13 Vector Calculus

13.1 Vector Fields

Suggested Time and Emphasis

1 class Essential material

Transparencies Available

- Transparency 56 (Figure 1, page 917)
- Transparency 57 (Figures 10–12, page 920)
- Transparency 58 (Exercises 11–14, graphs I–IV, page 923)
- Transparency 59 (Exercises 15–18, graphs I–IV, page 923)

Points to Stress

1. Two- and three-dimensional vector fields.
2. Vector fields can either be drawn "scaled," so that the lengths of the vectors are proportional to their magnitudes and the longest vectors in the field have a specified length, or "unscaled," so that the vectors appear at their true lengths.
3. Gradient fields in $\mathbb{R}^2$ and $\mathbb{R}^3$, and their relationships to level curves and surfaces.

Text Discussion

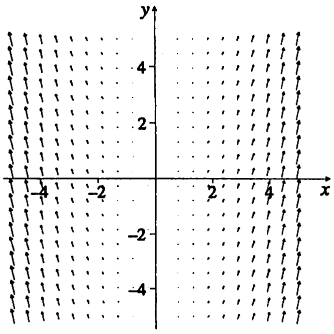
- Why is each vector in the vector field $\mathbf{F}(x,y) = -y\mathbf{i} + x\mathbf{j}$ tangent to the circle centered at the origin through the point (x,y)?
- Let $f(x,y)$ be a function of two variables, with level curves in the plane corresponding to $f(x,y) = k$. How is the gradient vector field ∇f related to these level curves? How does the length of ∇f vary with the spacing of the curves?

Materials for Lecture

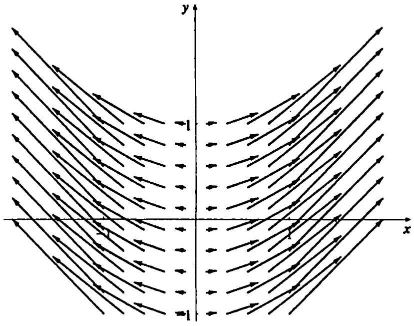
- Discuss various examples of vector fields on physical surfaces, such as wind speed and direction on the Earth, and temperature and altitude gradients.
- Point out that in order for $\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ to be continuous at (x,y), both P and Q must be continuous at (x,y). Thus, for example, $\mathbf{F}(x,y) = (x/|x|)\,\mathbf{i} + xy\mathbf{j}$ is not continuous at $(0,0)$ since $P(x,y) = x/|x|$ is not continuous at $(0,0)$. Also define what is meant by a non-vanishing vector field: a vector field in which the zero vector does not appear.

- Show pictures of some interesting vector fields in $\mathbb{R}^2$, such as those shown below, and describe the process of scaling.

1. $\mathbf{F}(x,y) = x\,\mathbf{i} + x^2\,\mathbf{j}$

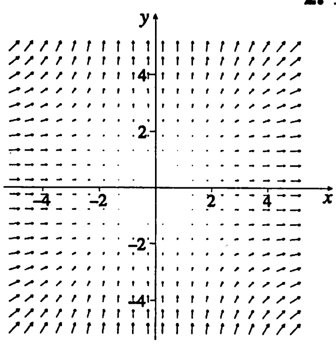


Scaled

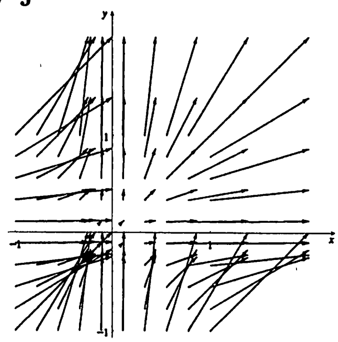


Unscaled

2. $\mathbf{F}(x,y) = x^2\,\mathbf{i} + y^2\,\mathbf{j}$

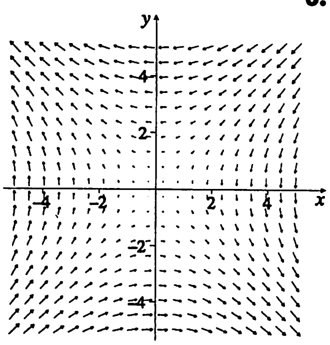


Scaled

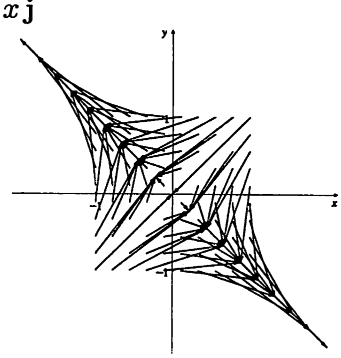


Unscaled

3. $\mathbf{F}(x,y) = -y\,\mathbf{i} - x\,\mathbf{j}$



Scaled



Unscaled

- Describe and sketch some elementary vector fields in $\mathbb{R}^3$:

  1. $\mathbf{F}(x,y,z) = -\frac{x}{2}\,\mathbf{i} - \frac{y}{2}\,\mathbf{j} - \frac{z}{2}\,\mathbf{k}$

  2. $\mathbf{F}(x,y,z) = -\frac{x}{\sqrt{x^2+y^2+z^2}}\,\mathbf{i} - \frac{y}{\sqrt{x^2+y^2+z^2}}\,\mathbf{j} - \frac{z}{\sqrt{x^2+y^2+z^2}}\,\mathbf{k}$, $(x,y,z) \neq (0,0,0)$

  Be sure to indicate what happens to these vector fields near the origin.

- Draw the contour map for $f(x,y) = x^2 + y^2$ and plot the gradient vector field $\nabla f(x,y) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$. Point out how the spacing between the contour curves $f(x,y) = k$ decreases, and $\nabla f(x,y)$ gets longer, as k increases. Explain the connection between spacing and length of $\nabla f(x,y)$.

Workshop/Discussion

- Define the gradient vector field for $f(x,y)$: $\mathbf{F}(x,y) = \nabla f(x,y) = f_x(x,y)\,\mathbf{i} + f_y(x,y)\,\mathbf{j}$. Compute ∇f for $f(x,y) = x^2 + y^2$ and show that the vectors in the gradient field are all orthogonal to the circles $f(x,y) = k$. Then similarly analyze ∇f for $f(x,y) = -x^2 + y$, for which the level curves are the parabolas $y = x^2 + k$.
- Sketch some interesting vector fields in $\mathbb{R}^2$:
  1. $\mathbf{F}(x,y) = x^2\,\mathbf{i} + x^3\,\mathbf{j}$
  2. $\mathbf{F}(x,y) = y^3\,\mathbf{i} + y^2\,\mathbf{j}$
  3. $\mathbf{F}(x,y) = (x+y)\,\mathbf{i} + (x-y)\,\mathbf{j}$. (Plot along the line $y = mx$ for various values of m.)
- The following is a good way to demonstrate continuous vector fields and their flow lines (or streamlines) as in Exercises 33 and 34.

  Have a student point at some other randomly selected student. Now have all the students who are sitting adjacent to the first student point in a direction similar to, but not equal to the first student's direction. Have their neighbors similarly point, until the entire lecture room becomes a continuous vector field. Now start in the middle of the room, with some random student, and walk along the flow line determined by the student-vector field, stressing that at all times you are walking in the direction in which the nearest student is pointing. (If this is too ignoble, have a student do it for you.) Demonstrate that starting at a different initial student can result in an entirely different path. Then challenge the students to try to make a vector field that forces you to walk in a circle, by pointing appropriately. Finally, have them do it again, this time pointing in a random direction, not worrying about their neighbors. Show that it is now (probably) impossible to walk through the hall, because there are points where there isn't a clear direction to follow. Point out that in a true vector field, the speed at which you walk would be determined by the length of the students' arms.

Group Work 1: Sketching Vector Fields

Solutions are included with this group work. We recommend either handing them out to the students at the end of the activity, or displaying them with an overhead projector.

Group Work 2: Gradient Fields and Level Curves

The students should choose obvious points for the level curves for $f(x,y) = \frac{1}{4}x^2 + \frac{1}{9}y^2$ (ellipses), and the level curves for $f(x,y) = \dfrac{y}{x+y}$, $x \neq -y$ (straight lines). For the latter function, note that $\nabla f = \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{(x+y)^2}$, and along the level curve $y = \dfrac{k}{1-k}x$, $k \neq 1$, $\nabla f = \dfrac{x}{k-1}\left[k\mathbf{i} + (k-1)\,\mathbf{j}\right]$.

Group Work 3 (Advanced): Points of Calm

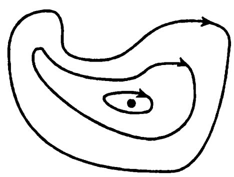
This is a difficult project which tries to show the non-existence of non-vanishing continuous vector fields on the sphere. The first part of this exercise is straightforward, the second is tricky, and the third is intended for a particularly motivated or talented group of students.

Set this activity up by having the students give examples of vector fields over the Earth, such as wind velocity or temperature gradients. Review the definition of a non-vanishing vector field, and give an intuitive idea of what is meant by a continuous vector field. A good example for part 2(a) is the function $2 + \sin(2\pi(x + y))$ or the function $3 + \sin(2\pi x) + \sin(2\pi y)$.

When the students are working on the second part, show them how one can create a torus out of the square by folding the sides together. Have the students figure out what kinds of vector fields on a square become continuous vector fields on a torus. Point out the basic topological idea that the vector field can now be viewed as a tangent vector field, since the torus becomes curved, but the tangent vectors stay "flat".

Part 3 is much harder than part 2. You may simply want to discuss what would happen if you tried to use an argument similar to the argument in part 2, that is, identifying the entire boundary with one point and trying to write a non-constant continuous function which lines up on the boundaries.

Another possible direction is to indicate that the answer to part 3 is "no," but that the proof is actually quite advanced. In an advanced class, you could provide an intuitive argument for the following special case: Assume that the solutions are a collection of nested closed curves, shrinking to a point as in the figure below. Since the solutions don't cross, you can keep on moving to the center point within all the nested closed curves. The vector field must vanish at this point; otherwise, the vector field would not be continuous there.



Conclude by discussing how this result shows that, at any given moment, there is at least one spot on the Earth at which no wind blows.

Homework Problems

Core Exercises: 2, 7, 11–14, 22, 25

Sample Assignment: 2, 6, 7, 11–14, 16, 19, 20, 22, 25, 27, 28, 29–32

Note: Problems 19, 20, 27, and 28 require a CAS.

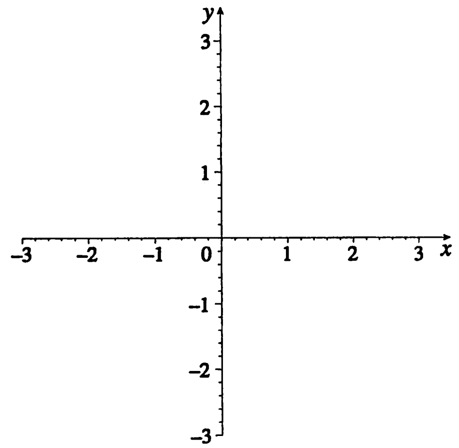
Exercise	C	A	N	G	V
1–10				×	
11–14					×
16					×
19		×		×	×
20		×		×	×

Exercise	C	A	N	G	V
22		×			
25		×		×	
27				×	×
28				×	×
29–32					×

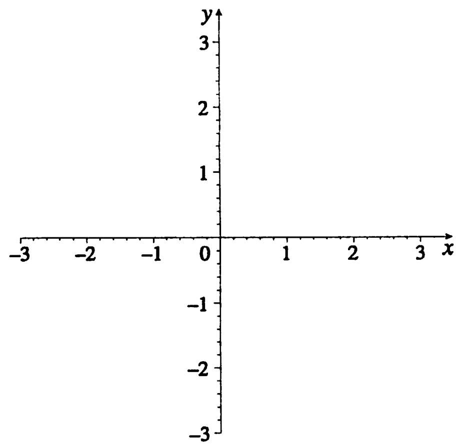
Group Work 1, Section 13.1
Sketching Vector Fields

Sketch each of the following vector fields.

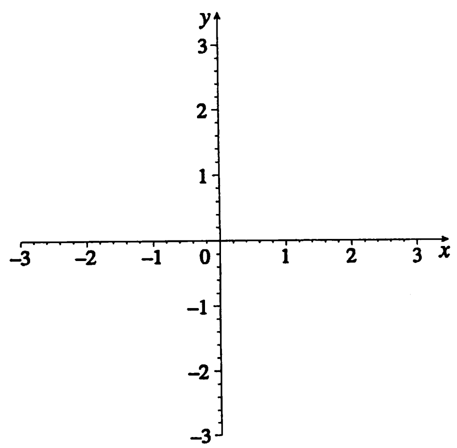
1. $x\,\mathbf{i} + y\,\mathbf{j}$



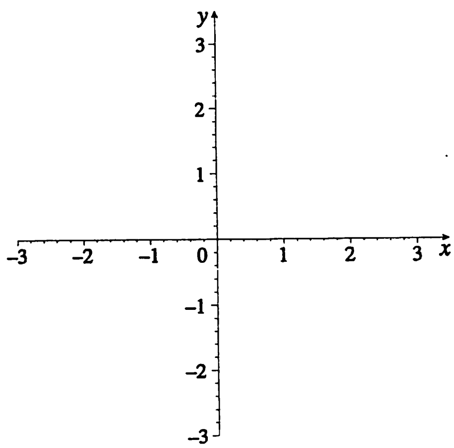
2. $y\,\mathbf{i} - x\,\mathbf{j}$



3. $\dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{(x^2 + y^2)^{1/2}}$

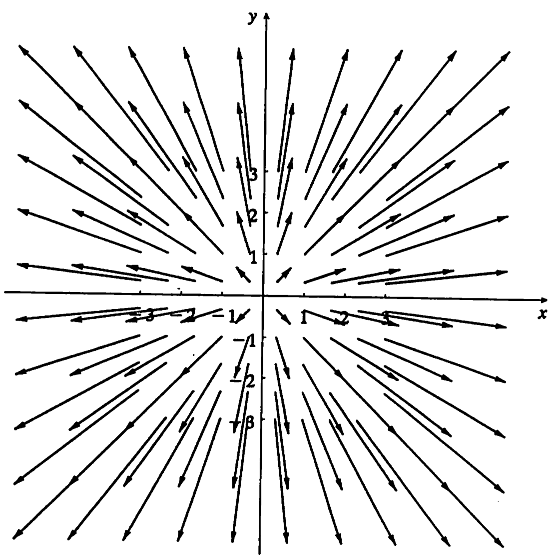


4. $y^2\,\mathbf{i} + x^2\,\mathbf{j}$

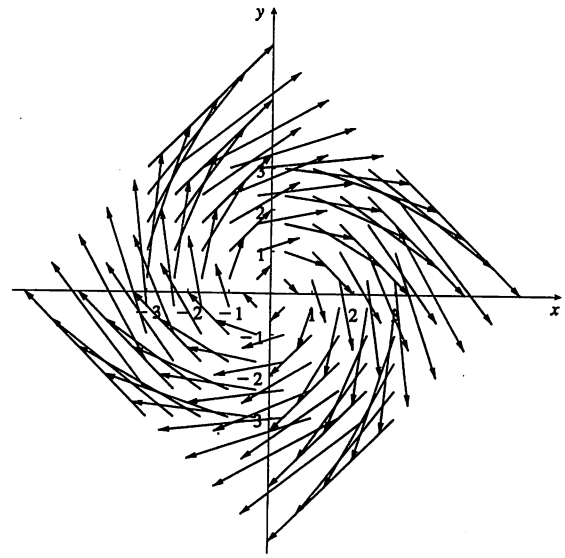


Group Work 1, Section 13.1
Sketching Vector Fields (Solutions)

