9 Vectors and the Geometry of Space

9.1 Three-Dimensional Coordinate Systems

Suggested Time and Emphasis

$\frac{1}{2}$–$\frac{3}{4}$ class Essential material

Points to Stress

1. The basics of points in three-dimensional space, including notation and the distance formula.
2. Equations of planes parallel to one of the coordinate planes.
3. The equation of a sphere.

Text Discussion

- Explain why the equation $y = x$ is the equation of a *plane* in three-dimensional space.
- How do we use the distance formula to get the equation of a sphere?

Materials for Lecture

- If possible, mark off one corner of the lecture room with electrical tape. Determine the coordinates of various students in the room, and/or find the equation of the plane of the chalkboard.
- Describe the unit cube $\{(x, y, z) \mid 0 \leq x, y, z \leq 1\}$ and its surfaces $\{(x, y, z) \mid x = 0, 0 \leq y \leq 1, 0 \leq z \leq 1\}$, $\{(x, y, z) \mid x = 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$, and so forth.
- Explain why the equation $z = \sqrt{R^2 - (x^2 + y^2)}$ represents the hemisphere $z \geq 0$, while the equation $x^2 + y^2 + z^2 = R^2$ represents a full sphere.
- Introduce the equation of a circular cylinder as an extension to $\mathbb{R}^3$ of the equation of the circle $x^2+y^2 = r^2$.
- Use inequalities to describe the quarter-unit sphere above the xy-plane and to the right of the yz-plane. Then use inequalities to describe the eighth of the unit sphere in the first octant.

Workshop/Discussion

- Describe some cylindrical surfaces such as $y = x^2$, $y = \sin x$, and $z = 1 - x^2$.
- Determine the curves given by the intersection of the sphere $x^2 + y^2 + z^2 = R^2$ with the various plane $x = k$, $y = k$, and $z = k$, indicating the necessary restrictions on k.
- Describe the set of points whose distance from each of $(0, 0, 1)$ and $(0, 0, -1)$ is 1. Then similarly discu the set of points whose distance from these points is 2, and then 0.5. Have the students explain why t set of all points equidistant in space from two given points is a plane.
- Describe the region in $\mathbb{R}^3$ determined by the equation $xy = 0$, then do the same for the inequality $xy >$

Group Work 1: Working with Surfaces in Three-Dimensional Space

Visualizing and sketching surfaces in three-dimensional space can be difficult for students. It is recommended that the answer key be distributed after the students have had time to work.

Group Work 2: Lines, Lines, Everywhere Lines

The idea is to help students visualize the triangular surface $x + y + z = 1, x, y, z \geq 0$. Note that at this stage, the students have not yet learned about the general equations of lines and planes. Question 3 is ambitious, and requires some formal thinking. Depending on the mathematical maturity of your class, you may want to either give a hint to the students, or perhaps do that part as a class. Questions 3 and 4 are printed on a separate page, so you can decide what to do based upon how the students are doing on the first two parts.

Group Work 3: Equidistance

Begin by having the students try to visualize and sketch in $\mathbb{R}^2$ the set of all points equidistant from a given point and a given line, starting with the set of all points equidistant from $(0, 2)$ and the x-axis.

Next, discuss how we can use algebraic equations to define an object described in words in this way. For example, the set of all points in the xy-plane equidistant from the point $(0, 1)$ and the line $y = -1$ must satisfy the equation $(y - 1)^2 + x^2 = (y + 1)^2$ or $y = \frac{1}{4}x^2$.

Now have the students compute an equation for the set of all points in space equidistant from the point $(0, 0, 2)$ and the xy-plane. They should graph their equation (a paraboloid) using technology to see if it matches their intuition. Then have them compute and graph the set of all points in space equidistant from the point $(0, 0, 2)$ and the x-axis (a parabolic cylinder). Note here that if P is the point (x, y, z), then the distance from P to the x-axis is $\sqrt{y^2 + z^2}$.

Discovery Project: Fun with Visualization

Ask the students to turn off their calculators and put down their pencils. Start by asking them to picture a plane in $\mathbb{R}^3$. Ask them how many regions in space are formed by a plane. (Answer: Two regions)

Then ask them how many regions are formed by *two* planes. Notice that there are two possible answers here. If the planes are parallel, they divide space into three regions, like a layer cake. If they cross, then four regions are formed. Have them get into groups to discuss how many regions can be made from three planes. For a shorter exercise, ask them to find only the maximum possible number of regions (8 regions). For a longer one, ask them to figure out all possible solutions (4, 6, 7, or 8 regions). If a group finishes ahead of the others, have them work on the case of four planes (maximum 15 regions.) Discuss the analogy between dividing space by planes and dividing a plane by lines.

Extended Lab Project: The Shapes of Things to Come

If the students have access to a CAS or calculator with three-dimensional graphing capabilities, have them look at surfaces of the form $Ax + By + Cz = 1$ and of the form $Ax^2 + By^2 + Cz^2 = 1$, varying the constants until they get a feel for the variety of shapes that each form can assume. Make sure that they try varying combinations of positive and negative constants. The goal is for the students to note that the first family is a collection of planes, and that the second takes on a variety of shapes.

Homework Problems

Core Exercises: 1, 5, 7, 9, 10, 14, 15, 18, 26

Sample Assignment: 1, 3, 4, 5, 7, 9, 10, 14, 15, 18, 21, 26, 28, 32, 33, 34

Note: Problem 1 of Focus on Problem Solving on page 703 would make a good project for motivated students.

Exercise	C	A	N	G	V
1					×
3	×				
4		×			×
5					×
7	×	×			
9		×			
10		×			

Exercise	C	A	N	G	V
14		×			
15		×			
18	×				×
19–28					×
32	×				
33	×			×	
34	×	×			

Group Work 1, Section 9.1
Working with Surfaces in Three-Dimensional Space

Provide a rough sketch and describe in words the surfaces described by the following five equations. The best way to do this exercise is to think first about the surfaces, before you calculate anything.

1. $x^2 + y^2 = 3^2$

2. $y^2 + z^2 = 1$

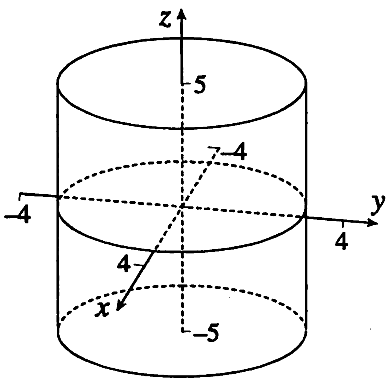
3. $z = y^2$

4. $xy = 1$

5. $x^2 + y^2 = z$

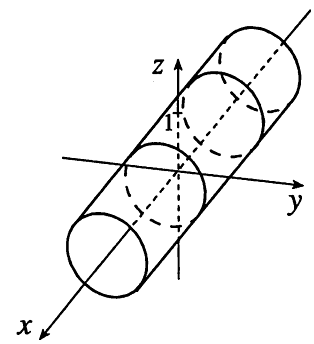
Group Work 1, Section 9.1
Working with Surfaces in Three-Dimensional Space (Solutions)

1. $x^2 + y^2 = 3^2$



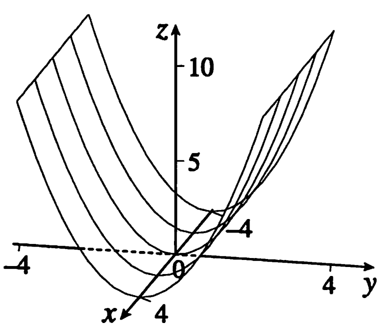
This is a right circular cylinder of radius 3 with axis the z-axis.

2. $y^2 + z^2 = 1$



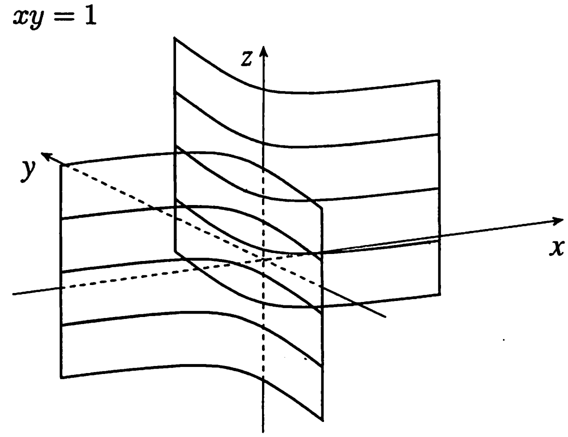
This is a right circular cylinder of radius 1 with axis the x-axis.

3. $z = y^2$



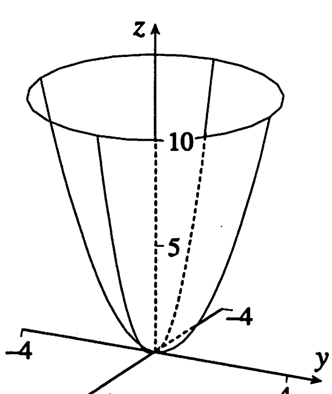
This is a parabolic cylinder parallel to the x-axis.

4. $xy = 1$



This is a hyperbolic cylinder parallel to the z-axis.

5. $x^2 + y^2 = z$



This is a paraboloid opening upward.

Group Work 2, Section 9.1
Lines, Lines, Everywhere Lines

Consider the surface S: $x + y + z = 1$, $x \geq 0$, $y \geq 0$, $z \geq 0$.

1. Show that every point on the line segment in the xy-plane, $x + y = 1$ ($x \geq 0$, $y \geq 0$) is also a point on this surface.

2. Show that every point on the line segments $x + z = 1$ ($x \geq 0$, $z \geq 0$) and $y + z = 1$ ($y \geq 0$, $z \geq 0$) in the xz- and yz-planes is also a point on this surface.

3. Given that two points $P_1\,(x_1, y_1, z_1)$ and $P_2\,(x_2, y_2, z_2)$ are on the surface S, show that their midpoint $P\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$ must also be a point on S.

4. Draw a picture of S.

9.2 Vectors

Suggested Time and Emphasis

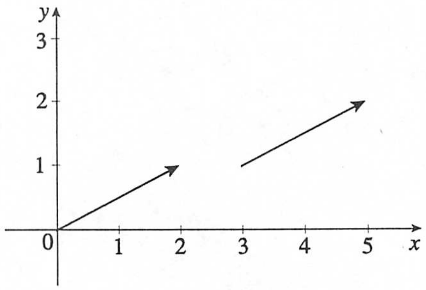
$\frac{1}{2}$–$\frac{3}{4}$ class   Essential material

Points to Stress

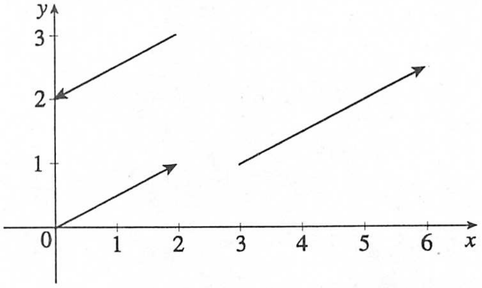
1. The basics of vectors, including the definition, length of vectors, vector addition and scalar multiplication.
2. The relationship between the vector representation $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, the point $P(a_1, a_2, a_3)$, and the position representation $\overrightarrow{OP}$.
3. The standard basis vectors **i**, **j**, and **k**, and general unit vectors.

Text Discussion

- The following two vectors have the same magnitude and direction. Are they the same vector, or two different vectors?



- Are the following three vectors parallel? (Some students can be confused by the definition of parallel vectors.)

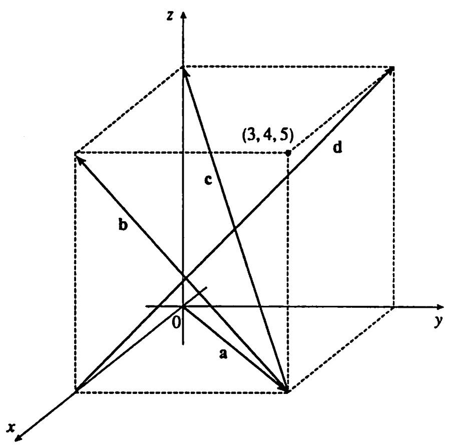


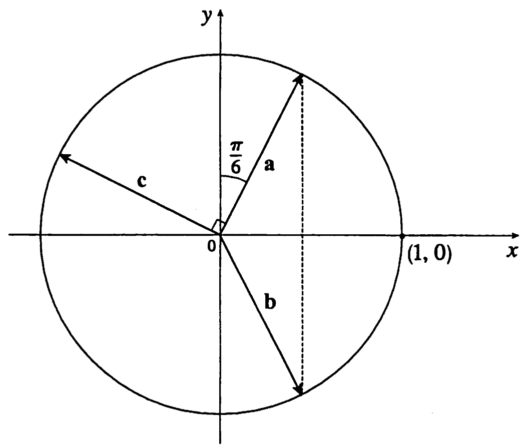
- If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ has $a_2 > 0$ and $a_3 < 0$, then the z-component of $-3\mathbf{a}$ has what sign?
- If $\overrightarrow{AB}$ is a representation of $\mathbf{a} = \langle a_1, a_2 \rangle$ and the initial point is $A\,(x, y)$, what are the coordinates of B?

