\right)$ in polar coordinates] and then breaking the double integral for the portion in the first quadrant into 2 pieces, one of which is just the area of the sector of angle $\frac{\pi}{3}$ of the unit disk.




- Give some different uses of polar integrals:

**1.** Compute $\iint_R \left(x^2+y^2\right)^2 \, dA$, where $R$ is the region enclosed by $x^2+y^2=4$ between the lines $y=x$ and $y=-x$.




**2.** Compute the volume between the cone $z^2=x^2+y^2$ and the paraboloid $z=4-x^2-y^2$.

## Group Work 1: The Polar Area Formula

## Group Work 2: Fun with Polar Area

Many students will write $\int_0^{3\pi}\int_0^{\theta} r \, dr \, d\theta$ for Problem 1(c). Try to get them to see for themselves that they must now subtract the area of the region that is counted twice in this expression. Note that in Problem 2 they are computing the volume of a hemisphere.

## Group Work 3: Fun with Polar Volume

If the students get stuck on Problem 2, point out that the intersection can be shown to be $x^2+y^2=\frac{3}{4}$, $z=\frac{1}{2}$, and hence the integral is over the region $x^2+y^2 \le \frac{3}{4}$. Problem 3 is challenging.

## Homework Problems

**Core Exercises:** 3, 4, 8, 17, 23, 29

**Sample Assignment:** 1, 3, 4, 6, 8, 9, 17, 21, 23, 29, 30, 32

| Exercise | C | A | N | G | V |
|---|---|---|---|---|---|
| 1–6 | | × | | | × |
| 8 | | × | | × | |
| 9 | | × | | | |
| 17 | | × | | | |
| 21 | | × | | | |

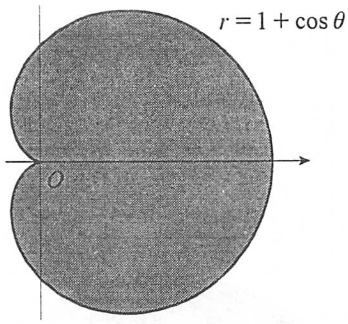
| Exercise | C | A | N | G | V |
|---|---|---|---|---|---|
| 23 | | × | | | |
| 29 | | × | | | |
| 30 | | × | | | |
| 32 | | × | | | |

## Group Work 1, Section 12.4
## The Polar Area Formula

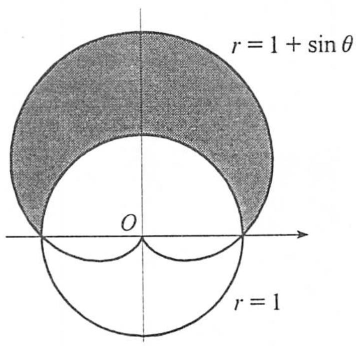
The following formula, used for finding the area of a polar region described by the polar curve $r = f(\theta)$, $a \le \theta \le b$, can be found in Appendix H.2, page A69:

$$A = \int_a^b \tfrac{1}{2} [f(\theta)]^2 \, d\theta$$

1. Use this formula to compute the area inside the cardioid $r = 1 + \cos\theta$.




2. Use the formula to find the area inside the cardioid $r = 1 + \sin\theta$ and outside the unit circle $r = 1$.

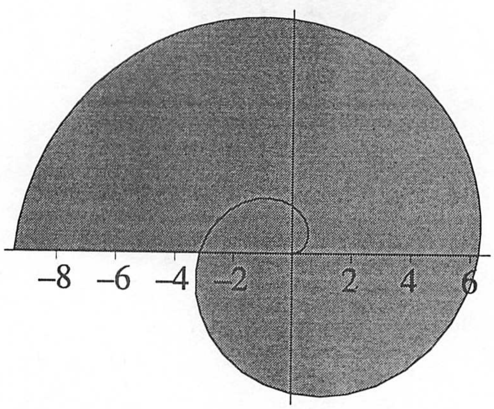


## Group Work 2, Section 12.4
## Fun with Polar Area

1. Sketch the following polar regions, and find their area:

(a) The region inside $r = 3\cos\theta$ and outside $\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$

(b) The region inside the curve $r = \theta$, $0 \le \theta \le 3\pi$



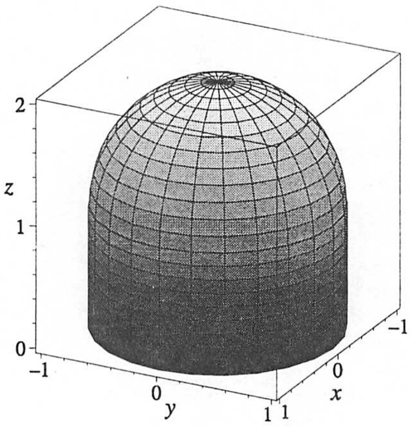

(c) The region between $\theta = \sqrt{2\pi r}$ and $\theta = r^2$ with $0 \le r \le \sqrt{2\pi}$

**2.** Compute $\iint_R f(x, y)\, dA$ if $f$ is the positive-valued function given implicitly by $x^2 + y^2 + z^2 = 4$ and $R$ is the region inside the circle $x^2 + y^2 = 4$.

**3.** Rewrite $\int_{-2}^{2} \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\, dy\, dx$ as a polar integral and evaluate it.

## Group Work 3, Section 12.4
## Fun with Polar Volume

**1.** Find the volume of the region bounded above by the upper hemisphere of the sphere $x^2+y^2+(z-1)^2=1$ and bounded below by the $xy$-plane.




**2.** Find the volume of the region bounded above by the sphere $x^2+y^2+z^2=1$ and below by the sphere $x^2+y^2+(z-1)^2=1$.

**3.** Find the volume of the ellipsoid $\dfrac{x^2}{9}+\dfrac{y^2}{9}+z^2=1$.

# 12.5 Applications of Double Integrals

## Suggested Time and Emphasis

$\frac{3}{4}$–1 class  Optional material

## Points to Stress

We recommend stressing only one of the following topics:

1. Density, mass, and centers of mass (for an engineering- or physics-oriented course)

2. Probability and expected values (for a course oriented toward biology or the social sciences)

## Text Discussion

- If a lamina has a uniform density, and an axis of symmetry, what information do we then have about the location of the center of mass?

- What is a logical reason that the total area under a joint density curve should be equal to one?

## Materials for Lecture

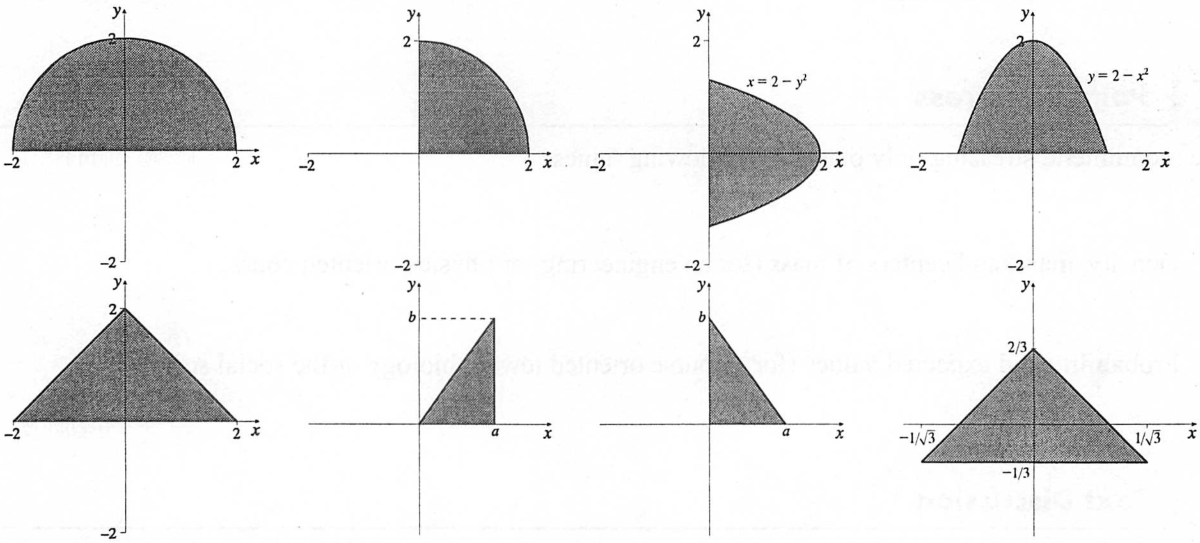
- Describe the general ideas behind continuous density functions, computations of mass, and centers of mass.

- Do one interesting mass problem. A good exercise is the mass over the unit disk if $\rho(x, y) = |x| + |y|$. This reduces to $4 \int_0^1 \int_0^{\sqrt{1-x^2}} (x + y)\, dy\, dx$ which, surprisingly, equals $\frac{8}{3}$.

- Describe the general idea of the joint density function of two variables. Similarly, describe the concept of expected value. If time permits, show that $f(x, y) = \dfrac{1}{2\pi} \dfrac{1}{(1 + x^2 + y^2)^{3/2}}$ describes a joint density function.

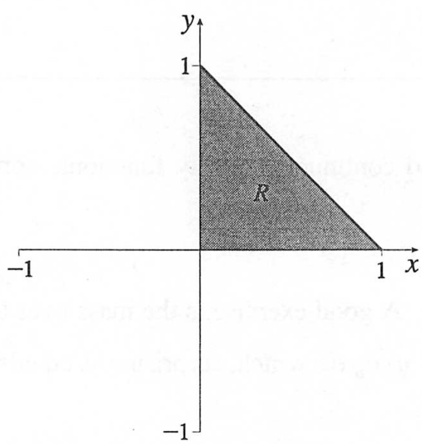
## Workshop/Discussion

- Define the centroid $(\overline{x}, \overline{y})$ of a plane region $R$ as the center of gravity, obtained by using a density of 1 for the entire region. So $\overline{x} = \dfrac{1}{A(R)} \iiint x\,dx\,dy$ and $\overline{y} = \dfrac{1}{A(R)} \iiint y\,dx\,dy$. Show the students how to find the centroid for two or three figures like the following:




Show the students that if $x = 0$ is an axis of symmetry for a region $R$, then $\overline{x} = 0$, and more generally, that $(\overline{x}, \overline{y})$ is on the axis of symmetry. Point out that if there are two axes of symmetry, then the centroid $(\overline{x}, \overline{y})$ is at their intersection.