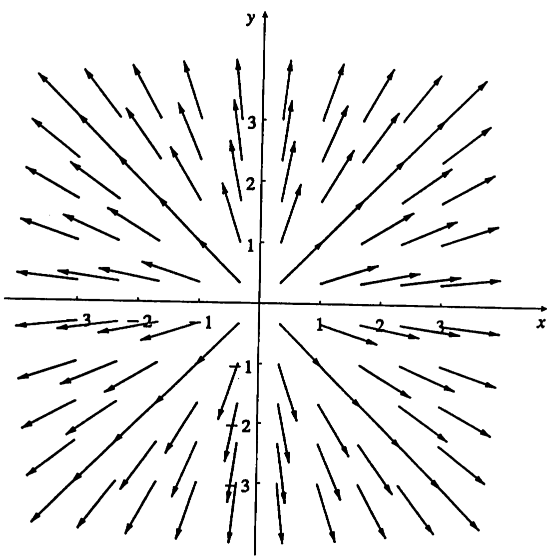
1. $x\,\mathbf{i} + y\,\mathbf{j}$



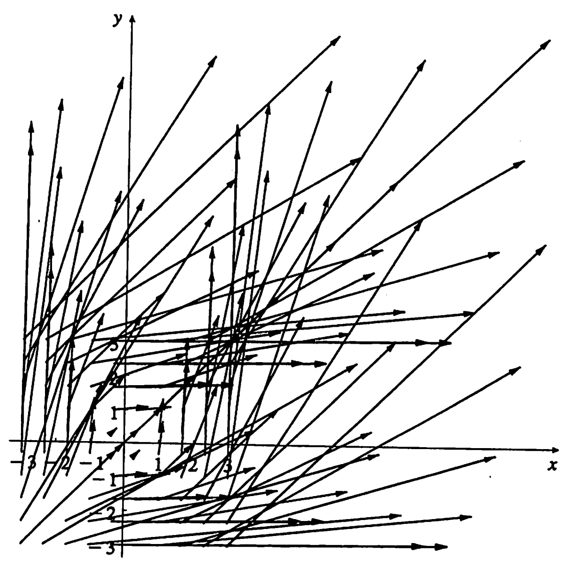
2. $y\,\mathbf{i} - x\,\mathbf{j}$



3. $\dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{(x^2 + y^2)^{1/2}}$



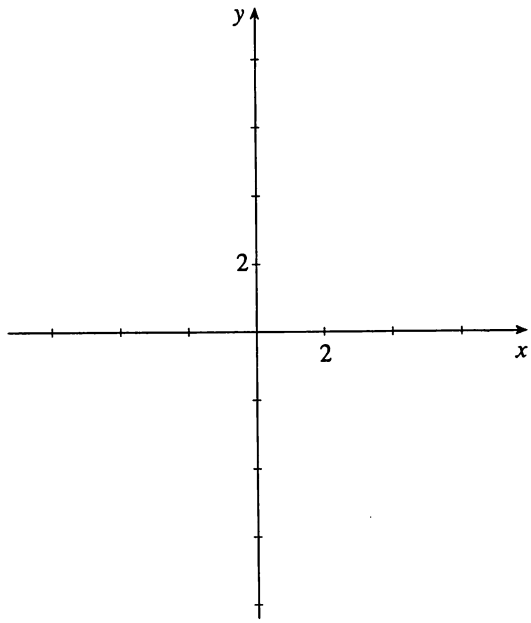
4. $y^2\,\mathbf{i} + x^2\,\mathbf{j}$



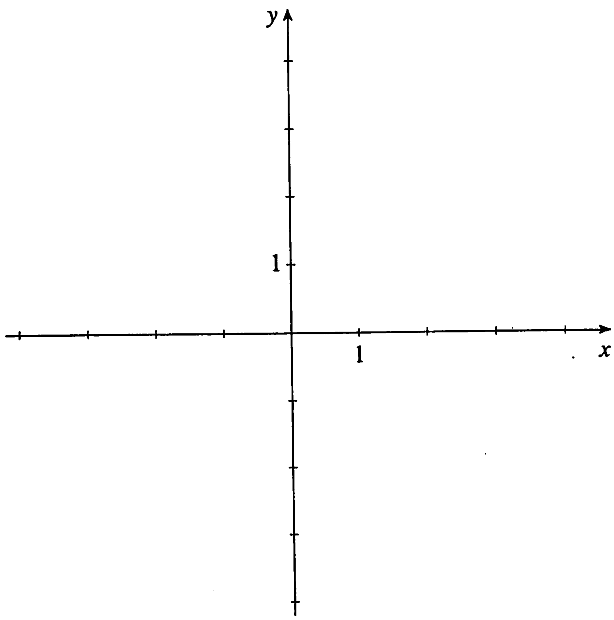
Group Work 2, Section 13.1
Gradient Fields and Level Curves

Compute the gradient fields for the following functions, and draw level curves $f(x,y) = k$ for the indicated values of k. Then sketch the gradient vector field at one or two points on each of these level curves.

1. $f(x,y) = \frac{x^2}{4} + \frac{y^2}{9}$; $k = 1, 2, 4$

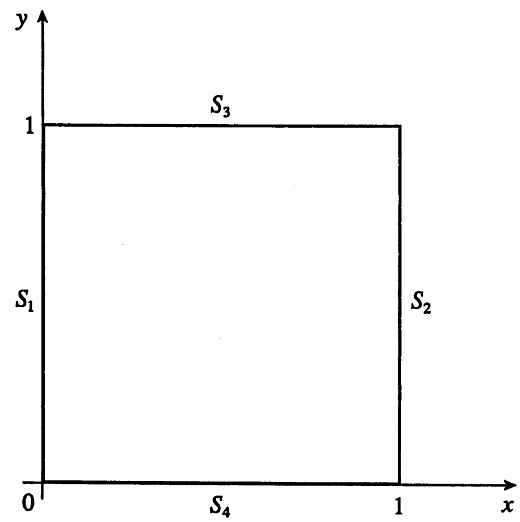


2. $f(x,y) = \frac{y}{x+y}$, $y \neq -x$; $k = \frac{1}{2}, \frac{3}{4}, 2$



Group Work 3, Section 13.1
Points of Calm

1. Draw a non-constant, non-vanishing, continuous vector field on the following unit square:



2. A torus (doughnut) can be obtained from a square by "gluing" the side S_1 to the side S_2, and then "gluing" S_3 to S_4.

(a) Describe a non-constant continuous function $f(x, y)$ such that $f(x, 0) = f(x, 1)$ for all x, and $f(0, y) = f(1, y)$ for all y. Notice that your function $f(x, y)$ can now be viewed as a continuous function on the torus.

(b) Describe a non-constant, non-vanishing, continuous tangent vector field on the torus.
Hint: Consider $\mathbf{F}(x, y) = f(x, y)(\mathbf{i} + \mathbf{j})$ where f is the function you found in part (a).

3. You have now described a non-constant, non-vanishing, continuous vector field on the torus. Is it possible to draw such a vector field on the unit sphere?

13.2 Line Integrals

Suggested Time and Emphasis

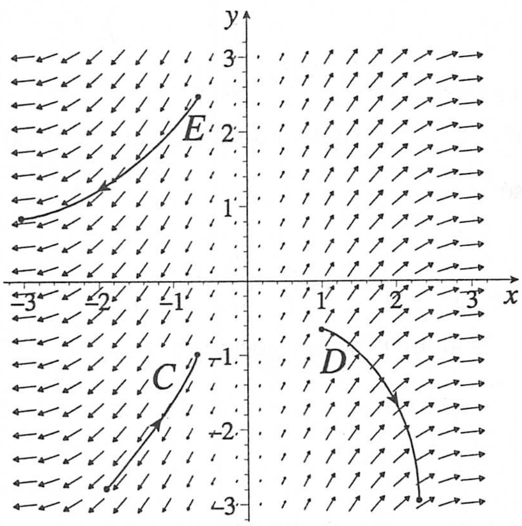
1–$1\frac{1}{2}$ classes   Essential Material

Points to Stress

1. The meaning of the line integral of a scalar function $f(x, y)$ along a curve C.
2. The meaning of $\int P\,dx + Q\,dy$ along a curve C.
3. Vector fields and work: the meaning of $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Text Discussion

- Place the three line integrals $\int_C \mathbf{F} \cdot d\mathbf{r}$, $\int_D \mathbf{F} \cdot d\mathbf{r}$, and $\int_E \mathbf{F} \cdot d\mathbf{r}$ in order from largest (most positive) to smallest (most negative).



Materials for Lecture

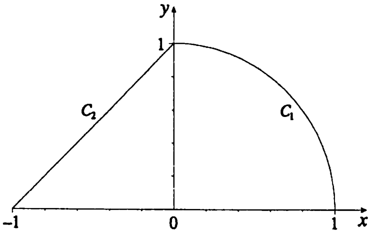
- Discuss the line integral of a scalar function as an extension of the ordinary single integral. Show in some detail why Formulas 3 and 9 actually work. In other words, partition the time interval, and show how the integral is approximated by the sum $\sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$. In particular, show again why

$$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\,\Delta t_i$$

  For the same formulas in three dimensions, simply show geometrically how the term under the square root involves Δz_i as well.

- Discuss the analytic and geometric interpretations of $\displaystyle\int_C f(x, y)\,ds = \int_a^b f(x(t), y(t))\,\frac{ds}{dt}\,dt$, where s is the arc length along the smooth curve C: $x = x(t)$, $y = y(t)$, $a \le t \le b$. This can be based on the area interpretation of $f(x,y) \geq 0$ along C, as shown in Figure 2 on page 925.

- Work through the following rich example: Consider the function $f(x,y) = x + y$ along the curve $C = C_1 \cup C_2$, where C_1 is parametrized by $x(t) = \cos t$, $y(t) = \sin t$, $0 \leq t \leq \frac{\pi}{2}$, and $C_2$is parametrized by $x(t) = -t$, $y(t) = 1 - t$, $0 \leq t \leq 1$.



$\int_C f(x,y)\,ds = \int_{C_1} (\cos t + \sin t)\sqrt{(-\sin t)^2 + (\cos t)^2}\,dt + \int_{C_2} [-t + (1-t)]\sqrt{(-1)^2 + (-1)^2}\,dt$

- If, instead of arc length, we just want to measure the (signed) distance traveled parallel to the x-axis, we can use the differential $dx = \dfrac{dx}{dt}\,dt$ for $x(t)$, and so

$$\int_C f(x,y)\,dx = \int_C f(x(t), y(t))\frac{dx}{dt}\,dt = \int_a^b f(x(t), y(t))\frac{dx}{dt}\,dt$$

Similarly, if we just want to measure the (signed) distance traveled parallel to the y-axis, we can use the differential $dy = \dfrac{dy}{dt}\,dt$, and so

$$\int_C f(x,y)\,dy = \int_C f(x(t), y(t))\frac{dy}{dt}\,dt = \int_a^b f(x(t), y(t))\frac{dy}{dt}\,dt$$

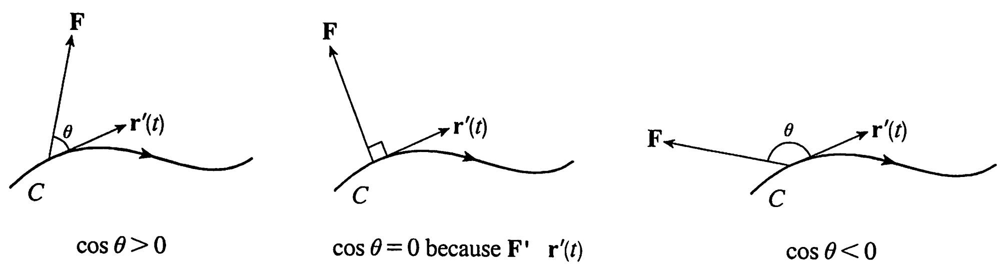
These are called the line integrals along C with respect to x and y. In the example above, we have

$$\int_C f(x,y)\,dx = \int_0^{\pi/2} (\cos t + \sin t)(-\sin t)\,dt + \int_0^1 (1-2t)\,dt = -\tfrac{1}{4}(\pi + 1)$$

and similarly $\int_C f(x,y)\,dy = \frac{1}{4}(\pi + 2)$.

- Show that if $\mathbf{F} = (x+y)\,\mathbf{i} + \mathbf{j}$, then the sum of the line integrals $\int_{-1}^{1} [(x+y)\,dx + dy]$ is not independent of path, by using the previous curve $C = C_1 \cup C_2$ and also using the line segment from $(1,0)$ to $(-1,0)$ as a curve C^*, parametrizing C^* as $x(t) = -t$, $y(t) = 0$, $-1 \leq t \leq 1$.

- In analyzing $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (\mathbf{F} \cdot \mathbf{T})\,ds$, show how the sign of $\mathbf{F} \cdot \mathbf{T}$ can be determined visually by looking at the angle θ between $\mathbf{F}$ and $\mathbf{T}$. Since $\mathbf{r}'(t)$ and $\mathbf{T}(t)$ are parallel and point in the same direction, the angle θ is also the angle between $\mathbf{F}$ and $\mathbf{r}'(t)$.



Here is an example treated algebraically using the previous curve $C = C_1 \cup C_2$ and $\mathbf{F}(x,y) = (-y+x)\,\mathbf{i} + y\,\mathbf{j}$: Along C_1, $\mathbf{F}\cdot\mathbf{r}'(t) = -\sin^2 t \le 0$, and along C_2, $\mathbf{F}\cdot\mathbf{r}'(t) = 3t-2$, which is positive for $0 \le t < \frac{2}{3}$, zero at $t = \frac{2}{3}$, and negative for $\frac{2}{3} < t \le 1$.

- If $\mathbf{F} = \mathbf{F}(x,y)$ is a vector field defined on a curve C parametrized by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, then we define $\int_C \mathbf{F}\cdot d\mathbf{r}$ as the line integral $\int_C (\mathbf{F}\cdot\mathbf{T})\,ds$, where $\mathbf{T}$ is a unit tangent to C at $(x(t), y(t))$. Recalling that $\mathbf{T}'(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ and $ds = |\mathbf{r}'(t)|\,dt$, and substituting, we get the useful equation

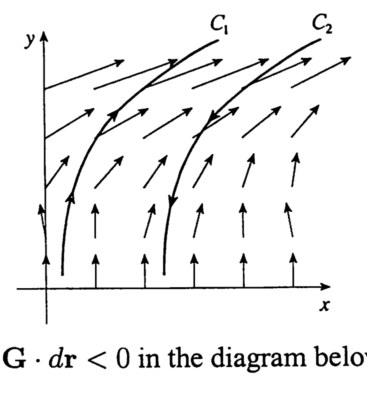
$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \frac{\mathbf{F}\cdot\mathbf{r}'(t)}{|\mathbf{r}'(t)|}|\mathbf{r}'(t)|\,dt = \int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_a^b F_1\frac{dx}{dt}\,dt + \int_a^b F_2\frac{dy}{dt}\,dt = \int_C F_1\,dx + F_2\,dy$$

  If we consider $\mathbf{F}$ to be a force on a particle, then we can interpret $\int_C \mathbf{F}\cdot d\mathbf{r}$ to be the work done by the field $\mathbf{F}$ as the particle moves along the curve C. Similar results hold in $\mathbb{R}^3$.

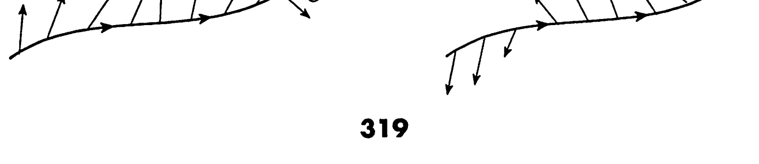
- Refer to Figure 12 and point out the difference between positive and negative work. The work done by the field in the figure is negative, but the work done by *you* is positive.

Workshop/Discussion

- Briefly discuss line integrals in $\mathbb{R}^3$.
- Consider the curve C parametrized by $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$, $0 \le t \le 1$. Draw the curve, then describe what is meant by $-\mathbf{r}$ and how replacing t by $1-t$ parametrizes $-\mathbf{r}$ as $(1-t)\,\mathbf{i} + (1-t)^2\,\mathbf{j}$, $0 \le t \le 1$..Explain why the curve given by this parametrization is denoted $-C$.
- Compute the integral $\int_C (x\cos y\,\mathbf{i} + \sin y\,\mathbf{j})\cdot d\mathbf{r}$ along the previous curve, obtaining $\frac{1}{2}\cos 1 + \sin 1 - 1$. Then compute $\int_C (x\cos y\,\mathbf{i} + \sin y\,\mathbf{j})\cdot d(-\mathbf{r}) = \int_0^1 (x\cos y\,\mathbf{i} + \sin y\,\mathbf{j})\cdot\left[(-\mathbf{r})'(t)\right]dt$ which turns out (after a u-substitution) to be $-\left(\frac{1}{2}\cos 1 + \sin 1 - 1\right)$, the negative of the line integral over $\mathbf{r}$. Conclude that $\int_{-C} \mathbf{F}\cdot d\mathbf{r} = -\int_C \mathbf{F}\cdot d\mathbf{r}$, in this case. Give an intuitive arguemnt as to why this is true in general.
- Consider the vector field $\mathbf{F}(x,y)$ and the curves C_1 and C_2 shown below. Explain why $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} > 0$ and $\int_{C_2} \mathbf{F}\cdot d\mathbf{r} < 0$.



