9 Vectors and the Geometry of Space

9.1 Three-Dimensional Coordinate Systems

1. Find the distance from the point $(1, 2, 3)$ to the origin $(0, 0, 0)$.

A) $\sqrt{13}$ B) $\sqrt{14}$ C) $\sqrt{15}$ D) 4
E) $\sqrt{17}$ F) $\sqrt{18}$ G) $\sqrt{19}$ H) $\sqrt{20}$

Answer: B)

2. Find the distance between the two points $(1, 2, 3)$ and $(4, 5, 6)$.

A) $3\sqrt{5}$ B) $2\sqrt{5}$ C) $4\sqrt{5}$ D) $5\sqrt{5}$
E) $3\sqrt{3}$ F) $2\sqrt{3}$ G) $4\sqrt{3}$ H) $5\sqrt{3}$

Answer: E)

3. Find the radius of the sphere whose equation is $x^2 + 2x + y^2 + z^2 = 4$.

A) 1 B) $\sqrt{2}$ C) $\sqrt{3}$ D) 2
E) $\sqrt{5}$ F) $\sqrt{6}$ G) $\sqrt{7}$ H) $2\sqrt{2}$

Answer: E)

4. Find the center of the sphere whose equation is $x^2 + 2x + y^2 - y + z^2 = 0$.

A) $(2, 1, 0)$ B) $(2, -1, 0)$ C) $(-2, 1, 0)$ D) $(-2, -1, 0)$
E) $\left(1, \frac{1}{2}, 0\right)$ F) $\left(-1, \frac{1}{2}, 0\right)$ G) $\left(1, -\frac{1}{2}, 0\right)$ H) $\left(-1, -\frac{1}{2}, 0\right)$

Answer: F)

5. Find the distance from the origin to the center of the sphere whose equation is $x^2 + 2x + y^2 + 4y + z^2 + 6z = 0$.

A) $\sqrt{6}$ B) $\sqrt{7}$ C) $\sqrt{8}$ D) 3
E) $\sqrt{12}$ F) $\sqrt{13}$ G) $\sqrt{14}$ H) $\sqrt{15}$

Answer: G)

6. What region of $\mathbb{R}^3$ is represented by the inequality $y^2 + z^2 \leq 1$?

A) points inside a circular cylinder with axis the x-axis
B) points inside a circular cylinder with axis the y-axis
C) points inside a circular cylinder with axis the z-axis
D) points inside a sphere of radius 1
E) points inside a sphere of radius $\sqrt{2}$
F) points inside a sphere of radius 2
G) points inside a sphere of radius $\sqrt{3}$
H) points inside a sphere of radius 3

Answer: A)

7. What region of $\mathbb{R}^3$ is represented by the equation $z = 3$?

A) line parallel to the xy-plane
B) line parallel to the xz-plane
C) line parallel to the yz-plane
D) plane parallel to the xz-plane
E) plane parallel to the xy-plane
F) line perpendicular to the xy-plane
G) line perpendicular to the xz-plane
H) line perpendicular to the yz-plane

Answer: E)

8. The center of the sphere $x^2 + y^2 + z^2 + 4x - 2y - 6z = 0$ is

A) $(1, 1, 1)$
B) $(0, 0, 0)$
C) $(4, -2, 9)$
D) $(2, -1, -3)$
E) $(-2, 1, 3)$
F) $(-2, -1, 3)$
G) $(2, 1, 3)$
H) $(-4, 2, 6)$

Answer: E)

9. Find the equation of the sphere, in standard form, one of whose diameters has $(-5, 2, 9)$ and $(3, 6, 1)$ as endpoints.

Answer: $x^2 + y^2 + z^2 + 2x + 4y - 10z + 14 = 0$

10. Find the lengths of the sides of the triangle ABC and determine whether the triangle is isosceles, a right triangle, both, or neither given: $A\,(5, 5, 1)$, $B\,(3, 3, 2)$, $C\,(1, 4, 4)$.

Answer: $AB = 3$; $BC = 3$; $CA = \sqrt{26}$; isosceles but not right

11. Find the equation of the sphere with center $C\,(-1, 2, 4)$ and radius $r = \frac{1}{2}$.

Answer: $(x + 1)^2 + (y - 2)^2 + (z - 4)^2 = \frac{1}{4}$

12. Describe in words the region of $\mathbb{R}^3$ represented by the inequality $1 \leq x^2 + y^2 + z^2 \leq 25$.

Answer: All points on and between the concentric spheres with radii one and five and center $(0, 0, 0)$.

13. Write an equation describing the set of points whose distance from the point $(3, 0, -1)$ is 6. Describe this set of points in words.

Answer: $(x - 3)^2 + y^2 + (z + 1)^2 = 36$; sphere of radius 6 centered at $(3, 0, -1)$.

14. Determine whether the points $A\,(2, 2, 4)$, $B\,(-1, 1, 2)$, and $C\,(8, 4, 8)$ lie on a straight line.

Answer: Yes

9.2 Vectors

15. Find the length of the vector $\langle 2, 2, 3 \rangle$.

A) $\sqrt{13}$
B) $\sqrt{14}$
C) $\sqrt{15}$
D) 4
E) $\sqrt{17}$
F) $\sqrt{18}$
G) $\sqrt{19}$
H) $\sqrt{20}$

Answer: E)

16. Find the length of the vector $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

A) $\sqrt{2}$
B) $\sqrt{3}$
C) 2
D) $\sqrt{5}$
E) $\sqrt{6}$
F) $\sqrt{7}$
G) $\sqrt{8}$
H) 3

Answer: H)

17. Find the length of the vector $\mathbf{a} + \mathbf{b}$, where $\mathbf{a} = \langle 1, 2 \rangle$ and $\mathbf{b} = \langle 3, 4 \rangle$.

A) $3\sqrt{19}$
B) $4\sqrt{7}$
C) $5\sqrt{7}$
D) $5\sqrt{3}$
E) $4\sqrt{19}$
F) $2\sqrt{5}$
G) $2\sqrt{13}$
H) $3\sqrt{17}$

Answer: G)

18. Find the length of the vector **a** − **b**, where **a** = $\langle 1,3\rangle$ and **b** = $\langle 5,2\rangle$.

A) $\sqrt{13}$ B) $\sqrt{17}$ C) $\sqrt{19}$ D) $\sqrt{21}$
E) $2\sqrt{13}$ F) $2\sqrt{17}$ G) $2\sqrt{19}$ H) $2\sqrt{21}$

Answer: B)

19. Find the length of the vector 2**a** + 3**b**, where **a** = $\langle 1,1\rangle$ and **b** = $\langle -1,2\rangle$.

A) $\sqrt{51}$ B) $\sqrt{53}$ C) $\sqrt{57}$ D) $\sqrt{65}$
E) $\sqrt{71}$ F) $\sqrt{73}$ G) $\sqrt{82}$ H) $\sqrt{87}$

Answer: D)

20. Find a unit vector that has the same direction as the vector $\langle 3,4\rangle$.

A) $\langle \frac{1}{3},\frac{2}{3}\rangle$ B) $\langle \frac{3}{5},\frac{4}{5}\rangle$ C) $\langle \frac{1}{3},\frac{1}{4}\rangle$ D) $\langle 4,3\rangle$
E) $\langle -\frac{1}{3},\frac{2}{3}\rangle$ F) $\langle -\frac{3}{5},\frac{4}{5}\rangle$ G) $\langle -\frac{1}{3},\frac{1}{4}\rangle$ H) $\langle -4,3\rangle$

Answer: B)

21. Find a unit vector that has the same direction as the vector $\langle 2,2,1\rangle$.

A) $\langle 1,1,1\rangle$ B) $\langle 1,0,0\rangle$ C) $\langle 0,1,0\rangle$ D) $\langle 0,0,1\rangle$
E) $\langle 1,1,0\rangle$ F) $\langle \frac{2}{3},\frac{2}{3},\frac{1}{3}\rangle$ G) $\langle \frac{1}{2},\frac{1}{2},1\rangle$ H) $\langle \frac{1}{3},\frac{1}{3},1\rangle$

Answer: F)

22. Find a unit vector that has the same direction as the vector $\langle 2,3,6\rangle$.

A) $\langle -\frac{1}{2},\frac{1}{3},-\frac{1}{6}\rangle$ B) $\langle \frac{1}{2},\frac{1}{3},\frac{1}{6}\rangle$ C) $\langle \frac{7}{2},\frac{7}{3},\frac{7}{6}\rangle$ D) $\langle \frac{2}{7},\frac{3}{7},\frac{6}{7}\rangle$
E) $\langle \frac{1}{3},\frac{1}{4},\frac{1}{5}\rangle$ F) $\langle -\frac{2}{3},\frac{3}{4},\frac{5}{6}\rangle$ G) $\langle \frac{2}{3},-\frac{3}{4},\frac{5}{6}\rangle$ H) $\langle \frac{2}{3},\frac{3}{4},-\frac{5}{6}\rangle$

Answer: D)

23. Find a unit vector that has the same direction as the vector $\langle 1,1,1\rangle$.

A) $\langle 1,1,1\rangle$ B) $\langle -1,-1,-1\rangle$ C) $\langle \sqrt{\frac{1}{3}},\sqrt{\frac{1}{3}},\sqrt{\frac{1}{3}}\rangle$ D) $\langle \sqrt{\frac{1}{2}},\sqrt{\frac{1}{2}},\sqrt{\frac{1}{2}}\rangle$
E) $\langle \frac{1}{3},\frac{1}{3},\frac{1}{3}\rangle$ F) $\langle \frac{1}{2},\frac{1}{2},\frac{1}{2}\rangle$ G) $\langle \frac{1}{4},\frac{1}{4},\frac{1}{4}\rangle$ H) $\langle 1,-1,1\rangle$

Answer: C)

24. Given **u** = 2**i** − **j** and **v** = 3**i** + 2**j**, find **u** − **v**.

Answer: −**i** − 3**j**

25. Given **u** = 2**i** − **j** and **v** = 3**i** + 2**j**, find |**u**|.

Answer: $\sqrt{5}$

26. Given the points $P(-2,5)$ and $Q(3,-3)$, find the unit vector in the direction of the displacement vector $\overrightarrow{PQ}$.

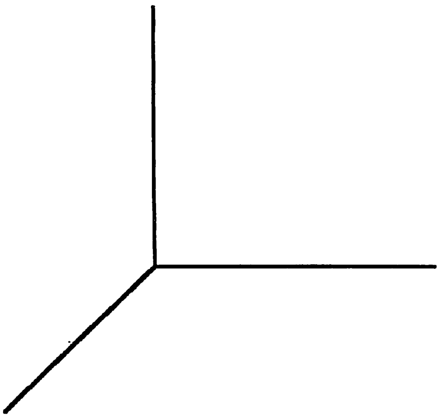
Answer: $\left\langle \frac{5}{\sqrt{89}},-\frac{8}{\sqrt{89}}\right\rangle$

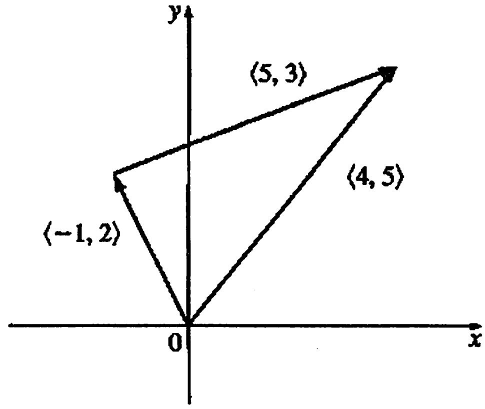
27. Find the coordinates of the point halfway between the midpoints of the vectors **a** = 3**i** − 5**j** + **k** and **b** = 5**i** + 3**j** + **k**.