- Consider the triangular region $R$ shown below, and assume that the density of an object with shape $R$ is proportional to the square of the distance to the origin. Set up and evaluate the mass integral for such an object, and then compute the center of mass $(\overline{x}, \overline{y})$.




## Group Work 1: Fun with Centroids

Have the students find the centroids for some of the eight regions described earlier in Workshop/Discussion.

## Group Work 2: Generating the Bivariate Normal Distribution

Go over, in detail, Exercise 32 from Section 12.4 (page 868), which involves computing the integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ using its double integral counterpart. Since these functions are related to the normal distribution and the bivariate normal distribution, the students can then actually show that $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x/\mu)^2/(2\sigma)} = 1$, as it should be. Show that this is also true for $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-((x-a)/\mu)^2/(2\sigma)}$ by noting that replacing $x$ by $x - a$ just corresponds to a horizontal shift of the integrand.

## Group Work 3: A Slick Model

This activity requires a CAS and is based on the results of Group Work 2.

## Homework Problems

**Core Exercises:** 1, 5, 10

**Sample Assignment:** 1, 5, 6, 9, 10, 14, 20, 24

| Exercise | C | A | N | G | V |
|---|---|---|---|---|---|
| 1 | | × | | | |
| 5 | | × | | | |
| 6 | | × | | | |
| 9 | | × | | | |

| Exercise | C | A | N | G | V |
|---|---|---|---|---|---|
| 10 | | × | | | |
| 14 | | × | | | |
| 20 | | × | | | |
| 24 | × | × | | | |

## Group Work 3, Section 12.5
## A Slick Model

An oil tanker has leaked its entire cargo of oil into the middle of the Pacific Ocean, far from any island or continent. The oil has spread out in all directions in a thin layer on the surface of the ocean. The slick can be modeled by the two-dimensional density function $K \exp\left(-2\frac{x^2+y^2}{w^2}\right)$, where $w$ is a fixed constant and the origin of the $xy$-plane represents the location of the tanker. Assuming that none of the oil evaporates, the density function must account for all of the oil and hence can be interpreted as a probability distribution.

**1.** Suppose $w = 2$. Find the value of $K$ which ensures that $K \exp\left(-2\frac{x^2+y^2}{w^2}\right)$ is a probability distribution.

**2.** Find the expected values $\mu_x$ and $\mu_y$ of $x$ and $y$ in the probability distribution from Problem 1. Interpret your answer geometrically.

**3.** Find the radius of the circle centered at the origin which contains exactly 99% of the oil in the slick.

## 12.6 Surface Area

### Suggested Time and Emphasis

$\frac{3}{4}$–1 class Recommended material, particularly if Stokes' Theorem and the Divergence Theorem are to be covered. However, it can be deferred to just before Section 13.6.

### Points to Stress

1. Review of parametric surfaces and their "partial derivatives" $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ along grid curves $u = u_0$, $v = v_0$ (Sections 10.5 and 11.4).

2. Computation of the area element $\left|\frac{\partial r}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right|$ of a parametric rectangle.

### Text Discussion

- Why does the area of the parallelogram determined by vectors $\Delta u\, \mathbf{r}_u^*$ and $\Delta v\, \mathbf{r}_v^*$ involve the cross product $\mathbf{r}_u^* \times \mathbf{r}_v^*$?

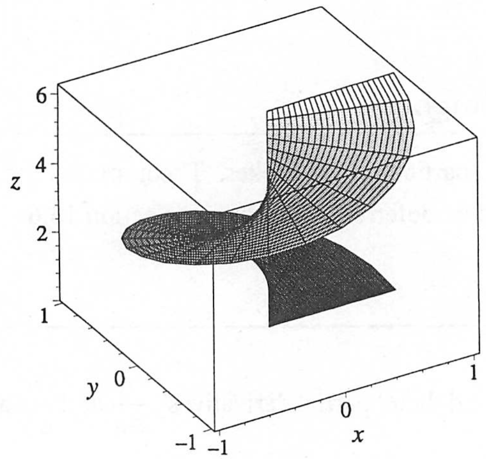
### Materials for Lecture

- Give an intuitive presentation developing the area element $\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy$ for the surface area of $z = f(x, y)$ as follows: If we have a plane $z = ax + by + c$, then the vectors $\mathbf{v}_1 = \langle -a, 0, 1\rangle$ and $\mathbf{v}_2 = \langle 0, -b, 1\rangle$ are in the plane, and $(\Delta x)\, \mathbf{v}_1$ and $(\Delta y)\, \mathbf{v}_2$ generate a small rectangle in the plane with area $|\mathbf{v}_1 \times \mathbf{v}_2|\, \Delta x\, \Delta y = \sqrt{1 + a^2 + b^2}\, \Delta x\, \Delta y = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, \Delta x\, \Delta y$. If we have a surface $z = f(x, y)$, then approximating a small part of the surface near a point by a small rectangle in the tangent plane at the point gives area $\approx \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, \Delta x\, \Delta y = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, \Delta x\, \Delta y$. So the surface area above a domain $D$ is $\iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy$.

- Give an intuitive presentation of the area element $|\mathbf{r}_u \times \mathbf{r}_v|\, dA$ for general parametric surfaces.

- Compute the surface area of the surface given in cylindrical coordinates by $z = \theta$ above the unit disk

$0 \le x^2 + y^2 \le 1$ and below $z = 2\pi$. You may use the polar coordinates $r$ and $\theta$ as parameters.




## Workshop/Discussion

- Set up an integral to compute the surface area of the surface $S$ obtained by rotating $y = x^2$, $0 \le x \le 2$ about the $x$-axis. Point out that this integral is hard to compute by hand, but a CAS can do it very easily.
- Find the area of the portion of the surface $z = 16 - x^2 - y^2$ that lies above the $xy$-plane.
- Set up an integral to compute the surface area of the portion of the cone $z = r$ lying above the region enclosed by the polar curve $r = \sqrt{\cos 2\theta}$, $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$.
- Show how to compute the surface area of the portion of the surface $z = xy$ inside the cylinder $x^2+y^2 = a^2$. Point out that some of the surface lies above the plane $z = 0$ (above points in the first and third quadrants) and the remainder lies below $z = 0$. Then show how the full surface integration as a single integral requires using cylindrical coordinates.

## Group Work 1: Setting up Surface Integrals

The first two problems are straightforward. After the students have started the third problem, hand each group "a hint sheet". Unbeknownst to them, different groups will get different hint sheets. After they are finished, they can present four different ways of solving the problem. If a group finishes early, give them the bonus problem. Depending on the students' facility with computation, and/or their access to a CAS, the instructor may want them to compute all the integrals, or just set them all up.

## Group Work 2: Surfaces of Revolution

This activity leads the students through computations of the surface areas of solids of revolution. If Section 10.5 has not been covered, students may need more of an introduction. Note that Problem 3 leads to an improper integral.

## Group Work 3: Time to Blow your Geographical Minds

The surface area of Wyoming is 96,988 square miles, and that of Colorado is 103,598 square miles. In this activity, the students compute these surface areas and obtain numbers that are much less. The reason for the discrepancy between their model and reality is that the model does not take mountains into account. If a hint is needed, perhaps show the students the states on a relief map or globe. Note that you may need to remind students how latitude and longitude are measured.

## Homework Problems

**Core Exercises:** 1, 6, 9, 18

**Sample Assignment:** 1, 2, 6, 9, 12, 13, 18, 22, 24

**Note:** Exercises 13(b), 18(c), and 18(d) require a CAS.

| Exercise | C | A | N | G | V |
|---|---|---|---|---|---|
| 1 | | × | | | × |
| 2 | | × | | | × |
| 6 | | × | | | × |
| 9 | | × | | | × |
| 12 | | × | | | |
| 13 | | × | × | | |
| 18 | | × | × | × | |
| 22 | | × | | | × |
| 24 | | × | | | |

## Group Work 1, Section 12.6
## Setting Up Surface Integrals

**1.** Set up the surface integral for the parametric surface $\mathbf{r}(u, v) = \langle 3u \sin v, 4u \cos v, u^3 \rangle$ over the domain $\{(u, v) \mid 0 \le u \le 5, 0 \le v \le \pi\}$

**2.** Compute the surface area of the part of the surface $z = y^2$ above the triangle with vertices $(0, 1, 0)$, $(1, 0, 0)$, and $(1, 1, 0)$.

**3.** Compute the surface area of a cone with height 4 and radius 3.

## Group Work 1, Section 12.6
## Setting Up Surface Integrals (Hint Sheet)

To compute the surface area of a cone with height 4 and radius 3, set up the shape as $z = f(x, y)$ and compute a surface integral.

## Group Work 1, Section 12.6

## Setting Up Surface Integrals (Hint Sheet)

To compute the surface area of a cone with height 4 and radius 3, set up the shape as $z = f(r, \theta)$ and compute a surface integral.

## Group Work 1, Section 12.6
## Setting Up Surface Integrals (Hint Sheet)

To compute the surface area of a cone with height 4 and radius 3, set up the shape in spherical coordinates ($\phi$ is constant) and compute a surface integral.

## Group Work 1, Section 12.6
## Setting Up Surface Integrals (Hint Sheet)

To compute the surface area of a cone with height 4 and radius 3, set up the shape as generated by a line segment rotated about the $z$-axis and compute a surface integral.

## Group Work 1, Section 12.6
## Setting Up Surface Integrals (Bonus Problem)

Set up the surface integral of the piece of the unit sphere above the polar circle $r = \sin\theta$. Compute the surface area. How would the answer have differed had we used $r = \cos\theta$?

## Group Work 2, Section 12.6
## Surfaces of Revolution

For certain surfaces there is a formula for the surface area which can be written as a single integral instead of as a double integral. These surfaces are the so-called surfaces of revolution which we get by rotating the graph of a function about one of the axes. The formula for a surface formed by rotating the graph of $y = f(x)$ about the $x$-axis is $A = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2}\, dx$.

**1.** Derive this formula using Formula 3 from Section 10.5 (page 740).

**2.** There is a similar formula for the area of a surface formed when the graph of $x = g(y)$ is rotated about the $y$-axis. Derive this formula.