- Explain why $\int_C \mathbf{F}\cdot d\mathbf{r} > 0$ and $\int_C \mathbf{G}\cdot d\mathbf{r} < 0$ in the diagram below.



- Compute the line integral of $\mathbf{F} = x^2 y\,\mathbf{i} + y^4\,\mathbf{j} + z^6\,\mathbf{k}$ along the curve $\mathbf{r}_1(t)$ given by $x = t^3$, $y = t$, $z = t^2$, $0 \le t \le 1$. Repeat for $\mathbf{r}_2(t)$, given by $x = t$, $y = t^2$, $z = t^3$, $0 \le t \le 1$. Note that both integrals go from $(0,0,0)$ to $(1,1,1)$, but the different paths led to different answers.
- Demonstrate that the value of a line integral is independent of the parametrization by considering the following parametrizations of the unit circle:
  1. $\boldsymbol{\alpha}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \le t \le 2\pi$
  2. $\boldsymbol{\beta}(t) = \langle \cos(2t), \sin(2t) \rangle$, $0 \le t \le \pi$
  3. $\boldsymbol{\gamma}(t) = \langle \cos(t^2), \sin(t^2) \rangle$, $0 \le t \le \sqrt{2\pi}$

  Show that for each parametrization, the unit circle is traversed once and the arc length is 2π.

Group Work 1: Fun With Line Integrals

This activity should give students an intuitive feel for line integrals.

Group Work 2: Computing Line Integrals

Group Work 3: Line Integrals over Circles and Ellipses

Group works 2 and 3 anticipate the material on conservative vector fields and independence of path developed in Section 13.3.

Group Work 4: I Sing the Field Electric!

Homework Problems

Core Exercises: 1, 6, 8, 13, 17, 31, 37

Sample Assignment: 1, 4, 6, 7, 8, 13, 17, 22, 25, 31, 34, 37

Note: Exercise 22 requires a CAS.

Exercise	C	A	N	G	V
1–12		×			
13					×
17		×			
22		×		×	
25		×			
31		×			
34		×			
37			×		

Group Work 1, Section 13.2
Fun With Line Integrals

Determine if the following line integrals $\int_C f(x,y)\,ds$ and $\int_C \mathbf{F}\cdot d\mathbf{r}$ are positive, negative or 0 either by graphical analysis or by direct computation.

1. $f(x,y) = \dfrac{y}{x^2+y^2}$; C is the top half of the unit circle, starting at $(-1,0)$ and moving clockwise.

2. $f(x,y) = \dfrac{y}{x^2+y^2}$; C is the bottom half of the unit circle, starting at $(1,0)$ and moving clockwise.

3. $f(x,y) = x^2 \sin \pi y$; C is the curve parametrized by $x = t$, $y = t^3$, $-1 \leq t \leq 1$.

4. $\mathbf{F}(x,y) = x\mathbf{i} - y\mathbf{j}$; C is the top half of the unit circle, starting at $(1,0)$ and moving counterclockwise.

5. $\mathbf{F}(x,y) = x\mathbf{i} - \dfrac{1}{\sqrt{x}}\mathbf{j}$; C is the part of the parabola $y = x^2$ starting at $(1,1)$ and ending at $(2,4)$.

Group Work 2, Section 13.2
Computing Line Integrals

1. Compute the line integral of $\mathbf{F} = x^2\,\mathbf{i} + y^4\,\mathbf{j} + z^6\,\mathbf{k} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ over the paths $\mathbf{r}_1$: $x = t^3$, $y = t$, $z = t^2, 0 \le t \le 1$ and $\mathbf{r}_2$: $x = t$, $y = t^2$, $z = t^3$, $0 \le t \le 1$.

2. Compute the line integral of $\mathbf{F} = x^2\,\mathbf{i} + y^4\,\mathbf{j} + z^6\,\mathbf{k}$ over the path $\mathbf{r}_3$: $x = t$, $y = t$, $z = t$, $0 \le t \le 1$.

3. Find $g(x, y, z)$ so that $\mathbf{F} = \nabla g$. **Hint:** Assume $g(x, y, z) = h(x) + l(y) + k(z)$.

4. Compute $\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$. Using the result from Problem 3, can you give a reason why Clairaut's Theorem could have been used to predict your answer?

5. Let $\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 0\mathbf{k}$, $0 \le t \le 2\pi$ be a parametrization of the unit circle. First make a conjecture as to the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$, and then compute it.

Group Work 3, Section 13.2
Line Integrals Over Circles and Ellipses

Let $\mathbf{F}(x, y) = x\,\mathbf{i} - y\,\mathbf{j}$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ if

1. C is the unit circle $\mathbf{r}$: $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$.

2. C lies along the unit circle, starting at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and moving counterclockwise to $(-1, 0)$.

3. C lies along the unit circle, starting at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and moving clockwise to $(-1, 0)$.

4. C is the ellipse $\mathbf{r}$: $x = \cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$.

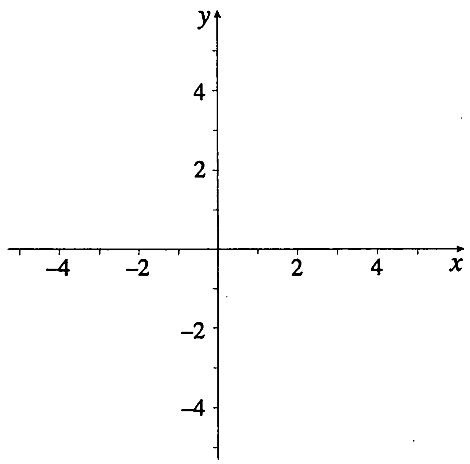
5. C is the ellipse $\mathbf{r}$: $x = 2\cos t$, $y = 3\sin t$, $0 \le t \le 2\pi$.

Group Work 4, Section 13.2
I Sing the Field Electric!

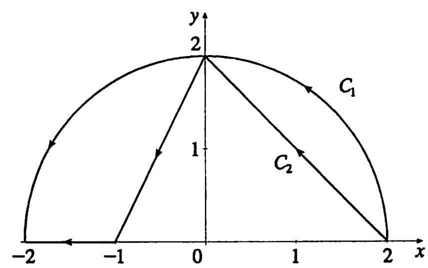
A charge q located at $(0, 0)$ creates an electric field at (x, y) given by

$$\mathbf{F} = \frac{K\,(x\,\mathbf{i} + y\,\mathbf{j})}{(x^2 + y^2)^{3/2}},\ K \text{ constant}$$

1. Draw this vector field in the spirit of Figures 4 and 8 in Section 13.1, and then calculate the work required to move a charge around the circle $x^2 + y^2 = 25$ in this field.



2. Calculate the work required to move a charge along the path C_1, the top half of $x^2 + y^2 = 4$.



3. Calculate the work required to move the charge along the path C_2, which consists of three line segments (see above).

13.3 The Fundamental Theorem for Line Integrals

Suggested Time and Emphasis

$1\text{–}1\frac{1}{4}$ classes   Essential Material

Points to Stress

1. The path independence of $\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$ under suitable conditions.

2. The equivalence of path independence to the condition that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C in the domain of $\mathbf{F}$.

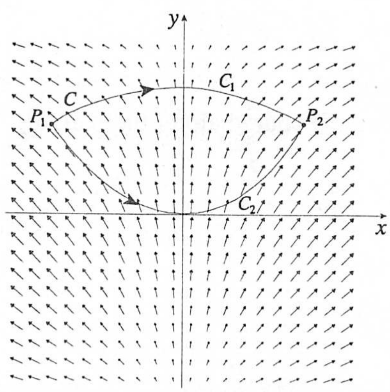
3. The equivalence of the following three conditions on a simply-connected domain:
   - Path independence
   - $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ being a conservative vector field ($\mathbf{F} = \nabla f$)
   - $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Text Discussion

- Is it true that every integral of $\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + (x - 2)\,\mathbf{j}$ is independent of path? Why or why not?
- Where is the Fundamental Theorem of Calculus used in the proof of Theorem 2 on page 936?

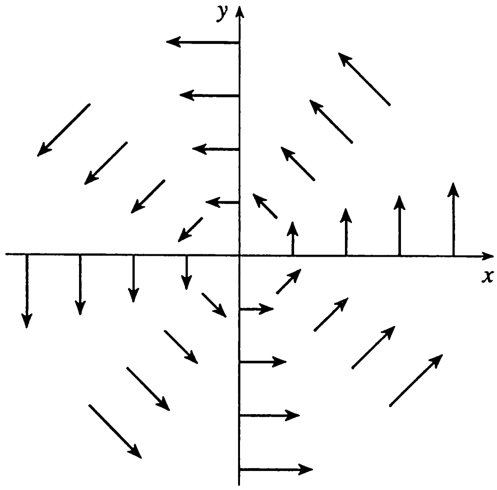
Materials for Lecture

- Give a proof for smooth curves of the Fundamental Theorem for line integrals, such as the one given on page 936.
- Give an outline of the proof that if every integral of $\mathbf{F}$ is independent of path and C is a closed curve in the domain of $\mathbf{F}$, then $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$. First write $C = C_1 \cup -C_2$ with each of C_1 and C_2 starting at P_1 and ending at P_2.



Then $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = -\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, so $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0$.

Explain why integrals of the vector field below are not generally independent of path, and hence that the field is not conservative.



- To show the geometry of conservative vector fields, look at the level sets of the potential function for some conservative vector fields, perhaps using Figure 9 on page 940. Explain why it is plausible that the line integral around a closed path is 0.

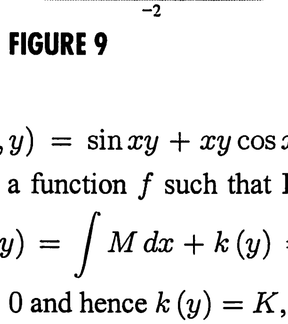


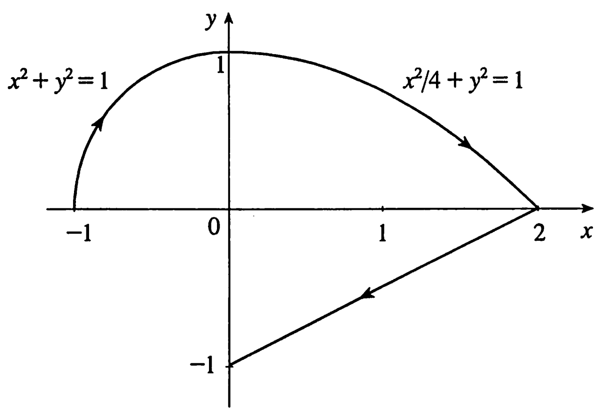
FIGURE 9

- Go through the following example:
Consider $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ with $M(x,y) = \sin xy + xy\cos xy$ and $N(x,y) = x^2\cos xy$. Verify that $\partial M/\partial y = \partial N/\partial x$. We want to find a function f such that $\mathbf{F} = \nabla f$. So assume that $M = \partial f/\partial x$. Then $f(x,y) = \int (\partial f/\partial x)\,dx + k(y) = \int M\,dx + k(y) = x\sin xy + k(y)$. Now $N = \partial f/\partial y = x^2\cos xy + k'(y)$. This gives $k'(y) = 0$ and hence $k(y) = K$, a constant. So $f(x,y) = x\sin xy + K$ is a function that satisfies $\nabla f = \mathbf{F}$.

- Repeat the same procedure with $M(x,y) = x^2y$ and $N(x,y) = xy^2$. This time $\partial M/\partial y \neq \partial N/\partial x$, and the procedure doesn't yield a $k(y)$ that works. So when $\mathbf{F}$ is not conservative, we cannot find a function f such that $\nabla f = \mathbf{F}$.

Workshop/Discussion

- Consider $\mathbf{F} = yz^2\,\mathbf{i} + xz^2\,\mathbf{j} + 2xyz\,\mathbf{k}$. Let $M = yz^2$, $N = xz^2$, and $R = 2xyz$. Check that $\partial M/\partial z = \partial R/\partial x$, $\partial M/\partial y = \partial N/\partial x$, and $\partial N/\partial z = \partial R/\partial y$, and so by Clairaut's Theorem, this is possibly a gradient field. Now try the procedure outlined in Materials for Lecture above:
$$f(x,y,z) = \int (\partial f/\partial z)\,dz = \int (2xyz)\,dz = xyz^2 + g(x,y)$$
and now
$$N = \partial f/\partial y = xz^2 + \partial g/\partial y = xz^2, \text{ by definition of } N$$
so $\partial g/\partial y = 0$, meaning that $g(x,y)$ is a function only of x, that is, $g(x)$. Then
$$M = \partial f/\partial x = yz^2 + g'(x) = yz^2$$
Thus $g(x) = k$ is a constant, and $f(x,y,z) = xyz^2 + k$.
- Show students why it is very easy to evaluate $\int_{C_1} \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = 2xy\,\mathbf{i} + x^2\,\mathbf{j}$ and C_1 is the curve shown below, by noting that $\mathbf{F}$ is conservative, and then either computing f such that $\nabla f = \mathbf{F}$ or by using the straight line path from $(-1,0)$ to $(0,-1)$.



Group Work 1: Think Before You Compute

In Problem 2, even after recognizing that $\mathbf{F}$ is conservative, the direct path from $(2,0,0)$ to $(0,1,2)$ is not the easiest choice for computations. For example, the path $(2,0,0) \longrightarrow (0,0,0) \longrightarrow (0,0,2) \longrightarrow (0,1,2)$ makes for a simpler calculation.

Group Work 2: Finding the Gradient Fields

Have each group try one of the first three problems below, and give Problem 4 to groups that finish early.
The following vector fields are conservative. Find the function $f(x,y)$ or $f(x,y,z)$ for which they are gradient fields.

1. $\mathbf{F}(x,y) = 3xy^2\,\mathbf{i} + 3x^2y\,\mathbf{j}$
2. $\mathbf{F}(x,y) = y\sin(xy)\,\mathbf{i} + x\sin(xy)\,\mathbf{j}$
3. $\mathbf{F}(x,y) = (2x+y)\,\mathbf{i} + (x+3y^2)\,\mathbf{j}$
4. $\mathbf{F}(x,y,z) = yze^{xyz}\,\mathbf{i} + xze^{xyz}\,\mathbf{j} + xye^{xyz}\,\mathbf{k}$

Extended Group Work 3: The Winding Number

Parts 1–5 of this extended activity can be done independently of the remaining parts, and may be suitable as a challenging in-class group work. The concept of a winding number is completely developed in this activity.

Homework Problems

Core Exercises: 1, 2, 4, 7, 11, 15, 23

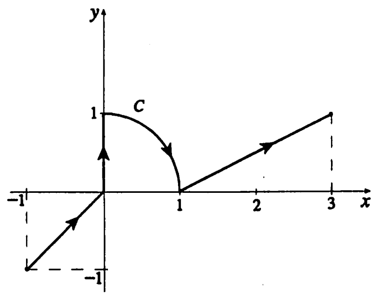
Sample Assignment: 1, 2, 4, 7, 10, 11, 12, 15, 21, 23, 27, 28

Exercise	C	A	N	G	V
1			×		×
2			×		
3–10		×			
11	×	×			×
12		×			

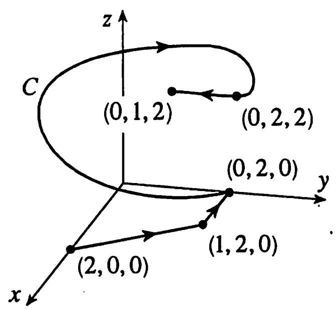
Exercise	C	A	N	G	V
15		×			
21		×			
23	×				×
27		×			
28		×			

Group Work 1, Section 13.3
Think Before You Compute

1. Compute $\int_C (ye^{xy}\,\mathbf{i} + xe^{xy}\,\mathbf{j}) \cdot d\mathbf{r}$ for the curve C shown below.



2. Compute $\int_C \left(yz^2\,\mathbf{i} + xz^2\,\mathbf{j} + 2xyz\,\mathbf{k}\right) \cdot d\mathbf{r}$ for the curve C shown below.