Answer: $(2,-\frac{1}{2},\frac{1}{2})$

28. Given $\langle -1, 2\rangle$ and $\langle 5, 3\rangle$, find their sum and illustrate geometrically.

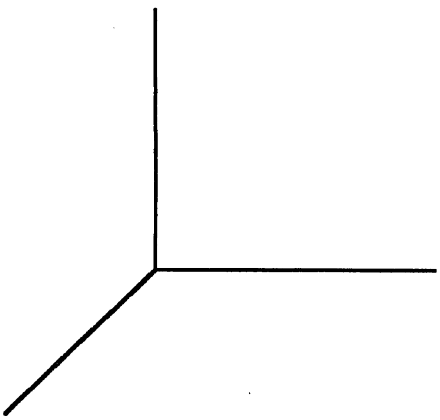
Answer:

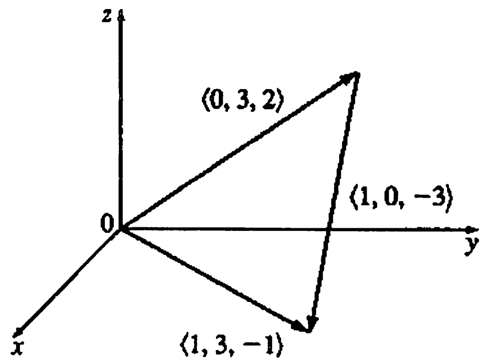




29. Given $\langle 0, 3, 2\rangle$ and $\langle 1, 0, -3\rangle$, find their sum and illustrate geometrically.

Answer:





30. Find $|\mathbf{a}|$, $\mathbf{a}+\mathbf{b}$, $\mathbf{a}-\mathbf{b}$, $2\mathbf{a}$, and $3\mathbf{a}+4\mathbf{b}$ given $\mathbf{a} = \langle -1, 2\rangle$ and $\mathbf{b} = \langle 4, 3\rangle$.
Answer: $|\mathbf{a}| = \sqrt{5}$, $\mathbf{a}+\mathbf{b} = \langle 3, 5\rangle$, $\mathbf{a}-\mathbf{b} = \langle -5, -1\rangle$, $2\mathbf{a} = \langle -2, 4\rangle$, and $3\mathbf{a}+4\mathbf{b} = \langle 13, 18\rangle$

31. Find $|\mathbf{a}|$, $\mathbf{a}+\mathbf{b}$, $\mathbf{a}-\mathbf{b}$, $2\mathbf{a}$, and $3\mathbf{a}+4\mathbf{b}$ given $\mathbf{a} = \langle 3, 2, -1\rangle$ and $\mathbf{b} = \langle 0, 6, 7\rangle$.
Answer: $|\mathbf{a}| = \sqrt{14}$, $\mathbf{a}+\mathbf{b} = \langle 3, 8, 6\rangle$, $\mathbf{a}-\mathbf{b} = \langle 3, -4, -8\rangle$, $2\mathbf{a} = \langle 6, 4, -2\rangle$, and $3\mathbf{a}+4\mathbf{b} = \langle 9, 30, 25\rangle$

32. Find $|\mathbf{a}|$, $\mathbf{a}+\mathbf{b}$, $\mathbf{a}-\mathbf{b}$, $2\mathbf{a}$, and $3\mathbf{a}+4\mathbf{b}$ given $\mathbf{a} = 2\mathbf{i}+3\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i}-2\mathbf{j}$.
Answer: $|\mathbf{a}| = \sqrt{13}$, $\mathbf{a}+\mathbf{b} = 5\mathbf{i}+\mathbf{j}$, $\mathbf{a}-\mathbf{b} = -\mathbf{i}+5\mathbf{j}$, $2\mathbf{a} = 4\mathbf{i}+6\mathbf{j}$, and $3\mathbf{a}+4\mathbf{b} = 18\mathbf{i}+\mathbf{j}$

33. Find a unit vector that has the same direction as $\langle 3, -5\rangle$.
Answer: $\left\langle \frac{3}{\sqrt{34}}, -\frac{5}{\sqrt{34}} \right\rangle$

34. Find a unit vector that has the same direction as $\langle 1, -4, 8\rangle$.
Answer: $\langle \frac{1}{9}, -\frac{4}{9}, \frac{8}{9}\rangle$

35. Find a unit vector that has the same direction as $2\mathbf{i}-4\mathbf{j}+7\mathbf{k}$.
Answer: $\frac{2}{\sqrt{69}}\mathbf{i} - \frac{4}{\sqrt{69}}\mathbf{j} + \frac{7}{\sqrt{69}}\mathbf{k}$

36. Which two of the following four vectors are parallel?
(a) $\langle 0, 8, 2\rangle$ (b) $\langle -1, 4, 1\rangle$ (c) $\langle -\frac{1}{2}, 2, \frac{1}{2}\rangle$ (d) $\langle \frac{7}{2}, 5, -\frac{7}{2}\rangle$
Answer: (b) and (c)

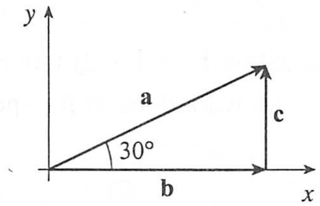
37. Find a unit vector in the direction opposite that of $\mathbf{a} = \langle 1, 3, -1 \rangle$.

Answer: $\left\langle -\frac{1}{\sqrt{11}}, -\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}} \right\rangle$

38. Find a vector of length 6 with the same direction as $\mathbf{a} = \langle 2, 2, -1 \rangle$

Answer: $\langle 4, 4, -2 \rangle$

39. Let $\mathbf{a}$ be the vector shown below. Find the horizontal component $\mathbf{b}$ and the vertical component $\mathbf{c}$ of $\mathbf{a}$ if $\mathbf{a} = 3$.



Answer: $\mathbf{b} = \left\langle \frac{3\sqrt{3}}{2} \right\rangle$, $\mathbf{c} = \left\langle \frac{3}{2} \right\rangle$

40. Is the quadrilateral $ABCD$ formed by the points $A(-1, 2, 0)$, $B(3, 1, 1)$, $C(1, 2, 1)$, and $D(-3, 3, 0)$ a parallelogram?

Answer: Yes

9.3 The Dot Product

41. Find the dot product of the vectors $\langle 1, 2 \rangle$ and $\langle 4, 5 \rangle$.

A) 14 B) 15 C) 16 D) 17
E) 18 F) 19 G) 20 H) 21

Answer: A)

42. Find the dot product of the vectors $\langle 1, 2, 3 \rangle$ and $\langle 3, 0, -7 \rangle$.

A) -24 B) -21 C) -18 D) -16
E) 24 F) 21 G) 18 H) 16

Answer: C)

43. What value of x will cause the two vectors $\langle x, 3 \rangle$ and $\langle 4, 5 \rangle$ to be orthogonal?

A) $-\frac{15}{4}$ B) $-\frac{4}{15}$ C) $-\frac{3}{20}$ D) $-\frac{20}{3}$
E) $\frac{15}{4}$ F) $\frac{4}{15}$ G) $\frac{3}{20}$ H) $\frac{20}{3}$

Answer: A)

44. Find the cosine of the angle between the two vectors $\langle 1, 2 \rangle$ and $\langle -2, 1 \rangle$.

A) $\frac{1}{\sqrt{2}}$ B) 0 C) $\frac{1}{\sqrt{3}}$ D) 1
E) $\frac{1}{\sqrt{5}}$ F) $\frac{1}{2}$ G) $\frac{4}{3}$ H) $\frac{3}{4}$

Answer: B)

45. Find the cosine of the angle between the two vectors $\langle 1, 2 \rangle$ and $\langle 2, 1 \rangle$.

A) $\frac{1}{\sqrt{2}}$ B) 0 C) $\frac{1}{\sqrt{3}}$ D) 1
E) $\frac{1}{\sqrt{5}}$ F) $\frac{1}{2}$ G) $\frac{4}{5}$ H) $\frac{3}{4}$

Answer: G)

46. Find the cosine of the angle between the two vectors $\langle 1,2\rangle$ and $\langle 3,4\rangle$.

A) $13/\sqrt{117}$ B) $13/\sqrt{119}$ C) $13/\sqrt{125}$ D) $13/\sqrt{132}$
E) $11/\sqrt{117}$ F) $11/\sqrt{119}$ G) $11/\sqrt{125}$ H) $11/\sqrt{132}$

Answer: G)

47. Find the cosine of the angle between the two vectors $\langle 1,1,1\rangle$ and $\langle 1,4,3\rangle$.

A) $8/\sqrt{41}$ B) $8/\sqrt{53}$ C) $8/\sqrt{69}$ D) $8/\sqrt{78}$
E) $4/\sqrt{41}$ F) $4/\sqrt{53}$ G) $4/\sqrt{69}$ H) $4/\sqrt{78}$

Answer: D)

48. A constant force with vector representation $\mathbf{F} = \mathbf{i} + 2\mathbf{j}$ moves an object along a straight line from the point $(2,4)$ to the point $(5,7)$. Find the work done in foot-pounds if force is measured in pounds and distance is measured in feet.

A) 6 B) 7 C) 8 D) 9
E) 10 F) 11 G) 12 H) 13

Answer: D)

49. A person pushing a lawnmower exerts a force of 30 pounds in the direction of the handle, which makes an angle of 30° with the ground. How much work is done in moving the lawnmower 20 feet?

A) $300\sqrt{3}$ B) $\frac{300}{\sqrt{3}}$ C) $200\sqrt{3}$ D) $\frac{200}{\sqrt{3}}$
E) 100 F) 200 G) 300 H) 400

Answer: A)

50. Let $\mathbf{a} = \langle 0,1,2\rangle$, $\mathbf{b} = \langle -1,-1,3\rangle$, $\mathbf{c} = \langle 2,4,-2\rangle$, and $\mathbf{d} = \langle 1,3,-3\rangle$. Find $\mathbf{a}\cdot\mathbf{d} - \mathbf{b}\cdot\mathbf{c}$.

Answer: 9

51. Let $\mathbf{a} = \langle 0,1,2\rangle$, $\mathbf{b} = \langle -1,-1,3\rangle$, $\mathbf{c} = \langle 2,4,-2\rangle$, and $\mathbf{d} = \langle 1,3,-3\rangle$. Find $(\mathbf{a}\cdot\mathbf{d})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\,\mathbf{d}$.

Answer: $\langle 15,39,-45\rangle$

52. Let $\mathbf{a} = \langle 0,1,2\rangle$, $\mathbf{b} = \langle -1,-1,3\rangle$, $\mathbf{c} = \langle 2,4,-2\rangle$, and $\mathbf{d} = \langle 1,3,-3\rangle$. Find $(2\mathbf{a}+\mathbf{b})\cdot(2\mathbf{c}-\mathbf{d})$.

Answer: -5

53. Determine all values for a such that the vectors $\mathbf{x} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{y} = \mathbf{i} + 2\mathbf{j} + a\mathbf{k}$ will form a 60° angle.

Answer: $a = \dfrac{-32 \pm 9\sqrt{11}}{7}$

54. Suppose $\mathbf{u} = \langle 1,2,3\rangle$ and $\mathbf{v} = \langle 1,1,-2\rangle$. Find two vectors $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{u} = \mathbf{a} + \mathbf{b}$, $\mathbf{a}$ is parallel to $\mathbf{v}$, and $\mathbf{b}$ is perpendicular to $\mathbf{v}$.

Answer: $\mathbf{a} = \left\langle -\frac{1}{2}, -\frac{1}{2}, 1\right\rangle$ and $\mathbf{b} = \left\langle \frac{3}{2}, \frac{5}{2}, 2\right\rangle$

55. Let $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{x} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$. Find $|\mathbf{a}|$ and $\mathbf{a}\cdot\mathbf{b}$.

Answer: $\sqrt{14}$; 3

56. Let $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$, and $\mathbf{c} = 2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$. Do these vectors form a right triangle? Show why or why not.

Answer: Yes, the vectors form a right triangle since the dot product of **a** and **c** is 0, and $\mathbf{b} + \mathbf{c} = \mathbf{a}$.

57. If $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, then $\mathbf{a}\cdot\mathbf{b}$ is

A) $2\sqrt{6}$ B) $3\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$ C) 10 D) -4 E) 4

Answer: -4

58. For which values of t are $\mathbf{a} = \langle t+2, t, t\rangle$ and $\mathbf{b} = \langle t-2, t+1, t\rangle$ orthogonal?

Answer: $t = -\frac{4}{3}$ or $t = 1$

59. Let $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$. Find the scalar projection of $\mathbf{a}$ onto $\mathbf{b}$.

Answer: $\frac{11}{\sqrt{14}}$

60. Find the projection of the vector $\langle 2, 3, -5\rangle$ along the vector $\langle -1, 1, 2\rangle$.

Answer: $\langle \frac{3}{2}, -\frac{3}{2}, -3\rangle$

61. What does the fact that $\mathbf{a} \cdot \mathbf{a} = 0$ imply about the vector $\mathbf{a}$?

Answer: $\mathbf{a} = \mathbf{0}$

62. Given two distinct nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, is it always true that $|\text{proj}_{\mathbf{a}}\, \mathbf{b}| = |\text{proj}_{\mathbf{b}}\, \mathbf{a}|$?

Answer: No

63. Find a value of a such that the vector $\langle a, 1, 1\rangle$ makes an angle of $45°$ with the vector $\langle 1, 2, 1\rangle$, or show that no such a exists.

Answer: $a = \frac{3 \pm \sqrt{15}}{2}$

64. Let $\mathbf{a} = \langle -1, 2, 2\rangle$ and suppose that $\mathbf{b}$ is a vector parallel to $\mathbf{a}$. Find $\text{proj}_{\mathbf{b}}\, \mathbf{a}$.