**3.** Use the formula from Problem 1 to find the surface area of a hemisphere of the unit sphere.
**Hint:** First find the formula for $0 \le x \le a$, and then take the limit as $a \to 1^-$.

**4.** Use the formula found in Problem 2 to set up an integral for the surface area of the ellipsoid $x^2 + \frac{1}{4}y^2 + z^2 = 1$ as a surface of revolution about the $y$-axis.

## Group Work 3, Section 12.6

## Time to Blow your Geographic Minds

The boundaries of Wyoming are given by the lines 41° North latitude, 45° North latitude, 104° West longitude, and 111° West longitude. The boundaries of Colorado are given by the lines 37° North latitude, 41° North latitude, 102° West longitude, and 109° West longitude.

1. Which state takes up more surface area on a globe?

2. Assume that the radius of the Earth is 3986 miles. What are the approximate areas of Wyoming and Colorado, in square miles?

3. Ask your instructor what the actual areas are. What do you think accounts for the discrepancy between these figures and your answers in Part 2?

# 12.7 Triple Integrals

## Suggested Time and Emphasis

$\frac{3}{4}$–1 class    Essential Material

## Points to Stress

1. The basic definition of a triple integral.
2. The various types of volume domain, and how to set up the volume integral based on a given domain.
3. Changing the order of integration in triple integrals.

## Text Discussion

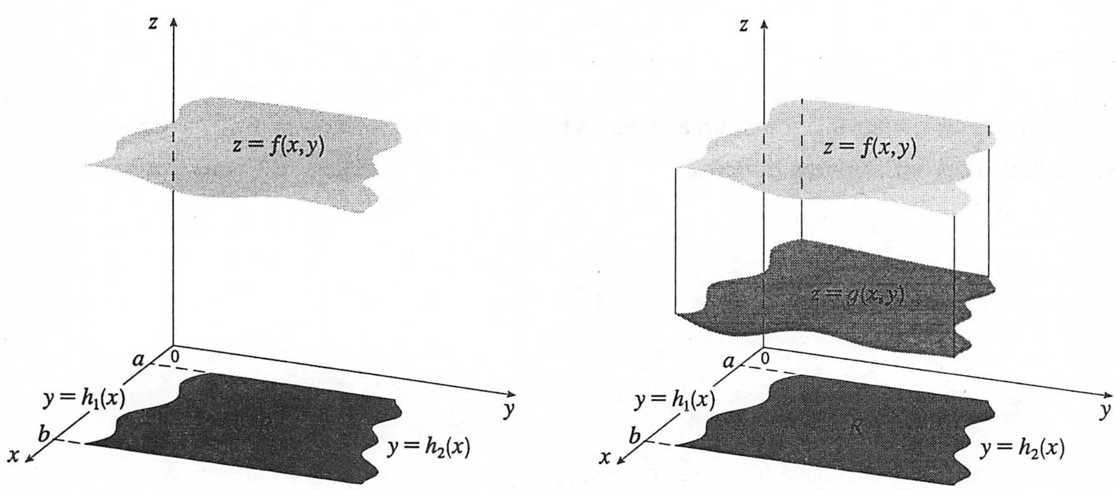
- Which variable ranges between two functions of the other two variables in a type 1 region? In a type 3 region?
- Give an example of a region that is both type 1 and type 2.

## Materials for Lecture

- One way to introduce volume integrals is by revisiting the concept of area, pointing out that area integrals can be viewed as double integrals (for example $\int_0^{10} f(x)\,dx = \int_0^{10}\int_0^{f(x)} dy\,dx$) and then showing how some volume integrals work by an analogous process. Set up a typical volume integral of a solid $S$ using double integrals and similarly transform it into a triple integral:

$$V = \iint_R f(x,y)\,dA = \int_a^b \int_{h_1(x)}^{h_2(x)} f(x,y)\,dy\,dx = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_0^{f(x,y)} dz\,dy\,dx$$

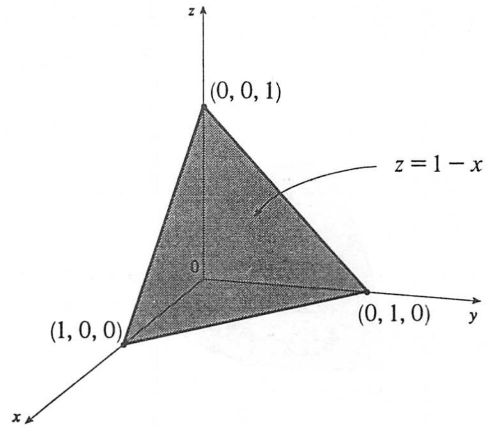
Then "move" the bottom surface of $S$ up to $z = g(x,y)$, so $S$ has volume $V = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g(x,y)}^{f(x,y)} dz\,dy\,dx$.



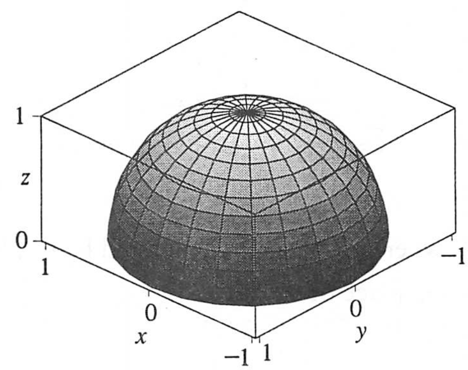

If we now have a function $k(x,y,z)$ defined on $S$, then the triple integral of $k$ over $S$ is

$$\iiint_S k(x,y,z)\,dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g(x,y)}^{f(x,y)} k(x,y,z)\,dz\,dy\,dx$$

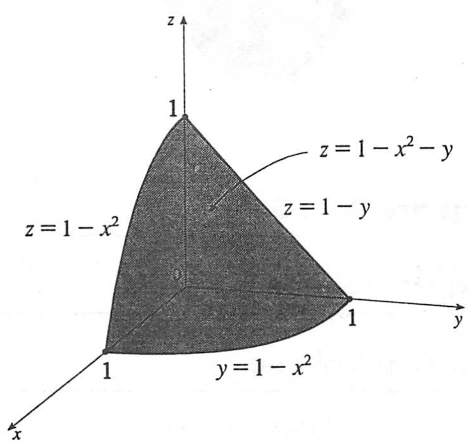
- Show the students why the region shown below is type 1, type 2, and type 3, and describe it in all three ways.




- Show how to identify the region of integration $E$ shown below for the volume integral $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$, and then rewrite the integral as an equivalent iterated integral of the form $\iiint_E f(x,y,z)\,dx\,dz\,dy$.

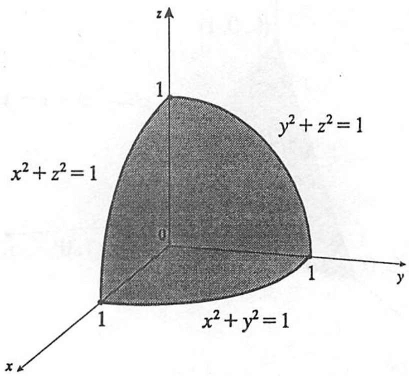



- Do a sample computation, such as the triple integral of $f(x,y,z) = z + xy^2$ over the volume $V$ bounded by the surface sketched below, in the first octant.

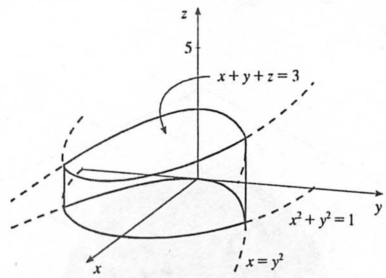


## Workshop/Discussion

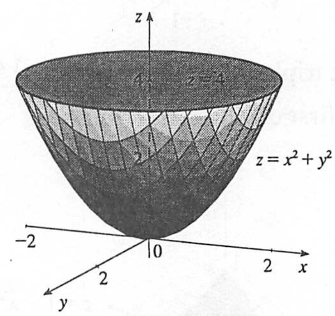
- Set up a triple integral for the volume $V$ of the piece of the sphere of radius 1 in the first octant, with different orders of integration.




- Sketch the solid whose volume is given by the triple integral $\int_{-\sqrt{(\sqrt{5}+1)/2}}^{\sqrt{(\sqrt{5}+1)/2}} \int_{y^2}^{\sqrt{1-y^2}} \int_0^{3-x-y} dz\, dy\, dx$. Such a sketch is shown below.




- Compute $\iiint_E \left(x^2+y^2\right)^{1/2} dV$, where $E$ is the solid pictured below, by first integrating with respect to $z$ and then using polar coordinates in place of $dx\,dy$.




## Group Work 1: The Square-Root Solid

## Group Work 2: Setting Up Volume Integrals

## Group Work 3: An Unusual Volume

This is a challenging group work for more advanced students. The idea is to show that just because a solid looks simple, the computation of its volume may be difficult. The line generated by $P_1$ and $P_2$ has equation

$z = \dfrac{L_1 - L_2}{2R}x + L_2$, and hence this equation, interpreted in three dimensions, is also the equation of the plane $S$. The integral $V(E) = \displaystyle\int_{-R}^{R} \int_{-\sqrt{R^2-(x-R)^2}}^{\sqrt{R^2-(x-R)^2}} \left( \frac{L_1 - L_2}{2R}x + L_2 \right) dy\,dx$ requires polar coordinates to solve by hand (since the bounding circle has equation $r = 2R\cos\theta$, $0 \le \theta \le \pi$) and also requires the students to remember how to integrate $\cos^2\theta$ and $\cos^4\theta$. Note that the problem can be simplified by moving the solid so that the $z$-axis runs through the center of $D$. Point out that a simple geometric solution can be obtained by replacing $S$ by the horizontal plane $z = \dfrac{L_1 + L_2}{2}$, thus giving a standard cylinder.