Materials for Lecture

- Emphasize that the position representation of $\mathbf{a} = \langle a_1, a_2 \rangle$ is $\overrightarrow{OP}$, where P is the point (a_1, a_2) and that the position representation $\overrightarrow{OP}$ for $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ has endpoint $P\,(a_1, a_2, a_3)$. Indicate why $\mathbf{i} + \mathbf{j}$ is not a unit vector, even though **i** and **j** are unit vectors.
- Present geometric examples of the parallelogram law (such as Figure 4 on page 653) and scalar multiplication (such as Figure 7 on page 654).

- For a vector **a**, discuss the vector line $l = \{c\mathbf{a} \mid c \in \mathbb{R}\}$. Then introduce the idea that a line is determined by a point and a vector.
- Foreshadow the process of resolving vectors into components by working the following two problems. In the first, start with a rectangular solid with one corner at $(0,0,0)$ and the other at $(3,4,5)$, and find the component representation $\langle x, y, z\rangle$ of each of the vectors labeled **a**, **b**, **c**, and **d** in the diagram. In the second, start with a unit circle, and an angle of $\frac{\pi}{6}$with respect to the y-axis, and attempt to find the component representations $\langle x, y\rangle$ of the vectors labeled **a**, **b**, and **c**.



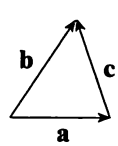


- Expand on Example 7 (page 658) by showing, in detail, how to break a two-dimensional vector into its horizontal and vertical components. One possible example is to calculate the work required to push a 200 lb object up a 30° incline (foreshadowing Section 9.3).

Workshop/Discussion

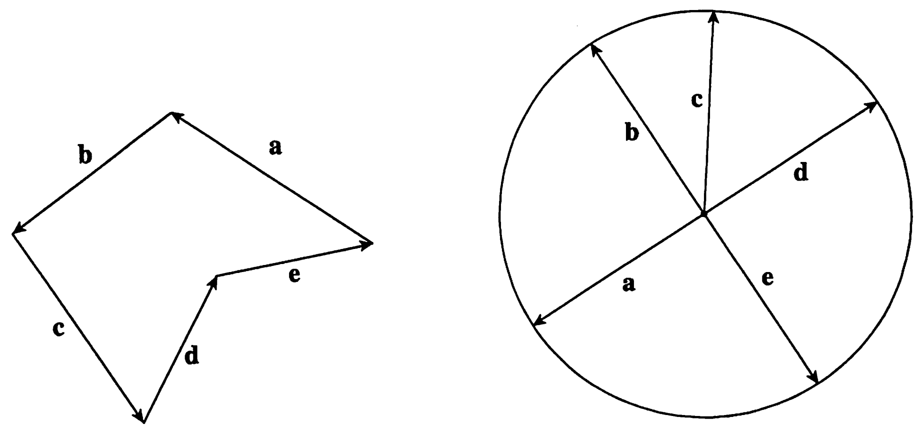
- Expand on the notion of a vector as a quantity with both magnitude and direction. If the students have a background in physics, make a list of quantities such as the ones below and have the students choose "vector" or "scalar" by a show of hands.

  Examples: speed, velocity, force, work, momentum, energy, friction
- Make sure to reinforce the fact that the location of the initial point of a vector can be chosen at will, and that the position vector uses the origin 0.
- Ask students to represent the vectors $\mathbf{i}+\mathbf{j}$, $\mathbf{i}+\mathbf{k}$, $\mathbf{j}+\mathbf{k}$, and $\mathbf{i}+\mathbf{j}+\mathbf{k}$ geometrically.
- Find unit vectors in the directions of $\langle 8,0,0\rangle$, $\langle 5,5,0\rangle$ and $\langle 1,2,3\rangle$, and explain what is happening geometrically. Then find a unit vector in the direction opposite that of $\langle -1,1,1\rangle$.
- Ask students to describe **c** in terms of **a** and **b**.



- Write **a** as $s\mathbf{b} + t\mathbf{c}$ where $\mathbf{b} = \langle 1,1\rangle$ and $\mathbf{c} = \langle -1,2\rangle$ and **a** is chosen by the students. Repeat for a different vector **a**. Do not stress the arithmetic for this introduction; instead emphasize the fact that it can be done for any choice of **a**. Follow up by letting $\mathbf{b} = \langle 2,4\rangle$ and $\mathbf{c} = \langle 1,2\rangle$, and ask the students to try to express $\langle 1, 1\rangle$ as a linear combination of $\mathbf{b}$ and $\mathbf{c}$. Conclude that some pairs of vectors in $\mathbb{R}^2$ have this "combining" property while others do not, and mention that we will get back to this concept.

- Draw the following two diagrams on the board, and compute $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} + \mathbf{e}$ for each of them. (Answers: $\mathbf{0}$, $\mathbf{c}$)



Group Work 1: The Position Vector

Group Work 2: Where Do They Point?

This group work extends the idea of adding several vectors without coordinates. For Problem 1, the students can be given the hint to first try computing $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} + \mathbf{e}$. The answers are $-2\mathbf{e}$ for Problem 1, $\left\langle -\frac{1}{2}r, -\frac{\sqrt{3}}{2}r \right\rangle$ for Problem 2(a), and $-(\mathbf{a} + \mathbf{d}) = \langle -r, 0\rangle$ for Problem 2(b), since $\mathbf{a} + \mathbf{b} = \langle r, 0\rangle$.

Group Work 3: The Return of Geometry

This is a challenging group work for stronger students. Question 1 is based on Exercise 37. It is best to have some groups do Question 1 first, and some start with Question 2. A group that finishes early can start on the other problem. Before starting the exercise, it may be helpful to remind the students of the definitions of "midpoint" and "bisect". An advanced class may not need the handout sheets with the diagrams; they may just need to have the problem statement written on the board. Other classes might need hints as they go along, given that proofs of this type will be new to most of the students.
After they have started Question 2, you may want to announce that $r\mathbf{a} + s\mathbf{b} = \mathbf{0}$ if and only if $r = s = 0$ (since $\mathbf{a}$ and $\mathbf{b}$ are not parallel).

Lab Project: Turning a Vector

Start by having the students use paper and pencil to rotate a vector $\langle a, b\rangle$ by 180°, 90°, and 45°. Then, have them use a CAS to explore general rotations and scaling of vectors (for example, replacing $\langle a, b\rangle$ by $\langle a + 1, b + 1\rangle$ and noticing that the resultant vector is not in general a scaling of $\langle a, b\rangle$).

Homework Problems

Core Exercises: 4(a), 4(d), 5, 6(a), 6(d), 6(f), 9, 12, 18, 19, 23, 24

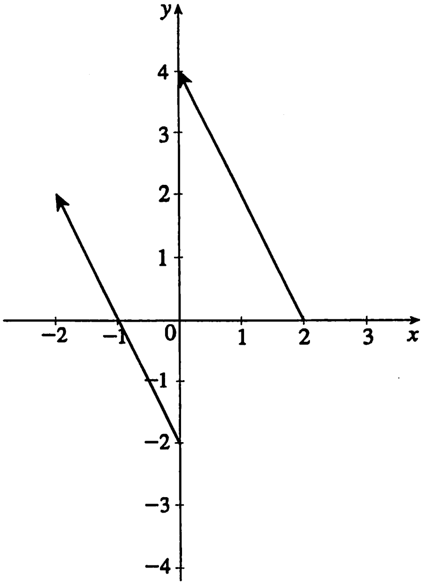
Sample Assignment: 4, 5, 6, 9, 12, 15, 18, 19, 23, 24, 33

Exercise	C	A	N	G	V
4					×
5					×
6					×
9		×			×
12		×			×
15		×			

Exercise	C	A	N	G	V
18		×			
19		×			
23		×		×	
24	×	×			
33	×				×

Group Work 1, Section 9.2
The Position Vector

Recall that a vector, say $\langle -2, 4 \rangle$, is always the same vector no matter where it is placed in space (or on the plane).



Also recall that we call a particular representation of a vector (namely the one starting at the origin) the *position vector*. Let **a** be the two-dimensional vector $\langle -2, 4 \rangle$.

1. Draw the position vector of **a** and find the angle that it makes with the positive x-axis.

2. Show that the tip of the position vector of $\dfrac{\mathbf{a}}{|\mathbf{a}|}$ is on the unit circle $x^2 + y^2 = 1$, and illustrate this fact on the diagram.

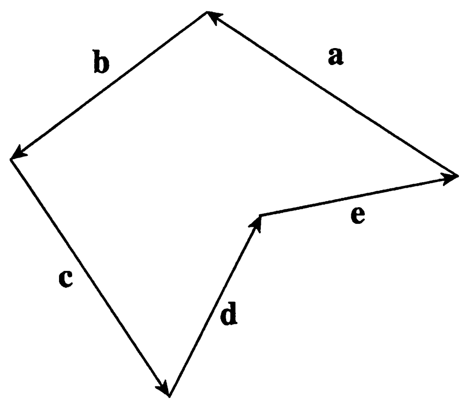
3. Now consider a general vector $\mathbf{b} \neq \mathbf{0}$ whose position vector makes an angle θ with the positive x-axis. Show that the unit vector in the direction of **b** is $\dfrac{\mathbf{b}}{|\mathbf{b}|} = \langle \cos\theta, \sin\theta \rangle$.

4. Explain why, no matter what $\mathbf{b}$ is ($\mathbf{b} \neq 0$), the tip of the position vector of $\frac{\mathbf{b}}{|\mathbf{b}|}$ is on the unit circle.

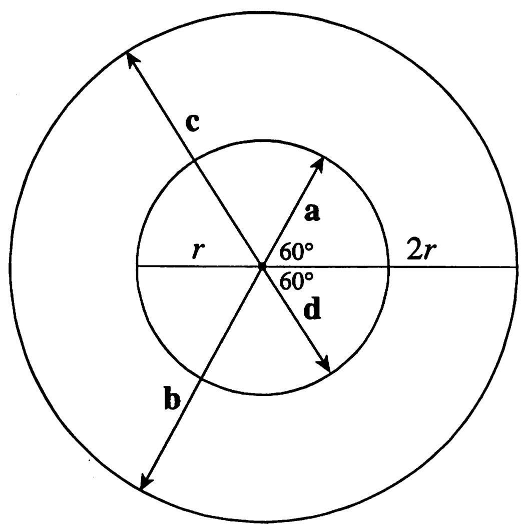
5. Consider the three-dimensional position vector $\mathbf{b} = \langle 1, 1, 1 \rangle$. Can you make a statement, similar to what was done in Problem 2, about the tip of the position vector of $\frac{\mathbf{b}}{|\mathbf{b}|}$? Which vector has greater magnitude, $\mathbf{b}$ or $\frac{\mathbf{b}}{|\mathbf{b}|}$? Answer the same questions for $\mathbf{c} = \langle \frac{1}{2}, 0, \frac{1}{3} \rangle$.

Group Work 2, Section 9.2
Where Do They Point?

1. Compute a + b + c + d − e for the following diagram.



2. (a) Compute the position representation $\overrightarrow{OP}$ for a + b in the following diagram. Give the coordinates of the point P.

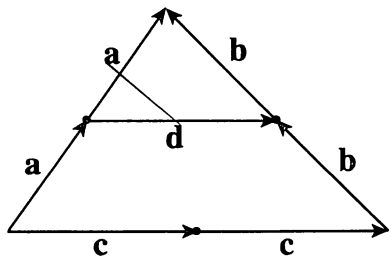


(b) Compute a + b + c + d.

Group Work 3, Section 9.2
The Return of Geometry

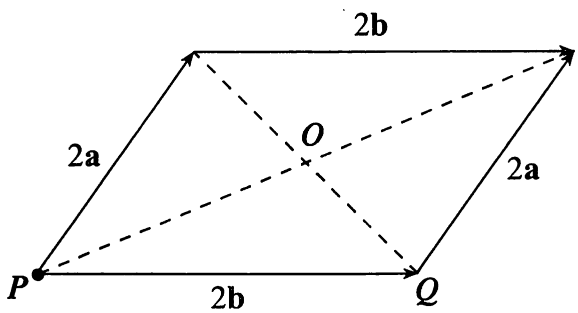
Vectors sometimes make it easier to prove geometric properties. In this exercise, you will get the chance to prove some useful geometric formulas using your knowledge of vectors.

1. Use vectors to show that the a line segment connecting the midpoints of two sides of a triangle is half as long as the opposite side, and also parallel to the opposite side.



Hint: You need to show that $\mathbf{d} = \mathbf{c}$. One way to do it is to express $\mathbf{c}$ in terms of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{d}$ in two different ways.