Answer: $\text{proj}_{\mathbf{b}}\, \mathbf{a} = \langle -1, 2, 2\rangle$

65. Suppose that $\mathbf{a}$ is perpendicular to both $\mathbf{b} = \langle 1, 2, 3\rangle$ and $\mathbf{c} = \langle -1, 0, 2\rangle$. What is $\mathbf{a} \cdot (2\mathbf{b} + 3\mathbf{c})$?

Answer: 0

66. If $\mathbf{a} = \langle x, 2, 5\rangle$ and $\mathbf{b} = \langle 3, y, 4\rangle$, find a relationship between x and y which guarantees that $\mathbf{a} \perp \mathbf{b}$.

Answer: $3x + 2y + 20 = 0$

9.4 The Cross Product

67. Find the cross product $\langle 1, 2, 0\rangle \times \langle 3, 4, 0\rangle$.

A) $\langle 0, 0, -2\rangle$ B) $\langle 0, 0, 2\rangle$ C) $\langle 3, 8, 0\rangle$ D) $\langle 8, 3, 0\rangle$
E) $\langle 0, 3, 8\rangle$ F) $\langle 0, 8, 3\rangle$ G) $\langle 3, 0, 8\rangle$ H) $\langle 8, 0, 3\rangle$

Answer: A)

68. Find the cross product $\langle 1, 2, 0\rangle \times \langle 0, 2, 1\rangle$.

A) $\langle 1, 2, 1\rangle$ B) $\langle -1, -2, -1\rangle$ C) $\langle 1, -2, 1\rangle$ D) $\langle -1, 2, -1\rangle$
E) $\langle 2, 1, 2\rangle$ F) $\langle -2, -1, -2\rangle$ G) $\langle 2, -1, 2\rangle$ H) $\langle -2, 1, -2\rangle$

Answer: G)

69. Find the cross product $\mathbf{a} \times \mathbf{b}$, where $\mathbf{a} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} + 3\mathbf{j}$.

A) $15\mathbf{i} - 8\mathbf{j} + 10\mathbf{k}$ B) $3\mathbf{i} + 11\mathbf{j} - 9\mathbf{k}$ C) $7\mathbf{i} + 13\mathbf{j} + \mathbf{k}$ D) $14\mathbf{i} + 5\mathbf{j} - 12\mathbf{k}$
E) $-7\mathbf{i} - 23\mathbf{j} + 33\mathbf{k}$ F) $-10\mathbf{i} + 27\mathbf{j} + 13\mathbf{k}$ G) $-12\mathbf{i} + 28\mathbf{j} - 17\mathbf{k}$ H) $18\mathbf{i} + 32\mathbf{j} + 7\mathbf{k}$

Answer: G)

70. Find the length of the cross product of the vectors $\langle 1, 1, 1\rangle$ and $\langle 1, 1, 2\rangle$.

A) $\frac{1}{\sqrt{5}}$ B) $\frac{1}{2}$ C) $\frac{1}{\sqrt{3}}$ D) $\frac{1}{\sqrt{2}}$
E) $\sqrt{5}$ F) 2 G) $\sqrt{3}$ H) $\sqrt{2}$

Answer: H)

71. Find the area of the triangle whose vertices are $\langle 1,2\rangle$, $\langle 4,6\rangle$, and $\langle -1,0\rangle$.

A) $\frac{1}{3}$ B) $\frac{1}{2}$ C) 1 D) $3/2$
E) $4/3$ F) 2 G) 3 H) 4

Answer: C)

72. Find a unit vector orthogonal to both of the vectors $\langle 1,-1,0\rangle$ and $\langle 1,2,3\rangle$.

A) $\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$ B) $\left\langle -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$ C) $\left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$ D) $\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle$
E) $\left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$ F) $\left\langle -\frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$ G) $\left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$ H) $\left\langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle$

Answer: D)

73. Find a unit vector orthogonal to the plane through the points $(1,0,0)$, $(0,1,0)$, and $(0,2,2)$.

A) $\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$ B) $\left\langle -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$ C) $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$ D) $\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$
E) $\left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$ F) $\left\langle -\frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$ G) $\left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$ H) $\left\langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle$

Answer: H)

74. Find the volume of the parallelepiped determined by the vectors $\langle 1,0,1\rangle$, $\langle 2,1,3\rangle$, and $\langle 1,1,1\rangle$.

A) 1 B) 2 C) 3 D) 4
E) 5 F) 6 G) 7 H) 8

Answer: A)

75. Find the scalar triple product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ of the vectors $\mathbf{a}=\langle 1,2,0\rangle$, $\mathbf{b}=\langle 0,3,4\rangle$, and $\mathbf{c}=\langle 5,0,6\rangle$.

A) 58 B) 60 C) 62 D) 64
E) 66 F) 68 G) 70 H) 72

Answer: A)

76. Find the torque vector τ of the force $\mathbf{F}=\langle 1,1,0\rangle$ acting on a rigid body at the point given by the position vector $\mathbf{r}=\langle 2,3,0\rangle$.

A) $\langle -1,0,0\rangle$ B) $\langle 0,-1,0\rangle$ C) $\langle 0,0,-1\rangle$ D) $\langle -1,1,0\rangle$
E) $\langle 0,-1,1\rangle$ F) $\langle 1,0,-1\rangle$ G) $\langle -1,1,1\rangle$ H) $\langle 1,-1,1\rangle$

Answer: C)

77. Find the cross product $\mathbf{a}\times\mathbf{b}$, where $\mathbf{a}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ and $\mathbf{b}=2\mathbf{i}-\mathbf{j}+\mathbf{k}$.

Answer: $5\mathbf{i}+5\mathbf{j}-5\mathbf{k}$

78. Consider the points $P=(1,2,3)$, $Q=(2,-1,0)$, and $R=(-1,4,1)$. Find the area of the triangle PQR.

Answer: $2\sqrt{14}$

79. Find the area of the parallelogram $PQRS$ with $P=(0,1,2)$, $Q=(4,1,1)$, and $R=(1,-1,3)$.

Answer: $\sqrt{93}$

80. Find the cross product $\mathbf{a}\times\mathbf{b}$, where $\mathbf{a}=\mathbf{i}-3\mathbf{j}+2\mathbf{k}$ and $\mathbf{b}=2\mathbf{i}-\mathbf{j}-3\mathbf{k}$.

Answer: $\mathbf{a}\times\mathbf{b}=11\mathbf{i}+7\mathbf{j}+5\mathbf{k}$

81. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors. Under what conditions is $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{a}$? When is $(2\mathbf{a})\times(3\mathbf{b})=6(\mathbf{a}\times\mathbf{b})$?

Answer: $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{a}$ when (1) $\mathbf{a}=\mathbf{b}$, (2) $\mathbf{a}=0$ or $\mathbf{b}=0$, (3) $\mathbf{a}$ and $\mathbf{b}$ are parallel; in all cases $\mathbf{a}\times\mathbf{b}=0$; $(2\mathbf{a})\times(3\mathbf{b})=6(\mathbf{a}\times\mathbf{b})$ always.

82. Use the property of the cross product that $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta$ to derive a formula for the distance d from a point P to a line l. Use this formula to find the distance from the origin to the line through $(2, 1, -4)$ and $(3, 3, -2)$.

Answer: Let $\mathbf{u}$ be a vector from l to P, $\mathbf{v}$ be a vector parallel to l and θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Note that $d = |\mathbf{u}| \sin\theta$. But $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta$. So $|\mathbf{u} \times \mathbf{v}| = |\mathbf{v}|\,d = \frac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{v}|} \Rightarrow d = \frac{\sqrt{173}}{3}$.

83. Given the vectors $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{w} = \mathbf{i} + \mathbf{j} + 4\mathbf{k}$, find a vector of length 2 which is orthogonal to both $\mathbf{u}$ and $\mathbf{w}$.

Answer: $\frac{10}{\sqrt{107}}\mathbf{i} - \frac{18}{\sqrt{107}}\mathbf{j} + \frac{2}{\sqrt{107}}\mathbf{k}$

84. Let $\mathbf{u} \times \mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Find $\mathbf{v} \times \mathbf{u}$, if it exists.

Answer: $\mathbf{v} \times \mathbf{u} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

85. Let $\mathbf{u} \times \mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Find $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$, if it exists.

Answer: 0

86. Let $\mathbf{u} \times \mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Find the area A of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$, if it exists.

Answer: $A = \sqrt{6}$

87. Let $\mathbf{a} = \langle 3, 0, -1\rangle$, $\mathbf{b} = \langle 4, 1, -1\rangle$, $\mathbf{c} = \langle 2, 1, 0\rangle$ and $\mathbf{d} = \langle -1, -2, -2\rangle$. Find: (a) $\mathbf{a} \times \mathbf{b}$ and (b) $\mathbf{c} \times \mathbf{d}$.

Answer: (a) $\langle 1, -1, 3\rangle$, (b) $\langle -2, 4, -3\rangle$

88. Find the area of the triangle with vertices $(1, 0, 2)$, $(2, 1, -3)$, and $(3, -2, 0)$.

Answer: $2\sqrt{14}$

89. For vectors $\mathbf{a}$ and $\mathbf{b}$, given that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|$, which of the following statements is true?

A) $\mathbf{a} = \mathbf{b}$
B) $\mathbf{a}$ and $\mathbf{b}$ are parallel
C) $\mathbf{a}$ and $\mathbf{b}$ are not parallel
D) $\mathbf{a}$ and $\mathbf{b}$ are perpendicular
E) $\mathbf{a}$ and $\mathbf{b}$ are not perpendicular

Answer: D)

90. Given $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$. Find $\mathbf{u} \times \mathbf{v}$ and show that $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

Answer: $\mathbf{u} \times \mathbf{v} = 3\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}$; to show orthogonality, show that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$ and $(\mathbf{v} \times \mathbf{u}) \cdot \mathbf{v} = 0$.

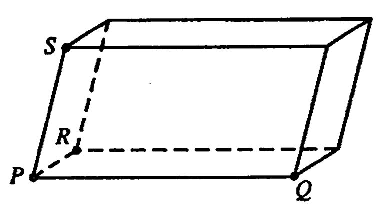
91. If vector $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, and vector $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$, find a vector perpendicular to both vectors.

Answer: Any scalar multiple of $18\mathbf{i} - 19\mathbf{j} - 11\mathbf{k}$ is perpendicular

92. Find the volume of the parallelepiped spanned by the vectors $2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, $3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, and $-\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

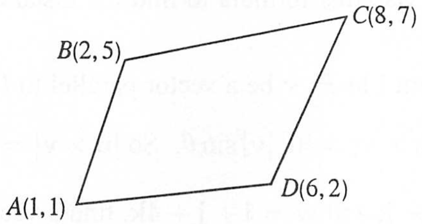
Answer: 64

93. Find the volume of the parallelepiped below given $P = (1, -3, 2)$, $Q = (3, -1, 3)$, $R = (2, 1, -4)$, and $S = (-1, 2, 1)$.



Answer: 91

94. Find the area of quadrilateral $ABCD$. Note that $ABCD$ is *not* a parallelogram.



Answer: $\frac{45}{2}$

95. Suppose that $\mathbf{a} \perp \mathbf{b}$, and let $\mathbf{c} = [\mathbf{a} \times (\mathbf{b} \times \mathbf{a})]$. Which of the following statements is true?

A) c is perpendicular to both a and b
B) c is perpendicular to neither a nor b
C) c is perpendicular to a
D) c is perpendicular to b

Answer: C)

96. Compute $|\mathbf{a} \times \mathbf{b}|$ if $|\mathbf{a}| = 3$, $|\mathbf{b}| = 7$, and $\mathbf{a} \cdot \mathbf{b} = 0$.

Answer: 21

97. If $\mathbf{b} = \langle 1, 0, 1 \rangle$, $\mathbf{c} = \langle 0, -1, 1 \rangle$, and $\mathbf{a} = \langle 2, -3, z \rangle$, find a value for z which guarantees that a, b, and c are coplanar.

Answer: $z = 5$

98. If $\mathbf{b} = \langle 1, -1, 2 \rangle$, $\mathbf{c} = \langle 2, 1, 3 \rangle$, and a is any vector, what is $[\mathbf{a} \times (\mathbf{b} \times \mathbf{c})] \cdot \mathbf{a}$?

Answer: 0

99. If we know that $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, which of the following statements is true?

A) $\mathbf{b} = \mathbf{c}$
B) $|\mathbf{b}| = |\mathbf{c}|$
C) b and c are parallel
D) a is parallel to $\mathbf{b} - \mathbf{c}$
E) a is a unit vector

Answer: D)

9.5 Equations of Lines and Planes

100. Find a vector v for a vector equation $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ of the line passing through the points $(1, 2, 3)$ and $(4, 5, 6)$.