Group Work 3, Section 13.3
The Winding Number

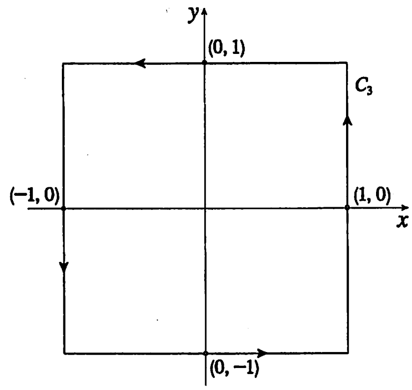
In this activity we consider the vector field $\mathbf{F}(x, y) = -\dfrac{y}{x^2+y^2}\,\mathbf{i} + \dfrac{x}{x^2+y^2}\,\mathbf{j}$.

1. Show that $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = 0$ where $\mathbf{F}$ is defined.

2. Compute $\oint_{C_1} \mathbf{F} \cdot d\mathbf{r}$ where C_1 is the unit circle centered at the origin, oriented counterclockwise.

3. Compute $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r}$ where C_2 is the circle $(x-2)^2 + y^2 = 1$, oriented counterclockwise.

4. Compute $\oint_{C_3} \mathbf{F} \cdot d\mathbf{r}$ where C_3 is the square shown below.



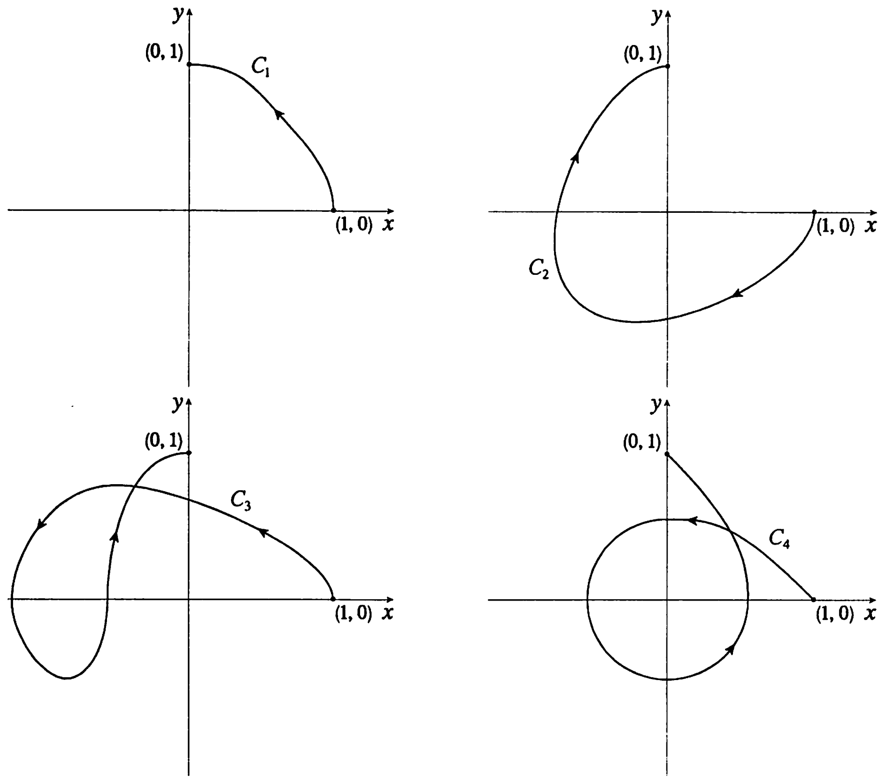
5. For what closed paths will you get zero for $\int_C \mathbf{F} \cdot d\mathbf{r}$, and under what conditions will you get a nonzero answer?

6. One meaning of $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for a closed curve C and any vector field $\mathbf{F}$ is the net circulation of $\mathbf{F}$ around C. Suppose we take an arbitrarily small path around a point (not the origin). What is the net circulation of $\mathbf{F}$ around this small path?

7. What is the net circulation of $\mathbf{F}$ around *any* path which encloses the origin?

8. Letting θ be the angle in polar coordinates for a point (x, y), show that $d\theta = -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$ and hence the vector field $\mathbf{F}$ is the gradient vector field for θ. Conclude that $\oint_C \mathbf{F} \cdot d\mathbf{r} = \theta(B) - \theta(A)$ where C connects the point A to the point B, and thus we can write $\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C d\theta$.

9. Use the previous result to calculate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for the following paths.



10. The number $\frac{1}{2\pi} \oint_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2\pi} \oint_C d\theta$ is called the **winding number** for any closed curve C. It measures the number of times C "winds" counterclockwise around the origin. Find parameterizations for closed paths with winding numbers of 1, 2, -1, and 4.

13.4 Green's Theorem

Suggested Time and Emphasis

1 class Essential Material

Points to Stress

1. The statement of Green's Theorem over a region D with boundary curve $C = \partial D$:

$$\oint_C P\,dx + Q\,dy = \oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

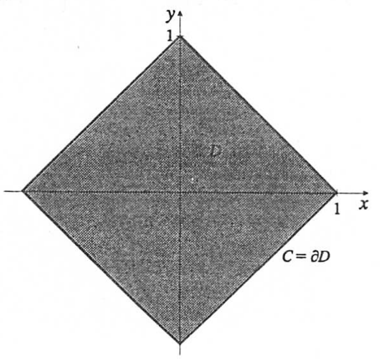
2. The extension of Green's Theorem to domains with holes.
3. The importance of Green's Theorem, in that it allows us to replace a difficult line integration by an easier area integration, or a difficult area integration by an easier line integration.

Text Discussion

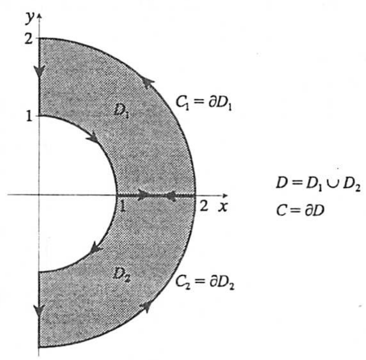
- If we know that $P(x, y) \equiv 0$ and $Q(x, y) \equiv 0$ on the boundary $C = \partial D$ of a region D, what is $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$?
- How do holes in a region affect $\oint_C P\,dx + Q\,dy$?
- Express $\oint_C y\,dx$ in terms of the region D enclosed by C.

Materials for Lecture

- Have a discussion of terminology: What is meant by "positive orientation". What is meant by "$C = \partial D$"?
- Give a careful statement of Green's Theorem. Indicate that its use is primarily to replace a difficult integral of one type (area or line) with a simpler integral of the other type.
- Go through some rich examples such as the following:



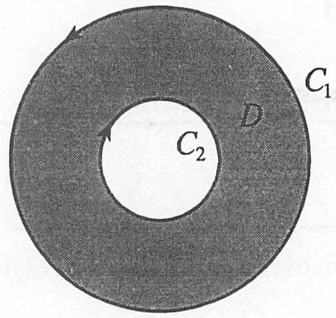
$\oint_C (x^4 + 2y)\,dx + (5x + \sin y)\,dy$
$= \iint_D 3\,dA = 6$, by geometry



$\oint_C (x^2 y)\,dx + (x^3 + 2xy^2)\,dy$
$= \oint_C = \oint_{C_1} + \oint_{C_2} = 2\iint_{D_1} (x^2 + y^2)\,dA$
Evaluating this integral using polar coordinates gives 15π.

- For arbitrary regions D, compute $\oint_{\partial D} -y\,dx + x\,dy$ using Green's Theorem, obtaining twice the area of D.

- Demonstrate Green's Theorem for regions with holes:

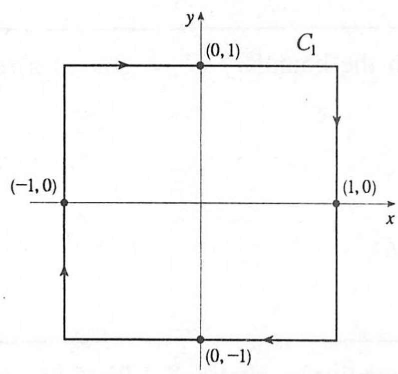


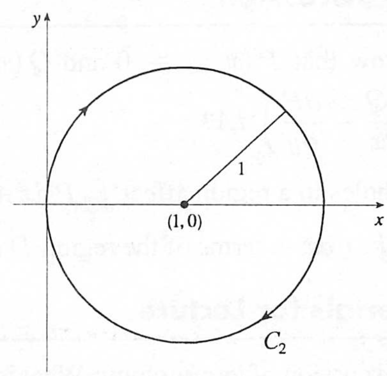
$$C = \partial D = C_1 \cup C_2$$

- Use Green's Theorem to set up a line integral to compute the area of the astroid $x^{2/3} + y^{2/3} = 1$.

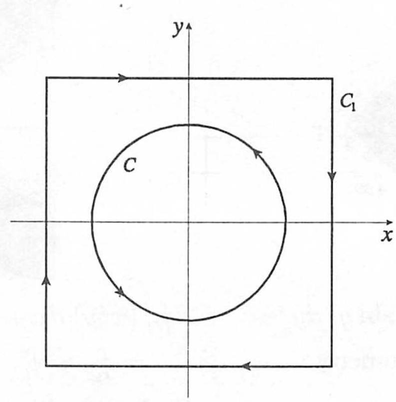
Workshop/Discussion

- Compute $\oint_C \left(y^2 - 2y + 2xy\right) dx + \left(x^2 + 3x + 2xy\right) dy$ for the following closed curves C_1 and C_2:



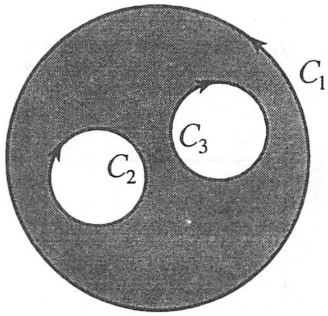


- Compute $\displaystyle\oint_{C_1} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$ in two ways: (a) by direct computation and (b) using Green's Theorem, where C_1 is the first closed curve shown above. This is equivalent to integrating around the curve C given by the unit circle oriented counterclockwise, since $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0$ in the region between the two curves.



- Suppose we know that $P(x,y) \equiv 1$ and $Q(x,y) \equiv 2$ on a boundary circle $C = \partial D$ of radius R. Ask students how to compute $\displaystyle\iint_D \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\right) dA$. Point out that on a closed curve C, $\int_C dx + 2\,dy = 0$.

- If time permits, show Green's Theorem for a region with 2 holes, showing that $C = C_1 \cup C_2 \cup C_3 = \partial D$ needs the positive orientation.



Group Work 1: Using Green's Theorem

These problems may be too hard for students to do without a few hints. Here are some hints that might prove helpful:

Problem 1: Green's Theorem can be used to show that the required integral is equal to $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{1} 4xy^3 \, dy \, dx$.

Problem 2: Green's Theorem can be used to replace the line integral with

$$\iint_D \left(6 - 3x^2 - 3y^2 + 6\right) dx\,dy = 3 \iint_D \left(4 - x^2 - y^2\right) dx\,dy$$

This integrand is positive until $x^2 + y^2 = 4$, and then remains negative. Thus letting C be the circle of radius 2 gives the maximum value of the integral, namely 24π.

An extension of Problem 2 is given in Problem 2 from Focus on Problem Solving after Chapter 13 (page 989).

Group Work 2: Green's Theorem and the Area of Plane Regions

In Problem 2, the natural parametrization does not give positive orientation, so we need to use $-C$ in its place.

Homework Problems

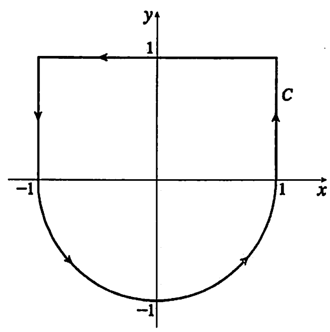
Core Exercises: 1, 4, 7, 11, 17, 19

Sample Assignment: 1, 4, 7, 8, 11, 14, 17, 19, 21, 27

Exercise	C	A	N	G	V
1		×		×	
4		×		×	
7		×		×	
8		×		×	
11		×		×	
14		×		×	
17		×		×	
19		×			
21	×	×			
27	×	×			

Group Work 1, Section 13.4
Using Green's Theorem

1. Compute $\oint_C \left(-\frac{xy^4}{2}\right) dx + \left(x^2y^3\right) dy$, where C is as follows:



2. What simple closed curve $C = \partial D$ gives the maximal value of $\oint_C \left(x^5 - 6y + y^3\right) dx + \left(y^4 + 6x - x^3\right) dy$?

Group Work 2, Section 13.4

Green's Theorem and the Area of Plane Regions

1. Let $C = \partial D$, where the area of the region D is A. Compute $\oint_C (a_1 x + a_2 y + a_3)\, dx + (b_1 x + b_2 y + b_3)\, dy$ where the a_i and b_i are constants.

2. Find the area under one arch of the cycloid with parametric equations $x(t) = 2(t - \sin t)$, $y(t) = 2(1 - \cos t)$, $0 \le t \le 2\pi$.

13.5 Curl and Divergence

Suggested Time and Emphasis

1 class   Essential Material

Points to Stress

1. The definition of curl: $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F}$
2. If $\mathbf{F}$ has continuous partial derivatives, $\mathbf{F}$ is conservative if and only if $\operatorname{curl}\mathbf{F} = 0$
3. The definition of divergence: $\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F}$
4. Physical interpretations of curl and divergence

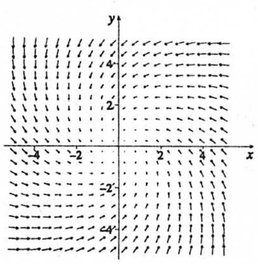
Text Discussion

- What theorem about mixed partial derivatives is used to show that if $\mathbf{F}$ is conservative, then $\operatorname{curl}\mathbf{F} = 0$?
- What do we know about $\operatorname{div}(\operatorname{curl}\mathbf{F})$?

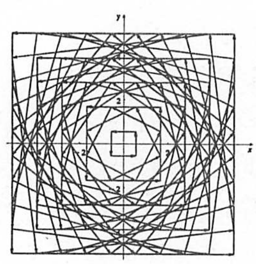
Materials for Lecture

- Make sure to point out that an expression such as $\dfrac{\partial}{\partial x}\mathbf{i}$ refers to an operator which, when applied to a function f, gives a vector, in this case $\dfrac{\partial f}{\partial x}\mathbf{i}$. Thus, ∇ maps a scalar function to its gradient, which is a vector function.
- Given $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, define $\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \nabla \times \mathbf{F}$ where ∇ is the operator $\dfrac{\partial}{\partial x}\mathbf{i} + \dfrac{\partial}{\partial y}\mathbf{j} + \dfrac{\partial}{\partial z}\mathbf{k}$ and $\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}$.
- Give examples illustrating rotation, and how it is reflected in the curl. Point out that if $\operatorname{curl}\mathbf{F} = 0$, $\mathbf{F}$ is called irrotational.

  1. $\mathbf{F} = (-x - y)\,\mathbf{i} + (x - y)\,\mathbf{j} + 0\,\mathbf{k}$. $\nabla \times \mathbf{F} = 2\,\mathbf{k}$, and the vector field is a rotation of each vector $x\,\mathbf{i} + y\,\mathbf{j}$ by $\frac{3}{4}\pi$ coupled with a stretch of $\sqrt{2}$.

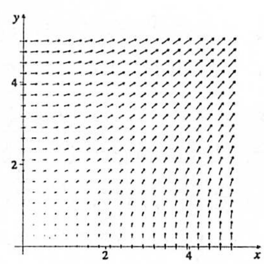


Scaled

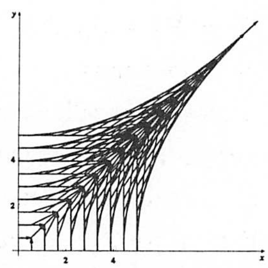


Unscaled

  2. $\mathbf{F} = y\,\mathbf{i} + x\,\mathbf{j}$. $\nabla \times \mathbf{F} = 0$, and the vector field has no rotation. Notice that $\mathbf{F}$ is conservative since $\dfrac{\partial M}{\partial y} = 1 = \dfrac{\partial N}{\partial x}$.