2. Use vectors to show that the diagonals of a parallelogram bisect each other



Hint: Let $\mathbf{c}$ be the vector from point P to point O, and let $\mathbf{d}$ be the vector from point Q to point O. Then there exist numbers k and l such that $\mathbf{c} = k(\mathbf{a} + \mathbf{b})$ and $\mathbf{d} = l(\mathbf{a} - \mathbf{b})$. Show that $k = l = 1$.

9.3 The Dot Product

Suggested Time and Emphasis

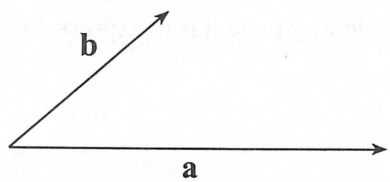
1 class Essential Material

Points to Stress

1. The algebraic and geometric formulations of the dot product.
2. The interpretation of the sign of $\mathbf{a} \cdot \mathbf{b}$ in terms of the angle between $\mathbf{a}$ and $\mathbf{b}$.
3. Orthogonal vectors.
4. Vector and scalar projections.

Text Discussion

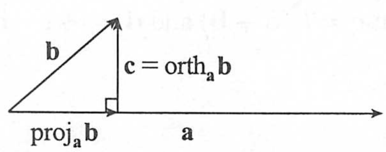
- Consider these two vectors:



Draw the vector projection of $\mathbf{b}$ onto $\mathbf{a}$, and the vector projection of $\mathbf{a}$ onto $\mathbf{b}$.

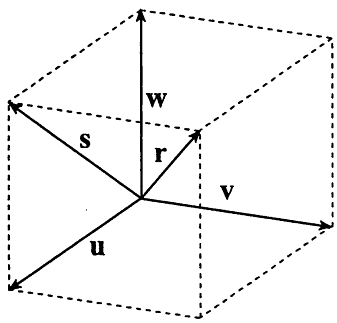
Materials for Lecture

- Another approach to the dot product is to introduce it algebraically using the coordinates of vectors ($\langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$). Assert that the result is the same even if the vectors are translated (this is clear) or rotated (this is less so, but special cases would be discussed below to justify the assertions.) Then show that if two vectors in $\mathbb{R}^2$ are rotated so that one is lying on the x-axis, we obtain $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$.
- Suppose $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$. Ask the class if it is reasonable to assume that $\mathbf{b} = \mathbf{c}$. Then show that this is not the case; that we only know that $\mathbf{b}$ and $\mathbf{c}$ have the same projection onto $\mathbf{a}$. Perhaps use the example where $\mathbf{a} = \langle 1, 2 \rangle, \mathbf{b} = \langle 2, 1 \rangle, \mathbf{c} = \langle 4, 0 \rangle$. Then convince the students that $\mathbf{a} \perp (\mathbf{b} - \mathbf{c})$ and give a geometric interpretation.
- Elaborate on Exercises 25 and 26 as follows: Referring to the figure below, conclude that if $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$, we can write $\mathbf{b} = \mathbf{c} + \operatorname{proj}_{\mathbf{a}} \mathbf{b}$, with $\mathbf{c} \perp \mathbf{a}$ and $\operatorname{proj}_{\mathbf{a}} \mathbf{b} \parallel \mathbf{a}$. $\mathbf{c}$ is called the orthogonal projection of $\mathbf{b}$ onto $\mathbf{a}$.



- The definition $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$ implies that the dot product is invariant under rotations. Illustrate this fact by choosing unit vectors $\mathbf{a}_1 = \left\langle \cos\frac{\pi}{6}, \sin\frac{\pi}{6} \right\rangle$, $\mathbf{b}_1 = \left\langle \cos\frac{3\pi}{2}, \sin\frac{3\pi}{2} \right\rangle$, $\mathbf{a}_2 = \left\langle \cos\left(\frac{\pi}{6} + \alpha\right), \sin\left(\frac{\pi}{6} + \alpha\right) \right\rangle$, and $\mathbf{b}_2 = \left\langle \cos\left(\frac{3\pi}{2} + \alpha\right), \sin\left(\frac{3\pi}{2} + \alpha\right) \right\rangle$, and showing that $\mathbf{a}_1 \cdot \mathbf{b}_1 = \mathbf{a}_2 \cdot \mathbf{b}_2$ regardless of α.

- Discuss Exercise 37. If possible, bring in a model tetrahedron.
- Compute **r**, **s**, and **r** · **s** if **u**, **v**, and **w** are mutually orthogonal unit vectors.



Workshop/Discussion

- Go over Exercise 1, perhaps allowing students to answer each question by a show of hands before going through the solutions.
- Compute the angle between the vectors $\langle -1, 1\rangle$ and $\langle \sqrt{3}, 1\rangle$.
- Compute values for a such that $\mathbf{a} = \langle 1, a, -1\rangle$ and $\mathbf{b} = \langle -1, 1, 1\rangle$ (a) make an angle of 45°, and (b) satisfy $\mathbf{a} \perp \mathbf{b}$.
- Go over two methods of solving Exercise 33, first using vectors, and then using calculus to minimize the distance from (x_1, y_1) to $\left(x_0, \dfrac{-ax_0 - c}{b}\right)$.
- Show that if $\mathbf{a} \perp \mathbf{b}$ and $\mathbf{a} \perp \mathbf{c}$ then $\mathbf{a} \perp (r\mathbf{b} + s\mathbf{c})$ for any real numbers r and s.
- Bring in a model square pyramid. Compute the angle between the faces, and then compute the angle between an edge and a face.

Group Work 1: The Right Stuff

Give each group of students a different set of three points, and have them use vectors to determine if they form a right triangle. They can do this either using dot products, or by calculating side lengths and using the Pythagorean Theorem. Perhaps have the students with the points in $\mathbb{R}^2$ carefully graph their points to provide a visual check. Point out that using the dot product is usually the easier method for points in $\mathbb{R}^3$.

Sample triples:

$(-2,-1), (-2,8), (8,-1)$	$(3,4), (3,12), (6,5)$
$(0,0), (10,7), (-14,20)$	$(2,1,2), (3,3,1), (2,2,4)$
$(-1,-2,-3), (0,0,-4), (-1,-1,-1)$	$(2,3,6), (3,4,7), (3,3,6)$

Group Work 2: The Regular Hexagon

If the students have trouble with this one, copy the figure onto the blackboard. Then draw a point at its center, and draw lines from this point to every corner point. This modified figure should make the exercise more straightforward.

Group Work 3: Gravity's Rainbow

Depending upon the background of a class, a detailed introduction to this exercise may be necessary. Students with a good physics background could anticipate the final answer. The idea is that since the constant gravitational force points downward, work is done against gravity only while the student is moving upward. In other words, horizontal movement does *no* work against gravity, and vertical movement results in the maximum work against gravity. The students should discover that the work done in going from point A to point D is independent of the path taken.

Homework Problems

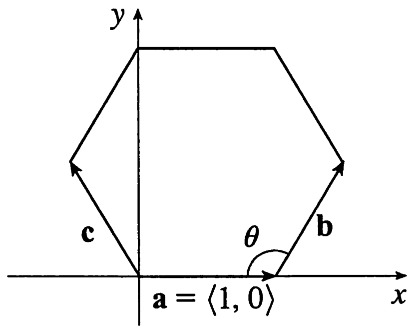
Core Exercises: 1, 3, 5, 9, 17(a), 17(d), 22, 26, 29

Sample Assignment: 1, 2, 3, 5, 6, 9, 10, 14, 17, 18, 22, 26, 29, 33, 35

Exercise	C	A	N	G	V
1	×				
2		×			
3–8		×			
9		×			×
10		×			×
14		×			
17		×			
18		×			
22		×			
26		×			×
29		×			
33	×	×			
35		×			×

Group Work 2, Section 9.3
The Regular Hexagon

Consider the following regular hexagon:



1. Compute $|\mathbf{a}|$, $|\mathbf{b}|$, and $|\mathbf{c}|$.

2. What is the angle θ?

3. What is $\mathbf{a} \cdot \mathbf{b}$?

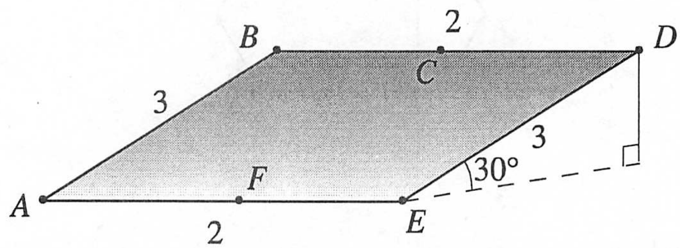
4. What is $\mathbf{a} \cdot \mathbf{c}$?

5. What are $\text{proj}_{\mathbf{a}}\, \mathbf{b}$ and $\text{proj}_{\mathbf{b}}\, \mathbf{c}$?

6. What is the x-component of $\mathbf{a} + \mathbf{b} + \mathbf{c}$?

Group Work 3, Section 9.3
Gravity's Rainbow

You are standing at the base of a hill. The hill is very broad and flat, and has a constant 30° incline. It is... Oh, what the heck, we won't lie to you, it's not a hill, it's just a plain old inclined plane. Let's start again. You are standing at the base of a big inclined plane, at the point marked A. You need to walk to a point D, two miles to the west, and three miles up the plane, as illustrated by the diagram below. The plane has a 30° incline.



We assume that gravity always points straight down, with a constant force G.

Now compute the amount of work done against gravity in walking from A to D along the following paths:

1. First walking up the hill to point B, and then walking across to point D.

2. First going across the plane to point E, and then up to point D.

3. First going halfway between A and E, to point F, then walking up to point C, and then over to point D.

4. First going to a point halfway between A and B, walking straight across, and then up to point D.

5. Walking along the diagonal between A and D. (Actually do the calculation.)

6. Walking along your own zany path going upward from A to D.

9.4 The Cross Product

Suggested Time and Emphasis

$\frac{3}{4}$–1 class   Essential material

Note: The scalar triple product comes up in the derivation of Kepler's Laws in Section 10.4, but can be omitted if this section is not to be covered.

Points to Stress

1. The cross product defined as a vector perpendicular to two given vectors, whose length is the area of the parallelogram determined by the vectors, and its algebraic interpretation.
2. The right-hand rule and properties of the cross product.
3. The scalar triple product as the volume of a parallelepiped.

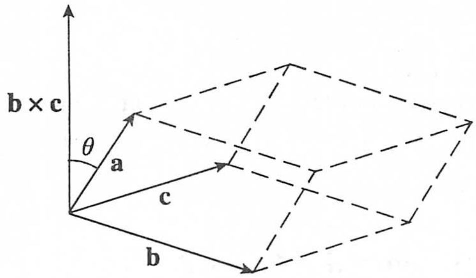
Text Discussion

- If $\mathbf{a} \perp \mathbf{b}$, what is the length of $\mathbf{a} \times \mathbf{b}$? What can we say about $(\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{a})$?
- Why is it that if $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$ then $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar?

Materials for Lecture

- Point out that one can define $\mathbf{a} \times \mathbf{b}$ as $\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$. From this definition, you can show directly that $\mathbf{a} \times \mathbf{b}$ is mutually orthogonal to $\mathbf{a}$ and $\mathbf{b}$, and that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$.
- Pose the problem of finding a vector perpendicular to two given vectors (without considering the length of the resultant vector). Show that there is an obvious solution for $\langle 1, 0, 0 \rangle$ and $\langle 0, 1, 0 \rangle$. Then look at the two vectors $\langle 1, 0, 1 \rangle$ and $\langle 0, 1, 1 \rangle$. Repeat for $\langle -3, 1, -7 \rangle$ and $\langle 0, -5, -5 \rangle$. Point out that in this case an alternate algebraic solution using dot products gives $-3x + y - 7z = -5y - 5z$, $y = -z$, $x = -\frac{8}{3}z$, and so $\mathbf{c} = \langle -\frac{8}{3}, -1, 1 \rangle$ works. Note that $\mathbf{c}$ is a scalar multiple of $\mathbf{a} \times \mathbf{b} = \langle -40, -15, -15 \rangle$. This suggests the general idea behind the cross product.
- Point out that while $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ have the same length, $\mathbf{b} \times \mathbf{a}$ points in the opposite direction to that of $\mathbf{a} \times \mathbf{b}$, by the right-hand rule. Thus $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$.
- If discussing torque, have a strong student stand on one side of a swinging door, while you stand on the other. Have the student stand relatively close to the hinge, while you stand towards the door edge. Have a pushing contest, and ask the students how a mere mathematics professor was able to best a mighty teenager. Alternatively, bring a bicycle wheel to class, and show how, when it is spinning, it is easy to translate, yet hard to twist. In the latter case, the twisting is acting against the torque that was set up by the spinning wheel.
- If $\mathbf{a} = \alpha\mathbf{b} + \beta\mathbf{c}$, with α and β constant, show that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$.