A) $\langle 1, 1, 0 \rangle$
B) $\langle 0, 1, 1 \rangle$
C) $\langle 1, 0, 1 \rangle$
D) $\langle 1, 1, 1 \rangle$
E) $\langle 0, 0, 1 \rangle$
F) $\langle 1, 0, 0 \rangle$
G) $\langle 0, 1, 0 \rangle$
H) $\langle 0, 0, 0 \rangle$

Answer: D)

101. For the line l passing through the points $(1, 0, 1)$ and $(2, 4, 7)$, what is the value of c in the following symmetric equation for l? $\dfrac{x-1}{1} = \dfrac{y-0}{4} = \dfrac{z-1}{c}$

A) 1
B) 2
C) 3
D) 4
E) 5
F) 6
G) 7
H) 8

Answer: F)

102. Find the value d for which the plane $x + y + 2z = d$ will pass through the point $(1, 2, 3)$.

A) 2 B) 3 C) 4 D) 5
E) 6 F) 7 G) 8 H) 9

Answer: H)

103. Find a unit vector perpendicular to the plane $x + 2y + 2z = 10$.

A) $\langle -\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \rangle$ B) $\langle -\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle$ C) $\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle$ D) $\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \rangle$
E) $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$ F) $\left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$ G) $\left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ H) $\langle -1, 0, 0 \rangle$

Answer: C)

104. Find the cosine of the angle between the two planes $x + 2y = 0$ and $x + 2z = 3$.

A) $\frac{2}{3}$ B) $\frac{3}{4}$ C) 1 D) $\frac{\sqrt{3}}{2}$
E) $\frac{\sqrt{3}}{4}$ F) $\frac{1}{2}$ G) $\frac{1}{\sqrt{2}}$ H) $\frac{1}{5}$

Answer: H)

105. Find the cosine of the angle between the two planes $x + y + z = 0$ and $x + 2y + 3z = 1$.

A) $4/\sqrt{35}$ B) $6/\sqrt{35}$ C) $7/\sqrt{35}$ D) $8/\sqrt{35}$
E) $5/\sqrt{42}$ F) $6/\sqrt{42}$ G) $7/\sqrt{42}$ H) $8/\sqrt{42}$

Answer: F)

106. Find direction numbers a, b, and c for the line of intersection of the two planes $x + y + z = 1$ and $x + z = 0$.

A) $1, 1, 0$ B) $1, 0, 1$ C) $0, 1, 1$ D) $1, -1, 1$
E) $1, 1, 1$ F) $1, -1, 0$ G) $1, 0, -1$ H) $0, 1, -1$

Answer: G)

107. Find an equation for the plane consisting of all points that are equidistant from the two points $(1, 1, 0)$ and $(0, 1, 1)$.

A) $x - z = 0$ B) $x - z = 1$ C) $x - y = 0$ D) $x - y = 1$
E) $x + z = 0$ F) $x + z = 1$ G) $x + y = 0$ H) $x + y = 1$

Answer: A)

108. Find the point at which the two lines $\mathbf{r} = \langle 1, 1, 0 \rangle + t \langle 1, -1, 2 \rangle$ and $\mathbf{r} = \langle 2, 0, 2 \rangle + s \langle -1, 1, 0 \rangle$ intersect.

A) $(1, 0, 1)$ B) $(2, 0, 2)$ C) $(1, 1, 1)$ D) $(2, 2, 2)$
E) $(1, 1, 0)$ F) $(2, 2, 0)$ G) $(0, 1, 1)$ H) $(0, 2, 2)$

Answer: B)

109. Let $A = (2, 1, 3)$, $B = (1, 2, -2)$, and $C = (-1, 3, 1)$. Find an equation of the plane through A, B, and C.

Answer: $8x + 13y + z = 32$

110. Give parametric equations for the line through the points $P(2, -1, -2)$ and $Q(3, 1, 4)$.

Answer: Two sets are: $x = 2 + t$, $y = -1 + 2t$, $z = -2 + 6t$ and $x = 3 + t$, $y = 1 + 2t$, $z = 4 + 6t$. Other sets of parametric equations also describe the line.

111. Let $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. Find an equation of the line parallel to $\mathbf{a} + \mathbf{b}$ and passing through the tip of $\mathbf{b}$.

Answer: $\mathbf{r}(t) = (1 + 3t)\,\mathbf{i} + (2 + 3t)\,\mathbf{j} + (3 + 2t)\,\mathbf{k}$

112. Find the cosine of the acute angle between the lines $x = 4 - 4t$, $y = 3 - t$, $z = 1 + 5t$ and $x = 4 - t$, $y = 3 + 2t$, $z = 1$.

Answer: $\frac{\sqrt{210}}{105}$

113. Find symmetric equations of the line passing through $(2, -3, 4)$ and parallel to the vector $\overrightarrow{AB}$, where A and B are the points $(-2, 1, 1)$ and $(0, 2, 3)$.

Answer: $\dfrac{x-2}{2} = \dfrac{y+3}{1} = \dfrac{z-4}{2}$

114. Find the distance between the following two lines.

$$\begin{aligned} x &= -t \\ y &= t \\ z &= 2t \end{aligned} \qquad \text{and} \qquad \begin{aligned} x &= 3+t \\ y &= 3t \\ z &= 5-4t \end{aligned}$$

Answer: $\frac{13}{\sqrt{30}}$

115. Let l and l' be two lines in space given by the equations

$$l: \begin{aligned} x &= 3+t \\ y &= 1-t \\ z &= 2t \end{aligned} \qquad l': \begin{aligned} x &= -1+t \\ y &= 2t \\ z &= 1+kt \end{aligned}$$

Find all values of k (if any) for which l and l' are parallel.

Answer: No value of k will work

116. Let l and l' be two lines in space given by the equations

$$l: \begin{aligned} x &= 3+t \\ y &= 1-t \\ z &= 2t \end{aligned} \qquad l': \begin{aligned} x &= -1+t \\ y &= 2t \\ z &= 1+kt \end{aligned}$$

Find all values of k (if any) for which l and l' are perpendicular.

Answer: $k = \frac{1}{2}$

117. Find the equation of the line through the points $(1, 1, -1)$ and $(2, -1, 1)$.

Answer: $x = 1 + t$, $y = 1 - 2t$, $z = -1 + 2t$

118. Find symmetric and parametric equations of the line that goes through the points $P(1, 2, 4)$ and $Q(3, -1, 6)$.

Answer: $x = 1 + 2t$, $y = 2 - 3t$, $z = 4 + 2t$; $\dfrac{x-1}{2} = \dfrac{y-2}{-3} = \dfrac{z-4}{2}$

119. Find an equation of the plane through the point $P = (2, 1, -4)$ and perpendicular to the line $x = 2 + 3t$, $y = 1 - 4t$, $z = 3 + 3t$.

Answer: $3x - 4y + 3z = -10$

120. Find the intersection point of the line $\mathbf{r} = \langle 1, 0, 2 \rangle + \langle 2, -2, 1 \rangle t$ and the plane $3x + 4y + 6z = 7$.

Answer: $(-3, 4, 0)$

121. Find the equation of the line containing the points $(2, -2, 3)$ and $(-5, -2, -3)$ in symmetric form.

Answer: $\dfrac{x-2}{7} = \dfrac{z-3}{6}$, $y = -2$

122. Consider the points $P = (1,2,3)$, $Q = (2,-1,0)$, and $R = (-1,4,1)$. Find an equation for the plane determined by P, Q, and R.

Answer: $3x + 2y - z = 4$

123. Find the equation of the plane containing the points $(5,3,1)$, $(1,8,4)$, and $(-1,3,-2)$.

Answer: $x + 2y - 2z = 9$

124. Determine an equation of the plane containing the following points: $(2,0,2)$, $(4,3,1)$, and $(0,1,3)$.

Answer: $-x + 2y + 4z - 6 = 0$

125. Find the normal vector to the plane which passes through the points $(1,0,0)$, $(0,0,1)$, and $(4,3,-2)$.

Answer: $\langle -3, 1, -3 \rangle$

126. Let L be the line given by $x = 2 - t$, $y = 1 + t$, and $z = 1 + 2t$. L intersects the plane $2x + y - z = 1$ at the point $P = (1,2,3)$. Find the angle L makes with the plane, to the nearest degree.

Answer: 30°

127. Let L be the line given by $x = 2 - t$, $y = 1 + t$, and $z = 1 + 2t$. L intersects the plane $2x + y - z = 1$ at the point $P = (1,2,3)$. Find parametric equations for the line through P which lies in the plane and is perpendicular to L.

Answer: $x = 1 + 3t$, $y = 2 - 3t$, $z = 3 + 3t$

128. Find the equation of the plane containing the lines $x = 4 - 4t$, $y = 3 - t$, $z = 1 + 5t$ and $x = 4 - t$, $y = 3 + 2t$, $z = 1$.

Answer: $10x - 5y + 7z - 32 = 0$

129. Given the points $A = (2,3,1)$, $B = (4,-1,5)$, and $O = (0,0,0)$, find the distance from O to the line through A and B.

Answer: $\frac{\sqrt{122}}{3}$

130. Find the distance between the two skew lines

$$L_1\colon \frac{x-1}{2} = \frac{y-3}{5} = \frac{z-1}{3} \quad \text{and} \quad L_2\colon \frac{x-2}{4} = \frac{y+1}{2} = \frac{z+2}{3}$$

Answer: $\frac{33\sqrt{373}}{373}$

131. Are the planes $3x - y + 5z = 13$ and $x + 7y - 2z = 4$ perpendicular to each other?

Answer: No

132. Find the angle between the lines $\frac{x-2}{1} = \frac{1-y}{3} = \frac{z-3}{1}$ and $\frac{x}{2} = \frac{y+3}{-1} = \frac{z-1}{2}$.

Answer: $\cos^{-1} \frac{7}{3\sqrt{11}} \approx 45.3°$ or 0.7904 radians

133. Find the distance between the point $(2,3,-1)$ and the line $\frac{x+1}{3} = \frac{y-2}{5} = \frac{z+1}{-1}$.

Answer: $\frac{\sqrt{110}}{5}$

134. Let $P = (1,3,2)$ and let L be the line with parametric equations $x = 2 - t$, $y = -1 + 2t$, $z = 3 + t$. Use the vector cross product to find the distance from P to L.

Answer: $\frac{\sqrt{66}}{3}$

135. Find a vector equation $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ for the line whose symmetric equation is $\frac{x-3}{2} = \frac{y+1}{-1}$, $z = 5$.

Answer: $\mathbf{r} = \langle 3, -1, 5 \rangle + t \langle 2, -1, 0 \rangle$

136. Do the two lines

$$\mathbf{x}_1(t) = \langle 1,1,3\rangle + t\langle -1,0,2\rangle \qquad \text{and} \qquad \mathbf{x}_2(s) = \langle -1,1,4\rangle + s\langle 2,0,1\rangle$$

intersect? If so, find the point of intersection.

Answer: Yes; $\left(\frac{1}{5}, 1, \frac{23}{5}\right)$

137. Find the point at which the following three planes intersect:

$$\begin{aligned} x - 2y + z &= 5 \\ 2x - y + z &= 1 \\ -2x + y + z &= 3 \end{aligned}$$

Answer: $\left(-\frac{8}{3}, -\frac{10}{3}, 1\right)$

138. Suppose $\mathbf{a}$ and $\mathbf{b}$ are vectors which lie in the plane $3x + 2y - 5z = 1$. Compute $(2\mathbf{a} + 3\mathbf{b}) \cdot \langle 3, 2, -5\rangle$.

Answer: 0

139. The lines l_1: $\mathbf{r}_1 = \langle 2,3,1\rangle + t\langle -1,1,1\rangle$ and l_2: $\mathbf{r}_2 = \langle 3,1,2\rangle + s\langle -1,2,-1\rangle$ both contain the point $P(2,3,1)$. Find the value of s which gives the point of intersection P, and then compute the angle θ between the two lines.

Answer: $s = 1$, $\theta \approx 61.9°$

140. The planes P_1: $x + 2y + 3z = 2$ and P_2: $-2x + 3y + 2z = -4$ both contain the point $P(2,0,0)$. Find a vector equation $\mathbf{r} = \overrightarrow{OP_0} + t\mathbf{d}$ for the line of intersection of these planes.