## Homework Problems

**Core Exercises:** 2, 5, 10, 26, 29

**Sample Assignment:** 2, 5, 8, 10, 13, 16, 20, 21, 26, 29, 37, 45

**Note:** Problem 7 from Focus on Problem Solving (page 914) can be assigned here as a project for an advanced student or group of students.

| Exercise | C | A | N | G | V |
|---|---|---|---|---|---|
| 2 | | × | | | |
| 5 | | × | | | |
| 8 | | × | | | |
| 10 | | × | | | × |
| 13 | | × | | | × |
| 16 | | × | | | × |

| Exercise | C | A | N | G | V |
|---|---|---|---|---|---|
| 20 | | | × | | |
| 21 | | × | | | |
| 26 | | × | | | |
| 29 | | × | | | × |
| 37 | | × | | | |
| 45 | | × | | | |

## Group Work 1, Section 12.7
## The Square-Root Solid

Consider the volume integral over the solid $S$ given by $V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} dz\, dy\, dx$.

**1.** Identify the solid $S$ by drawing a picture.

**2.** Rewrite the volume integral as $V = \iiint_S dx\, dy\, dz$.

**3.** Rewrite the volume integral as $V = \iiint_S dz\, dx\, dy$.

**4.** Starting with the original iterated integral, compute the volume by any means at your disposal.

## Group Work 2, Section 12.7
## Setting Up Volume Integrals

Set up volume integrals for the following solids:

**1.** $S$ is the solid formed by hollowing out a square cylinder of side length 1 centered along the $z$-axis through the solid sphere $x^2 + y^2 + z^2 = 4$.

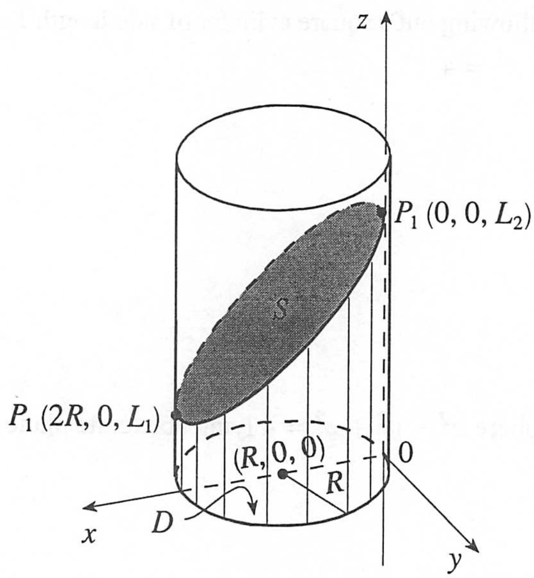
**2.** $S$ is the part of the solid sphere $x^2 + y^2 + z^2 = 4$ lying above the plane $z = 1$.

**3.** $S$ is the solid formed by drilling a cylindrical hole of radius 1 centered along the $z$-axis through the solid sphere $x^2 + y^2 + z^2 = 4$.

**4.** $S$ is the solid formed by hollowing out a square cylinder of side length 1 centered along the $y$-axis through the solid sphere $x^2 + y^2 + z^2 = 4$.

# Group Work 3, Section 12.7
## An Unusual Volume

Consider the solid $E$ shown below.




**1.** Find the equation of the plane $S$, parallel to the $y$-axis, which forms the top cap of $E$.

**2.** Set up a volume integral for $E$ of the form $V(E) = \iiint_E dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} dz\right] dA$.

**3.** Compute $V(E)$ by any means at your disposal. (**Hint:** Try polar coordinates.)

## Discovery Project: Volumes of Hyperspheres

Problems 1 and 2 review computations that the students may already know. Problem 4 is optional for this project, but it is highly recommended. To extend this project, students can be asked to find a book or article that discusses hyperspheres, and add some geometric discussion of these objects to their reports.

## 12.8 Triple Integrals in Cylindrical and Spherical Coordinates

### Suggested Time and Emphasis

$\frac{3}{4}$–1 class    Essential Material

### Transparencies Available

- Transparency 54 (Figure 7, page 895)
- Transparency 55 (Discovery Project: The Intersection of Three Cylinders, page 901)

### Points to Stress

1. The basic shapes of cylindrical and spherical rectangular solids
2. Volume integrals in cylindrical and spherical coordinates

### Text Discussion

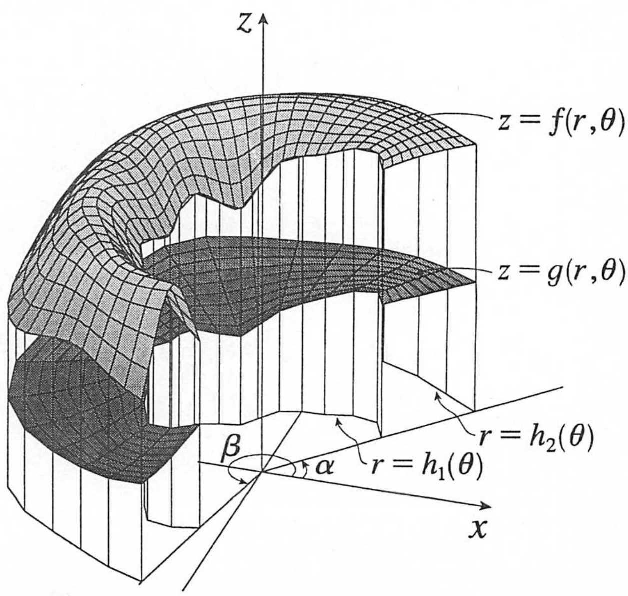
- Does the region of Example 2 have an axis of symmetry? If so, what does it say about the choice of using cylindrical coordinates?
- Why should we choose to use spherical coordinates in Example 4?

### Materials for Lecture

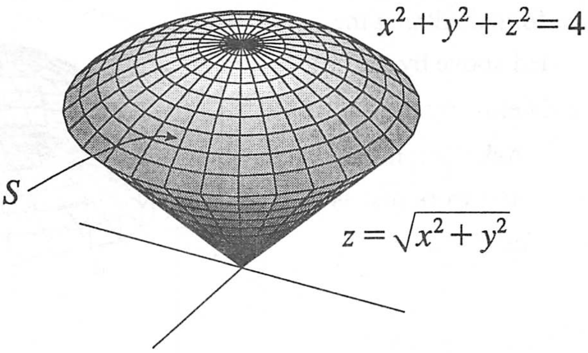
- Convert a typical cylindrical volume integral of a solid $S$ computed using double integrals into a triple integral:

$$V = \iint_R f(r,\theta)\, r\, dr\, d\theta = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r,\theta)\, r\, dr\, d\theta = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_0^{f(r,\theta)} r\, dz\, dr\, d\theta$$

Move the bottom surface of $S$ up to $z = g(r,\theta)$, as pictured at right. The volume now becomes $\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{g(r,\theta)}^{f(r,\theta)} r\, dz\, dr\, d\theta$. The basic volume element is given in Figure 3 of the text. Conclude with the situation where we have $h(r,\theta,z)$ defined on $S$. Then the triple integral of $h$ on $S$ is $\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{g(r,\theta)}^{f(r,\theta)} h(r,\theta,z)\, r\, dz\, dr\, d\theta$.



- Develop a straightforward example such as the region depicted below:

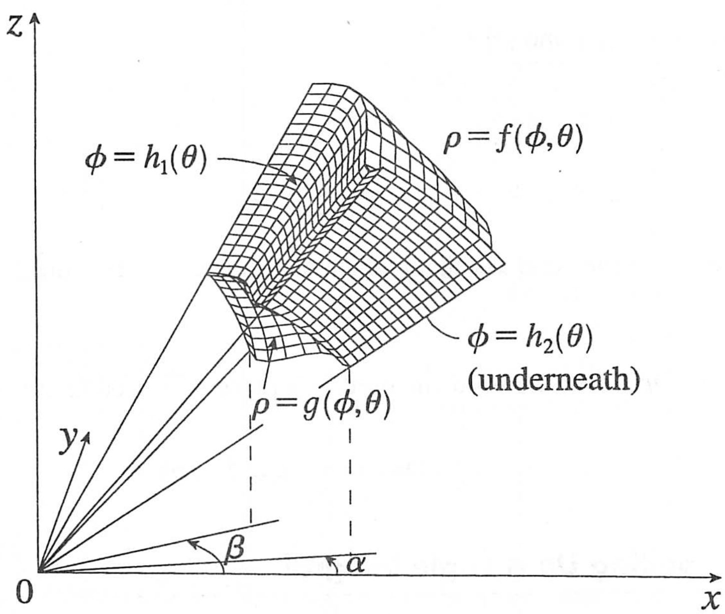



The Capped Cone

Set this volume up as a triple integral in cylindrical coordinates, and then find the volume. (The computation of this volume integral is not that hard, and can be assigned to the students.) Conclude by setting up the volume integral of $h(r,\theta,z)=rz$ over this region.

- Draw a basic spherical rectangular solid $S$ and compute that its volume is approximately $\rho^2 \sin\phi\,\Delta\rho\,\Delta\phi\,\Delta\theta$ if $\Delta\rho$, $\Delta\phi$, and $\Delta\theta$ are small. Calculate the volume of the solid pictured below to be

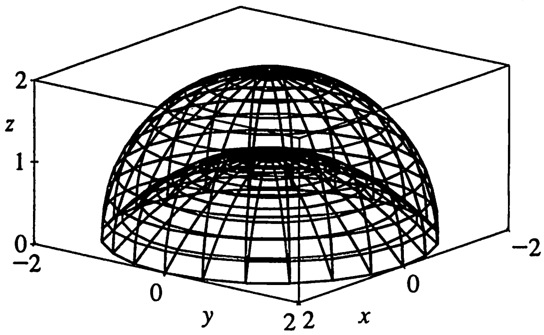
$$V=\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{g(\phi,\theta)}^{f(\phi,\theta)} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$




As usual, if there is a function $l(\rho,\phi,\theta)$ on $S$, then the triple integral is $\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{g(\phi,\theta)}^{f(\phi,\theta)} l(\rho,\phi,\theta)\,\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$.

- Redo "The Capped Cone" example, this time in spherical coordinates. (This is similar to Example 4 in the text.)

- Indicate to the students why using spherical coordinates is a good choice for calculating the volume of $E$, the solid bounded above by the sphere $x^2 + y^2 + z^2 = 4$ and below by the paraboloid $4z = 4 - x^2 - y^2$. Ask them if they think cylindrical coodinates would work just as well. Then compute $V(E)$ using either method.