- Explain the geometry involved in computing the volume of a parallelepiped spanned by vectors a, b, and c:



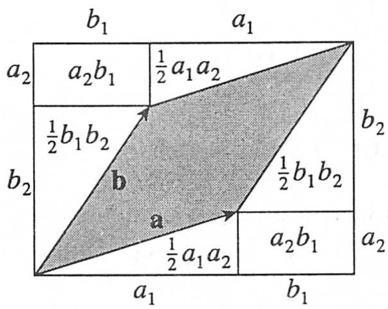
$$V = (\text{area of the base}) \times (\text{height}) = |\mathbf{b} \times \mathbf{c}|\,|\text{proj}_{\mathbf{b}\times\mathbf{c}}\,\mathbf{a}|$$
$$= |\mathbf{b} \times \mathbf{c}|\,|\mathbf{a}|\,|\cos\theta| = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

Conclude that a, b, and c are coplanar if and only if the triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$.

- The solid discussed above was originally called a *parallelepipedon* from the Greek words: *para* (beside), *allele* (other), *epi* (upon), and *pedon* (ground). It meant that there was always a face that was parallel to the one on the ground.)

Workshop/Discussion

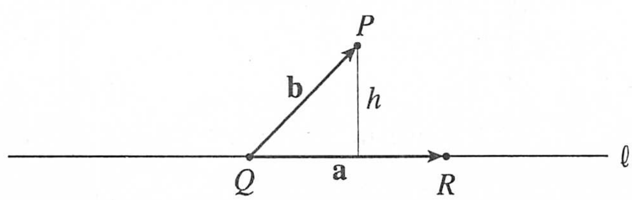
- Give a geometric proof (without using cross products) that the area of the parallelogram defined by $\mathbf{a} = \langle a_1, a_2, 0\rangle$ and $\mathbf{b} = \langle b_1, b_2, 0\rangle$ is $a_1b_2 - a_2b_1$, that is, $|\mathbf{a} \times \mathbf{b}|$.



Shaded Area $= (a_1 + b_1)(a_2 + b_2) - 2a_2b_1 - b_1b_2 - a_1a_2 = a_1b_2 - a_2b_1$

- Find the set of all position vectors mutually perpendicular to non-collinear vectors $\langle a_1, a_2, 0\rangle$ and $\langle b_1, b_2, 0\rangle$. Note that the resultant set of vectors determines a line in space (the z-axis, in fact). Then have the students try to determine what the set of all vectors perpendicular to the single vector $\langle 1, 2, 3\rangle$ will look like. (They can just try to visualize the answer.) Show how the set of all of these vectors from a given base point forms a plane in space.

- Discuss the distance from a point to a line (Exercise 27) this way:



Let h be the distance from P to $\overrightarrow{QR} = \mathbf{a}$. Pause to allow the students to find the area of the parallelogram determined by a and b in terms of h. ($A = h\,|\mathbf{a}|$) From that fact, derive that the distance from P to l is $\frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|}$.

- Show that if $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, then a is parallel to $\mathbf{b} - \mathbf{c}$. Then use geometry to show that if a is parallel to $\mathbf{b} - \mathbf{c}$, the area of the parallelogram determined by a and b equals the area of the parallelogram determined by a and c.

Group Work 1: Messing with the Cross Product

Four different problems are given for the group work. Give each group one problem to solve. If a group finishes early, give them another one to work out. After every group has finished their original problem, give them a few minutes to practice, and then have them come up and state their problem and demonstrate their solution.

Note: Problem 4 is more difficult than the others; assign it to groups accordingly. It may be helpful to point out that $(\mathbf{b} - \mathbf{c}) \parallel \mathbf{a}$ implies that a lies in the plane determined by b and c.

Group Work 2: A Matter of Shading

Lab Project 1: Cross Product Properties

With a computer algebra system, have the students check the distributive property of cross products: $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$ and show the non-associativity of $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ and the non-commutativity of $\mathbf{a} \times \mathbf{b}$ by writing the vectors as $\langle a_1, a_2, a_3 \rangle$, $\langle b_1, b_2, b_3 \rangle$, and $\langle c_1, c_2, c_3 \rangle$ and having the system perform the relevant computations.

Lab Project 2: Exploring the Triple Product

For this project, you should use a CAS which can easily compute and graphically illustrate cross products. Consider the triple product $\mathbf{n} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. Use two fixed vectors $\mathbf{b} = \langle 1, -1, 2 \rangle$ and $\mathbf{c} = \langle 2, 1, 3 \rangle$.

1. Treating all vectors as position vectors, let the students check, using several different vectors a, that n is in the plane through $\langle 0, 0, 0 \rangle$ generated by b and c.
2. Have the students explain why the fact illustrated in point 1 is true.
3. Let them explore how n changes with various conditions on a, b, and c, for example:
   (a) $\mathbf{a} \perp \mathbf{b}$ (Result: n is parallel to b)
   (b) $\mathbf{a} \parallel \mathbf{b} \times \mathbf{c}$ (Result: $\mathbf{n} = 0$)
   (c) $\mathbf{a} = \mathbf{b}$ (Result: n is parallel to $\operatorname{orth}_{\mathbf{b}} \mathbf{c}$)
   (d) $\mathbf{a} \parallel \mathbf{b}$ (Result: n is perpendicular to a and n is parallel to $\operatorname{orth}_{\mathbf{b}} \mathbf{c}$)

   Note: the orthogonal projection $\operatorname{orth}_{\mathbf{b}} \mathbf{a}$ is introduced in Exercise 25 of Section 9.3.
4. Have the students verify Formula 8 (page 673) for the vector triple product by checking that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$ when $\mathbf{a} = \langle -2, -1, 2 \rangle$, $\mathbf{b} = \langle 1, -1, 2 \rangle$, and $\mathbf{c} = \langle 2, 1, 3 \rangle$.

Homework Problems

Core Exercises: 1, 2, 5, 7, 13, 16, 17, 21, 24, 33

Sample Assignment: 1, 2, 5, 7, 11, 12, 13, 16, 17, 21, 24, 25, 28, 33

Exercise	C	A	N	G	V
1	×				
2		×			×
5		×			×
7		×			
11		×			
12		×			×
13		×			

Exercise	C	A	N	G	V
16		×			
17		×			
21		×			
24		×			
25		×			
28	×	×			×
33	×				×

Group Work 1, Section 9.4
Messing with the Cross Product

1. Consider the vector $\mathbf{c} = \mathbf{a} \times (\mathbf{a} \times \mathbf{b})$. Is $\mathbf{c} \perp \mathbf{a}$? Is $\mathbf{c} \perp \mathbf{b}$? Either prove or give a counterexample.

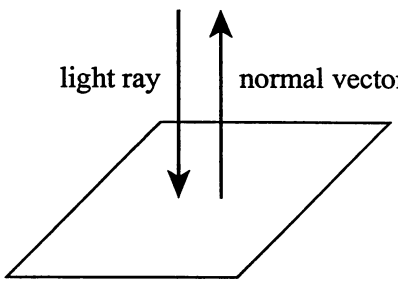
2. Suppose that $\mathbf{j} \times \mathbf{a} = \mathbf{i}$. Give two possible solutions for $\mathbf{a}$, and discuss other possible solutions.

3. Given $\mathbf{a} = \langle 1, 2, 3 \rangle$ and $\mathbf{b} = \langle 1, -1, -1 \rangle$, sketch the collection of all position vectors $\mathbf{c}$ satisfying $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$.

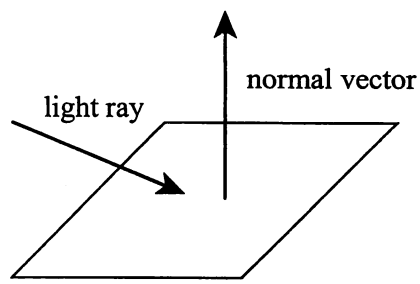
4. Suppose $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$. Show that $\mathbf{a}$ is perpendicular to $(\mathbf{b} - \mathbf{c}) \times (\mathbf{a} + \mathbf{b})$.
Hint: Why is $\mathbf{b} - \mathbf{c}$ parallel to $\mathbf{a}$?

Group Work 2, Section 9.4
A Matter of Shading

One way to accurately render three-dimensional objects on a computer screen involves using the dot and cross products. In order to determine how to shade a piece of a surface we need to determine the angle at which rays from the light source hit the surface. To determine this angle, we compute the dot product of the light vector with the vector perpendicular to the surface at the particular point, called the normal vector.
If the light ray hits the surface straight on, that is, has an angle of $0°$ with the normal, then this piece of the surface will appear bright. On the other hand, if the light comes in on an angle, this piece of the surface will not appear as bright.



This piece appears bright.



This piece appears dimmer.

Suppose the light source is placed directly above the xy-plane, so that the light rays come in parallel to the vector $\langle 0, 0, -1 \rangle$. At what angle (to the normal) do the light rays hit a triangle bounded by the points

1. $(3, 2, 4)$, $(2, 5, 3)$, and $(1, 2, 6)$?

2. $(3, 5, 2)$, $(3, 3, 1)$, and $(1, 3, 1)$?

3. Suppose we are standing above the light source looking down on the xy-plane. Which of these two regions will appear brighter to us?

Discovery Project: The Geometry of a Tetrahedron

The three parts of this project can be assigned independently of each other. The temptation will be for the instructor to start giving diagrams and hints too soon. Students will probably not be familiar with how to get started on this type of exercise, but the "getting started" process is one of the most important things they will learn from this project.

Note that three very different solutions to Problem 3 are given in the solutions manual, and only one of them uses the result of Problem 1.

9.5 Equations of Lines and Planes

Suggested Time and Emphasis

1 class Essential Material

Points to Stress

1. Three ways to describe a line:

- Vector (parametric) equations (starting with point P_0 on the line and direction vector $\mathbf{d}$): $\mathbf{r} = \overrightarrow{OP_0} + t\mathbf{d}$.
- Symmetric equations: $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$.
- Two-point vector equation (starting with two points P_0 and Q_0 on the line): $\mathbf{r} = \overrightarrow{OP_0} + t\,\overrightarrow{P_0Q_0}$.

2. Three ways to describe a plane:

- Vector equation (starting with point P_0 and normal vector $\mathbf{n}$): $\mathbf{n} \cdot \left(\mathbf{r} - \overrightarrow{OP_0}\right) = 0$.
- Scalar equation (starting with point P_0): $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ or $ax + by + cz + d = 0$.
- Parametric equations (starting with point P_0 and two direction vectors $\mathbf{a}$ and $\mathbf{b}$): $\mathbf{r} = \overrightarrow{OP_0} + t\mathbf{a} + s\mathbf{b}$.

Text Discussion

- When specifying the equation of a line in space, the text says that you need a point on the line and a vector parallel to the line. Why can't you determine a line in space simply by using one vector?
- In two dimensions, we can specify a line just by writing the one equation $y = mx + b$. In three dimensions, we can specify a line by the three equations $x = x_0 + at$, $y = y_0 + bt$, and $z = z_0 + ct$. Is there a way that we can write two equations for a two-dimensional line?

Materials for Lecture

- An overall theme for this section could be that a "line" is determined by a point and a direction, and a "plane" is determined by a point and a normal vector, or a point and two directions.
- Review parametric representation of lines in $\mathbb{R}^2$, and then generalize to $\mathbb{R}^3$. Recall that in two dimensions, a line can be determined by a point and a slope. Ask for the slope of the line between the points $(0, 0)$ and $(1, 2)$. Start with the line $y = mx + b$ and write it parametrically as $x = t, y = mt + b$. Then write the vector equation with $\mathbf{r}_0 = \langle 0, b\rangle$ and $\mathbf{d} = \langle 1, m\rangle$, a vector whose direction has slope m. Next ask for the slope of the line between $(0, 0, 0)$ and $(1, 2, 3)$. Note that there is no answer — we lose the idea of "slope" when going from two to three dimensions. So a vector is our only way of specifying direction in three dimensions.
- Discuss the overdetermined system which comes up in Example 3. Note the possibilities:
  1. The lines are the same (infinitely many solutions)
  2. The lines intersect (one solution)
  3. The lines are parallel or skew (no solution)

- Discuss two ways to find the equation of a plane containing three non-collinear points P, Q, and R. O way is to form the normal vector $\mathbf{n} = \mathbf{a} \times \mathbf{b}$, where $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{b} = \overrightarrow{PR}$, and use the vector equati Another approach is to use the parametric equation $\mathbf{r} = \overrightarrow{OP} + t\mathbf{a} + s\mathbf{b}$. This latter equation requires l computation. Verify that points given by the parametric equation also satisfy the vector equation.

- Redo Example 5 [find the plane passing through the points $P(1,3,2)$, $Q(3,-1,6)$, and $R(5,2,0)$] us parametric equations.

- Use Example 7 (page 680) to show how to solve algebraically for the line of intersection of two planes $y + z = 1 - x$, $-2y + 3z = 1 - x$, and then compute the vector equation of the line by setting $x = t$ a solving the 2×2 system, which gives $x = t$, $y = \frac{2}{5} - \frac{2}{5}t$, $z = \frac{3}{5} - \frac{3}{5}t$.

Workshop/Discussion

- Review lines in two dimensions. For example, ask the students to draw the line passing through $(1,2)$ the direction of $\mathbf{i} - 2\mathbf{j}$ and then write parametric equations for this line. Stress that these equations are unique.