Answer: $\mathbf{r} = \langle 2,0,0\rangle + t\langle -5,-8,7\rangle$

9.6 Functions and Surfaces

141. Identify the surface $x = y^2 + 2z^2$.

A) ellipsoid but not a sphere
B) hyperboloid of one sheet
C) hyperboloid of two sheets
D) cylinder
E) sphere
F) elliptic but not circular paraboloid
G) cone
H) circular paraboloid (figure of revolution)

Answer: F)

142. Identify the surface $x^2 + y^2 + z^2 = 4$.

A) ellipsoid but not a sphere
B) hyperboloid of one sheet
C) hyperboloid of two sheets
D) cylinder
E) sphere
F) elliptic but not circular paraboloid
G) cone
H) circular paraboloid (figure of revolution)

Answer: E)

143. Identify the surface $x^2 = y^2 + z^2$.

A) ellipsoid but not a sphere
B) hyperboloid of one sheet
C) hyperboloid of two sheets
D) cylinder
E) sphere
F) elliptic but not circular paraboloid
G) cone
H) circular paraboloid (figure of revolution)

Answer: G)

144. Identify the surface $2 = y^2 + z^2$.

A) ellipsoid but not a sphere
B) hyperboloid of one sheet
C) hyperboloid of two sheets
D) cylinder
E) sphere
F) elliptic but not circular paraboloid
G) cone
H) circular paraboloid (figure of revolution)

Answer: D)

145. Identify the surface $x^2 + y^2 + z^2 = 3$.

A) ellipsoid but not a sphere
B) hyperboloid of one sheet
C) hyperboloid of two sheets
D) cylinder
E) sphere
F) elliptic but not circular paraboloid
G) cone
H) circular paraboloid (figure of revolution)

Answer: E)

146. Identify the surface $x^2 - y^2 + z^2 = 10$.

A) ellipsoid but not a sphere
B) hyperboloid of one sheet
C) hyperboloid of two sheets
D) cylinder
E) sphere
F) elliptic but not circular paraboloid
G) cone
H) circular paraboloid (figure of revolution)

Answer: B)

147. Identify the surface $-x^2 + y^2 - z^2 = 10$.

A) ellipsoid but not a sphere
B) hyperboloid of one sheet
C) hyperboloid of two sheets
D) cylinder
E) sphere
F) elliptic but not circular paraboloid
G) cone
H) circular paraboloid (figure of revolution)

Answer: C)

148. Identify the surface $x = y^2 + z^2$.

A) ellipsoid but not a sphere
B) hyperboloid of one sheet
C) hyperboloid of two sheets
D) cylinder
E) sphere
F) elliptic but not circular paraboloid
G) cone
H) circular paraboloid (figure of revolution)

Answer: H)

149. Identify the trace of the surface $x = y^2 + z^2$ in the plane $x = 1$.

A) ellipse but not a circle
B) parabola
C) hyperbola
D) circle
E) two parallel straight lines
F) two intersecting straight lines
G) point
H) straight line

Answer: D)

150. Identify the trace of the surface $x = y^2 + z^2$ in the plane $x = 0$.

A) ellipse but not a circle
B) parabola
C) hyperbola
D) circle
E) two parallel straight lines
F) two intersecting straight lines
G) point
H) straight line

Answer: G)

151. Identify the trace of the surface $x = y^2 + z^2$ in the plane $y = 0$.

A) ellipse but not a circle
B) parabola
C) hyperbola
D) circle
E) two parallel straight lines
F) two intersecting straight lines
G) point
H) straight line

Answer: B)

152. Identify the trace of the surface $x = y^2 + z^2$ in the plane $z = 1$.

A) ellipse but not a circle
B) parabola
C) hyperbola
D) circle
E) two parallel straight lines
F) two intersecting straight lines
G) point
H) straight line

Answer: B)

153. Identify the trace of the surface $x^2 = y^2 + z^2$ in the plane $z = 0$.

A) ellipse but not a circle
B) parabola
C) hyperbola
D) circle
E) two parallel straight lines
F) two intersecting straight lines
G) point
H) straight line

Answer: F)

154. Identify the trace of the surface $x = y^2 + z^2$ in the plane $x = y$.

A) ellipse but not a circle
B) parabola
C) hyperbola
D) circle
E) two parallel straight lines
F) two intersecting straight lines
G) point
H) straight line

Answer: D)

155. Identify the trace of the surface $x = 2y^2 + 3z^2$ in the plane $x = 1$.

A) ellipse but not a circle
B) parabola
C) hyperbola
D) circle
E) two parallel straight lines
F) two intersecting straight lines
G) point
H) straight line

Answer: A)

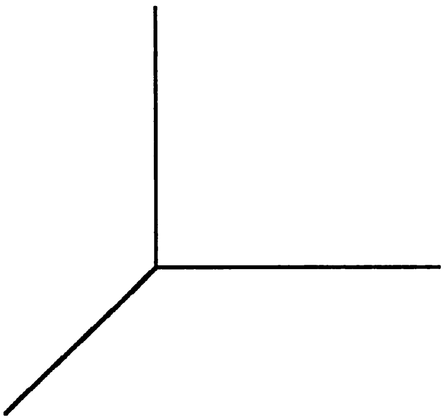
156. Let S be the quadric surface given by $x^2 + 2y^2 + 2x - z = 0$. What kind of surface is S?

Answer: elliptic paraboloid

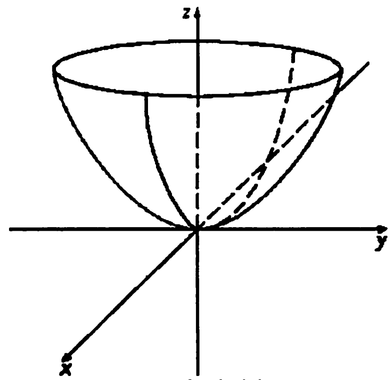
157. Let S be the quadric surface given by $x^2 + 2y^2 + 2x - z = 0$. What are the traces of S in each of the three coordinate planes?

Answer: xy-plane: ellipse; yz-plane: parabola; xz-plane: parabola

158. Sketch the graph of $z = x^2 + y^2$ in $\mathbb{R}^3$, and name the surface.

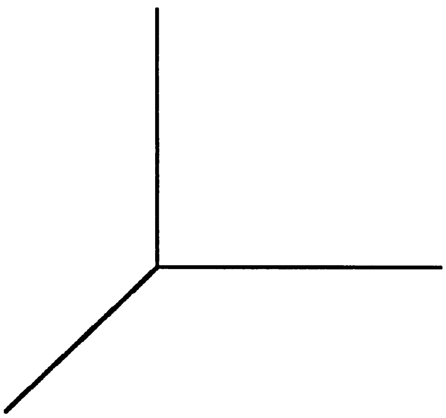


Answer:

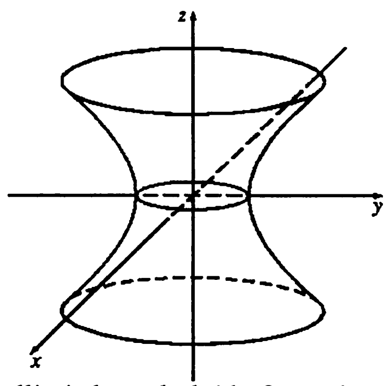


paraboloid

159. Sketch the graph of $\frac{x^2}{4} + y^2 - z^2 = 1$ in $\mathbb{R}^3$, and name the surface.



Answer:



elliptic hyperboloid of one sheet

160. Which of the following is not a quadric surface?

A) $x^2 + z^2 = 1$ B) $z = x^2 + y^2$ C) $y = x^3 + z$
D) $z = x^2 - y^2$ E) $x^2 + y^2 + z^2 = 1$

Answer: C)

161. Find the coordinates of the point(s) of intersection of the line $x = 1 - t$, $y = 1 - t$, $z = 4t$ and the surface $z = x^2 + 2y^2$.

Answer: $(-2, -2, 12)$, $\left(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}\right)$

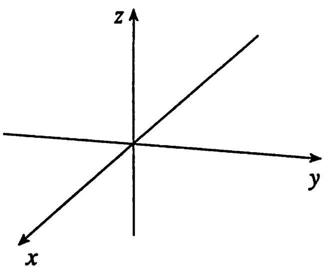
162. Describe the trace of the surface $z = 4x^2 = 0$ in the plane $z = 1$.

Answer: $x = \pm\frac{1}{2}$; two straight lines

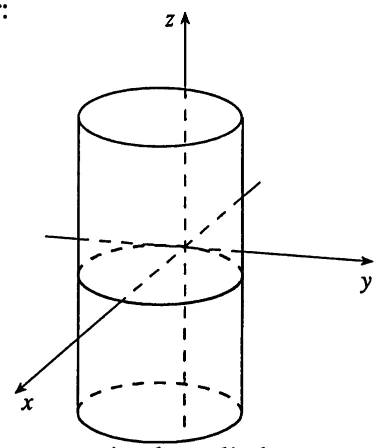
163. Describe the vertical traces $x = 0$ and the horizontal traces $z = -1$ (if any) for the surfaces $z = x^2 + y$ and $z^2 = x^2 + y^2$.

Answer: $z = x^2 + y^2$: $x = 0 \Rightarrow z = y^2$, a parabola; $z = -1 \Rightarrow x^2 + y^2 = -1$, no trace.
$z^2 = x^2 + y^2$: $x = 0 \Rightarrow z = \pm y$, two straight lines; $z = -1 \Rightarrow 1 = x^2 + y^2$, a circle

164. Sketch and identify the quadratic surface given by $x^2 + y^2 - 2x = 0$.

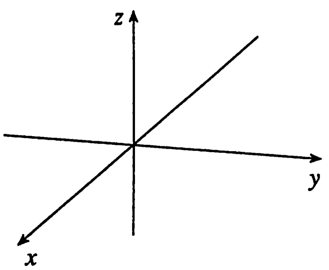


Answer:

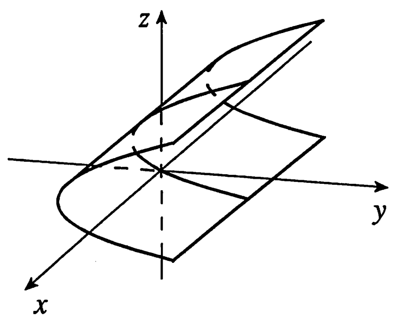


circular cylinder

165. Sketch and identify the quadric surface given by the equation $z^2 - 2z - y = 0$.



Answer:



parabolic cylinder

166. Let $f(x, y) = x \sin y$. Find $f\left(2, \frac{\pi}{3}\right)$.

A) $\sqrt{3}$ B) $\sqrt{2}$ C) $\frac{\sqrt{3}}{2}$ D) $\frac{\sqrt{2}}{2}$
E) $\frac{1}{2}$ F) $\frac{1}{3}$ G) 1 H) 0

Answer: A)

167. Let $f(x, y) = x^2 + 2xy + y^2$. If $x = 2$, find $f(x, 2x)$.

A) 12 B) 16 C) 24 D) 28
E) 32 F) 36 G) 42 H) 48

Answer: F)

168. Let $f(x, y) = x \sin y$. If $x = \pi$, find $f(x, x/2)$.

A) $\frac{\pi}{6}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{3}$ D) $\frac{\pi}{2}$
E) $\frac{2\pi}{3}$ F) $\frac{3\pi}{4}$ G) π H) 2π

Answer: G)

169. Let $f(x, y) = \left(x^2 + y\right)^3$. If $x = 1$, find $f(x, 2x)$.

A) 1 B) 2 C) 3 D) 4
E) 8 F) 9 G) 16 H) 27

Answer: H)

170. Find the domain of the function $f(x,y) = \sqrt{x - y^2}$.

A) All points on or to the left of $x = y^2$
B) All points on or to the right of $x = y^2$
C) All points to the left of $x = y^2$
D) All points to the right of $x = y^2$
E) All points on or to the left of $x = 0$
F) All points on or to the right of $x = 0$
G) All points to the left of $x = 0$
H) All points in the xy-plane

Answer: B)

171. Find the range of the function $f(x,y) = \sqrt{x - y^2}$.

A) $(0, \infty)$
B) $[0, \infty)$
C) $(-\infty, 0)$
D) $(-\infty, \infty)$
E) $(1, \infty)$
F) $[1, \infty)$
G) $(\sqrt{2}, \infty)$
H) $[\sqrt{2}, \infty)$

Answer: B)

172. Find the domain of the function $f(x,y) = e^{x - y^2}$.

A) All points on or to the left of $x = y^2$
B) All points on or to the right of $x = y^2$
C) All points to the left of $x = y^2$
D) All points to the right of $x = y^2$
E) All points on or to the left of $x = 0$
F) All points on or to the right of $x = 0$
G) All points to the left of $x = 0$
H) All points in the xy-plane

Answer: H)

173. Find the range of the function $f(x,y) = e^{x - y^2}$.

A) $(0, \infty)$
B) $[0, \infty)$
C) $(-\infty, \infty)$
D) $(-\infty, 0)$
E) $(1, \infty)$
F) $[1, \infty)$
G) $(\sqrt{2}, \infty)$
H) $[\sqrt{2}, \infty)$

Answer: A)