## Workshop/Discussion

- Give a geometric description of the solid $S$ whose volume is given in spherical coordinates by $V = \int_0^{\pi} \int_{\pi/4}^{\pi/3} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$, and then show the students how to write the volume of $S$ as a triple integral in cylindrical coordinates.

- Give the students some three-dimensional regions and ask them which coordinate system would be most convenient for computing the volume of that region. Examples:

  1. $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 4 \text{ and } -2 \leq z \leq 3\}$

  2. $\{(x, y, z) \in \mathbb{R}^3 \mid z^2 + y^2 \leq 4 \text{ and } |x| \leq 1\}$

  3. $\{(x, y, z) \in \mathbb{R}^3 \mid z^2 \geq x^2 + y^2 \text{ and } x^2 + y^2 + z^2 \leq 1\}$

  4. $\{(x, y, z) \in \mathbb{R}^3 \mid \frac{1}{4} \leq x^2 + y^2 + z^2 \leq 9, x \geq 0, y \geq 0, \text{ and } z \geq 0\}$

- Give the students some integrands and ask them which coordinate system would be most convenient for integrating that integrand.

  1. $f(x, y, z) = 1/(x^2 + y^2)$ over the solid enclosed by a piece of a circular cylinder $x^2 + y^2 = a$

  2. $f(x, y, z) = e^{2x^2 + 2y^2 + 2z^2}$ over a solid between a cone and a sphere

## Group Work 1: Setting Up a Triple Integral

If a group finishes Problems 1–4 quickly, have them choose one of their integrals and compute it, and explain why they made the choice they did. Note that for Problem 5, the solid is a truncated piece of the cone $z = -r + R$ in cylindrical coordinates.

## Group Work 2: A Partially Eaten Sphere

Notice that you are removing "ice cream cones" both above and below the $xy$-plane.

## Homework Problems

**Core Exercises:** 1, 5, 8, 19, 28, 29, 30, 33

**Sample Assignment:** 1, 4, 5, 8, 14, 15, 19, 26, 28, 29, 30, 33

**Note:**
- Exercise 28(b) requires a CAS, or the students can be instructed to sketch the torus by hand.
- Exercise 31 makes a good lab project.
- Exercise 33 requires significant thought.

| Exercise | C | A | N | G | V |
|---|---|---|---|---|---|
| 1 | | × | | × | × |
| 4 | | × | | × | × |
| 5 | | × | | | × |
| 8 | | × | | | |
| 14 | | × | | | |
| 15 | | × | | | |

| Exercise | C | A | N | G | V |
|---|---|---|---|---|---|
| 19 | | × | | | |
| 26 | | × | | | × |
| 28 | | × | | × | |
| 29 | | × | | | × |
| 30 | | × | | | × |
| 33 | × | × | | | |

## Group Work 1, Section 12.8
## Setting Up a Triple Integral

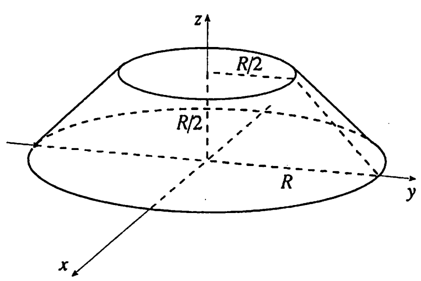
**1.** Set up the triple integral $\iiint_S (x^2 + y^2)\, dV$ in rectangular coordinates, where $S$ is a solid sphere of radius $R$ centered at the origin.

**2.** Set up the same triple integral in cylindrical coordinates.

**3.** Set up the same triple integral in spherical coordinates.

**4.** Set up an integral to compute the volume of the solid inside the cone $x^2 = y^2 + z^2$, $|x| \leq 1$, in a coordinate system of your choice.

**5.** Compute $\iiint_E z\,(x^2 + y^2)\, dV$, where $E$ is the solid shown below.



## Group Work 2, Section 12.8
## A Partially Eaten Sphere

**1.** Compute the volume of the solid $S$ formed by starting with the sphere $x^2 + y^2 + z^2 = 9$, and removing the solid bounded below by the cone $z^2 = 2\left(x^2 + y^2\right)$.

**2.** Set up the triple integral $\iiint_S yz\, dV$ in the same coordinate system you used for Problem 1.

## Applied Project: Roller Derby

This project has a highly dramatic outcome, in that a real race can be run whose results are predicted by mathematics. An in-class demonstration of the "roller derby" can be done before this project is assigned. If this project is to be assigned, it should be assigned in its entirety, in order to predict the outcome of the race.

## Discovery Project: The Intersection of Three Cylinders

This discovery project extends the problem of finding the volume of two intersecting cylinders given in Exercise 62 in Section 6.2. This is a very thought-provoking project for students with good geometric intuition. There is value to be gained from having students work on it, even if they don't wind up getting a correct answer. A good solution is given in the Complete Solutions Manual.

# 12.9 Change of Variables in Multiple Integrals

## Suggested Time and Emphasis

$1\text{–}1\frac{1}{4}$ classes Recommended Material (essential if Chapter 17/*16* is to be covered)

## Points to Stress

1. Reason for change of variables: to reduce a complicated multiple integration problem to a simpler integral or an integral over a simpler region in the new variables

2. What happens to area over a change in variables: The role of the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$

3. Various methods to construct a change of variables

## Text Discussion

- What is the area of the image $R$ of the unit square $S$ [with opposite corners at $(0,0)$ and $(1,1)$] in the $uv$-plane under the transformation $x = u + 2v$, $y = -6u - v$?

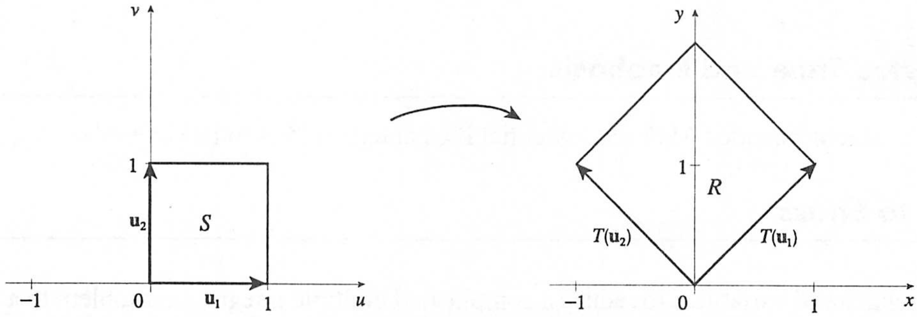
## Materials for Lecture

- One good way to begin this section is to discuss $u$-substitution from a geometric point of view. For example, $\int (\sin^2 x) \cos x \, dx$ is a somewhat complicated integral in $x$-space, but using the change of coordinate $u = \sin x$ reduces it to the simpler integral $\int u^2 \, du$ in $u$-space. If the students are concurrently taking physics or chemistry, discuss how the semi-logarithmic paper that they use is an example of this type of coordinate transformation.

- Review the fact that to integrate $f(x,y)$ over a region $R$ in the $xy$-plane, if we have a change of variables $\mathbf{r}(u,v)$ which transforms the rectangle $R_1$ in $uv$-space to $R$, then we have $\iint_R f(x,y)\,dA = \iint_{R_1} F(u,v)\,|r_u \times r_v|\,du\,dv$, where $F(u,v) = f(x(u,v), y(u,v))$ and $r_u \times r_v = C\mathbf{k}$, where $C$ is the Jacobian determinant $\begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial y}{\partial u}\dfrac{\partial x}{\partial v}$, denoted by $\dfrac{\partial(x,y)}{\partial(u,v)}$. Thus, $|r_u \times r_v| = \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$.

- Note that it is very important that we take the absolute value of the Jacobian determinant. For example, point out that the Jacobian determinant for spherical coordinates is always negative (see Example 4). Another example of a negative Jacobian is the transformation $x = u + 2v$, $y = 3u + v$, which takes $\langle 1,0\rangle$ to $\langle 1,3\rangle$ and $\langle 0,1\rangle$ to $\langle 2,1\rangle$.

- Consider the linear transformation $x = u - v$, $y = u + v$. This takes the unit square $S$ in the $uv$-plane into a square with area 2.

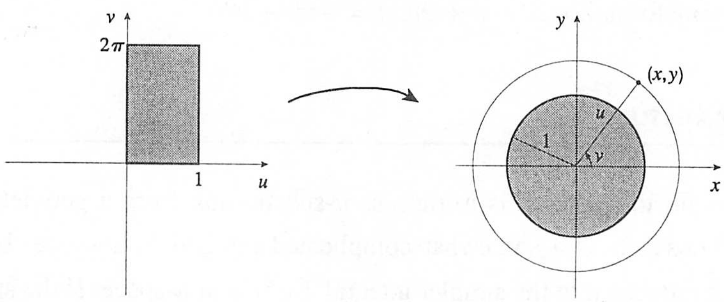



Note that the Jacobian of this transformation is $\dfrac{\partial(x,y)}{\partial(u,v)} = 2$. In general, we have

$$A(R) = \iint_R 1\,dA = \iint_S \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| A(S)$$

So for linear transformations, the Jacobian is the determinant of the matrix of coefficients, and the absolute value of this determinant describes how area in $uv$-space is magnified in $xy$-space under the transformation $T$.

- Pose the problem of changing the rectangle $[0,1] \times [0,2\pi]$ in the $uv$-plane into a disk in the $xy$-plane by a change of variable $\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j}$. Show that $x = u\cos v$, $y = u\sin v$ will work.




Then $u^2 = x^2 + y^2$ and $\tan v = y/x$, so $u$ can be viewed as the distance to the origin and $v$ is the angle with the positive $x$-axis. This implies that the $u$, $v$ transformation is really just the polar-coordinate transformation. The grid lines $u = c \ge 0$ go to circles, and $v = c$ go to rays. Therefore $uv$-rectangles go to polar rectangles in $xy$-space. Perhaps note that trying to look at this transformation in reverse leads to problems at the origin. See if the students can determine what happens to the line $u = 0$. Also note that there are other transformations that work, such as $x = u\sin v$, $y = u\cos v$.