Scaled



Unscaled

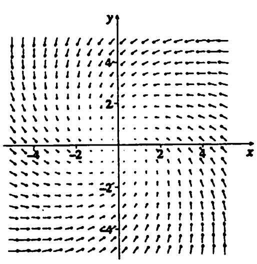
- If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + 0\,\mathbf{k}$ is an extension to $\mathbb{R}^3$ of the two-dimensional field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, then $\operatorname{curl}\mathbf{F} = \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\mathbf{k}$ and so Green's Theorem can be written as $\oint_{C=\partial D} \mathbf{F}\cdot d\mathbf{r} = \iint_D (\operatorname{curl}\mathbf{F}\cdot\mathbf{k})\,dA$.

- Point out how Clairaut's Theorem shows that $\operatorname{curl}\nabla f = \mathbf{0}$, and also that $\operatorname{div}(\operatorname{curl}\mathbf{F}) = \nabla\cdot(\nabla\times\mathbf{F}) = 0$.

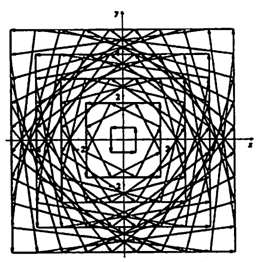
Workshop/Discussion

- The text shows that $\mathbf{F} = \nabla f(x,y,z)$ gives $\operatorname{curl}\mathbf{F} = 0$. Point out that the converse is also true under "normal" circumstances. First use the vector field $\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$ and show that $\mathbf{F} = \nabla(xyz)$. Then note that $\mathbf{F} = y^2e^{xyz}(1+xyz)\,\mathbf{i} + xye^{xyz}(2+xyz)\,\mathbf{j} + x^2y^3e^{xyz}\,\mathbf{k}$ has $\nabla\times\mathbf{F} = 0$. So $\mathbf{F} = \nabla f$. Then $f = \displaystyle\int \frac{\partial f}{\partial z}\,dz + k(x,y) = xy^2e^{xyz} + k(x,y)$. Now compute that $k(x,y) = k$, a constant, and so $f(x,y,z) = xy^2e^{xyz} + k$.

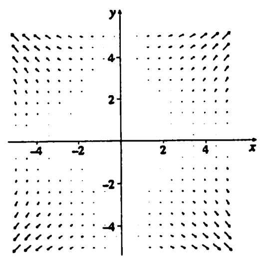
- If $\mathbf{F}_1 = (-x-y)\,\mathbf{i} + (x-y)\,\mathbf{j}$, $\nabla\cdot\mathbf{F}_1 = -1-1 = -2$. Thus the flow is tending to compress and is not diverging anywhere. If $\mathbf{F}_2 = xy^2\,\mathbf{i} + yx^2\,\mathbf{j}$, $\nabla\cdot\mathbf{F}_2 = x^2+y^2$ which is greater than zero if $(x,y) \neq (0,0)$. So in this case the flow is tending to diverge everywhere except at the origin. If $\nabla\cdot\mathbf{F} = 0$, then $\mathbf{F}$ is neither tending to compress nor tending to diverge, and $\mathbf{F}$ is called incompressible. Point out that for any vector field $\mathbf{F}$, $\operatorname{curl}\mathbf{F}$ is incompressible. Note that $\nabla\times\mathbf{F}_1 = 2\mathbf{k}$ and $\nabla\times\mathbf{F}_2 = \mathbf{0}$, both of which are clearly incompressible.



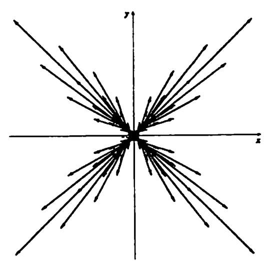
$\mathbf{F}_1$ (scaled)



$\mathbf{F}_1$ (unscaled)



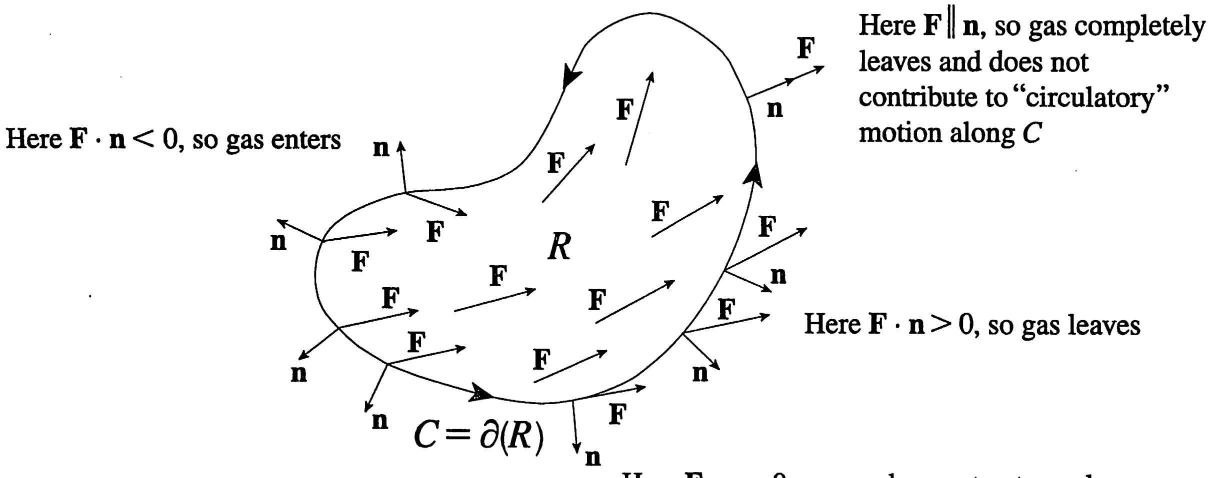
$\mathbf{F}_2$ (scaled)



$\mathbf{F}_2$ (unscaled)

- State the divergence form of Green's Theorem:
If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, then $\oint_{C=\partial D}(\mathbf{F}\cdot\mathbf{n})\,ds = \iint_D \operatorname{div}\mathbf{F}\,dA$.
The following is a physical interpretation of the theorem. Picture a gas in a thin box, all of whose particles are moving parallel to the xy-plane. Suppose that we can approximate the box by a plane, and consider a region R in the plane with boundary $C = \partial R$. At any point (x,y), if $\mathbf{F}(x,y)$ represents the velocity vector of the gas, then $\operatorname{div}\mathbf{F}(x,y)$ measures the net movement from (x,y). By summing up (integrating) $\operatorname{div}\mathbf{F}(x,y)$ over the region R, we get the net change in the amount of gas contained in R. But another way to measure the net change is to stand on C, and measure how much gas leaves at each point. Here you need the normal component $\mathbf{F}\cdot\mathbf{n}$ of $\mathbf{F}$ to C, where $\mathbf{n}$ is a unit normal to C. This is precisely another

statement of Green's Theorem, using div **F** (x, y).



- Do Exercise 33 to illustrate the relationships between curl **F** and rotations.

Group Work 1: Gradient Fields Revisited

Problem 2 shows that the result of Problem 1(b) is always true. Problem 2 is a somewhat abstract exercise, suitable for more advanced students.

Group Work 2: Divergence and Curl

Group Work 3: An Essential, Incompressible Fluid

Homework Problems

Core Exercises: 1, 5, 7, 10, 13, 19, 20

Sample Assignment: 1, 5, 7, 8, 10, 13, 16, 19, 20, 23, 24, 26

Exercise	C	A	N	G	V
1		×			
5		×			
7				×	
8				×	
10	×	×			
13		×			

Exercise	C	A	N	G	V
16		×			
19		×			
20		×			
23		×			
24		×			
26		×			

Group Work 1, Section 13.5
Gradient Fields Revisited

1. Let $\mathbf{F} = -2x\,\mathbf{i} - 3y\,\mathbf{j} + 5z\,\mathbf{k}$.

(a) Compute $\nabla \cdot \mathbf{F}$ and give a geometric description of $\mathbf{F}$.

(b) Is $\mathbf{F}$ a gradient vector field? If so, find $f(x, y, z)$ such that $\mathbf{F} = \nabla f$.

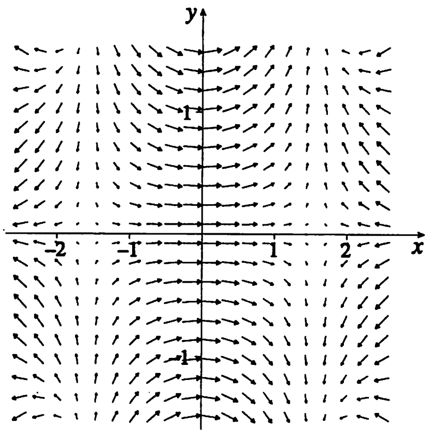
2. Let $\mathbf{F} = P(x)\,\mathbf{i} + Q(y)\,\mathbf{j} + R(z)\,\mathbf{k}$.

(a) Is $\mathbf{F}$ always a gradient vector field?

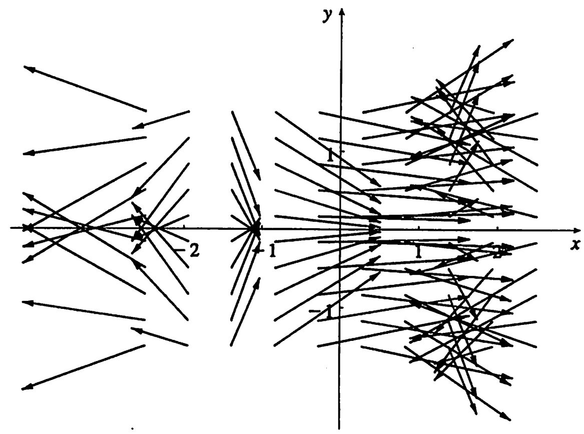
(b) Explain how you would find $f(x, y, z)$ such that $\mathbf{F} = \nabla f$, if you had explicit functions $P(x)$, $Q(y)$, and $R(z)$.

Group Work 2, Section 13.5
Divergence and Curl

Consider the vector field $f(x, y) = 2\cos x\,\mathbf{i} + \sin xy\,\mathbf{j}$ shown below.



Scaled



Unscaled

1. Find formulas for the divergence and curl of f.

2. Show that the divergence is 0 everywhere along the y-axis. How is this apparent in the graph?

3. Find the divergence at $\left(\frac{\pi}{4}, 1\right)$, $\left(-\frac{\pi}{4}, 1\right)$, $\left(\frac{\pi}{4}, -1\right)$, and $\left(-\frac{\pi}{4}, -1\right)$. How can the signs of the answers be seen in the graph?

4. Find the curl at $\left(\frac{\pi}{3}, 1\right)$ and at $\left(\frac{2\pi}{3}, 1\right)$. Relate the sign difference in your answers to the direction of the curl.

Group Work 3, Section 13.5
An Essential, Incompressible Fluid

Water is an essentially incompressible fluid, that is, the divergence of a velocity field representing a flow of water is 0. For each of the following vector fields, compute $\nabla \cdot \mathbf{F}$ and determine if $\mathbf{F}$ could represent the velocity vector field for water flowing. Then compute curl $\mathbf{F}$ and describe the axis of rotation (direction of the curl) of the fluid at the origin and at $(1, 1, 1)$.

1. $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + xz\,\mathbf{j} - yz\,\mathbf{k}$

2. $\mathbf{F}(x, y, z) = (2x - y)\,\mathbf{i} + (2z - y)\,\mathbf{j} + (2x - z)\,\mathbf{k}$

3. $\mathbf{F}(x, y, z) = \dfrac{1}{y^2 + z^2}\,\mathbf{i} - \dfrac{2xy}{(y^2 + z^2)^2}\,\mathbf{j} - \dfrac{2xz}{(y^2 + z^2)^2}\,\mathbf{k}$

13.6 Surface Integrals

Suggested Time and Emphasis

$1\text{–}1\frac{1}{4}$ classes   Essential Material

Points to Stress

1. The definition of the surface integral of a scalar function $f(x, y, z)$ viewed as an extension of the surface area integral.

2. The intuitive idea of an oriented surface with orientation given by a unit normal vector. The concept of positive orientation.

3. The surface integral of a vector field over an oriented surface

Text Discussion

- What do we know about the unit normal vector to a closed surface if that surface has positive orientation?

- Give an intuitive explanation as to why it isn't possible to choose an orientation for the Möbius strip.

Materials for Lecture

- Describe the meaning of the surface integral $\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, dA$ for $f(x, y, z)$ defined over the parametric surface $S = \mathbf{r}(u, v)$, $(u, v) \in D$.

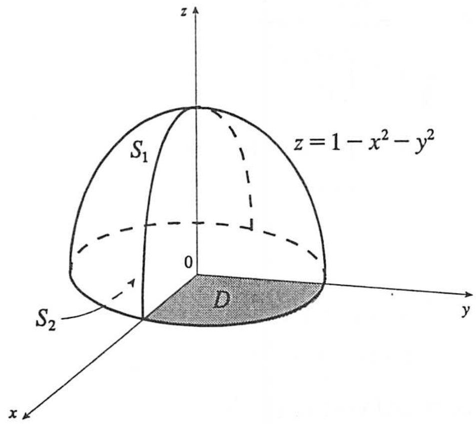
- If S is given by $z = g(x, y)$, show how the surface integral $\iint_S f(x, y, z)\, dS$ becomes

$$\iint_D f(x, y, g(x, y)) \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dA$$

- Do an extended example such as the following: Let $S = S_1 + S_2$ as in the figure (S_2 is the unit disk in the xy-plane). We want to compute $\iint_S xy\, dS = \iint_{x^2+y^2 \le 1} xy\sqrt{1 + 4x^2 + 4y^2}\, dA + \iint_{x^2+y^2 \le 1} xy\, dA$ [here $g(x, y) = 1 - x^2 - y^2$ for the first integral, $g(x, y) = 0$ for the second.] Using polar coordinates, we compute that this integral is $0 + 0 = 0$. We now use $f(x, y) = |xy|$ and some symmetry arguments. If the shaded region is D, then the surface integral becomes

$$4\left(\iint_D xy\sqrt{1 + 4x^2 + 4y^2}\, dA + \iint_D xy\, dA\right)$$

which is not 0.



- Give examples of oriented surfaces with upward orientation and closed surfaces with positive (outward) orientation (indicated by unit normals). For example, the paraboloid $z = x^2+y^2$ has $\mathbf{N} = -2x\,\mathbf{i}-2y\,\mathbf{j}+\mathbf{k}$. The upward unit normal is $\mathbf{n} = \dfrac{\mathbf{N}}{|\mathbf{N}|}$. At $(1,1,2)$, $\mathbf{n} = \frac{1}{3}\left(-2\,\mathbf{i}-2\,\mathbf{j}+\mathbf{k}\right)$. Notice that this vector points inward to the paraboloid. The outward unit normal is $\mathbf{n}_1 = -\mathbf{n}$. Use the ellipsoid $\dfrac{x^2}{2}+\dfrac{y^2}{3}+z^2 = 1$ for a positively-oriented closed surface.

- Return to the ellipsoid $\dfrac{x^2}{2}+\dfrac{y^2}{3}+z^2 = 1$. Parametrize the surface by $x = \sqrt{2}\sin u \sin v$, $y = \sqrt{3}\cos u \sin v$, $z = \cos v$. Then $\mathbf{r}_u = \dfrac{\partial x}{\partial u}\mathbf{i}+\dfrac{\partial y}{\partial u}\mathbf{j}+\dfrac{\partial z}{\partial u}\mathbf{k} = \sqrt{2}\cos u \sin v\,\mathbf{i} - \sqrt{3}\sin u \sin v\,\mathbf{j}$ and $\mathbf{r}_v = \dfrac{\partial x}{\partial v}\mathbf{i}+\dfrac{\partial y}{\partial v}\mathbf{j}+\dfrac{\partial z}{\partial v}\mathbf{k} = \sqrt{2}\sin u \cos v\,\mathbf{i} - \sqrt{3}\cos u \cos v\,\mathbf{j} - \sin v\,\mathbf{k}$. The unit normal vector $\mathbf{n} = \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$ is difficult to compute in general. However, it is reasonable to compute at certain points; the point $\left(\frac{\pi}{4},\frac{\pi}{4}\right)$ in uv-space gives the point $\left(\frac{1}{\sqrt{2}},\frac{\sqrt{3}}{2},\frac{1}{\sqrt{2}}\right)$ on the ellipsoid, and there we have outward unit normal $\mathbf{n} = \frac{1}{\sqrt{17}}\left(\sqrt{3}\,\mathbf{i}+\sqrt{2}\,\mathbf{j}+2\sqrt{3}\,\mathbf{k}\right)$.