- Discuss intersecting lines, perpendicular lines, and parallel lines. Find the angle between a pair intersecting lines.

- Go through Example 8 carefully, emphasizing the geometry.

- Define the parametric equation of a plane through the origin generated by vectors $\mathbf{a}$ and $\mathbf{b}$: $\mathbf{r} = s\mathbf{a} +$ Verify that this is a plane by showing that $\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b}) = 0$ for all $\mathbf{r}$. Parametrize the plane with equat $4x - y + 2z = 8$, by finding three points on the plane [for example, $(2,0,0)$, $(0,-8,0)$, and $(0,0,$ using the method of setting two of the coordinates equal to zero and computing the third] and obtain $\mathbf{r} = \langle 2,0,0 \rangle + s\langle -2,-8,0 \rangle + t\langle -2,0,4 \rangle$. Alternatively, write the equation as $z = -2x + \frac{1}{2}y + 4$ a show that a typical vector in this plane is $\mathbf{r} = \left\langle x, y, -2x + \frac{1}{2}y + 4 \right\rangle = x\langle 1,0,-2 \rangle + y\left\langle 0,1,\frac{1}{2} \right\rangle + \langle 0,0,$ This is another parametric equation, with parameters x and y.

- Give the students some parts of Exercise 1 to try in groups. Poll the groups before giving out any answe Ask students with conflicting opinions to try to explain their answers to each other.

Group Work 1: The Match Game

This is a pandemonium-inducing game. Each group is given two points on a straight line, parametric equati for a different line, a symmetric equation for a third line, and a vector equation for a fourth line. Each line described in each of the four ways. The first group to find four descriptions for the same line wins some fo of prize.

The activity works best if the students can walk around, showing each other their descriptions and trying find matches.

For the convenience of the teacher, each row contains a winning combination. Make sure that each team st with descriptions from different rows.

Category A	Category B	Category C	Category D
The line between $(0,0,1)$ and $(1,2,1)$	$\mathbf{r} = \langle 2,4,1\rangle + t\langle 1,2,0\rangle$	$x = 2t$, $y = 4t$, $z = 1$	$\frac{x-1}{2} = \frac{y-2}{4}, z = 1$
The line between $(0,-3,3)$ and $(3,3,0)$	$\mathbf{r} = \langle 1,-1,2\rangle + t\langle 1,2,-1\rangle$	$x = 2+t$, $y = 1+2t$, $z = 1-t$	$\frac{x-1}{2} = \frac{y+1}{4} = \frac{z-2}{-2}$
The line between $(1,3,2)$ and $(1,-1,6)$	$\mathbf{r} = \langle 1,2,3\rangle + t\langle 0,-1,1\rangle$	$x = 1$, $y = -t$, $z = 5+t$	$x = 1, \frac{y-1}{-2} = \frac{z-4}{2}$
The line between $(0,0,4)$ and $(12,8,8)$	$\mathbf{r} = \langle 9,6,7\rangle + t\langle -3,-2,-1\rangle$	$x = 6-6t$, $y = 4-4t$, $z = 6-2t$	$\frac{x-3}{3} = \frac{y-2}{2} = \frac{z-5}{1}$
The line between $(5,0,7)$ and $(-2,-7,0)$	$\mathbf{r} = \langle 3,-2,5\rangle + t\langle -1,-1,-1\rangle$	$x = 2-2t$, $y = -3-2t$, $z = 4-2t$	$\frac{x}{2} = \frac{y+5}{2} = \frac{z-2}{2}$
The line between $(-3,3,-9)$ and $(3,-3,9)$	$\mathbf{r} = \langle 0,0,0\rangle + t\langle -1,1,-3\rangle$	$x = -1+t$, $y = 1-t$, $z = -3+3t$	$\frac{x+2}{-2} = \frac{y-2}{2} = \frac{z+6}{-6}$
The line between $(-4,2,1)$ and $(-11,1,-1)$	$\mathbf{r} = \langle 3,3,3\rangle + t\langle 7,1,2\rangle$	$x = 10+7t$, $y = 4+t$, $z = 5+2t$	$\frac{x+4}{14} = \frac{y-2}{2} = \frac{z-1}{4}$

▲ Group Work 2: Da Planes! Da Planes!

For this activity, each group gets a different point in space. The groups have to find equations of planes containing this point perpendicular to various directions, such as the x-axis, or the line $y = x$. As a follow-up question, have them try to determine if two given lines are skew.

▲ Group Work 3: Planes from Points

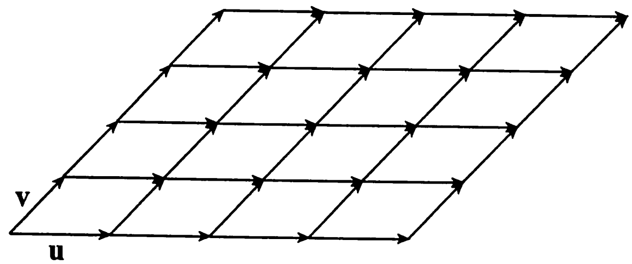
Give each group two sets of three points each, one non-collinear set and one collinear set. Ask the students to give a parametric equation of the unique plane containing the points. For the second set of points this is a trick question, since collinear points do not determine a plane.

Group Work 4: The Moving Plane

Group Work 5: The Spanning Set

The purpose of this activity is to give the students a sense of how two non-parallel vectors in two dimensions span the entire xy-plane.

Start by giving each student or group of students a sheet of regular graph paper and a transparent grid of parallelograms formed by two vectors **u** and **v**.



Next give each student a point (x, y) in the plane. By placing the grid over the graph paper they should estimate values of r and s such that $\langle x, y\rangle = r\mathbf{u} + s\mathbf{v}$. Repeat for several other points (including some for which one or both of r and s will be negative) until the students have convinced themselves that every point in the plane can be expressed in this manner.

Now repeat the activity with different vectors **u** and **v**, perhaps using the same points as before.

As a wrap-up, give the students specific vectors **u** and **v**, such as $\mathbf{u} = \langle 3, 1\rangle$ and $\mathbf{v} = \langle -1, -2\rangle$, and have them determine values of r and s for several points. See if they can find general formulas for r and s in terms of the point (x, y). What goes wrong algebraically if **u** and **v** are parallel?

Lab Project: Equations of Planes

Use a computer graphing program (for example, Maple, Mathematica, or Matlab) for this lab.

Goal: By the end of this lab, students should be able to find and visualize the plane given by three points, or by a point and the normal to the plane. Along the way they will also learn about parametrically represented planes and implicitly defined graphs. This is a good chance for the students to learn how to use their software effectively. Make sure they graph using boxed axes.

1. Give the students three non-collinear points in space. They should choose one of the points as a base point, and from that point find two direction vectors which lie in the plane.
2. The students should use the base point and direction vectors to define the same plane parametrically. They should also graph the plane on the computer as a parametrically defined surface.
3. Now have the students use the two direction vectors to find a vector normal to the plane. They should then find a linear equation of the plane, and plot the plane as the graph of a surface.
4. As a discussion question, give the students three points which each have the same y-coordinate k, so the plane $y = k$ is not the graph of a function of x and y. Discuss how to graph this plane, noting that some graphing programs can graph it implicitly, while others cannot. Make the point that the parametric equation for the plane is really no different than for any other plane. This can lead to a discussion of the many different equivalent representations of the same plane.

Homework Problems

Core Exercises: 1, 2, 8, 17, 20, 24, 34, 37

Sample Assignment: 1, 2, 8, 12, 17, 20, 22, 24, 26, 30, 34, 35, 37, 47, 52, 54

Note: Problem 3 of Focus on Problem Solving (page 703) would make a good, challenging project for motivated students.

Exercise	C	A	N	G	V
1					×
2		×			
8		×			
12		×			
17		×			
19–28		×			
30		×			

Exercise	C	A	N	G	V
34		×			
35		×			
37		×			×
47		×			
52		×			
54					×

Group Work 2, Section 9.5
Da Planes! Da Planes!

Consider the point __________ in $\mathbb{R}^3$.

1. Find an equation of a line that contains your point and the origin.

2. Find an equation of a line that contains your point and the point $(1, -1, 1)$.

3. Now find an equation of the plane that contains the two lines you've just found.

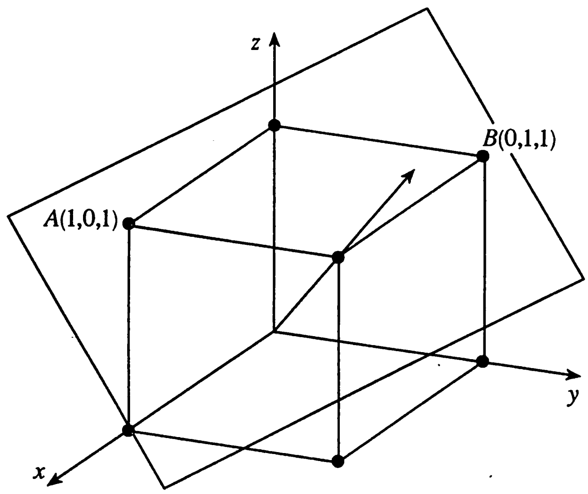
4. Find an equation of the plane that contains your point and is perpendicular to the x-axis.

5. Find an equation of the plane that contains your point and is perpendicular to the line $y = x$ in the xy-plane.

6. Finally, find an equation of a plane that does *not* contain your point.

Group Work 4, Section 9.5
The Moving Plane

Consider the unit box $\{(x, y, z) \mid 0 \le x, y, z \le 1\}$ and a plane which is perpendicular to the diagonal line from $(0,0,0)$ to $(1,1,1)$.



Start moving the plane along the diagonal until it hits the points $A\,(1,0,1)$ and $B\,(0,1,1)$ simultaneously.

1. What is the equation of the plane P, where P is perpendicular to the diagonal line from $(0,0,0)$ to $(1,1,1)$ and contains the points $A\,(1,0,1)$ and $B\,(0,1,1)$?

2. Show that the plane P goes through the point $C\,(1,1,0)$.

3. Describe the figure made by the intersection of P and the unit box.

4. Compute the area of the figure found in Problem 3.

9.6 Functions and Surfaces

Suggested Time and Emphasis

$\frac{3}{4}$–1 class   Essential material

Points to Stress

1. A function of two variables as a rule assigning a real number to every point in its domain, and the definition and shape of the domain of such a function.

2. The representation of graphs of functions of two variables as surfaces in $\mathbb{R}^3$, and the uses of horizontal traces to describe these surfaces.

3. Quadric surfaces as the graphs of second-degree polynomials in x, y, and z.

Text Discussion

- If f is a function of two variables and $f(3,4) = -1$, give the coordinates of a point on the graph of f.

- What are the vertical traces of the surface $z = 4x^2 + y^2$? What are the horizontal traces for $z > 0$? For $z < 0$?

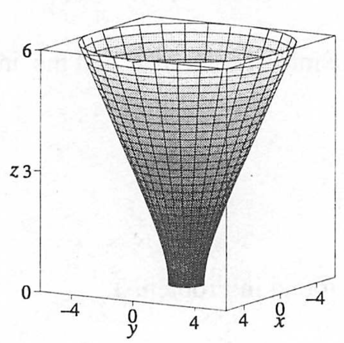
- Why is the quadric surface $x^2 + \frac{1}{9}y^2 + \frac{1}{4}z^2 = 1$ not the graph of a function $z = f(x,y)$?

Materials for Lecture

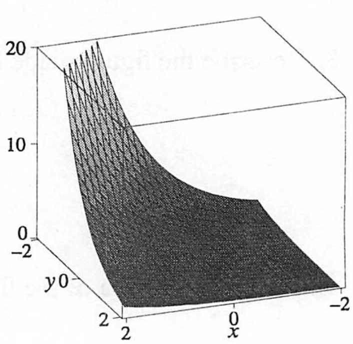
- Revisit the function $f(x,y) = x \ln(y^2 - x)$. Sketch the domain, as done in Figure 2 (page 686) of the text. Then go on to sketch the set of points (x,y) where $f(x,y) = 0$, $f(x,y) > 0$, $f(x,y) < 0$.

- Use mathematical reasoning and traces to describe the domains and the graphs of the functions at right, perhaps later putting up transparencies of the solutions to verify the reasoning.

$f(x,y) = \sqrt{x^2 + y^2 - 1}$



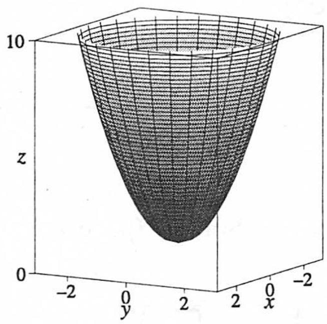
$f(x,y) = e^{x-y}$



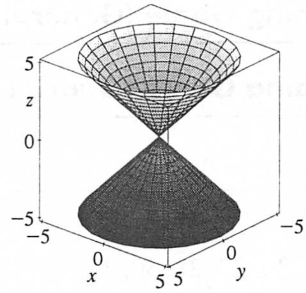
- Sketch the surface $x^2 + y^2 - z^2 = 1$ by looking at traces in horizontal and vertical planes.