- Pose the problem of changing a rectangle into the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} \le 1$. Posit that the answer might again be of the form $x = c_1 u\cos v$, $y = c_2 u\sin v$ and solve $\dfrac{x^2}{c_1^2} + \dfrac{y^2}{c_2^2} = u^2$. So chose $c_1 = 2$, $c_2 = 3$ and then the rectangle $[0,1] \times [0,2\pi]$ maps into the specified ellipse. This time, $\frac{2}{3}\tan v = \dfrac{y}{x}$ so $\tan v = \dfrac{3y}{2x}$. The grid line $u = c > 0$ goes to the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = c$ and $v = k$ goes to the ray $y = \frac{2}{3}x\tan k$. This means that $(u,v)$ gives an elliptical coordinate system.

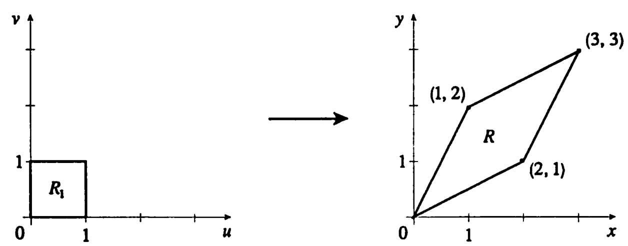
## Workshop/Discussion

- Show that for the polar-coordinate representation discussed in the lecture suggestions above, the Jacobian is equal to $u$, and we get $u\,du\,dv$ as previously computed.

- For the "elliptical" coordinates described in the Materials for Lecture, the Jacobian is $\begin{vmatrix} 2\cos v & 3\sin v \\ -2u\sin v & 3u\cos v \end{vmatrix} = 6u$, leading to $6u\,du\,dv$. Using this new coordinate system to compute $\iint_R x^2\,dA$ where $R$ is the region bounded by the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$, we have $\iint_R x^2 dA = \int_0^{2\pi}\int_0^1 (2u\cos v)^2\,|6u|\,du\,dv = \int_0^{2\pi} 6\cos^2 v\,dv = 6\pi$.

- Show how to compute the volume of the solid $S$ bounded by the ellipsoid $\frac{x^2}{2}+\frac{y^2}{3}+\frac{z^2}{4}=1$. In the $xy$-plane, we have the elliptic region $R$ bounded by $\frac{x^2}{2}+\frac{y^2}{3}=1$. The change of variables $x=\sqrt{2}u$, $y=\sqrt{3}v$ maps the unit disk $D$: $u^2+v^2\le 1$ to $R$. So we get $V = 2\iint_D 2\sqrt{1-(u^2+v^2)}\sqrt{6}\,du\,dv$. Now using polar coordinates on this $uv$-integral, we get $V = 4\sqrt{6}\int_0^{2\pi}\int_0^1 \sqrt{1-r^2}r\,dr\,d\theta = \frac{8\sqrt{6}}{3}\pi$. Another approach is to map the unit ball $u^2+v^2+w^2\le 1$ to the ellipsoid using $x=\sqrt{2}u$, $y=\sqrt{3}v$, $z=2w$ and then $V = \iiint_B 2\sqrt{6}\,dV = \frac{8\sqrt{6}}{3}\pi$, using the change of variables formula for triple integrals.

- Describe how to find a transformation which maps the $uv$-plane as follows:




Show how it is sufficient to check what happens to $\langle 0,1\rangle$ and $\langle 1,0\rangle$.

- Consider the change of variables $x=u^2-v^2$, $y=2uv$ described in Example 1 of the text. Show that the grid lines $u=a$ give the parabolas $x=a^2-\frac{y^2}{4a^2}$, and the grid lines $v=b$ give the parabolas $x=\frac{y^2}{4b^2}-b^2$.

## Group Work 1: Many Changes of Variables

## Group Work 2: Transformed Parabolas

## ▲ Homework Problems

**Core Exercises:** 2, 9, 11, 17

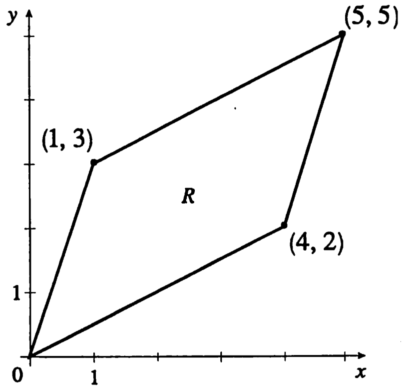
**Sample Assignment:** 2, 5, 9, 10, 11, 14, 17, 19

| Exercise | C | A | N | G | V |
|---|---|---|---|---|---|
| 2 | | × | | | |
| 5 | | × | | | |
| 9 | | × | | × | × |
| 10 | | × | | × | × |

| Exercise | C | A | N | G | V |
|---|---|---|---|---|---|
| 11 | | × | | | |
| 14 | | × | | | |
| 17 | | × | × | | |
| 19 | | × | | | × |

## Group Work 1, Section 12.9
## Many Changes of Variables

**1.** Find $\iint_R xy^2\,dA$ where $R$ is given below.



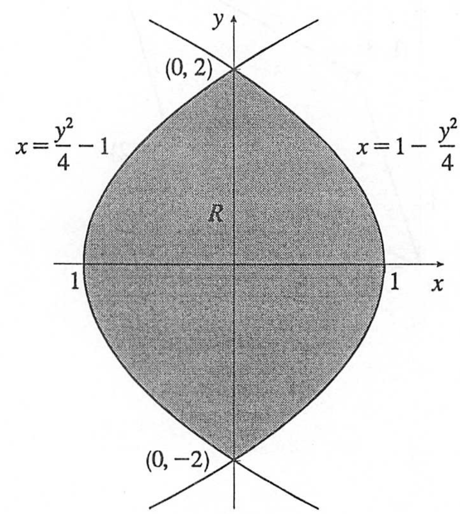

**2.** Find a mapping $T$ which maps the triangle bounded by $(0,0)$, $(0,1)$, and $(1,0)$ to the triangle bounded by $(0,0)$, $(5,2)$, and $(5,-2)$. What is the Jacobian of $T$? What is the area of $R$? Compute $\iint_R xy\,dA$.

**3.** Find a mapping $T$ which maps the unit disc $u^2 + v^2 \le 1$ onto the region $R$ enclosed by the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$. Compute $\iint_R x^2\,dA$ and $\iint_R y^2\,dA$.

## Group Work 2, Section 12.9
## Transformed Parabolas

Consider the change of variables $x = u^2 - v^2$, $y = 2uv$ described in Example 1 of the text.

**1.** Compute $\iint_R x^2\,dA$, where $R$ is the region shown below.
**Hint:** How is $\iint_R x^2\,dA$ related to $\iint_{R_1} x^2\,dA$, where $R_1$ is the portion of $R$ above the $x$-axis?




**2.** Let $S$ be the rectangle $\{(u, v) \mid 0 \le u \le 1, 0 \le v \le 1\}$. What is the image $T$ in the $xy$-plane of $S$ under this change of variables?

**3.** What is the area of $T$?

# 12 Sample Exam

Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

**1.** Consider the function $f(x,y) = x^y$ on the rectangle $[1,2] \times [1,2]$.

(a) Approximate the value of the integral $\int_1^2 \int_1^2 f(x,y)\,dy\,dx$ by dividing the region into four squares and using the function value at the lower left-hand corner of each square as an approximation for the function value over that square.

(b) Does the approximation give an overestimate or an underestimate of the value of the integral? How do you know?

**2.** Given that $\int_0^{\pi/2} \frac{dx}{1+\sin^2 x} = \frac{\pi}{2\sqrt{2}}$,

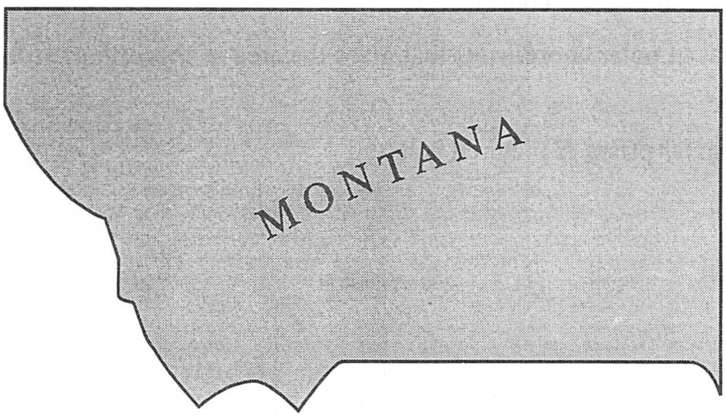
(a) evaluate the double integral

$$\int_0^{\pi/2} \int_0^{\pi/2} \frac{1}{\left(1+\sin^2 x\right)\left(1+\sin^2 y\right)}\,dx\,dy$$

(b) evaluate the triple integral

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_{1/(1+\sin^2 x)}^{1/(1+\sin^2 y)} dz\,dx\,dy$$

**3.** Consider the region below:




(a) Divide the region into smaller regions, all of which are Type I.

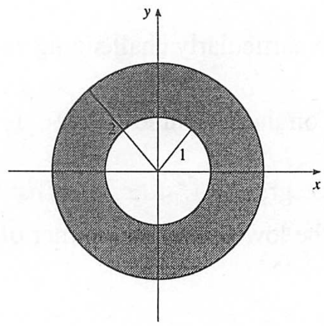
(b) Divide the region into smaller regions, all of which are Type II.

**4.** Rewrite the integral

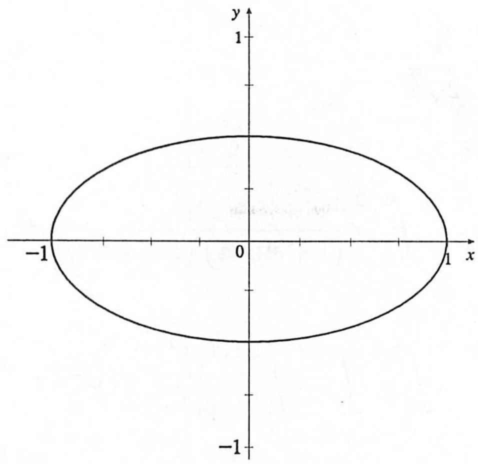
$$\int_0^{2\pi} \int_0^1 r^2\,dr\,d\theta$$

in rectangular coordinates.