174. Find the domain of the function $f(x,y) = \ln(x - y^2)$.

A) All points on or to the left of $x = y^2$
B) All points on or to the right of $x = y^2$
C) All points to the left of $x = y^2$
D) All points to the right of $x = y^2$
E) All points on or to the left of $x = 0$
F) All points on or to the right of $x = 0$
G) All points to the left of $x = 0$
H) All points in the x-y plane

Answer: D)

175. Find the range of the function $f(x,y) = \ln(x - y^2)$.

A) $(0, \infty)$
B) $[0, \infty)$
C) $(-\infty, \infty)$
D) $(-\infty, 0)$
E) $(1, \infty)$
F) $[1, \infty)$
G) $(\sqrt{2}, \infty)$
H) $[\sqrt{2}, \infty)$

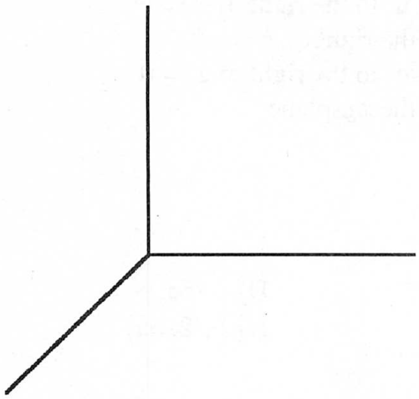
Answer: C)

176. Identify the graph of the function $f(x,y) = 3 - x^2 - y^2$.

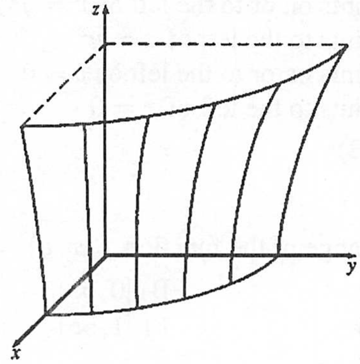
A) Cone
B) Paraboloid
C) Ellipsoid
D) Hyperboloid of one sheet
E) Hyperboloid of two sheets
F) Hyperbolic cylinder
G) Elliptic cylinder
H) Parabolic cylinder

Answer: B)

177. For the function $z = \sqrt{x^2 + y^2 - 1}$, sketch the portion of the surface in the first octant.



Answer:



178. Describe the vertical traces $x = k$ and $y = k$ and the horizontal traces $z = k$ for the function $f(x, y) = x^2 - y^2$.

Answer: $x = k$: $z = k^2 - y^2$, parabola; $y = k$: $z = x^2 - k^2$, parabola; $z = k$: $x^2 - y^2 = k$, hyperbola

9.7 Cylindrical and Spherical Coordinates

179. Convert $(1, \pi, 1)$ from cylindrical coordinates to rectangular coordinates.

A) $(1,1,1)$ B) $(-1,1,1)$ C) $(1,-1,1)$ D) $(1,1,-1)$
E) $(-1,0,1)$ F) $(0,-1,1)$ G) $(1,1,-1)$ H) $(0,1,1)$

Answer: E)

180. Convert $(2, \frac{\pi}{4}, 3)$ from cylindrical coordinates to rectangular coordinates.

A) $(1,1,3)$ B) $(0,2,3)$ C) $(2,0,3)$ D) $(\sqrt{2}, \sqrt{2}, 3)$
E) $(-1,-1,3)$ F) $(0,-2,3)$ G) $(-2,0,3)$ H) $(-\sqrt{2}, -\sqrt{2}, 3)$

Answer: D)

181. Convert $(1,1,1)$ from rectangular coordinates to cylindrical coordinates.

A) $(\sqrt{2}, \frac{\pi}{2}, 1)$ B) $(\sqrt{2}, \frac{\pi}{4}, 1)$ C) $(1, \frac{\pi}{2}, 1)$ D) $(1, \frac{\pi}{4}, 1)$
E) $(1, \frac{\pi}{2}, \sqrt{2})$ F) $(1, \frac{\pi}{4}, \sqrt{2})$ G) $(1, \frac{\pi}{2}, 2)$ H) $(1, \frac{\pi}{4}, 2)$

Answer: B)

182. Convert $(1, \sqrt{3}, \sqrt{3})$ from rectangular coordinates to cylindrical coordinates.

A) $(1, \frac{\pi}{3}, \sqrt{3})$ B) $(1, \frac{\pi}{6}, \sqrt{3})$ C) $(\sqrt{3}, \frac{\pi}{3}, 1)$ D) $(\sqrt{3}, \frac{\pi}{6}, 1)$
E) $(2, \frac{\pi}{3}, \sqrt{3})$ F) $(2, \frac{\pi}{6}, \sqrt{3})$ G) $(\sqrt{3}, \frac{\pi}{3}, 2)$ H) $(\sqrt{3}, \frac{\pi}{6}, 2)$

Answer: E)

183. Convert $(1, \pi, \pi)$ from spherical coordinates to rectangular coordinates.

A) $(0,0,-1)$ B) $(0,0,1)$ C) $(0,1,-1)$ D) $(1,0,-1)$
E) $(1,1,-1)$ F) $(1,0,1)$ G) $(0,1,1)$ H) $(1,1,1)$

Answer: A)

184. Convert $\left(1, \frac{\pi}{4}, \frac{\pi}{4}\right)$ from spherical coordinates to rectangular coordinates.

A) $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ B) $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ C) $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ D) $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}\right)$
E) $\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ F) $\left(\frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}\right)$ G) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ H) $(1, 0, 0)$

Answer: B)

185. Convert $\left(1, 1, \sqrt{2}\right)$ from rectangular coordinates to spherical coordinates.

A) $\left(\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{4}\right)$ B) $\left(2, \frac{\pi}{4}, \frac{\pi}{4}\right)$ C) $\left(\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4}\right)$ D) $\left(2, \frac{\pi}{2}, \frac{\pi}{4}\right)$
E) $\left(\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{2}\right)$ F) $\left(2, \frac{\pi}{4}, \frac{\pi}{2}\right)$ G) $\left(\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$ H) $\left(2, \frac{\pi}{2}, \frac{\pi}{2}\right)$

Answer: B)

186. Convert $\left(1, \sqrt{3}, 2\right)$ from rectangular coordinates to spherical coordinates.

A) $\left(2, \frac{\pi}{6}, \frac{\pi}{4}\right)$ B) $\left(4, \frac{\pi}{6}, \frac{\pi}{4}\right)$ C) $\left(\sqrt{2}, \frac{\pi}{6}, \frac{\pi}{4}\right)$ D) $\left(\sqrt{8}, \frac{\pi}{6}, \frac{\pi}{4}\right)$
E) $\left(2, \frac{\pi}{3}, \frac{\pi}{4}\right)$ F) $\left(4, \frac{\pi}{3}, \frac{\pi}{4}\right)$ G) $\left(\sqrt{2}, \frac{\pi}{3}, \frac{\pi}{4}\right)$ H) $\left(\sqrt{8}, \frac{\pi}{3}, \frac{\pi}{4}\right)$

Answer: H)

187. Describe the surface whose equation in cylindrical coordinates is $r = 3$.

A) Cylinder with vertical axis B) Cylinder with horizontal axis
C) Sphere D) Vertical plane or half-plane
E) Horizontal plane or half-plane F) Paraboloid
G) Cone or half-cone with vertical axis H) Cone or half-cone with horizontal axis

Answer: A)

188. Describe the surface whose equation in cylindrical coordinates is $z = 3$.

A) Cylinder with vertical axis B) Cylinder with horizontal axis
C) Sphere D) Vertical plane or half-plane
E) Horizontal plane or half-plane F) Paraboloid
G) Cone or half-cone with vertical axis H) Cone or half-cone with horizontal axis

Answer: E)

189. Describe the surface whose equation in cylindrical coordinates is $\theta = 3$.

A) Cylinder with vertical axis B) Cylinder with horizontal axis
C) Sphere D) Vertical plane or half-plane
E) Horizontal plane or half-plane F) Paraboloid
G) Cone or half-cone with vertical axis H) Cone or half-cone with horizontal axis

Answer: D)

190. Describe the surface whose equation in cylindrical coordinates is $z = 4r$.

A) Cylinder with vertical axis B) Cylinder with horizontal axis
C) Sphere D) Vertical plane or half-plane
E) Horizontal plane or half-plane F) Paraboloid
G) Cone or half-cone with vertical axis H) Cone or half-cone with horizontal axis

Answer: G)

191. Describe the surface whose equation in cylindrical coordinates is $z = r^2$.

A) Cylinder with vertical axis B) Cylinder with horizontal axis
C) Sphere D) Vertical plane or half-plane
E) Horizontal plane or half-plane F) Paraboloid
G) Cone or half-cone with vertical axis H) Cone or half-cone with horizontal axis

Answer: F)

192. Describe the surface whose equation in spherical coordinates is $\rho = 3$.

A) Cylinder with vertical axis
B) Cylinder with horizontal axis
C) Sphere
D) Vertical plane or half-plane
E) Horizontal plane or half-plane
F) Paraboloid
G) Cone or half-cone with vertical axis
H) Cone or half-cone with horizontal axis

Answer: C)

193. Describe the surface whose equation in spherical coordinates is $\theta = 3$.

A) Cylinder with vertical axis
B) Cylinder with horizontal axis
C) Sphere
D) Vertical plane or half-plane
E) Horizontal plane or half-plane
F) Paraboloid
G) Cone or half-cone with vertical axis
H) Cone or half-cone with horizontal axis

Answer: D)

194. Describe the surface whose equation in spherical coordinates is $\phi = 3$.

A) Cylinder with vertical axis
B) Cylinder with horizontal axis
C) Sphere
D) Vertical plane or half-plane
E) Horizontal plane or half-plane
F) Paraboloid
G) Cone or half-cone with vertical axis
H) Cone or half-cone with horizontal axis

Answer: G)

195. If $P = (1, 1, \sqrt{2})$ in rectangular coordinates, find the spherical coordinates of P.

Answer: $(2, \frac{\pi}{4}, \frac{\pi}{4})$

196. If $Q = (1, \frac{\pi}{2}, 3)$ in cylindrical coordinates, find rectangular coordinates of Q.

Answer: $(0, 1, 3)$

197. Convert the point $(1, -\sqrt{3}, -2)$ to cylindrical and spherical coordinates.

Answer: Cylindrical coordinates: $(2, \frac{-\pi}{3}, -2)$; spherical coordinates: $(2\sqrt{2}, \frac{-\pi}{3}, \frac{3\pi}{4})$

198. Convert the point $(0, -5, 0)$ to cylindrical and spherical coordinates.

Answer: Cylindrical coordinates: $(5, \frac{3\pi}{2}, 0)$; spherical coordinates: $(5, \frac{3\pi}{2}, \frac{\pi}{2})$

199. Find rectangular and cylindrical equations for the surface whose equation in spherical coordinates is $\rho = 3$. Describe the surface.

Answer: Rectangular: $x^2 + y^2 + z^2 = 9$; cylindrical: $r^2 + z^2 = 9$

200. Find rectangular and spherical equations for the surface whose equation in cylindrical coordinates is $\theta = \frac{\pi}{4}$. Describe the surface.

Answer: Rectangular: $x = y$;spherical: $\theta = \frac{\pi}{4}$

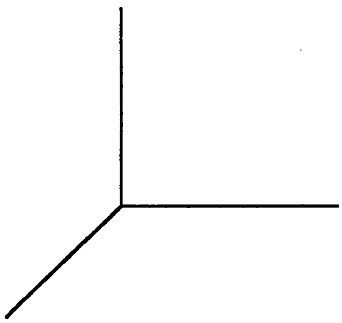
201. Find cylindrical and spherical equations for the surface whose equation in rectangular coordinates is $x = 2$. Describe the surface.

Answer: Cylindrical: $r \cos \theta = 2$;spherical: $\rho \sin \phi \cos \theta = 2$

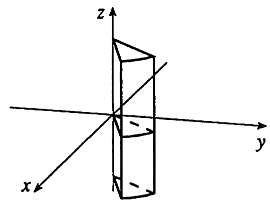
202. Describe in words the solid represented in spherical coordinates by the inequality $2 \leq \rho \leq 5$.

Answer: solid sphere of radius 5 centered at origin, with hollow ball inside of radius 2

203. Describe in words or sketch the solid represented in cylindrical coordinates by the inequalities $\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$, $0 \le r \le 2$, $-1 \le z \le 1$.