- Give examples of surface integrals $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA$ for a vector field $\mathbf{F}$ over the parametric surface $S = \mathbf{r}(u,v)$ where $(u,v)\in D$. For example, letting $\mathbf{F} = y^2\,\mathbf{i} - z^2\,\mathbf{j}+\mathbf{k}$ over the ellipsoid $\dfrac{x^2}{2}+\dfrac{y^2}{3}+z^2 = 1$ gives

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_{0\le u,v\le 2\pi}\left(3\cos^2 u\sin^2 v\,\mathbf{i} - \cos^2 v\,\mathbf{j}+u\,\mathbf{k}\right)\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA\\ &= \iint_{0\le u,v\le 2\pi}\left(3\sqrt{3}\sin u\cos^2 u\sin^4 v - \sqrt{2}\cos u\cos^2 v\sin^2 v+\sqrt{6}\sin v\cos v\right)dA\end{aligned}$$

This integral will be easy to evaluate when we learn the Divergence Theorem.

Workshop/Discussion

- Point out that the general unit normal vector for a parametrized surface $S = \mathbf{r}(u,v)$ is $\dfrac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}$, and for the surface $z = g(x, y)$ becomes $\dfrac{-\dfrac{\partial g}{\partial x}\mathbf{i} - \dfrac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\dfrac{\partial g}{\partial x}\right)^2 + \left(\dfrac{\partial g}{\partial y}\right)^2}}$.

- Consider $\iint_S \mathbf{F} \cdot d\mathbf{S}$ with $\mathbf{F} = y^2\,\mathbf{i} - z^2\,\mathbf{j} + \mathbf{k}$, where S is the piece of the paraboloid $z = x^2 + y^2$ above the unit disk. Perhaps just set up the integral

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_{x^2+y^2 \le 1} \left[-2xy^2 - \left(x^2 + y^2\right)(-2y) + 1\right] dA \\ &= \iint_{x^2+y^2 \le 1} \left(2x^2y - 2xy^2 + 2y^3 + 1\right) dA \end{aligned}$$

  and note that the answer is π, since the first three terms integrate to zero by symmetry.
- Compute both the upward and outward unit normals to the closed surface $S = S_1 + S_2$, where S_1 is the piece of the plane $3x + 2y + 4z = 0$ inside the sphere $x^2 + y^2 + z^2 = 4$, and S_2 is the part of that sphere above S_1. It is important to be able to set up these unit normal vectors for Stokes' Theorem and the Divergence Theorem.
- Let $\mathbf{F} = 5x\,\mathbf{i} + 3y\,\mathbf{j} + 2z\,\mathbf{k}$ and let S be the surface $x^2 + y^2 + z^2 = 4$. Compute $\iint_S (\mathbf{F} \cdot \mathbf{n})\, dS$, where $\mathbf{n}$ is the outward unit normal vector.

Group Work 1: Up and Out

Group Work 2: The Flux of a Vector Field

Homework Problems

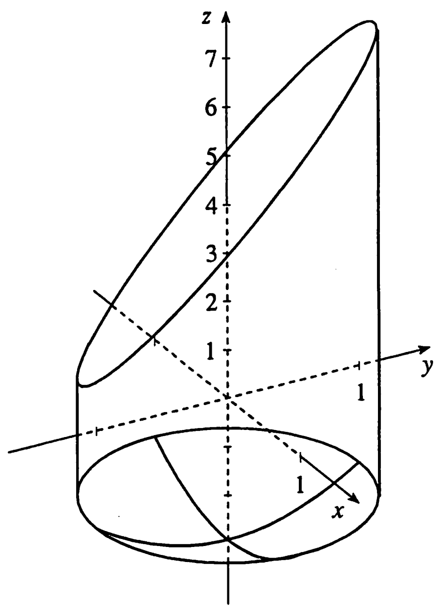
Core Exercises: 1, 7, 18, 25, 38

Sample Assignment: 1, 6, 7, 12, 13, 18, 22, 25, 34, 38

Exercise	C	A	N	G	V
1			×		
5–18		×			
22		×			
25		×			
34		×			
38		×			

Group Work 1, Section 13.6
Up and Out

1. Consider the piecewise smooth surface $S = S_1 \cup S_2 \cup S_3$ which is bounded on top by S_1: $z = 2x + 3y + 4$, on the bottom by S_3: $z = x^2 + y^2 - 3$ and on the side by S_2: $x^2 + y^2 = 1$ (see picture below).



(a) Compute unit normal vector fields $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ that are outward on S_1, S_2, and S_3.

(b) Compute unit normal vector fields $\mathbf{m}_1$, $\mathbf{m}_2$, and $\mathbf{m}_3$ that are upward on S_1 and S_3, and inward on S_2.

2. Now consider the surface $S = S_1 \cup S_2$ where S_1 is given by $z = x^4 + y^4$, and S_2 is the piece of the plane $z = x + 2$ inside S_1.

(a) Compute unit normal vector fields $\mathbf{n}_1$ and $\mathbf{n}_2$ that are upward everywhere.

(b) Compute unit normal vector fields $\mathbf{m}_1$ and $\mathbf{m}_2$ that are outward everywhere.

(c) If we wanted to "walk" around the intersecting curve in a counterclockwise orientation, which choices of the surface normal vector fields on $\mathbf{S}_1$ and $\mathbf{S}_2$ are both consistent with this assignment of orientation?

Group Work 2, Section 13.6
The Flux of a Vector Field

Consider the sphere $x^2 + y^2 + z^2 = R^2$ as a level surface of the function $G(x, y, z) = x^2 + y^2 + z^2$.

1. Compute the gradient $\nabla G(x, y, z)$ to this surface.

2. Does $\nabla G(x, y, z)$ point inward or outward from the surface of the sphere?

3. Compute an outward unit normal vector $\mathbf{n}$ to the sphere.

Now consider the vector field $\mathbf{F}(x, y, z) = \dfrac{1}{(x^2 + y^2 + z^2)^{3/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$.

4. Where is $\mathbf{F}$ defined?

5. Compute the flux $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (\mathbf{F} \cdot \mathbf{n})\, dS$, where S is the sphere $x^2 + y^2 + z^2 = R^2$.

13.7 Stokes' Theorem

Suggested Time and Emphasis

1 class Essential Material

Points to Stress

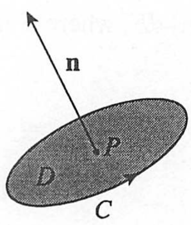
1. The statement of Stokes' Theorem
2. The connection between the curl and the circulation of a velocity field

Text Discussion

- Why can Stokes' Theorem be regarded as a three-dimensional version of Green's Theorem?
- Is it possible for a closed oriented curve C to be the boundary of more than one smooth oriented surface?

Materials for Lecture

- Stress the meaning of oriented smooth surfaces and bounding simple closed curves (with the notation $C = \partial S$). Use the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$ and the top half of the ellipsoid $x^2 + y^2 + \frac{z^2}{4} = 1$ to illustrate that a closed curve (here the circle $x^2 + y^2 = 1$) can be the boundary of many oriented smooth surfaces.
- State Stokes' Theorem in the case where S is the surface $z = f(x, y)$ with upper unit normal $\mathbf{n}$ and boundary curve $C = \partial S$.
- Verify Stokes' Theorem for $\mathbf{F} = z^2 y\,\mathbf{i} + 2x\,\mathbf{j} + x^2 y z^3\,\mathbf{k}$ on S, where S is the top half of the sphere $x^2 + y^2 + z^2 = 4$. Obtain $\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \oint_{C=\partial S} \mathbf{F} \cdot d\mathbf{r}$, where C: $x^2 + y^2 = 4$, or $\oint_{C=\partial S} z^2 y\,dx + 2x\,dy + x^2 y z^3\,dz = \oint_{x^2+y^2=4} 2x\,dy = 2\,(\text{area of a circle of radius } 2) = 8\pi$, by Green's Theorem. Also, if S is the top half of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{9} = 1$, then we still get $\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = 8\pi$, since $\partial S = C$ is the same circle, $x^2 + y^2 = 4$.
- The following is an intuitive justification of why the curl is a measure of circulation per unit area. Let $\mathbf{v} = \mathbf{v}(x, y, z)$ be the velocity of a fluid flow. Define the circulation of $\mathbf{v}$ around a circle C as $\oint_C (\mathbf{v} \cdot \mathbf{T})\,ds$. Point out that for velocities of a given magnitude, the circulation measures the extent to which $\mathbf{v}$ maintains the direction of the unit tangent vector $\mathbf{T}$, which is to say the extent to which the flow is rotating in the direction of C. Now take a point P within the flow, and let D be a very small disk centered at P with unit normal $\mathbf{n}$ at P.

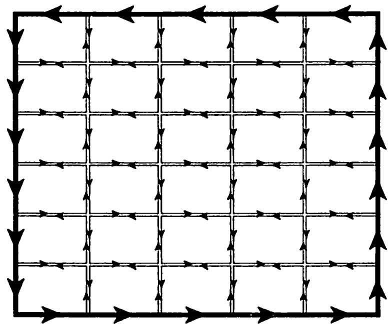


Let C be the boundary of D, positively oriented. By Stokes' Theorem, the circulation of $\mathbf{v}$ around C is approximately equal to the average $\mathbf{n}$-component of $\operatorname{curl}\mathbf{v}$ on D times the area of D. It follows that the average **n**-component of curl **v** on D equals

$$\frac{\text{the circulation of } v \text{ around } C}{\text{the area of } D}$$

Now vary D by letting the radius shrink to zero. This process describes at each point P the component of curl **v** in the direction of **n** as the circulation of **v** per unit area in the plane normal to **n**.

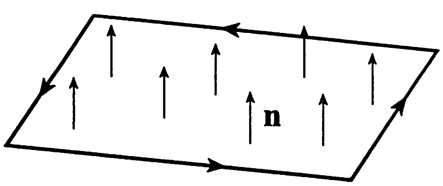
- Let S be a surface and let $C = \partial S$. The following is an intuitive explanation of the equation $\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$. Consider the region R chopped up into a bunch of smaller regions as shown, and suppose we want to integrate $\int_R (\nabla \times \mathbf{F})\, dA$ for some field $\mathbf{F}$.

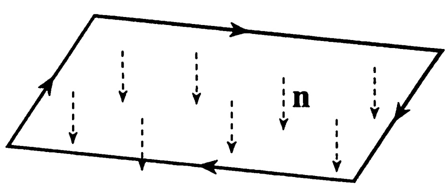


Since $\nabla \times \mathbf{F}$ is the curl of the field, which measures the local rotation of the field, the arrows in the small rectangles above represent the curl of $\mathbf{F}$. Notice that all of the rotations inside the square cancel out, and the only rotation left is the part along the boundary. So, when we compute $\int_S (\nabla \times \mathbf{F})\, dA$, we are really measuring the rotational movement along the boundary. However, there is another way to measure the movement along the boundary: a line integral. The work done by the field in moving a particle along the boundary is given by $\oint_C \mathbf{F} \cdot d\mathbf{r}$. Therefore $\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$.

Workshop/Discussion

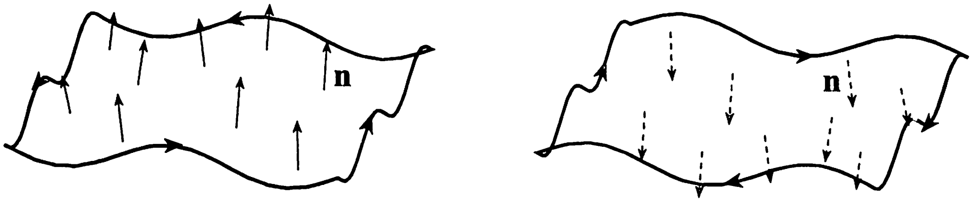
- Use the following to discuss the idea of an oriented surface with a positively oriented boundary. Start with a planar region R, such as $0 \le x \le \pi$, $0 \le y \le \pi$. Thinking of this region as lying in $\mathbb{R}^3$, there are clearly two ways to continuously assign normal vectrs, either in the positive z-direction or the negative z-direction. For each case, we get a different positive orientation on $C_1 = \partial R$, as shown below.





Now suppose we "wrinkle" the surface slightly. Let S be the surface $z = \frac{1}{5}\sin x + \frac{1}{5}\cos y$, $0 \le x \le \pi$, $0 \le y \le \pi$. Although the normal vectors no longer all point in the same direction, there are still two distinct ways to continuously assign the normal vectors, each one giving a different positive orientation on

$C_2 = \partial S$.



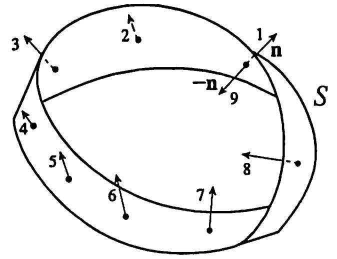
- Use Stokes' Theorem to show that $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = -2\pi$, where S is the surface formed by the lower hemisphere of $x^2+y^2+z^2=1$ and $\mathbf{F} = z^2y\,\mathbf{i} + 2x\,\mathbf{j} + x^2yz^3\,\mathbf{k}$. Explain how the negative result arises from the orientation given to the boundary circle $x^2+y^2=1$.

- Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = -y^3\,\mathbf{i} + x^3\,\mathbf{j} + \cos z^3\,\mathbf{k}$ and C is the curve generated by the intersection of the cylinder $x^2+y^2=4$ and the plane $x+y+z=1$. One approach is to create a surface S such that $C = \partial S$. To do this, choose S to be the portion of the plane with normal $\mathbf{N} = \mathbf{i}+\mathbf{j}+\mathbf{k}$ and unit normal $\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$ over the circle $x^2+y^2=4$. Then by Stokes' Theorem,

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S (\operatorname{curl}\mathbf{F}\cdot\mathbf{n})\,dS = \iint_{x^2+y^2\le 4}(\operatorname{curl}\mathbf{F}\cdot\mathbf{n})\,dA = \iint_{x^2+y^2\le 4} 3\left(x^2+y^2\right)dA \\ &= 3\iint_{x^2+y^2\le 4}\left(x^2+y^2\right)dx\,dy = 3\int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta \\ &= 24\pi\end{aligned}$$

  Note that trying to compute this integral without Stokes' Theorem is very difficult.

- Show that the Möbius strip S is not orientable, as follows. If you start at P with unit normal $\mathbf{n}_1$ on S_1 and move around continuously in the direction indicated, you need to chose $-\mathbf{n}_1$ for consistency, a contradiction.



Group Work 1: The Silo

Group Work 2: Plane Surfaces

Problems 1 and 2 are straightforward. In Problem 3, students need to realize that because we don't have a formula for $C = \partial S$, since it is arbitrary, the only possible way to compute the line integral is to use Stokes' Theorem. Since $\operatorname{curl}\mathbf{F} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, this surface integral turns out to be a very easy calculation.

Homework Problems

Core Exercises: 1, 2, 8

Sample Assignment: 1, 2, 5, 8, 9, 11, 17

Note:
- Exercise 11 parts (b) and (c) require a CAS.
- Exercise 16 makes an excellent group work. Students may need the hint that we need to find conditions on **F** so that Stokes' Theorem applies.

Exercise	C	A	N	G	V
1	×				
2		×			
5		×			×
8		×			
9		×			
11		×		×	
17		×			

Group Work 1, Section 13.7
The Silo

Let S be the surface formed by capping the piece of the cylinder $x^2 + y^2 = 2$, $0 \le z \le 4$ with the top half of the sphere $x^2 + y^2 + (z-4)^2 = 2$.

1. Draw a rough sketch of S.

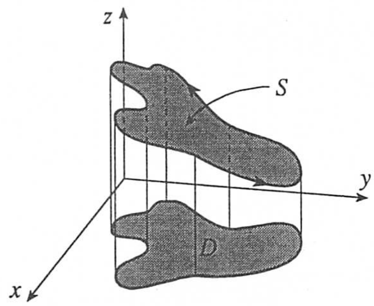
2. Show that the outward normal gives a smooth orientation to S.

3. What is $C = \partial S$? Parametrize C so that it has a positive orientation with respect to the outward normal.

4. Evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \left(zx + z^2y + x\right)\mathbf{i} + \left(z^3yx + y\right)\mathbf{j} + z^4x^2\,\mathbf{k}$.

Group Work 2, Section 13.7
Plane Surfaces

Consider the surface S formed by the piece of the plane $ax + by + cz + d = 0$ above the region D in the xy-plane with area A_D.