**5.** Evaluate $\iint_D \cos\left(x^2+y^2\right)dA$, where $D=\left\{(x,y)\mid 1\le x^2+y^2\le 4\right\}$ is a washer with inner radius 1 and outer radius 2.



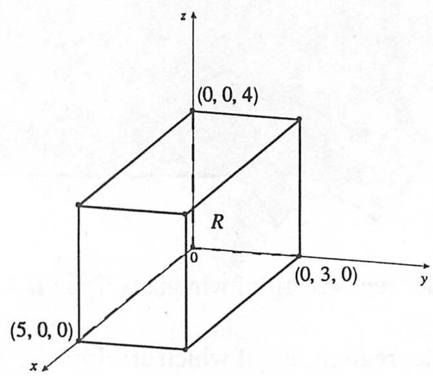

**6.** Consider the ellipse $x^2+2y^2=1$.




(a) Rewrite the equation in polar coordinates.

(b) Write an integral in polar coordinates that gives the area of this ellipse. *Note:* Your answer will not look simple.

**7.** Consider the rectangular prism $R$ pictured below:




Compute $\iiint_R 10\,dV$ and $\iiint_R x\,dV$.

**8.** (a) Compute

$$\int_0^1\int_{-1}^1\int_0^{xy} 1\,dz\,dx\,dy$$

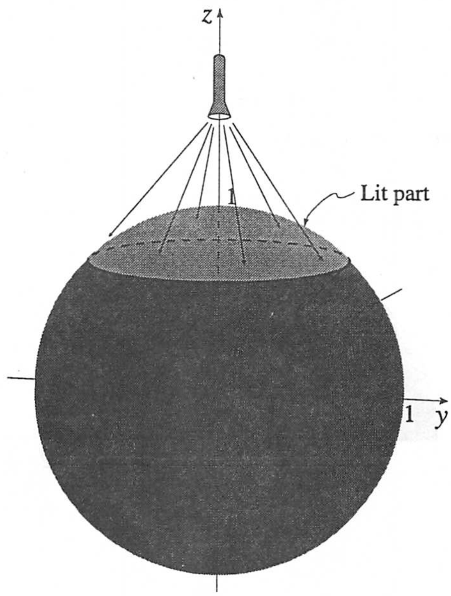
and give a geometric interpretation of your answer.

(b) Compute

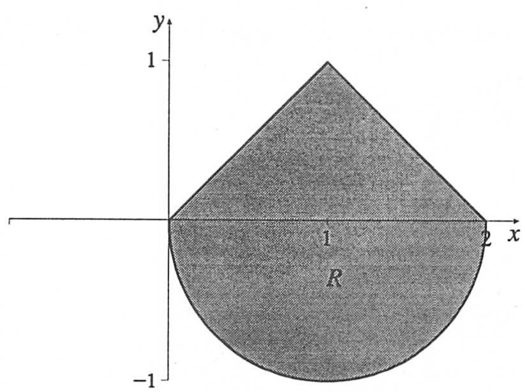
$$\int_0^1 \int_{-1}^1 \int_0^{|xy|} 1 \, dz \, dx \, dy$$

and give a geometric interpretation of your answer.

**9.** A light on the $z$-axis, pointed at the origin, shines on the sphere $\rho = 1$ such that $\frac{1}{4}$ of the total surface area is lit. What is the angle $\phi$?




**10.** Consider the region $R$ enclosed by $y = x$, $y = -x + 2$, $y = -\sqrt{1 - (x-1)^2}$:

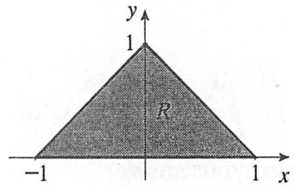



Set up the following integrals as one or more iterated integrals, but do not actually compute them:

(a) $\iint_R (x + y) \, dy \, dx$

(b) $\iint_R (x + y) \, dx \, dy$

**11.** Consider the region $R$ enclosed by $y = x + 1$, $y = -x + 1$, and the $x$-axis.



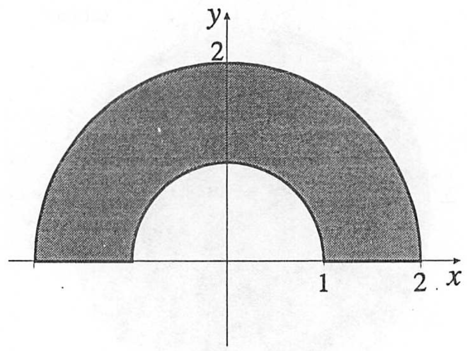

(a) Set up the integral $\iint_R xy\, dx\, dy$ in polar coordinates.

(b) Compute the integral $\iint_R xy\, dx\, dy$ using any method you know.

**12.** Consider the double integral

$$\iint_R \frac{1}{9 - (x^2 + y^2)^{3/2}}\, dA$$

where $R$ is given by the region between the two semicircles pictured below:

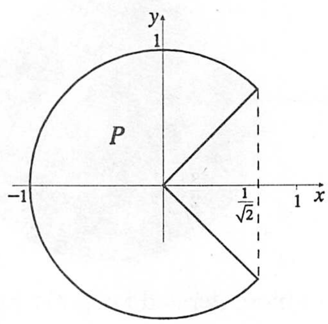



(a) Compute the shaded area.

(b) Show that the function $\dfrac{1}{9 - (x^2 + y^2)^{3/2}}$ is constant on each of the two bounding semicircles.

(c) Give a lower bound and an upper bound for the double integral using the above information.

**13.** Observe the following Pac-Man:




(a) Describe him in polar coordinates.

(b) Evaluate $\iint_{\text{Pac-Man}} x\, dA$ and $\iint_{\text{Pac-Man}} y\, dA$.

**14.** Set up and evaluate an integral giving the surface area of the parametrized surface $x = u + v$, $y = u - v$, $z = 2u + 3v$, $0 \leq u \leq 1$, $0 \leq v \leq 1$.

**15.** Consider the triple integral

$$\int_0^1 \int_{y^3}^{\sqrt{y}} \int_0^{xy} dz\, dx\, dy$$

representing a solid $S$. Let $R$ be the projection of $S$ onto the plane $z = 0$.

(a) Draw the region $R$.

(b) Rewrite this integral as $\iiint_S dz\, dy\, dx$.

**16.** Consider the transformation $T$: $x = 2u + v$, $y = u + 2v$.

(a) Describe the image $S$ under $T$ of the unit square $R = \{(u, v) \mid 0 \le u \le 1, 0 \le v \le 1\}$ in the $uv$-plane using a change of coordinates.

(b) Evaluate $\iint_S (3x + 2y)\, dA$

**17.** What is the volume of the following region, described in spherical coordinates: $1 \le \rho \le 9$, $0 \le \theta \le \frac{\pi}{2}$, $\frac{\pi}{6} \le \phi \le \frac{\pi}{4}$?

**18.** Consider the transformation $x = v \cos 2\pi u$, $y = v \sin 2\pi u$.

(a) Describe the image $S$ under $T$ of the unit square $R = \{(u, v) \mid 0 \le u \le 1, 0 \le v \le 1\}$.

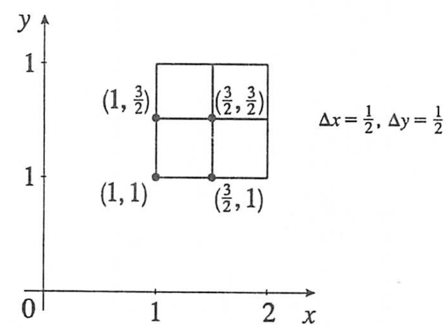
(b) Find the area of $S$.

**19.** Consider the function $f(x, y) = ax + by$, where $a$ and $b$ are constants. Find the average value of $f$ over the region $R = \{(x, y) \mid -1 \le x \le 1, -1 \le y \le 1\}$.

## 12 Sample Exam Solutions

**1.** $f(x, y) = x^y$

(a)



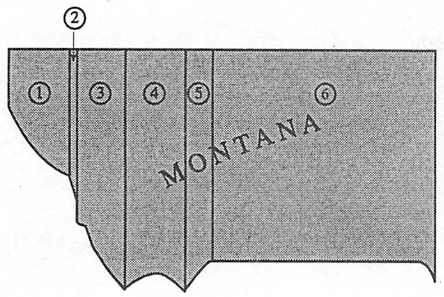

$$\begin{aligned}\int_1^2 \int_1^2 f(x, y)\, dy\, dx &\approx f(1, 1) \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} + f\left(\tfrac{3}{2}, 1\right) \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} + f\left(1, \tfrac{3}{2}\right) \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} + f\left(\tfrac{3}{2}, \tfrac{3}{2}\right) \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} \\ &= \tfrac{1}{4}\left[1 + \tfrac{3}{2} + 1 + \left(\tfrac{3}{2}\right)^{3/2}\right] \approx \tfrac{1}{4}(5.3375) \approx 1.344\end{aligned}$$

(b) This estimate is an underestimate since the function is increasing in the $x$- and $y$-directions as $x$ and $y$ go from 1 to 2.