- Use mathematical reasoning and traces to describe the following quadric surfaces:

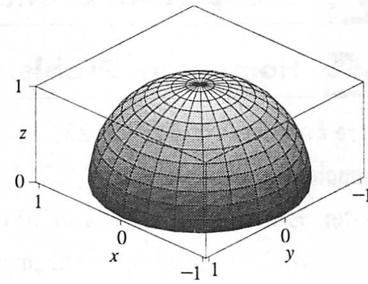
1. $z = x^2 + y^2 + 1$, a paraboloid with vertex at $(0, 0, -1)$



2. $z^2 = x^2 + y^2$, a cone



3. $z = \sqrt{1 - x^2 - y^2}$, the top hemisphere of the sphere $x^2 + y^2 + z^2 = 1$



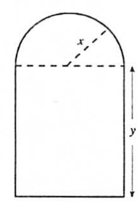
Workshop/Discussion

- Domain calculations often involve solving inequalities (sometimes nontrivial ones), and thus are usually not so simple for the students. Go over some examples, such as calculating domains for the following functions:

$$f(x, y) = \frac{\sqrt{x^2 + y^3}}{x^2 + 3x - 8} \qquad f(x, y) = -2\cos 2x + y$$

$$f(x, y) = \sin\left(\sqrt{1 - (x^2 + y^2)}\right) \qquad f(x, y) = \exp\left(\frac{x + y}{xy}\right)$$

- Let A be the area of the Norman window shown at right. Lead the students to see that A can be expressed as a function of two variables x and y. Have them figure out the domain of A and use level curves to determine what the graph of A looks like.



- Examine the hyperboloid of two sheets $-x^2 - y^2 + \frac{1}{2}z^2 = 1$. Show what conditions are needed on z to ensure that there are horizontal traces. Show that the horizontal trace at $z = k \geq \sqrt{2}$ is a circle of radius $\sqrt{\frac{1}{2}z^2 - 1}$, and that the vertical traces are hyperbolas.
- Compare the graphs of the surfaces $z^2 = x^2 + y^2$ and $z = x^2 + y^2$. Once they see the difference visually, ask the students to use vertical traces to illustrate this difference algebraically.
- Examine the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Ask students what conditions on a, b, and c ensure that all horizontal traces are circles. What conditions ensure that all vertical traces are circles? If both horizontal and vertical traces are circles, what does that say about the ellipsoid? If time permits, have the students vary a, b, and c in the ellipsoid equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Make sure they try varying the relative magnitudes of a, b, and c ($a < b < c$, $a < c < b$, $a = b < c$, and so on). Also make sure to point out that when $a = b = c$, we have a sphere of radius a.

Group Work 1: Staying Cool

Group Work 2: The Matching Game (General Functions)

Group Work 3: The Matching Game (Quadric Surfaces)

Homework Problems

Core Exercises: 4, 12, 15, 23, 24, 27

Sample Assignment: 1, 4, 7, 12, 15, 23, 24, 26, 27, 28, 30, 33

Note: Exercises 27, 28, and 30 require a CAS. Problem 5 from Focus on Problem Solving (page 703) would make a good optional project to assign to motivated students.

Exercise	C	A	N	G	V
1	×				
4		×		×	
7		×		×	
12				×	
15					×
23					×

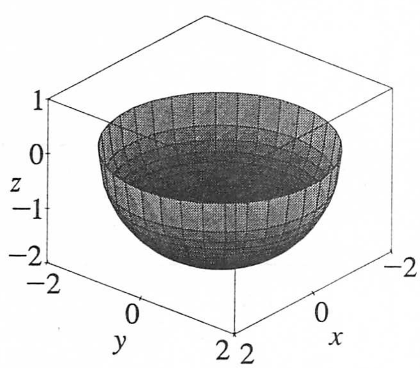
Exercise	C	A	N	G	V
24		×			×
26		×			×
27					×
28				×	
30				×	
33	×	×			

Group Work 3, Section 9.6
The Matching Game (Quadric Surfaces)

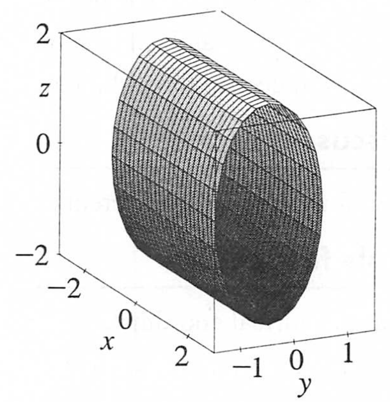
Match each function with its graph. Give reasons for your choices.

1. $x^2 + y^2 + \frac{1}{4}z^2 = 1$ **2.** $z = -\sqrt{4 - x^2 - y^2}$ **3.** $y^2 + \frac{1}{4}z^2 = 1$

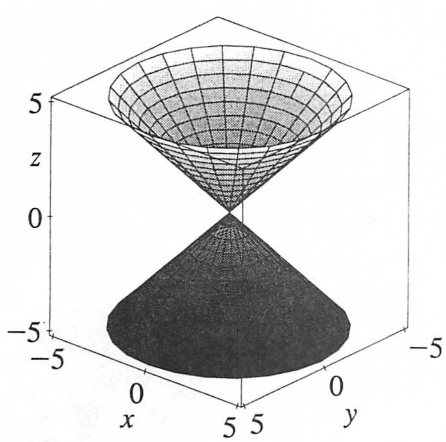
4. $\frac{1}{9}z^2 - \frac{1}{4}y^2 = 1$ **5.** $\frac{1}{4}x^2 - y^2 - z^2 = 1$ **6.** $|z| = \sqrt{x^2 + y^2}$



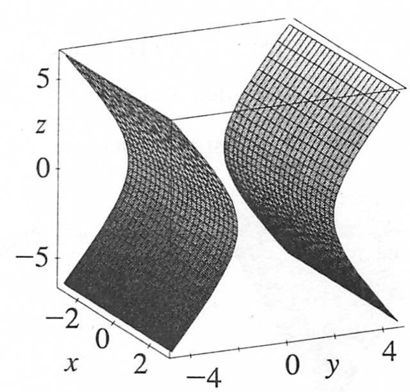
I



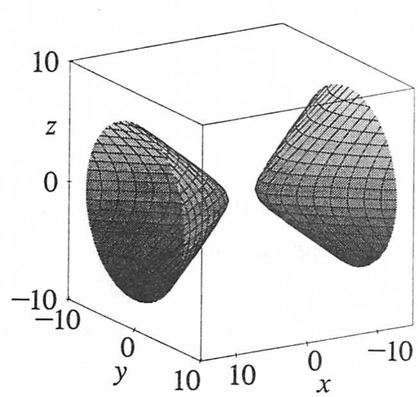
II



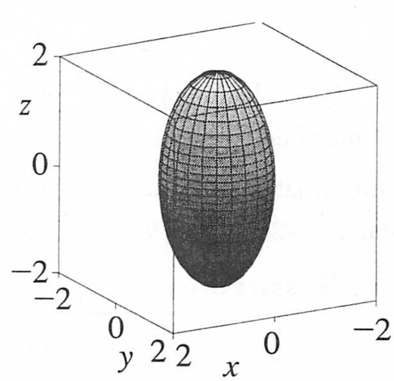
III



IV



V



VI

9.7 Cylindrical and Spherical Coordinates

Suggested Time and Emphasis

1 class Essential Material

Points to Stress

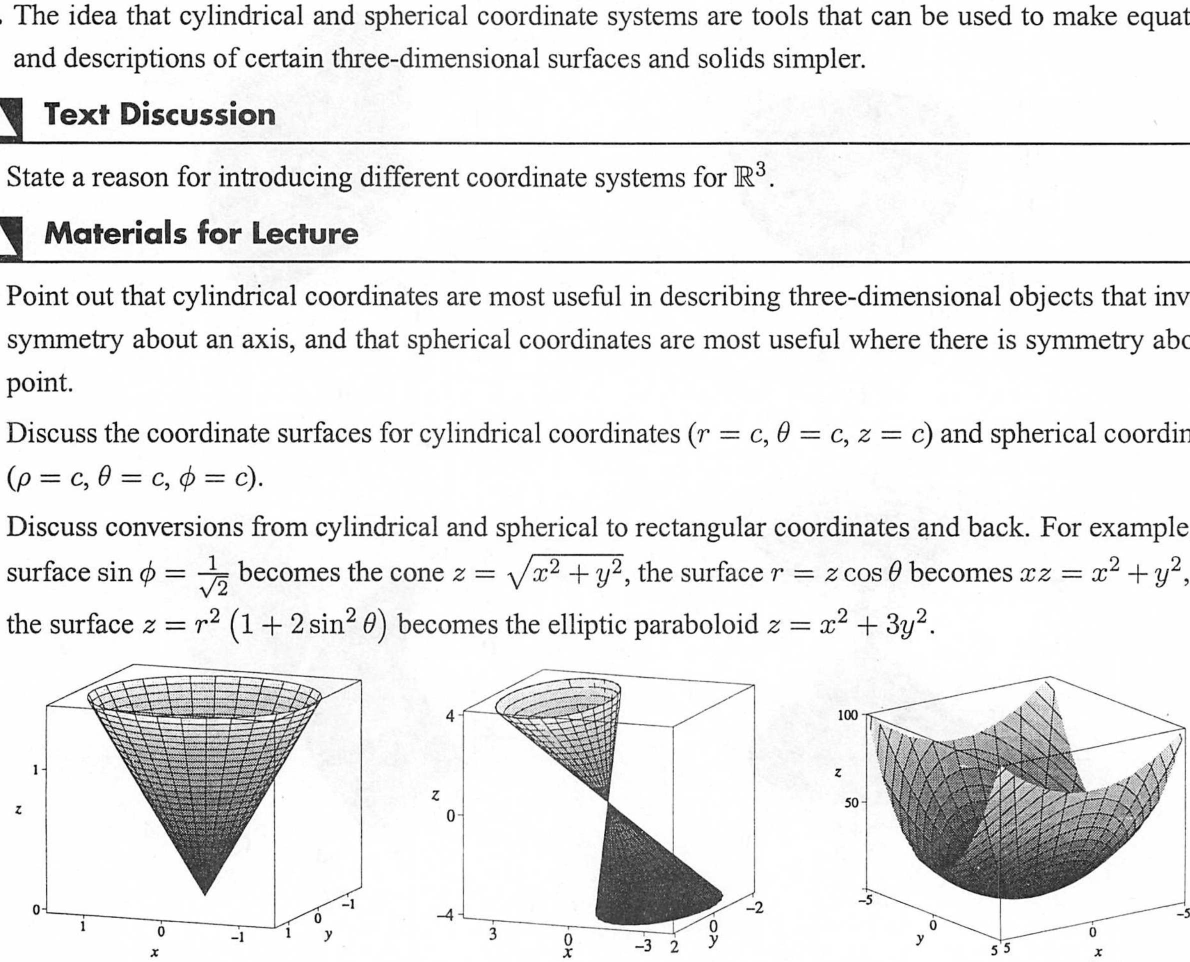
1. The basic formulas for cylindrical and spherical coordinates as extensions of polar coordinates in $\mathbb{R}^2$.
2. The idea that cylindrical and spherical coordinate systems are tools that can be used to make equations and descriptions of certain three-dimensional surfaces and solids simpler.

Text Discussion

- State a reason for introducing different coordinate systems for $\mathbb{R}^3$.

Materials for Lecture

- Point out that cylindrical coordinates are most useful in describing three-dimensional objects that involve symmetry about an axis, and that spherical coordinates are most useful where there is symmetry about a point.
- Discuss the coordinate surfaces for cylindrical coordinates ($r = c$, $\theta = c$, $z = c$) and spherical coordinates ($\rho = c$, $\theta = c$, $\phi = c$).
- Discuss conversions from cylindrical and spherical to rectangular coordinates and back. For example, the surface $\sin\phi = \frac{1}{\sqrt{2}}$ becomes the cone $z = \sqrt{x^2 + y^2}$, the surface $r = z\cos\theta$ becomes $xz = x^2 + y^2$, and the surface $z = r^2\left(1 + 2\sin^2\theta\right)$ becomes the elliptic paraboloid $z = x^2 + 3y^2$.



$\sin\phi = \frac{1}{\sqrt{2}}$ $r = z\cos\theta$ $z = r^2\left(1 - \sin^2\theta\right)$

- As an example of a change from rectangular to spherical coordinates, use the circular paraboloid $z = x^2 + y^2$, which becomes $\cos\phi = \rho\sin^2\phi$.
- Identify the somewhat mysterious surface $\rho\cos\phi = \rho^2\sin^2\phi\cos 2\theta$ given in spherical coordinates by using the formula $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and changing to rectangular coordinates.

Workshop/Discussion

- Describe in terms of cylindrical coordinates the surface of rotation formed by rotating the cone $z = 1/x$ about the z-axis, noting that there is an axis of symmetry. Point out that the equations come from simply replacing x by r.