Answer:



wedge of a circular cylinder with axis the z-axis

204. Describe the surface whose equation in cylindrical coordinates is $\theta = \frac{\pi}{2}$.

Answer: Vertical plane

205. Describe the surface whose equation in cylindrical coordinates is $r = \sin\theta$.

Answer: Circular cylinder with axis the z-axis

206. Describe the surface whose equation in spherical coordinates is $\sin\phi = \cos\phi$.

Answer: Half-cone with axis the z-axis

10 Vector Functions

10.1 Vector Functions and Space Curves

1. Find $\lim_{t\to 0} \langle 2t, \cos t, e^t \rangle$.

A) $\langle 2,1,0\rangle$ B) $\langle 0,0,1\rangle$ C) $\langle 2,0,1\rangle$ D) $\langle 2,1,1\rangle$
E) $\langle 1,1,1\rangle$ F) $\langle 1,1,0\rangle$ G) $\langle 1,0,1\rangle$ H) $\langle 0,1,1\rangle$

Answer: H)

2. Find $\lim_{t\to 1} \left[(2t+1)\,\mathbf{i} + e^t\,\mathbf{j} + \arctan t\,\mathbf{k}\right]$.

A) $\mathbf{i}+e\mathbf{j}+\frac{\pi}{4}\mathbf{k}$ B) $\mathbf{i}+e\mathbf{j}+\frac{\pi}{2}\mathbf{k}$ C) $\mathbf{i}+\frac{\pi}{4}\mathbf{k}$ D) $\mathbf{i}+\frac{\pi}{2}\mathbf{k}$
E) $3\mathbf{i}+e\mathbf{j}+\frac{\pi}{4}\mathbf{k}$ F) $3\mathbf{i}+e\mathbf{j}+\frac{\pi}{2}\mathbf{k}$ G) $3\mathbf{i}+\frac{\pi}{4}\mathbf{k}$ H) $3\mathbf{i}+\frac{\pi}{2}\mathbf{k}$

Answer: E)

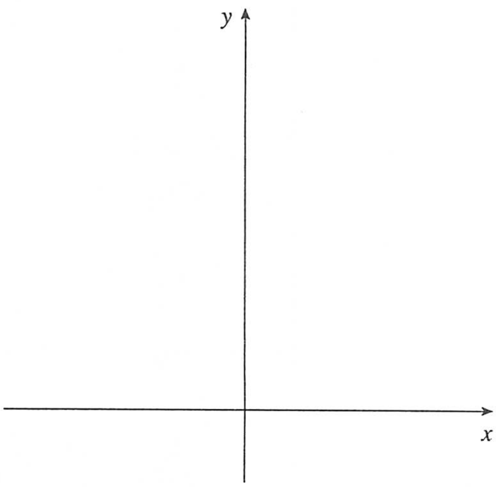
3. Let $\mathbf{r}(t) = 2t^2\,\mathbf{i} + \dfrac{1}{3t^3}\,\mathbf{j}$. Find corresponding parametric equations.

Answer: $x = 2t^2,\ y = \dfrac{1}{3t^3}$

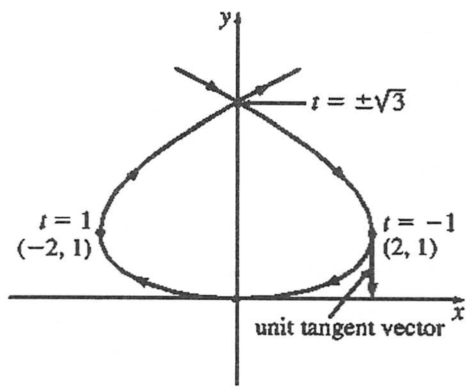
4. The position of a particle at time t is given parametrically by $y = t^2$ and $x = \frac{1}{3}\left(t^3 - 3t\right)$. Show that the particle crosses the y-axis three times.

Answer: The particle crosses the y-axis when $x = 0$. That occurs at $t = -\sqrt{3}$, 0, and $\sqrt{3}$

5. Consider the curve in the xy-plane defined parametrically by $x = t^3 - 3t$, $y = t^2$, $z = 0$. Sketch a rough graph of the curve.



Answer:



6. Find a space curve which parametrizes the intersection of the paraboloid $z = -x^2 - y^2$ and the plane $y = x$.

Answer: $\langle t, t, -2t^2\rangle$

7. The graphs of which three of the following four vector functions lie along the line $y = 4 - x$?

A) $x = 6 - t^2, y = t^2 - 2$
B) $x = \cos^2 t, y = \sin^2 t + 3$
C) $x = 2e^t, y = 2e^t - 8$
D) $x3 = \ln(1/t), y = \ln(e^4 t)$

Answer: A), B), and D)

8. Consider the paraboloid $z = x^2 + y^2$. Which of the following space curves have as their range points lying on this paraboloid?

A) $\mathbf{r}(t) = \langle t\sin t, t\cos t, t\rangle, t \in \mathbb{R}$
B) $\mathbf{r}(t) = \langle t^2, t\sin t, t\cos t\rangle, t \in \mathbb{R}$
C) $\mathbf{r}(t) = \langle t\cos t, t\sin t, t^2\rangle, t \in \mathbb{R}$
D) $\mathbf{r}(t) = \langle \sqrt{t}\sin t, t, \sqrt{t}\cos t\rangle, t > 0$
E) $\mathbf{r}(t) = \langle \sqrt{t}\sin t, \sqrt{t}\cos t, t\rangle, t > 0$

Answer: C) and E)

9. Find a space curve which parametrizes the intersection of the paraboloid $z = x^2 + y^2$ and the plane $y - z = 0$.

Answer: $\mathbf{r}(t) = \langle \frac{1}{2}\cos t, \frac{1}{2} + \frac{1}{2}\sin t, \frac{1}{2} + \frac{1}{2}\sin t\rangle, 0 \le t \le 2\pi$

10.2 Derivatives and Integrals of Vector Functions

10. Find the derivative of the vector function $\mathbf{r}(t) = \langle t, 1/t, e^t\rangle$ when $t = 1$.

A) $\langle 0,1,1\rangle$
B) $\langle 1,0,1\rangle$
C) $\langle 1,1,e\rangle$
D) $\langle 0,0,e\rangle$
E) $\langle -1,1,e\rangle$
F) $\langle 1,-1,e\rangle$
G) $\langle 1,1,-1\rangle$
H) $\langle 1,1,1\rangle$

Answer: F)

11. Find the derivative of the vector function $\mathbf{r}(t) = t\,\mathbf{i} + \sin t\,\mathbf{j}$ when $t = 0$.

A) $\mathbf{i}$
B) $\mathbf{j}$
C) $-\mathbf{i}$
D) $-\mathbf{j}$
E) $-\mathbf{i}+\mathbf{j}$
F) $\mathbf{i}-\mathbf{j}$
G) $-\mathbf{i}-\mathbf{j}$
H) $\mathbf{i}+\mathbf{j}$

Answer: H)

12. Find the tangent vector $\mathbf{r}'(t)$ of the function $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$ when $t = 1$.

A) $\langle 1,0,0\rangle$
B) $\langle 0,0,1\rangle$
C) $\langle 0,1,0\rangle$
D) $\langle 1,2,3\rangle$
E) $\langle 0,2,3\rangle$
F) $\langle 1,2,6\rangle$
G) $\langle 0,1,1\rangle$
H) $\langle 1,1,1\rangle$

Answer: D)

13. Find the tangent vector $\mathbf{r}'(t)$ of the function $\mathbf{r}(t) = \sin t\,\mathbf{i} - \cos t\,\mathbf{j}$ when $t = \frac{\pi}{3}$.

A) $\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$
B) $\frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$
C) $-\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$
D) $-\frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$
E) $\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$
F) $\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$
G) $-\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$
H) $-\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$

Answer: A)

14. Find the unit tangent vector $\mathbf{T}(t)$ to the curve $\mathbf{r}(t) = \langle \sin t, 2t, t^2\rangle$ when $t = 0$.

A) $\left\langle -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, 0\right\rangle$
B) $\left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, 0\right\rangle$
C) $\left\langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right\rangle$
D) $\left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right\rangle$
E) $\left\langle -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}, 0\right\rangle$
F) $\left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}, 0\right\rangle$
G) $\left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right\rangle$
H) $\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right\rangle$

Answer: H)

15. Find the unit tangent vector $\mathbf{T}(t)$ to the curve $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$ when $t = 0$.

A) $\mathbf{i}$ B) $\mathbf{j}$ C) $\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ D) $\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$
E) $\frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$ F) $\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}$ G) $\frac{2}{\sqrt{5}}\mathbf{i} - \frac{1}{\sqrt{5}}\mathbf{j}$ H) $\frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}$
Answer: A)

16. Evaluate the integral $\int_0^1 \left(t\mathbf{i} - 2t^2\mathbf{j}\right) dt$.

A) $\mathbf{i}$ B) $\mathbf{j}$ C) $\mathbf{i} - \mathbf{j}$ D) $\mathbf{i} - \frac{1}{2}\mathbf{j}$
E) $\frac{1}{2}\mathbf{i} - \mathbf{j}$ F) $\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$ G) $\frac{1}{2}\mathbf{i}$ H) $\frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j}$
Answer: H)

17. Find $\mathbf{r}(1)$ if $\mathbf{r}'(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j}$ and $\mathbf{r}(0) = \mathbf{i}$.

A) $\frac{1}{3}\mathbf{i} + \frac{1}{4}\mathbf{j}$ B) $\frac{4}{3}\mathbf{i} + \frac{1}{4}\mathbf{j}$ C) $\frac{2}{3}\mathbf{i} + \frac{1}{4}\mathbf{j}$ D) $\mathbf{i} + \frac{1}{4}\mathbf{j}$
E) $\frac{1}{3}\mathbf{i} + \frac{3}{4}\mathbf{j}$ F) $\frac{4}{3}\mathbf{i} + \frac{3}{4}\mathbf{j}$ G) $\frac{2}{3}\mathbf{i} + \frac{3}{4}\mathbf{j}$ H) $\mathbf{i} + \frac{3}{4}\mathbf{j}$
Answer: B)

18. Find the unit tangent vector to the curve $\mathbf{r}(t) = \left\langle e^{2t}\cos t, e^{2t}\sin t, e^{2t}\right\rangle$ at the point where $t = \frac{\pi}{2}$.

A) $\left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right\rangle$ B) $\left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right\rangle$ C) $\left\langle -\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right\rangle$ D) $\left\langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right\rangle$
E) $\left\langle \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}}\right\rangle$ F) $\left\langle -\frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right\rangle$ G) $\left\langle \frac{1}{\sqrt{14}}, -\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right\rangle$ H) $\left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right\rangle$
Answer: C)

19. Let $\mathbf{r}(t) = \left\langle t^3, t^4 + 3\right\rangle$. At the point $(1, 4)$, the unit tangent vector is

A) $3\mathbf{i} + 4\mathbf{j}$ B) $\frac{1}{\sqrt{3}}\mathbf{i} - \frac{4}{3\sqrt{3}}\mathbf{j}$ C) $\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$ D) $\frac{2}{\sqrt{2}}\mathbf{i} - \frac{2}{\sqrt{2}}\mathbf{j}$ E) $-\frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$
Answer: C)

20. Let $\mathbf{r}(t) = 2t^2\,\mathbf{i} + \dfrac{1}{3t^3}\mathbf{j}$. Find $\mathbf{r}'(t)$.

Answer: $\mathbf{r}'(t) = 4t\,\mathbf{i} - \dfrac{1}{t^4}\mathbf{j}$

21. Let $\mathbf{u}(t) = 2t\,\mathbf{i} + \sin t\,\mathbf{j} - \cos t\,\mathbf{k}$ and $\mathbf{v}(t) = \mathbf{i} + t^2\,\mathbf{j} - t\,\mathbf{k}$. Find $\dfrac{d}{dt}\left[\mathbf{u}(t) \times \mathbf{v}(t)\right]$.

Answer: $\dfrac{d}{dt}\left[\mathbf{u}(t) \times \mathbf{v}(t)\right] = \left(t\cos t - \sin t - t^2\sin t\right)\mathbf{i} + (4t + \sin t)\,\mathbf{j} + \left(6t^2 - \cos t\right)\mathbf{k}$

22. Find an expression for $\dfrac{d}{dt}\left[(\mathbf{u}(t) \times \mathbf{v}(t)) \cdot \mathbf{w}(t)\right]$.

Answer: $(\mathbf{u}'(t) \times \mathbf{v}(t)) \cdot \mathbf{w}(t) + (\mathbf{u}(t) \times \mathbf{v}'(t)) \cdot \mathbf{w}(t) + (\mathbf{u}(t) \times \mathbf{v}(t)) \cdot \mathbf{w}'(t)$

23. Let $x = t^3$ and $y = 3t^2$. Find the point on the curve closest to $(0, 3)$.

Answer: $\left(\pm\left(\sqrt{15} - 3\right)^{3/2}, 3\sqrt{15} - 9\right)$

24. Find the point(s) on the curve described by the vector function $\mathbf{r}(t) = \left\langle t^2 + 2t, t^3 - 12t\right\rangle$ where the tangent vector is horizontal or vertical.