**2.** (a)
$$\int_0^{\pi/2} \int_0^{\pi/2} \frac{1}{\left(1 + \sin^2 x\right)\left(1 + \sin^2 y\right)}\, dx\, dy = \left(\int_0^{\pi/2} \frac{dx}{1 + \sin^2 x}\right)\left(\int_0^{\pi/2} \frac{dy}{1 + \sin^2 y}\right) = \left(\frac{\pi}{2\sqrt{3}}\right)^2 = \frac{\pi^2}{12}$$

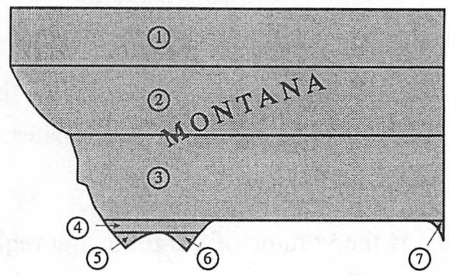
(b) $$\int_0^{\pi/2}\int_0^{\pi/2}\int_{1/(1+\sin^2 x)}^{1/(1+\sin^2 y)} dz\,dx\,dy = \int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{1}{1+\sin^2 y}-\frac{1}{1+\sin^2 x}\right)dx\,dy$$
$$= \int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{1}{1+\sin^2 y}\right)dx\,dy - \int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{1}{1+\sin^2 x}\right)dx\,dy$$
$$= \frac{\pi^2}{4\sqrt{3}} - \frac{\pi^2}{4\sqrt{3}} = 0$$

**3.** (a)



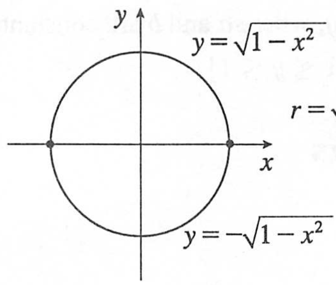

Type I

(b)




Type II

**4.** $\int_0^{2\pi}\int_0^1 r^2\,dr\,d\theta = \int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\sqrt{x^2+y^2}\,dy\,dx$




**5.** $\int_0^{2\pi}\int_1^2 \cos\left(r^2\right) r\,dr\,d\theta = \pi\left(\sin 4 - \sin 1\right)$

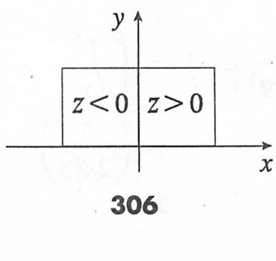
**6.** $x^2 + 2y^2 = 1$

(a) $r^2\left(\cos^2\theta + 2\sin^2\theta\right) = r^2\left(1+\sin^2\theta\right) = 1, r \ge 0$

(b) $$\int_0^{2\pi}\int_0^{1/\sqrt{1+\sin^2\theta}} r\,dr\,d\theta$$

**7.** Since the parallelepiped has volume 60, we have $\iiint_R 10\,dV = 600$.
$\iiint_R x\,dV = 12\int_0^5 x\,dx = 12\left(\frac{25}{2}\right) = 150$

**8.** (a) $\int_0^1\int_{-1}^1\int_0^{xy} 1\,dz\,dx\,dy = \int_0^1\int_{-1}^1 xy\,dx\,dy = \int_0^1\left[\frac{1}{2}x^2y\right]_{-1}^1 dy = 0$. The region between $z=0$ and $z=xy$ in the first quadrant is above the $xy$-plane, while a symmetric region is below the $xy$-plane in the second quadrant.



(b) $\int_0^1 \int_{-1}^1 \int_0^{|xy|} 1\,dz\,dx\,dy = 2\int_0^1 \int_0^1 \int_0^{xy} dz\,dx\,dy = 2\int_0^1 \left[\frac{1}{2}x^2y\right]_0^1 dy = 2\int_0^1 \frac{1}{2}y\,dy = \frac{1}{2}y^2\Big]_0^1 = \frac{1}{2}$.
This is the total volume between $z = 0$ and $z = xy$. Because we take the absolute value, the volumes do not cancel.

**9.** Since the surface area is $4\pi$, we need to find $\phi$ so that the area lit is $\pi$.
$\pi = \int_0^\phi \int_0^{2\pi} \sin\phi\,d\theta\,d\phi = 2\pi\int_0^\phi \sin\phi\,d\phi = 2\pi(-\cos\phi + \cos 0)$, so $\frac{1}{2} = 1 - \cos\phi \Rightarrow \cos\phi = \frac{1}{2}$
$\Rightarrow \phi = \frac{\pi}{3}$.

**10.** (a) $\int_0^1 \int_{-\sqrt{1-(x-1)^2}}^{x} (x+y)\,dy\,dx + \int_1^2 \int_{-\sqrt{1-(x-1)^2}}^{2-x} (x+y)\,dy\,dx$

(b) $\int_{-1}^0 \int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} (x+y)\,dx\,dy + \int_0^1 \int_y^{2y} (x+y)\,dx\,dy$. Note that the circular part of the curve is $y = -\sqrt{1-(x-1)^2}$ or $x = 1 \pm \sqrt{1-y^2}$.

**11.** (a) $\int_0^{\pi/2} \int_0^{1/(\sin\theta+\cos\theta)} r^3 \sin\theta\cos\theta\,dr\,d\theta + \int_{\pi/2}^{\pi} \int_0^{1/(\sin\theta-\cos\theta)} r^3 \sin\theta\cos\theta\,dr\,d\theta$

(b) 0

**12.** $\displaystyle\iint_R \frac{1}{9-(x^2+y^2)^{3/2}}\,dA$

(a) $\frac{1}{2}(4\pi - \pi) = \frac{3\pi}{2}$

(b) Since the semicircles satisfy $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, we have on $x^2 + y^2 = 1$,
$\dfrac{1}{9(x^2+y^2)^{3/2}} = \dfrac{1}{8}$ and on $x^2 + y^2 = 4$, $\dfrac{1}{9(x^2+y^2)^{3/2}} = 1$.

(c) A lower bound is the minimum value times the area, that is, $\frac{1}{8} \cdot \frac{3\pi}{2} = \frac{3\pi}{16}$.
An upper bound is the maximum value times the area, that is, $1 \cdot \frac{3\pi}{2} = \frac{3\pi}{2}$.

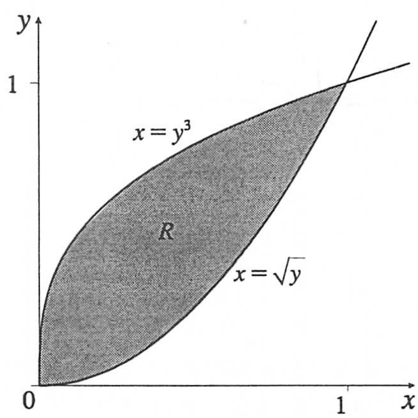
**13.** (a) $\left\{(r,\theta) \mid 0 \le r \le 1, \frac{\pi}{4} \le \theta \le \frac{7\pi}{4}\right\}$

(b) $\iint_{\text{Pac-Man}} x\,dA = \int_0^1 \int_{\pi/4}^{7\pi/4} r^2\cos\theta\,d\theta\,dr = \int_0^1 \left[r^2\sin\theta\right]_{\pi/4}^{7\pi/4} dr = -\sqrt{2}\int_0^1 r^2\,dr = -\frac{\sqrt{2}}{3}$

$\iint_{\text{Pac-Man}} y\,dA = \int_0^1 \int_{\pi/4}^{7\pi/4} r^2\sin\theta\,d\theta\,dr = \int_0^1 \left[-r^2\cos\theta\right]_{\pi/4}^{7\pi/4} dr = 0$

**14.** $\mathbf{F}(u,v) = (u+v)\,\mathbf{i} + (u-v)\,\mathbf{j} + (2u+3v)\,\mathbf{k}$, and so $\mathbf{F}_u = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{F}_v = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Then $\mathbf{F}_u \times \mathbf{F}_v = 5\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, and the surface area is $\int_0^1 \int_0^1 |\mathbf{F}_u \times \mathbf{F}_v|\,du\,dv = \int_0^1 \int_0^1 \sqrt{30}\,du\,dv = \sqrt{30}$.

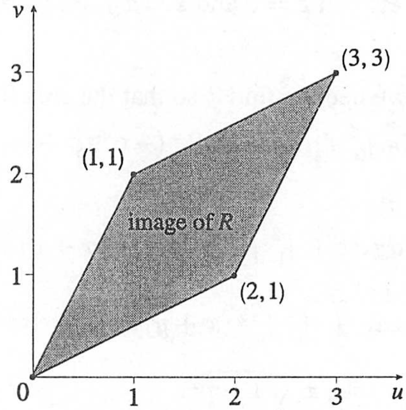
**15.** (a)




(b) $\int_0^1 \int_{y^3}^{\sqrt{y}} \int_0^{xy} dz\,dx\,dy = \int_0^1 \int_{x^2}^{\sqrt[3]{x}} \int_0^{xy} dz\,dy\,dx$

**16.** $x = 2u + v, y = u + 2v$

(a)

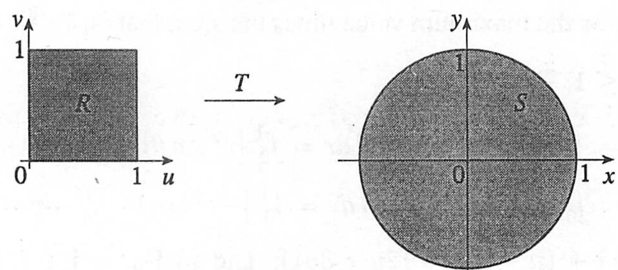



(b) The Jacobian is $\begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$, so

$$\begin{aligned}\iint_S (3x+2y)\,dA &= \int_0^1 \int_0^1 [3(2u+v) + 2(u+2v)]\,3\,du\,dv \\ &= 3\int_0^1 \left[3u^2 + 3uv + u^2 + 4uv\right]_0^1 dv \\ &= 3\int_0^1 (3 + 3v + 1 + 4v)\,dv = 3\left[4v + \tfrac{7}{2}v^2\right]_0^1 = \tfrac{45}{2}\end{aligned}$$

**17.** $$\begin{aligned}\int_1^9 \int_0^{\pi/2} \int_{\pi/6}^{\pi/4} \rho^2 \sin\phi\,d\phi\,d\theta\,d\rho &= \tfrac{\pi}{2}\int_1^9 \left[-\rho^2\cos\phi\right]_{\pi/6}^{\pi/4} d\rho = \tfrac{\pi}{2}\int_1^9 \tfrac{\sqrt{3}-\sqrt{2}}{2}\rho^2\,d\rho \\ &= \tfrac{\sqrt{3}-\sqrt{2}}{2}\left[\tfrac{1}{3}\rho^3\right]_1^9 = \tfrac{182}{3}\left(\sqrt{3}-\sqrt{2}\right)\pi\end{aligned}$$

**18.** (a)




$T$ maps the unit square in the $uv$-plane to the unit circle in the $xy$-plane.

(b) The area of $S$ is $\pi$.

**19.** $$f_{\text{ave}} = \frac{\int_{-1}^{1}\int_{-1}^{1}(ax+by)\,dy\,dx}{\int_{-1}^{1}\int_{-1}^{1} 1\,dy\,dx} = \frac{\int_{-1}^{1} 2ax\,dx}{4} = 0$$