1. Show that the surface area of S is $A_S = \frac{A_D}{|c|}\sqrt{a^2 + b^2 + c^2}$.

2. Show that a unit normal $\mathbf{n}$ to S is $\mathbf{n} = \frac{a\mathbf{i} + b\mathbf{j} + c\mathbf{k}}{\sqrt{a^2 + b^2 + c^2}}$.

3. Consider the plane $2x + 3y + 4z + 5 = 0$ and S the surface over D as above with boundary curve $C = \partial S$ having positive orientation.

(a) If $\mathbf{F}(x, y, z) = (x^3 + z + 2y)\,\mathbf{i} + 2x\,\mathbf{j} + (-4x + y)\,\mathbf{k}$, compute curl $\mathbf{F}$.

(b) Compute $\oint_C \left[(x^3 + z + 2y)\,\mathbf{i} + 2x\,\mathbf{j} + (-4x + y)\,\mathbf{k}\right] \cdot d\mathbf{r}$ in terms of A_D.
Hint: Can you use Stokes' Theorem here?

Writing Project: Three Men and Two Theorems

The story behind Green's and Stokes' Theorems turns out to be quite fascinating. This project should be thoroughly enjoyable for any student interested in the history of mathematics.

13.8 The Divergence Theorem

Suggested Time and Emphasis

1 class Essential Material

Points to Stress

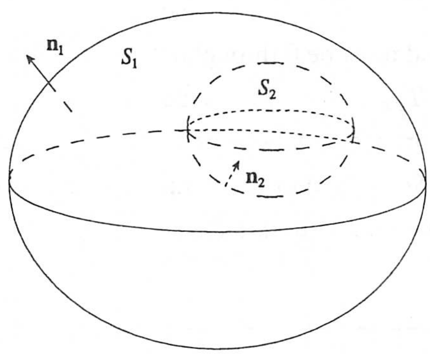
1. The meaning of a simple closed solid region R and its boundary surface $S = \partial R$
2. A careful statement of the Divergence Theorem.

Text Discussion

- Why is it that if we apply the Divergence Theorem to the region between two surfaces with one inside the other, we choose a normal for the inner surface that points toward the inside of the inner surface?
- "Source" and "sink" are defined on page 982. Give some intuitive reasons why these names are appropriate.

Materials for Lecture

- Provide a statement of the Divergence Theorem and stress the importance of an outward positive orientation.
- Point out that if $\mathbf{F}$ is incompressible, then $\operatorname{div}\mathbf{F} = 0$ and hence $\iint_{S=\partial R} \mathbf{F} \cdot d\mathbf{S} = 0$.
- Note that the value of the Divergence Theorem is that it allows us to reduce a surface integral to a triple integral.
- State the extension of the Divergence Theorem to regions between two closed surfaces, as shown:



- Check again that $\mathbf{F} = r(y,z)\,\mathbf{i} + s(x,z)\,\mathbf{j} + t(x,y)\,\mathbf{k}$ is incompressible on any simple closed region R and so, by the Divergence Theorem, $\iint_{S=\partial R} \mathbf{F} \cdot d\mathbf{S} = 0$.
- Perhaps give the following intuitive interpretation of $\nabla \cdot \mathbf{F}$:
  Choose a point P and surround it by a closed ball N with small radius r. According to the Divergence Theorem, the flux of $\mathbf{v}$ out of N is given by $\iiint_N (\operatorname{div}\mathbf{v})\,dx\,dy\,dz$. Thus, the Average Value Theorem tells us that the flux of $\mathbf{v}$ out of N is the average divergence of $\mathbf{v}$ on N times the volume of N. Dividing by the volume gives the average divergence of $\mathbf{v}$ on N to be $\dfrac{\text{flux of } \mathbf{v} \text{ out of } N}{\text{volume of } N}$. Letting the radius of the

ball shrink to 0 says that the divergence of $\mathbf{v}$ at P is $\lim_{r\to 0} \dfrac{\text{flux of } \mathbf{v} \text{ out of } N}{\text{volume of } N}$. In other words, divergence can be regarded as flux per unit volume. Now view $\mathbf{v}$ as the velocity of a fluid in steady-state motion. A positive divergence at a point indicates a net flow of liquid away from that point, since $\operatorname{div} \mathbf{v} > 0$ at P means that for some ball N, the flux out of N is positive. Similarly, a negative divergence indicates a net flow of liquid toward the point.

Points at which the divergence is positive are called sources; points at which the divergence is negative are called sinks. If the divergence of $\mathbf{v}$ is 0 throughout, then the flow has no source and no sink, and $\mathbf{v}$ is called incompressible.

- If there is time, recall the Laplacian $\nabla^2 = \operatorname{div} \cdot \nabla$ defined in Section 13.5. We can now use the Divergence Theorem and the following argument to show that a steady-state temperature distribution T satisfies $\nabla^2 T = 0$: Assume that on the surface of a hotplate, a temperature distribution is maintained which varies from point to point, but does not change over time. (We do *not* assume that the hot-plate is two dimensional. It is a three-dimensional piece of metal with a heating element on one side, and insulation on the other.) Then, in many cases, the temperature distribution inside the metal of the plate will also reach a steady state, again independent of time. Let $T(x, y, z)$ be the temperature at (x, y, z) over this solid. We will show that T satisfies the partial differential equation $\nabla^2 T = T_{xx} + T_{yy} + T_{zz} = 0$.

  At each point in the solid, ∇T points in the direction of most rapid increase in temperature. Since heat flows from warmer to cooler regions, the heat flows in the direction of $-\nabla T$. We will assume that the rate of flow (as a function of time) is proportional to the magnitude of the vector $-\nabla T$.

  Pick any point (x, y, z) and let R be a solid ball of metal containing (x, y, z), with surface S. Since the temperature is in a steady state in the entire region, heat neither enters nor leaves R. Since the flow is parallel to ∇T, this means that $\int_S (\nabla T) \cdot \mathbf{n}\, dS = 0$. By the Divergence Theorem, $\int_R \nabla \cdot \nabla T\, dV = 0$, or

$$\int_R \nabla \cdot \left(\frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k} \right) dV = \int_R \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) dV = 0$$

  We can conclude that the integrand must be 0 throughout R, and since R can be chosen to be any ball, we conclude that the Laplacian of T, $T_{xx} + T_{yy} + T_{zz}$, is identically 0 throughout the solid. So the problem of finding the temperature distribution in a metal object such as a hotplate reduces to the problem of finding a solution to Laplace's equation $\nabla^2 T = 0$ (a harmonic function) that satisfies certain conditions on the boundary. Give examples of harmonic functions such as $T(x, y, z) = e^x \cos y + z$.

Workshop/Discussion

- Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = xz\,\mathbf{i} + yx\,\mathbf{j} + xyz\,\mathbf{k}$ and S is the surface of the unit cube. (The Divergence Theorem gives $\frac{5}{4}$.)
- Show that if $\mathbf{F}$ and $\mathbf{G}$ are given as

$$\begin{aligned}\mathbf{F} &= (8x + 3y)\,\mathbf{i} + (5x + 4z - 2y)\,\mathbf{j} + \left(9y^2 - \sin x + 7z\right)\mathbf{k} \\ \mathbf{G} &= (12y + 8z)\,\mathbf{i} + (e^z + \sin x + 9y)\,\mathbf{j} + \left(xy^2 e^{xy} + 4z\right)\mathbf{k}\end{aligned}$$

  then $\iint_{S=\partial R} \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{G} \cdot d\mathbf{S}$, where S is the surface of a region R for which the Divergence Theorem holds.

- Evaluate $\iint_{S=\partial R} (x + y^2 + 2z)\,dS$, where R is the solid sphere $x^2 + y^2 + z^2 \le 4$. Note that to apply the Divergence Theorem, we need to "guess" a vector field $\mathbf{F}$ such that $\mathbf{F} \cdot \mathbf{n} = x + y^2 + 2z$. Set $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + W\,\mathbf{k}$. Since the outward unit normal vector on S is $\mathbf{n} = \frac{1}{2}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$, we have $\frac{1}{2}xP = x$, $\frac{1}{2}yQ = y^2$, and $\frac{1}{2}zW = 2z$. Thus we need $P = 2$, $Q = 2y$, and $W = 4z$. So one natural choice is $\mathbf{F} = 2\mathbf{i} + 2y\,\mathbf{j} + 4z\,\mathbf{k}$. Then $\nabla \cdot \mathbf{F} = 2$ and $\iint (x + y^2 + 2z)\,dS = \iiint_R 2\,dV = \frac{32}{3}\pi$.
- Discuss harmonic functions and give some additional examples, such as $f(x,y,z) = 2x^2 + 3y^2 - 5z^2$ and $g(x,y,z) = e^{\sqrt{2}z} \sin x \cos y$.

Group Work 1: A Handy Way to Find Flux

By using the Divergence Theorem and noting that $\operatorname{div} \mathbf{F} > 0$, the students can answer the first question without having to compute an integral.

Group Work 2: Finding Surface Integrals

The surface in this activity is similar to the surface in Group Work 1, making it a good supplement to that exercise. The volume of the region R can easily be shown to be π using geometry.

Group Work 3: The Position Vector

This activity is a good warmup to Exercise 18 in the text.

Group Work 4: When Are Surface Integrals Always Zero?

Problem 1 of this activity is related to Exercise 19, since the vector field is a scalar multiple of $\mathbf{E}(x)$ defined on page 984. Problem 2 is easily answered using the Divergence Theorem.

Homework Problems

Core Exercises: 1, 2, 7, 12

Sample Assignment: 1, 2, 7, 8, 9, 12, 17, 22, 23

Exercise	C	A	N	G	V
1					×
2		×			×
7–15		×			
17		×			
22		×			
23		×			

Group Work 1, Section 13.8
A Handy Way to Find Flux

Consider $\mathbf{F} = \frac{xy^2}{2}\,\mathbf{i} + \frac{y^3}{6}\,\mathbf{j} + zx^2\,\mathbf{k}$ over the surface S, where S is the cylinder $x^2 + y^2 = 1$ capped by the planes $z = \pm 1$.

1. Is the net flux of **F** from the surface positive or negative?

2. What is the value of the flux?

Group Work 2, Section 13.8
Finding Surface Integrals

Compute

$$\iint_{S=\partial R} \mathbf{F} \cdot d\mathbf{S}$$

where

$$\mathbf{F} = (x - z)\,\mathbf{i} + (y - x)\,\mathbf{j} + (z - y)\,\mathbf{k}$$

and S is the cylinder $x^2 + y^2 = 1$ capped by the planes $2z = 1 - x$ and $2z = x - 1$.

Group Work 3, Section 13.8
The Position Vector

Let $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, the position vector at (x, y, z).

1. Compute $\iint_{S=\partial R} \mathbf{F} \cdot \mathbf{n}\, dS$ for any region R.

2. Find a vector field $\mathbf{G}$ such that $\iint_{S=\partial R} \mathbf{G} \cdot \mathbf{n}\, dS$ is equal to the volume of R, for any region R.

Group Work 4, Section 13.8
When Are Surface Integrals Always Zero?

Let U be the solid interior of a closed surface S, and assume that the origin does not lie in the set U or on its boundary S.

1. Show that $\iint_S \frac{\mathbf{x}}{|\mathbf{x}|^3} \cdot d\mathbf{S} = 0$, where $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

2. If S is the surface of the sphere $x^2 + y^2 + (z-2)^2 = 1$, then is it true that $\iint_S \frac{\mathbf{x}}{|\mathbf{x}|^2} \cdot d\mathbf{S} = 0$?

13 Sample Exam

Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

1. Match up each entry in the first column to one in the second. Note that a given entry in the second column can be used once, more than once, or not at all.

If a vector field $\mathbf{F}$ is the gradient of some scalar function, then $\mathbf{F}$ is ______________.	conservative
If a curve C is the union of a finite number of smooth curves, then C is ______________.	curl
If $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D, then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is ______________ in D.	divergence
If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, $\mathbf{F}$ is defined everywhere in $\mathbb{R}^2$ and $\partial P/\partial y = \partial Q/\partial x$, then $\mathbf{F}$ is ______________.	flux
If a curve C doesn't intersect itself anywhere between its endpoints, then C is ______________.	irrotational
If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ then $\nabla \times \mathbf{F}$ is called the ______________.	path independent
If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ then $\nabla \cdot \mathbf{F}$ is called the ______________.	piecewise smooth
If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S then $\iint_S \mathbf{F} \cdot d\mathbf{S}$ is the ______________.	simple
If $\mathbf{F}$ is a vector field and $\operatorname{curl} \mathbf{F} = 0$ at a point P, then $\mathbf{F}$ is ______________ at P.	simply-connected

2. Consider the oriented surface S for $z \geq 0$, consisting of the portion of the surface of the paraboloid $z = 4 - (x^2 + y^2)$ above the xy-plane and with outward normal.

(a) What is the boundary curve $C = \partial S$ and what direction is its positive orientation?

(b) What surface S_1 in the xy-plane with what assignment of a normal has the same boundary curve $C = \partial S_1$ with the same orientation?

(c) Compute $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, if $\mathbf{F} = (xe^z - 3y)\,\mathbf{i} + \left(ye^{z^2} + 2x\right)\mathbf{j} + \left(x^2y^2z^3\right)\mathbf{k}$.

3. Parametrize the boundary curve $C = dS$ of the surface S: $\dfrac{x^2}{9} + \dfrac{y^2}{9} + \dfrac{z^2}{16} = 1$, $z \le 0$, so that it has positive orientation with respect to S.

4. (a) Find a counterclockwise parametrization of the ellipse $x^2 + \dfrac{y^2}{4} = 1$.

(b) Compute the double integral

$$\iint_{0 \le x^2 + y^2/4 \le 1} 3x^2 y \, dA$$

Hint: Can you find a vector function $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ such that $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 3x^3 y$?

5. Consider $\mathbf{F}(x, y, z) = -\dfrac{y}{x^2 + y^2}\mathbf{i} + \dfrac{x}{x^2 + y^2}\mathbf{j} + z\,\mathbf{k}$.

(a) Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is the circle $x^2 + y^2 = 1$ in the xy-plane, oriented counterclockwise.

(b) Show that $\operatorname{curl} \mathbf{F} = \langle 0, 0, 0 \rangle$ everywhere that $\mathbf{F}$ is defined.

(c) Indicate why you cannot use Stokes' Theorem on this problem. [That is, explain why your answers to (a) and (b) don't contradict one another.]

6. (a) Use the Divergence Theorem to show that, for a closed surface S with an outward normal which encloses a solid region B,

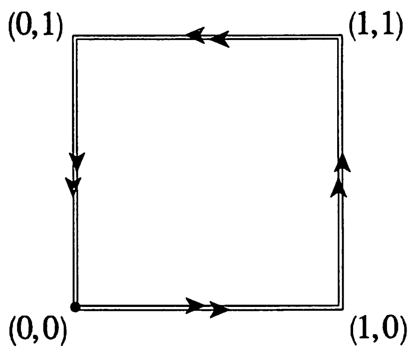
$$\text{Volume}(B) = \iint_S \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F}(x, y, z) = \langle x, 0, 0 \rangle$.

(b) Use part (a) to show that the volume enclosed by the unit sphere is $\frac{4}{3}\pi$.

(c) Compute $\iint \mathbf{F} \cdot d\mathbf{S}$ if $\mathbf{F}(x, y, z) = \langle 3x, 4y, 5z \rangle$.

7. Compute the work done by the vector field $\mathbf{F}(x, y) = \left(\sin x + xy^2\right)\mathbf{i} + \left(e^y + \frac{1}{2}x^2\right)\mathbf{j}$ in $\mathbb{R}^2$, where C is the path that goes around the unit square twice.



8. Consider the vector field $\mathbf{F}(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$.

(a) Compute $\operatorname{curl} \mathbf{F}$.

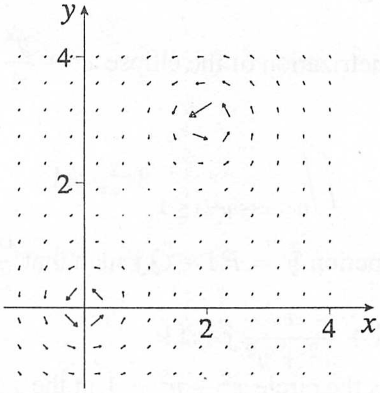
(b) If C is any path from $(0, 0, 0)$ to (a_1, a_2, a_3) and $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = \mathbf{a} \cdot \mathbf{a}$.