Group Work 1, Section 9.6
Staying Cool

Let $T(x,y)$ be the temperature in a 10 ft $\times$ 10 ft room on a winter night, where one corner of the room is at $(0,0)$ and the opposite corner is at $(10,10)$ For each of the following functions T,

(a) Draw or describe in words a graph of the temperature function.

(b) Describe the likely floor locations of the heat vents.

(c) Suppose you like to sleep with a temperature of 70° or less. Where would you put the bed?

1. $T(x,y) = 78 - \frac{1}{10}\left[x^2 + (y-5)^2\right]$

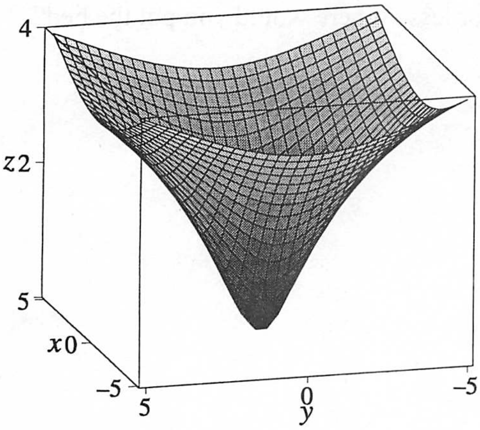
2. $T(x,y) = \frac{1}{2}x - y + 75$

Group Work 2, Section 9.6
The Matching Game (General Functions)

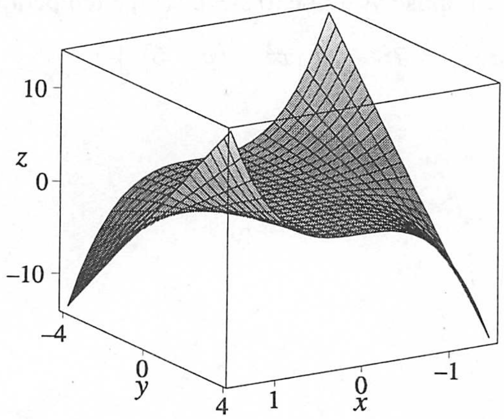
Match each function with its graph. Give reasons for your choices.

1. $f(x,y) = \dfrac{1}{x+1} + \sin y$ **2.** $f(x,y) = \sqrt{4 - x^2 - y^2}$ **3.** $f(x,y) = \cos\left(x + y^2\right)$

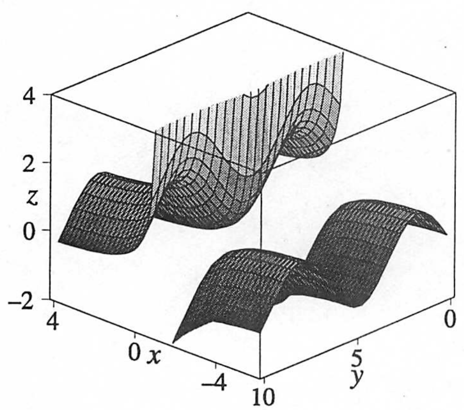
4. $f(x,y) = \ln\left(x^2 + y^2 + 1\right)$ **5.** $f(x,y) = x^2\sqrt{y}$ **6.** $f(x,y) = x^3 y$



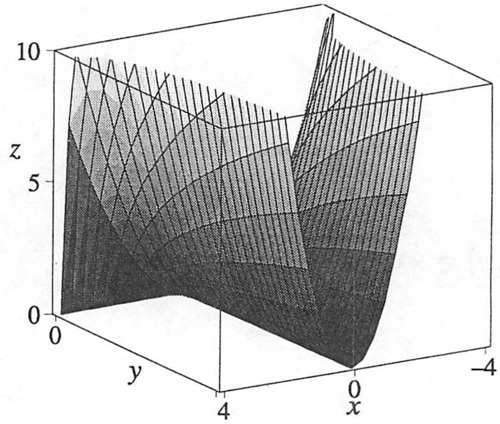
I



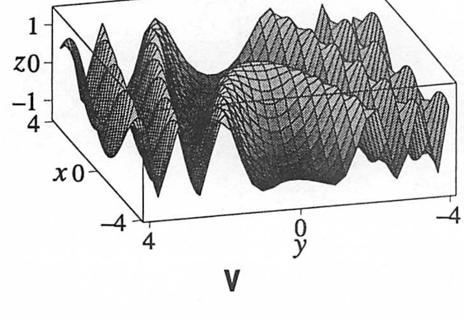
II



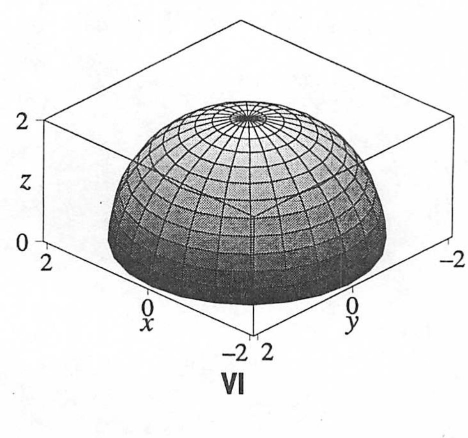
III



IV



V



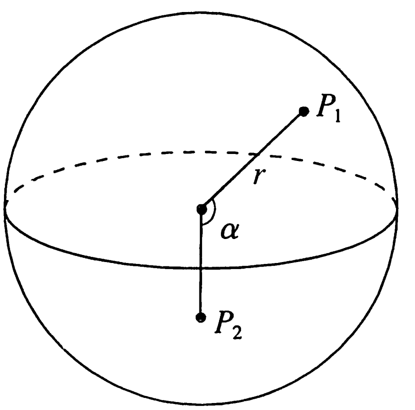
VI

- Calculate the intersection of the surfaces $z = x^2 + y^2$ and $z = x$, first in rectangular coordinates, then in cylindrical coordinates. (Here is a case where the rectangular coordinates are the easiest to visualize, even though there is an $x^2 + y^2$ term.)
- Describe the intensity from a point light source in terms of spherical coordinates.
- Describe the coordinates of all position vectors with length 3 in each of the three coordinate systems. Repeat for the coordinates of all points on a circular cylinder of radius 2 with central axis the z-axis, and the coordinates of all points on the line through the origin with direction vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$. Explain which coordinate system is best suited to represent each object.
- Elaborate on Exercise 34 in the following way (this idea can also be used as a group work):

  1. Have the students read the exercise.
  2. Show one approach to the solution:

     (a) Note that the distance along an arc of a circle is given by the formula $d = r\alpha$, where r is the radius and α is the angle which subtends the arc.

     (b) Draw the following picture:



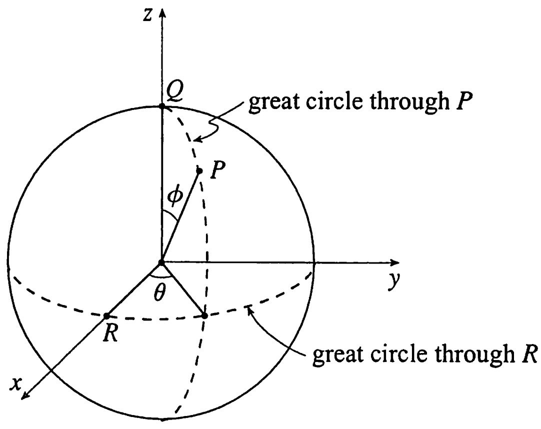
     (c) Assuming that $r = 1$, write the points P_1 and P_2 in spherical coordinates $[(1, \theta_1, \phi_1)$ and $(1, \theta_2, \phi_2)]$, and derive the rectangular representations $P_1 = (\sin\phi_1 \cos\theta_1, \sin\phi_1 \sin\theta_1, \cos\phi_1)$ and $P_2 = (\sin\phi_2 \cos\theta_2, \sin\phi_2 \sin\theta_2, \cos\phi_2)$.

     (d) Conclude that $\cos\alpha = \overrightarrow{OP_1} \cdot \overrightarrow{OP_2}$.

     (e) Use the formula in part (d) to find the answer to Exercise 34.

  3. Compute the Great Circle distance between the students' current location and a city with the same latitude on another continent.
  4. Discuss how latitude and longitude can be translated to θ and ϕ.

- Bring a beach ball and a tape measure to class. Set up the following problem: given an arbitrary orientation of the coordinate axes and a point P on the beach ball, find its spherical coordinates.



The students should quickly see that we cannot measure ϕ and θ directly, because the ball is opaque. However, given points Q and R where the positive z- and x-axes intersect the sphere, we can calculate these quantities by measuring along great circles (as indicated in the diagram).

Group Work 1: Surfaces

Group Work 2: Describe Me!

Homework Problems

Core Exercises: 3, 6, 8, 10, 13, 18, 23, 26

Sample Assignment: 1, 2, 3, 4, 6, 7, 8, 10, 13, 14, 18, 19, 23, 26

Exercise	C	A	N	G	V
1	×				
2	×				
3		×			×
4		×			×
6		×			
7		×			×
8		×			×

Exercise	C	A	N	G	V
10		×			
13					×
14					×
18		×			×
19		×			×
23		×			
26					×

Group Work 1, Section 9.7
Surfaces

1. Using inequalities in some coordinate system, describe the solid that remains when a hole of radius 1 is drilled parallel to the z-axis through the cylinder $x^2 + y^2 = 4$.

2. Consider the piece of surface cut from the paraboloid $z = x^2 + y^2$ by the plane $z = x$.

(a) Describe the bounding curve in rectangular coordinates. What does the curve look like when projected onto the xy-plane?

(b) Show (by translating into cylindrical coordinates) that the projection of the bounding curve in the plane $z = 0$ is $r^2 = r\cos\theta$. (That is, $r = 0$ or $r = \cos\theta$.)

(c) Show that a parametrization of the bounding curve can be given in rectangular coordinates by $\langle \cos^2\theta, \cos\theta\sin\theta, \cos^2\theta \rangle$, $0 \le \theta \le \pi$.

Group Work 2, Section 9.7
Describe Me!

Sketch, or describe in words, the following surfaces whose equations are given in cylindrical coordinates:

1. $r = \theta$

2. $z = r$

3. $z = \theta$

Sketch, or describe in words, the following surfaces whose equations are given in spherical coordinates:

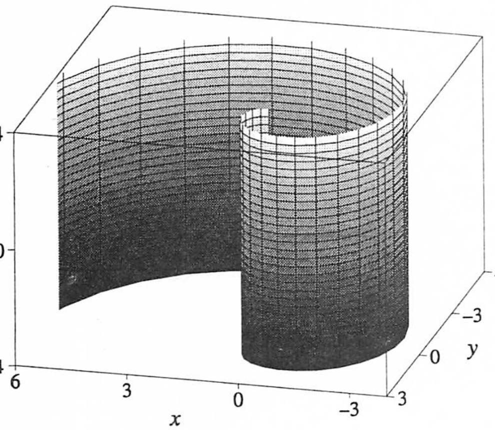
4. $\theta = \frac{\pi}{4}$

5. $\phi = \frac{\pi}{4}$

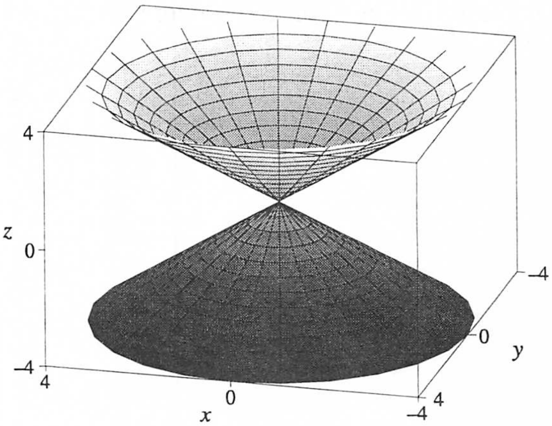
6. $\rho = \phi$

Group Work 2, Section 9.7
Describe Me! (Solutions)