9. Consider the vector fields $\mathbf{F}(x, y) = \dfrac{-y}{x^2 + y^2}\mathbf{i} + \dfrac{x}{x^2 + y^2}\mathbf{j}$ and

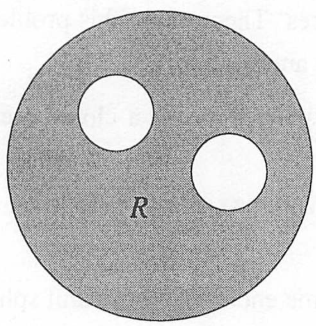
$$\mathbf{G}(x, y) = \frac{-(y-3)}{(x-2)^2 + (y-3)^2}\mathbf{i} + \frac{x-2}{(x-2)^2 + (y-3)^2}\mathbf{j}.$$

(a) Given that $\operatorname{curl} \mathbf{F}(x, y) = \mathbf{0}$ for $(x, y) \ne (0, 0)$, compute $\operatorname{curl} \mathbf{G}(x, y)$ for $(x, y) \ne (2, 3)$.

(b) Below is the plot of the vector field $\mathbf{F}(x, y) + \mathbf{G}(x, y)$. Describe where this vector field is defined. Describe where it is irrotational.



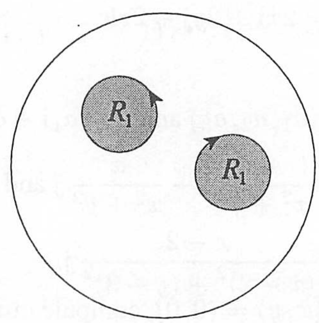
10. Consider the shaded region below.



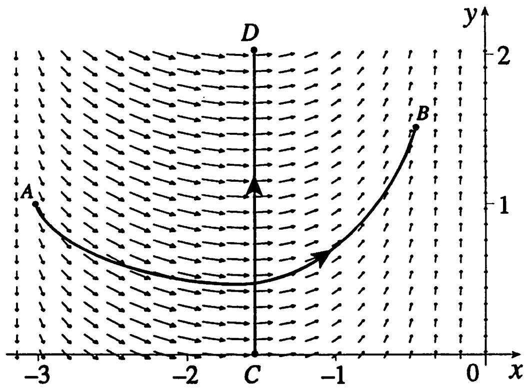
(a) Draw arrows on the boundaries ∂R of R to give it a positive orientation.

(b) If the outer circle has radius 4 and the two smaller circles have radius 1, evaluate $\frac{1}{2}\left(\int_{\partial R} y\,dx - x\,dy\right)$.

(c) Compute $\frac{1}{2}\left(\int_{\partial R_1} y\,dx - x\,dy\right)$, where R_1 is the new shaded region in the figure below. Each smaller circle has radius 1.



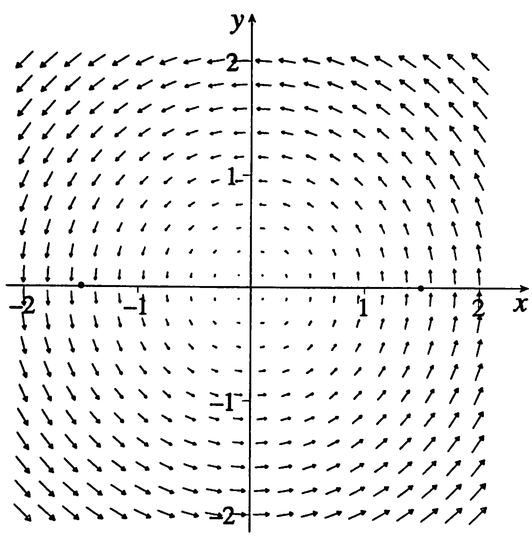
11. Consider the following vector field.**F**.



(a) Is the line integral of **F** along the path from A to B positive, negative, or zero? How do you know?

(b) Is the line integral of **F** along the path from C to D positive, negative, or zero? How do you know?

12. Consider the vector field below.

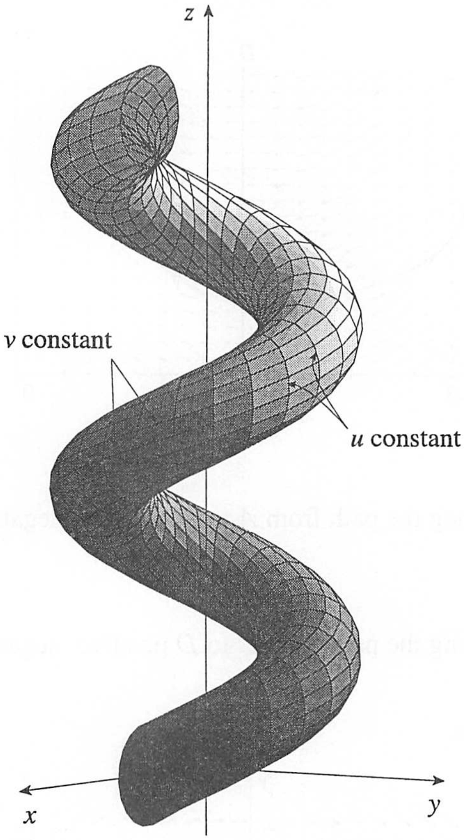


(a) Draw and label a curve C_1 from $(-1.5, 0)$ to $(1.5, 0)$ such that $\int_{C_1} \mathbf{F} \cdot d\mathbf{s} > 0$.

(b) Draw and label a curve C_2 from $(-1.5, 0)$ to $(1.5, 0)$ such that $\int_{C_2} \mathbf{F} \cdot d\mathbf{s} < 0$.

(c) Draw and label a curve C_3 from $(-1.5, 0)$ to $(1.5, 0)$ such that $\int_{C_3} \mathbf{F} \cdot d\mathbf{s} \approx 0$.

13. The following parametric surface has grid curves which can be shown to be circles when u is constant.



$$x = (2 + \sin v)\cos u \qquad y = (2 + \sin v)\sin u \qquad z = u + \cos v$$

(a) Find the center and radius of the circle at $u = \frac{\pi}{2}$.

(b) Find the normal vector to S at the point P generated when $u = v = \frac{\pi}{2}$.

14. Consider the surfaces S_1: $\dfrac{x^2}{9} + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$, $z \geq 0$ and S_2: $4z = 9 - x^2 - y^2$, $z \geq 0$. Let $\mathbf{F}$ be any vector field with continuous partial derivatives defined everywhere. Show that $\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

15. Set up and evaluate the integral for the surface area of the parametrized surface

$$x = u + v \qquad y = u - v \qquad z = 2u + 3v$$
$$0 \leq u \leq 1 \qquad 0 \leq v \leq 1$$

13 Sample Exam Solutions

1. Conservative; piecewise smooth; path independent; conservative; simple; curl; divergence; flux; irrotational

2. (a) C is the circle of radius 2 centered at the origin in the xy-plane. It has positive orientation if it is parametrized in the counterclockwise direction as viewed from above.

(b) If S_1 is the disk of radius 2 centered at the origin with upward normal, then $C = \partial S_1$ with the same orientation.

(c) By Stokes' Theorem, $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_{C=\partial S} \mathbf{F} \cdot d\mathbf{r} = \oint_{C=\partial S_1} \mathbf{F} \cdot d\mathbf{r} = \iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$. Since $z = 0$ on S_1, $\mathbf{F} = (x - 3y)\,\mathbf{i} + (y + 2x)\,\mathbf{j} + 0\mathbf{k}$, $\mathbf{n} = \mathbf{k}$, and $\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x - 3y & y + 2x & 0 \end{vmatrix} = 5\mathbf{k}$. So $\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{x^2+y^2 \leq 4} (\operatorname{curl} \mathbf{F} \cdot \mathbf{k})\, dA = 5\left(\text{area of disk } x^2 + y^2 \leq 4\right) = 20\pi$.

3. $\mathbf{r}(\phi, \theta) = 3 \sin \phi \cos \theta\, \mathbf{i} + 3 \sin \phi \sin \theta\, \mathbf{j} + 4 \cos \phi\, \mathbf{k}$

4. (a) $x^2 + \dfrac{y^2}{4} = 1$ can be parametrized counterclockwise by $\mathbf{F}(t) = \langle \cos t, 2 \sin t \rangle$, $0 \leq t \leq 2\pi$.

(b) Note that if $\mathbf{F} = 0\mathbf{i} + x^3 y \mathbf{j}$, then $\partial Q/\partial x - \partial P/\partial y = 3x^2 y$. So

$$\begin{aligned} \iint_{0 \leq x^2+y^2/4 \leq 1} 3x^2 y\, dA &= \int_{\text{boundary}} P\, dx - Q\, dy = \int_0^{2\pi} \cos^3 2 + 2 \sin t\,(2 \cos t\, dt) \\ &= 4 \int_0^{2\pi} \cos^4 t \sin t\, dt \quad (\text{Let } u = \cos t,\ du = -\sin t\, dt) \\ &= -4 \int_{-1}^{1} u^4\, du = 0 \end{aligned}$$

This can also be found directly, as follows:

$$\int_0^1 \int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} 3x^2 y\, dy\, dx = \int_0^1 \left[\tfrac{3}{2} x^2 y^2\right]_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} dx = 0$$

5. $\mathbf{F}(x, y, z) = -\dfrac{y}{x^2 + y^2}\,\mathbf{i} + \dfrac{x}{x^2 + y^2}\,\mathbf{j} + z\,\mathbf{k}$.

(a) $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$, $0 \leq t \leq 2\pi$.
$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle -\sin t, \cos t, 0 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle\, dt = \int_0^{2\pi} 1\, dt = 2\pi$

(b) $\operatorname{curl} \mathbf{F}(x, y, z) = \left\langle 0 - 0, 0 - 0, \dfrac{y^2 - x^2}{(x^2 + y^2)^2} - \dfrac{y^2 - x^2}{(x^2 + y^2)^2} \right\rangle = \langle 0, 0, 0 \rangle$ everywhere except the z-axis (where $\mathbf{F}$ is undefined).

(c) Since $\mathbf{F}$ is not defined along the z-axis, we cannot find a surface such that C is its boundary and $\mathbf{F}$ is defined everywhere on the surface.
Another reason: If $P = -\dfrac{y^2}{x^2 + y^2}$, then $\dfrac{\partial P}{\partial y} = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}$, which does not have a limit at $(0, 0)$ and is discontinuous there.

6. (a) If we assume an outward normal, then by the Divergence Theorem,
$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_B \operatorname{div} \mathbf{F} \cdot dV = \iiint_B dV$ (since $\operatorname{div} \mathbf{F} = 1$), which is simply the volume of B.

(b) Parametrize the sphere by $r(\theta, \phi) = \langle \cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi \rangle$. Then
$r_\theta \times r_\phi = \left\langle \sin^2 \phi \cos \theta, \sin^2 \phi \sin \theta, -\sin \phi \cos \phi \right\rangle$, which points outward, and
$f(r(\theta, \phi)) = \langle \cos \theta \sin \phi, 0, 0 \rangle$, so
$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{\pi} \int_0^{2\pi} \cos^2 \theta \sin^3 \phi\, d\theta\, d\phi = \int_0^{2\pi} \cos^2 \theta\, d\theta \int_0^{\pi} \sin^3 \phi\, d\phi = \frac{4}{3}\pi$.

(c) $\operatorname{div} \mathbf{F} = 12$, so $\iint_S \mathbf{F} \cdot d\mathbf{S} = 12 \cdot \text{Volume}(B) = 16\pi$.

7. Using Green's Theorem with $P = \sin x + xy^2$ and $Q = e^y + \frac{1}{2}x^2$, we get

$$\text{Work} = \int_C P\,dx + Q\,dy = 2\int_{\text{Square}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 2\int_0^1 \int_0^1 (x - 2xy)\,dx\,dy$$

$$= 2\int_0^1 \left[\tfrac{1}{2}x^2 - x^2 y\right]_0^1 dy = 2\int_0^1 \left(\tfrac{1}{2} - y\right) dy = 2\left[\tfrac{1}{2}y - \tfrac{1}{2}y^2\right]_0^1 = 0$$

8. (a) $\operatorname{curl} \mathbf{F} = 0$

(b) Let $f(x, y, z) = x^2 + y^2 + z^2$. Then $\mathbf{F} = \nabla f$ and by the Fundamental Theorem for line integrals, $\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{a}) - f(\mathbf{0}) = a_1^2 + a_2^2 + a_3^2 = \mathbf{a} \cdot \mathbf{a}$.

9. (a) If $\mathbf{G} = P\mathbf{i} + Q\mathbf{j}$, then computation gives

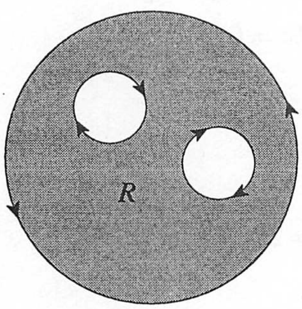
$$\operatorname{curl} \mathbf{G} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} = \left[\frac{(y-3)^2 - (x-2)^2}{(x-2)^2 + (y-3)^2} - \frac{(y-3)^2 - (x-2)^2}{(x-2)^2 + (y-3)^2}\right]\mathbf{k} = \mathbf{0} \text{ for}$$

$(x, y) \neq (2, 3)$.

Or: $\operatorname{curl} \mathbf{G} = \mathbf{0}$ since the vector field $\mathbf{G}$ is just $\mathbf{F}$ translated to the right 2 units and up 3 units.

(b) $\mathbf{F} + \mathbf{G}$ is defined at all points except $(0, 0)$ and $(2, 3)$, since $\mathbf{F}$ is not defined at $(0, 0)$ and $\mathbf{G}$ is not defined at $(2, 3)$. At all other points, $\operatorname{curl}(\mathbf{F} + \mathbf{G}) = \operatorname{curl} \mathbf{F} + \operatorname{curl} \mathbf{G} = \mathbf{0}$, and $\mathbf{F} + \mathbf{G}$ is irrotational.

10. (a)



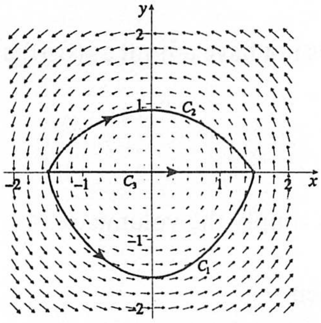
(b) $\frac{1}{2}\left(\int_{\partial R} y\,dx - x\,dy\right) = \text{area}(R) = \pi \cdot 4^2 - 2 \cdot \pi \cdot 1^2 = 14\pi$

(c) $\frac{1}{2}\left(\int_{\partial R_1} y\,dx - x\,dy\right) = 0$, since the two smaller circles have equal areas and opposite orientations.

11. (a) Since $\mathbf{F}$ points in almost the same direction as vectors tangent to the path from A to B, $\mathbf{F}(t) \cdot \mathbf{r}'(t) > 0$ everywhere along the path, and hence the line integral $\int \mathbf{F} \cdot d\mathbf{r} > 0$.

(b) Since $\mathbf{F}$ is perpendicular to the path from C to D at every point, we have $\mathbf{F}(t) \cdot \mathbf{r}'(t) = 0$ everywhere along the path, and hence the line integral $\int \mathbf{F} \cdot d\mathbf{r} = 0$.

12.



13. (a) When $u = \frac{\pi}{2}$, $x = 0$, $y = 2 + \sin v$, and $z = \frac{\pi}{2} + \cos v$, so the center is $\left(0, 2, \frac{\pi}{2}\right)$ and the radius is 1.

(b) The normal vector at $P\left(0, 3, \frac{\pi}{2}\right)$ is $3\mathbf{j}$.

14. Both surfaces have the same boundary curve C: $x^2 + y^2 = 9$, $z = 0$. By Stokes' Theorem, $\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

15. $\mathbf{F}(u, v) = \langle u + v, u - v, 2u + 3v \rangle \Rightarrow \mathbf{F}_u = \langle 1, 1, 2 \rangle$, $\mathbf{F}_v = \langle 1, -1, 3 \rangle$, and $\mathbf{F}_u \times \mathbf{F}_v = \langle 5, -1, -2 \rangle$. Thus the surface area is $\int_0^1 \int_0^1 |\mathbf{F}_u \times \mathbf{F}_v|\,du\,dv = \int_0^1 \int_0^1 \sqrt{30}\,du\,dv = \sqrt{30}$.