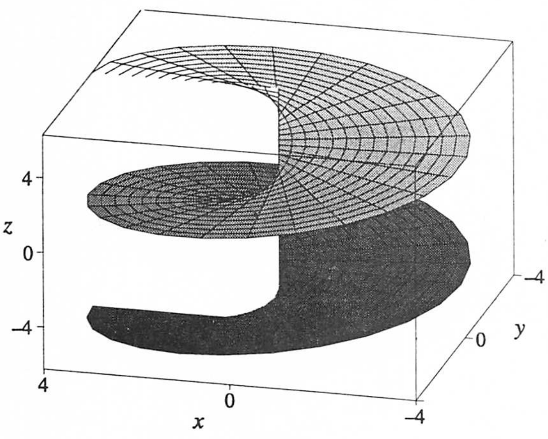
1. $r = \theta$



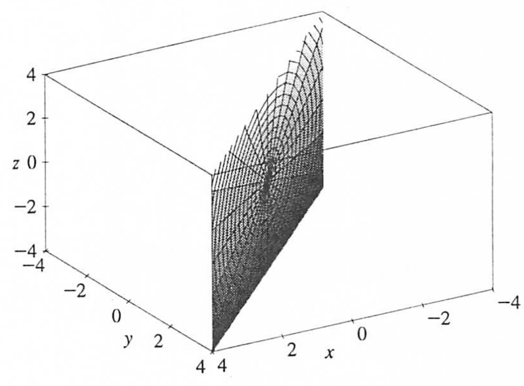
2. $z = r$



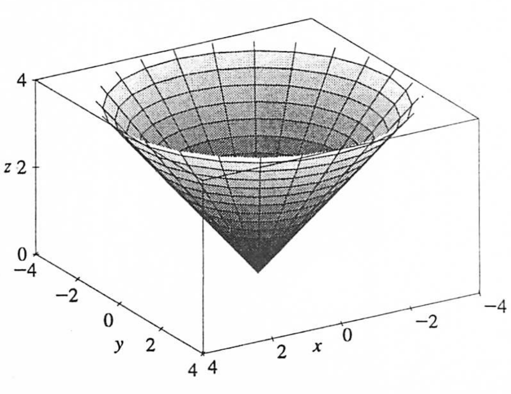
3. $z = \theta$



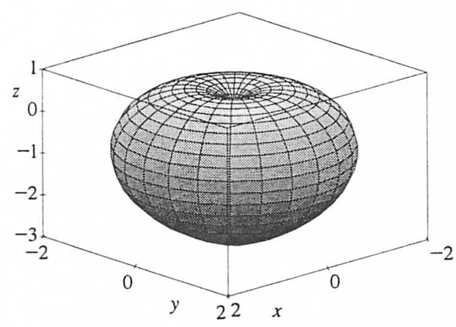
4. $\theta = \frac{\pi}{4}$



5. $\phi = \frac{\pi}{4}$



6. $\rho = \phi$



Laboratory Project: Families of Surfaces

This project serves as an introduction both to the variety of shapes that can be obtained by varying the parameters in a family of surfaces, and to the use of a computer to investigate these surfaces. If this project is assigned, it is recommended that the third part be assigned to all students, and then the first or second parts also assigned if there is time and interest.

9 Sample Exam

Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

1. (a) Consider the two surfaces $\rho = 3\csc\phi$ (given in spherical coordinates) and $r = 3$ (given in cylindrical coordinates). Are they the same surface, or are they different surfaces? Explain your answer.

(b) Consider the two surfaces $\sin\phi = \cos\phi$ (given in spherical coordinates) and $z = \sqrt{r^2}$ (given in cylindrical coordinates). Are they the same surface, or are they different surfaces? Explain your answer.

2. Describe the following in rectangular coordinates:

(a) The intersection of the surfaces given in cylindrical coordinates by $\theta = \frac{\pi}{4}$ and $\theta = \frac{2\pi}{3}$

(b) The intersection of the surfaces given in spherical coordinates by $\rho = 1$ and $\theta = \frac{\pi}{2}$

3. Find a pair of values x, y with $0 < x < 1$, $0 < y < 1$ such that $f(x,y) \geq 8$, or show that no such values x and y can exist, for each of the following functions of two variables:

(a) $f(x,y) = \dfrac{1}{x} + \dfrac{1}{y}$

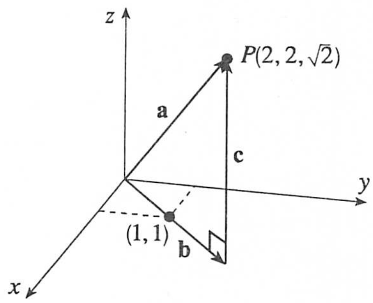
(b) $f(x,y) = 2^{xy+2.5}$

(c) $f(x,y) = \dfrac{1}{x^2+y^2+1}$

(d) $f(x,y) = \cos\left(2x + 3y + \ln\left(x^2+y^2\right)\right)$

(e) $f(x,y) = \dfrac{1}{x-y+1}$

4. Let $\mathbf{a} = \overrightarrow{OP}$, where P is the point $(2, 2, \sqrt{2})$. Compute the vectors $\mathbf{b}$ and $\mathbf{c}$.



5. Let $\mathbf{a} = (x+y)\,\mathbf{i} + 2\mathbf{j} + y\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + (4x+y+1)\,\mathbf{j} + 4\mathbf{k}$.

(a) Find values of x and y such that $\mathbf{a} \perp \mathbf{b}$.

(b) Find values of x and y such that $\mathbf{a} \parallel \mathbf{b}$. (*Hint:* Assume that $c\mathbf{a} = \mathbf{b}$ for some value c.)

6. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be three vectors in the plane $3x - 5y + 6z = 7$. Compute $(-\mathbf{a} + 4\mathbf{b} - 7\mathbf{c})\cdot(-3\mathbf{i} + 5\mathbf{j} - 6\mathbf{k})$.

7. Let r, s, and t be distinct non-zero vectors in space. Which of the following must be true, which might be true, and which cannot be true? Justify your answers.

(a) If $\mathbf{r} \parallel \mathbf{s}$ and $\mathbf{s} \parallel \mathbf{t}$, then $\mathbf{r} \parallel \mathbf{t}$.

(b) If $\mathbf{r} \perp \mathbf{s}$ and $\mathbf{s} \perp \mathbf{t}$, then $\mathbf{r} \perp \mathbf{t}$.

(c) If $\mathbf{r} \times (\mathbf{s} \times \mathbf{t}) = \mathbf{0}$ and $\mathbf{s} \times \mathbf{t} \neq \mathbf{0}$, then $\mathbf{r} \perp (\mathbf{s} + \mathbf{t})$.

(d) If $\mathbf{r} \cdot (\mathbf{s} \times \mathbf{t}) = 0$ and $\mathbf{s} \times \mathbf{t} \neq \mathbf{0}$, then $\mathbf{r} \perp (\mathbf{s} + \mathbf{t})$.

8. Suppose we have three distinct unit vectors a, b, and c which satisfy the following conditions:

(i) $\mathbf{b} \times \mathbf{c} \neq \mathbf{0}$ (ii) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{0}$

Which of the following must be true, which might be true, and which cannot be true? Justify your answers.

(a) b is perpendicular to c.

(b) a is perpendicular to c.

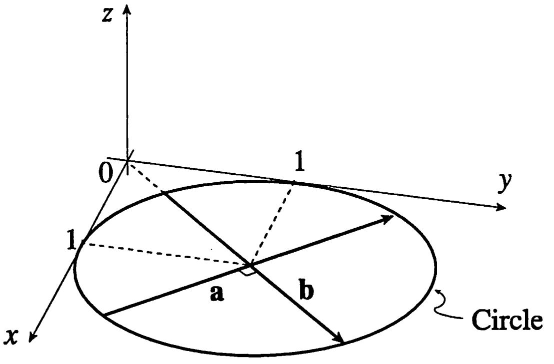
(c) a is perpendicular to b.

9. Describe and sketch the surfaces in space defined by the following equations.

(a) $y = -z + 1$

(b) $x^2 + y^2 = 3$

10. Referring to the diagram below, give the component representation of each vector.



(a) a

(b) b

(c) $\mathbf{a} \times \mathbf{b}$

(d) $\mathbf{a} + \mathbf{b}$

(e) $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} \times \mathbf{b})$

11. Let $\mathbf{N} = \mathbf{i} - \mathbf{j} + \mathbf{k}$.

(a) What is the equation of the plane P containing $(0, 0, 0)$ with normal vector N?

(b) Find two unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ in the plane P which are not parallel to one another.

(c) What is the relationship between N and $\mathbf{u}_1 \times \mathbf{u}_2$?

12. (a) Show that the line given by $x = t, y = 3t - 2, z = -t$ intersects the plane $x + y + z = 1$

(b) Find a point of intersection.

13. Consider the plane $x + y + z = 0$.

(a) Give three distinct points with integer coordinates that lie on this plane.

(b) Find the area of the triangle formed by those three points.

14. A particle moves in such a way that its path traces out the circle $x^2 + y^2 = 4$, $z = 3$.

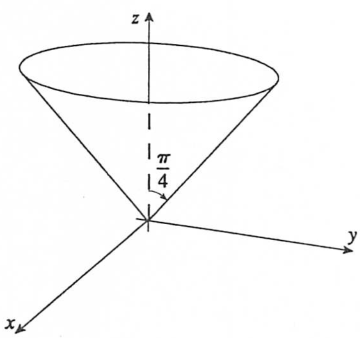
(a) Write an equation of the curve traced out by the particle in cylindrical coordinates.

(b) Write an equation of the curve traced out by the particle in spherical coordinates.

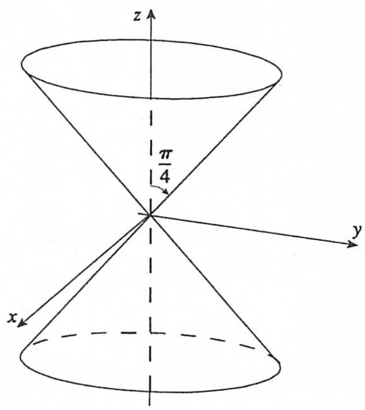
9 Sample Exam Solutions

1. (a) $\rho = 3\csc\phi \Rightarrow \rho\sin\phi = 3$ or $r = 3$ in cylindrical coordinates. These are the same surface, a cylinder of radius 3.

(b) The surfaces are different. The surface $\cos\phi = \frac{1}{\sqrt{2}}$, $\rho \geq 0$ is a single cone, and the surface $z = r$ is a double cone.



$\cos\phi = \frac{1}{\sqrt{2}}, \rho \geq 0$



$z = r$

2. (a) The intersection of two half-planes, namely, the z-axis

(b) The circle $y^2 + z^2 = 1$ in the yz-plane ($x = 0$)

3. (a) $x = 0.25$, $y = 0.25$ gives $f(x, y) = 8$.

(b) $x = 0.9$, $y = 0.9$ gives $f(x, y) \approx 9.9$.

(c) $\frac{1}{2} \leq \frac{1}{x^2 + y^2 + 1} \leq 1$

(d) $|\cos w| \leq 1$ for any w

(e) $x = 0.09$, $y = 0.99$ gives $f(x, y) = 10$

4. $\mathbf{b} = \langle 2, 2, 0\rangle$, $\mathbf{c} = \langle 0, 0, \sqrt{2}\rangle$

5. (a) $x = -1$, $y = 1$ (among others) (b) $x = -2$, $y = 8$

6. 0

7. (a) True

(b) Might be true; false if $\mathbf{r} = \mathbf{t}$

(c) True: $\mathbf{r}$ is parallel to $\mathbf{s} \times \mathbf{t}$

(d) False: $\mathbf{r} = \mathbf{s} + \mathbf{t} \perp \mathbf{s} \times \mathbf{t}$, but $\mathbf{r} \cdot (\mathbf{s} + \mathbf{t}) \neq 0$

8. (a) Might be true

(b) True

(c) True

9. (a) $y + z = 1$, a plane

(b) $x^2 + y^2 = 3$, a cylinder of radius $\sqrt{3}$

10. (a) $\mathbf{a} = -\sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}$

(b) $\mathbf{b} = \sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}$

(c) $\mathbf{a} \times \mathbf{b} = -4\mathbf{k}$

(d) $\mathbf{a} + \mathbf{b} = 2\sqrt{2}\,\mathbf{j}$

(e) $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} \times \mathbf{b}) = 8\sqrt{2}\,\mathbf{i}$

11. (a) $x - y + z = 0$

(b) $\mathbf{u}_1 = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$, $\mathbf{u}_2 = \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$

(c) $\mathbf{v} \parallel \mathbf{u}_1 \times \mathbf{u}_2$

12. (a) The line is L: $\mathbf{r}(t) = \langle 0, -2, 0 \rangle + t \langle 1, 3, -1 \rangle$ and the normal to the plane is $\langle 1, 1, 1 \rangle$. Since $\langle 1, 3, -1 \rangle \cdot \langle 1, 1, 1 \rangle = 3 \neq 0$, these vectors are not perpendicular and thus the line intersects the plane.

(b) When $t = 1$, $\mathbf{r}(1) = \langle 1, 1, -1 \rangle$ is in the plane $x + y + z = 1$.

13. (a) $P_1 = (1, -1, 0)$, $P_2 = (1, 0, -1)$, $P_3 = (0, -1, 1)$

(b) If $\mathbf{a} = \overrightarrow{P_1P_2} = \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \overrightarrow{P_1P_3} = -\mathbf{i} + \mathbf{k}$, then the area of the triangle is $\frac{1}{2}\,|\mathbf{a} \times \mathbf{b}| = \frac{\sqrt{3}}{2}$

14. (a) Cylindrical coordinates: $\mathbf{s}(t) = \langle 2, t, 3 \rangle$, $0 \leq t \leq 2\pi$

(b) Spherical coordinates: $\mathbf{w}(t) = \left\langle \sqrt{13}, t, \arccos \frac{3}{\sqrt{13}} \right\rangle$, $0 \leq t \leq 2\pi$