Answer: Horizontal at $(0, 16)$ and $(8, -16)$, vertical at $(-1, 11)$

25. Let $\mathbf{r}(t) = \langle \cos 2t, \sin 2t, 3\rangle$. Compute the tangent vector $\mathbf{r}'(t)$ and show that it is always orthogonal to $\mathbf{r}(t)$.

Answer: $\mathbf{r}'(t) = \langle -2\sin 2t, 2\cos 2t, 0\rangle$; $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$

26. Given $\mathbf{r}(t) = \langle t^2, t^6, t \rangle$, find the unit tangent vector at $t = 1$ and parametric equations for the tangent line through the tip of $\mathbf{r}(1)$.

Answer: $\mathbf{T}(1) = \frac{1}{\sqrt{41}} \langle 2, 6, 1 \rangle$; $x = 1 + 2t$, $y = 1 + 6t$, $z = 1 + t$

27. If $\mathbf{u}(t) = \langle -\sqrt{t} \sin t, t, t^{2/3} \rangle$ and $\mathbf{v}(t) = \langle -\sqrt{t} \sin t, \cos^2 t, -t^{1/3} \rangle$, compute $\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t))$ and $\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{u}(t))$.

Answer: $\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = 0$, $\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{u}(t)) = \sin^2 t + 2t \sin t \cos t + 2t + \frac{4}{3} t^{1/3}$

10.3 Arc Length and Curvature

28. Find the length of the curve $\mathbf{r}(t) = \langle 2t, \sin t, \cos t \rangle$, $0 \le t \le 2\pi$.

A) $2\pi\sqrt{2}$ B) $\pi\sqrt{10}$ C) $2\pi\sqrt{3}$ D) $\pi\sqrt{14}$
E) 4π F) $3\pi\sqrt{2}$ G) $2\pi\sqrt{5}$ H) $\pi\sqrt{22}$

Answer: G)

29. Find the length of the curve $\mathbf{r}(t) = \langle t, 2t, 2t \rangle$, $0 \le t \le 2$.

A) 4 B) 6 C) 8 D) 10
E) 12 F) 14 G) 16 H) 18

Answer: B)

30. Find the unit tangent vector $\mathbf{T}(t)$ to the curve $\mathbf{r}(t) = \langle \sin t, t, \cos t \rangle$ when $t = 0$.

A) $\langle -1, 0, 0 \rangle$ B) $\langle 0, 0, -1 \rangle$ C) $\langle 1, 0, 0 \rangle$ D) $\langle 0, 0, 1 \rangle$
E) $\left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ F) $\left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$ G) $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$ H) $\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$

Answer: G)

31. Find the unit tangent vector $\mathbf{T}(t)$ to the curve $\mathbf{r}(t) = \langle t, 2t, t^2 \rangle$ when $t = 1$.

A) $\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle$ B) $\langle 1, 0, 0 \rangle$ C) $\langle 0, 1, 0 \rangle$ D) $\langle 0, 0, 1 \rangle$
E) $\left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ F) $\left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$ G) $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$ H) $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

Answer: A)

32. Find the unit normal vector $\mathbf{N}(t)$ to the curve $\mathbf{r}(t) = \langle \sin t, t, \cos t \rangle$ when $t = 0$.

A) $\langle -1, 0, 0 \rangle$ B) $\langle 0, 0, -1 \rangle$ C) $\langle 1, 0, 0 \rangle$ D) $\langle 0, 0, 1 \rangle$
E) $\left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ F) $\left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$ G) $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$ H) $\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$

Answer: B)

33. Find the unit normal vector $\mathbf{N}(t)$ to the curve $\mathbf{r}(t) = \langle t, 2t, t^2 \rangle$ when $t = 1$.

A) $\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle$ B) $\langle 1, 0, 0 \rangle$ C) $\langle 0, 1, 0 \rangle$ D) $\left\langle -\frac{2}{3\sqrt{5}}, -\frac{4}{3\sqrt{5}}, \frac{5}{3\sqrt{5}} \right\rangle$
E) $\left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ F) $\left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$ G) $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$ H) $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

Answer: D)

34. Find the curvature κ of the curve $y = 2x^2$ at $x = 0$.

A) 0	B) $\frac{1}{8}$	C) $\frac{1}{4}$	D) $\frac{1}{2}$
E) 1	F) 2	G) 4	H) 8

Answer: G)

35. Find the curvature κ of the curve $\mathbf{r}(t) = \langle 2\sin t, 2, 2\cos t\rangle$ when $t = \frac{\pi}{2}$.

A) 0	B) $\frac{1}{8}$	C) $\frac{1}{4}$	D) $\frac{1}{2}$
E) 1	F) 2	G) 4	H) 8

Answer: D)

36. Find the curvature κ of the curve $\mathbf{r}(t) = \langle \sin 2t, 3t, \cos 2t\rangle$ when $t = \frac{\pi}{2}$.

A) $\frac{3}{13}$	B) $\frac{4}{13}$	C) $\frac{6}{13}$	D) $\frac{8}{13}$
E) $\frac{1}{3}$	F) $\frac{4}{9}$	G) $\frac{2}{3}$	H) $\frac{8}{9}$

Answer: B)

37. Find the length of the curve $\mathbf{r}(t) = \left\langle \sin 2t, \cos 2t, 2t^{3/2}\right\rangle$, $0 \le t \le 1$.

A) $\frac{2}{27}\left(13\sqrt{13} - 8\right)$	B) $\frac{13}{9}$	C) $\frac{13\sqrt{13}-6}{27}$	D) $\frac{16}{9}$
E) $\frac{10}{27}\left(\sqrt{13} - 2\right)$	F) $\frac{19}{9}$	G) $\frac{4}{9}\left(7\sqrt{7} - 2\right)$	H) $\frac{22}{9}$

Answer: A)

38. Find the unit tangent and the unit normal to the graph of the vector function $\mathbf{r}(t) = \left\langle t^2 - 2, 2t - t^3\right\rangle$ at $t = 1$.

Answer: Unit tangent: $\frac{1}{\sqrt{5}}\langle 2, -1\rangle$; unit normal: $\frac{1}{\sqrt{5}}\langle 1, 2\rangle$

39. Find the curvature of $\mathbf{r}(t) = \left\langle t, t^2, t^3\right\rangle$ at $t = 2$.

Answer: $\kappa = \frac{2\sqrt{181}}{161\sqrt{161}}$

40. Consider the curve in the xy-plane defined parametrically by $x = t^3 - 3t$, $y = t^2$, $z = 0$. Find the unit tangent vector at $t = -1$.

Answer: $\mathbf{T}(t) = -\mathbf{j}$

41. Consider the curve in the xy-plane defined parametrically by $x = t^3 - 3t$, $y = t^2$, $z = 0$. Find all points where the curve has vertical or horizontal tangents.

Answer: Horizontal tangent at $(0, 0)$; vertical tangents at $(2, 1)$ and $(-2, 1)$

42. Suppose C is the curve given by the vector function $\mathbf{r}(t) = \left\langle t, t^2, 1 - t^2\right\rangle$. Find the unit tangent vector, the unit normal vector, and the curvature of C at the point where $t = 1$.

Answer: $\mathbf{T}(1) = \frac{1}{3}\langle 1, 2, -2\rangle$, $\mathbf{N}(1) = \frac{1}{\sqrt{18}}\langle -4, 1, -1\rangle$, $\kappa = \frac{2\sqrt{2}}{27}$

43. Find the length of arc along the curve $\mathbf{x}(t) = \langle 5t, \sin t, \cos t\rangle$ from $t = 0$ to $t = 2$.

Answer: $2\sqrt{26}$

44. Find the length of the circular helix described by $x = 2\cos t$, $y = 2\sin t$, $z = \sqrt{5}t$, $0 \le t \le 2\pi$.

Answer: 6π

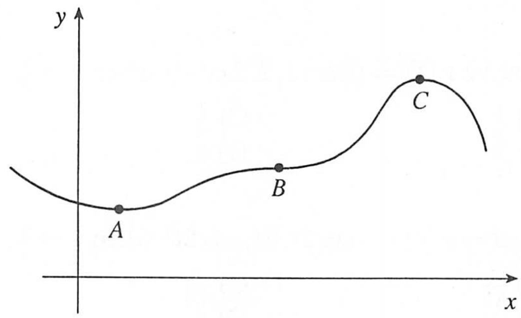
45. Find the center of the osculating circle of the parabola $y = x^2$ at the origin.

Answer: $\left(0, \frac{1}{2}\right)$

46. Find the center of the osculating circle of the curve described by $x = 4\sin t$, $y = 3t$, $z = 4\cos t$ at $(0, 0, 4)$.

Answer: $(0, 0, -2.25)$

47. Consider $\mathbf{r}(t)$, the vector function describing the curve shown below. Put the curvatures at A, B, and C in order from smallest to largest.



Answer: B, A, C

48. Show that if $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ are parallel at some point on the curve described by $\mathbf{r}(t)$, then the curvature at that point is 0. Give an example of a curve $\mathbf{r}(t)$ for which $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ are *always* parallel.

Answer: $\mathbf{r}'$ and $\mathbf{r}''$ are parallel $\Rightarrow$ $\mathbf{r}' \times \mathbf{r}'' = 0$ $\Rightarrow$ $\kappa = \dfrac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = 0$. Example: $\mathbf{r} = \langle t, t, t \rangle$

49. Consider $y = \sin x$, $-\pi < x < \pi$. Determine graphically where the curvature is maximal and minimal.

Answer: Minimal at $x = 0$, maximal at $x = \pm\frac{\pi}{2}$

50. Find the equation of the plane normal to $\mathbf{r}(t) = \left\langle e^t \sin \frac{\pi}{2}t, e^t \cos \frac{\pi}{2}t, t^2 \right\rangle$ when $t = 1$.

Answer: $ex - \frac{\pi}{2}y + 2z = e^2 + 2$.

10.4 Motion in Space

51. Let the position function of a particle be $\mathbf{r}(t) = \langle t^2, 2t, e^t \rangle$. Find the velocity of the particle when $t = 1$.

A) $\langle 2, 2, 1 \rangle$	B) $\langle 2, 2, e \rangle$	C) $\langle 2, 0, 1 \rangle$	D) $\langle 2, 0, e \rangle$
E) $\langle 1, 1, 1 \rangle$	F) $\langle 1, 1, e \rangle$	G) $\langle 1, 0, 1 \rangle$	H) $\langle 1, 0, e \rangle$

Answer: B)

52. Let the position function of a particle be $\mathbf{r}(t) = \langle t^2, 2t, e^t \rangle$. Find the acceleration of the particle when $t = 0$.

A) $\langle 2, 2, 1 \rangle$	B) $\langle 2, 2, e \rangle$	C) $\langle 2, 0, 1 \rangle$	D) $\langle 2, 0, e \rangle$
E) $\langle 1, 1, 1 \rangle$	F) $\langle 1, 1, e \rangle$	G) $\langle 1, 0, 1 \rangle$	H) $\langle 1, 0, e \rangle$

Answer: C)

53. Let the position function of a particle be $\mathbf{r}(t) = \langle t^2, t, t^2 \rangle$. Find the speed of the particle when $t = 1$.

A) 1	B) 2	C) 3	D) 4
E) $\sqrt{8}$	F) $\sqrt{10}$	G) $\sqrt{13}$	H) $\sqrt{14}$

Answer: C)

54. Let the acceleration of a particle be $\mathbf{a}(t) = t\mathbf{i}$, and let its velocity when $t = 0$ be $\mathbf{v}(0) = \mathbf{i} + \mathbf{k}$. Find its velocity when $t = 1$.

A) $\frac{1}{2}\mathbf{i} + \mathbf{k}$ B) $2\mathbf{i} + \mathbf{k}$ C) $3\mathbf{i} + \mathbf{k}$ D) $\frac{3}{2}\mathbf{i} + \mathbf{k}$
E) $\frac{1}{2}\mathbf{i}$ F) $2\mathbf{i}$ G) $3\mathbf{i}$ H) $\frac{3}{2}\mathbf{i}$

Answer: D)

55. Let the acceleration of a particle be $\mathbf{a}(t) = t\mathbf{i}$, and let its velocity when $t = 0$ be $\mathbf{v}(0) = \mathbf{i} + \mathbf{k}$. Find its speed when $t = 2$.

A) $\sqrt{5}$ B) $\sqrt{6}$ C) $\sqrt{7}$ D) $\sqrt{8}$
E) 3 F) $\sqrt{10}$ G) $\sqrt{11}$ H) $\sqrt{12}$

Answer: F)

56. Let the velocity of a particle be $\mathbf{v}(t) = \mathbf{i} + t\mathbf{j}$, and let its position when $t = 0$ be $\mathbf{r}(0) = \mathbf{j} + 2\mathbf{k}$. Find its position when $t = 2$.

A) $3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ B) $2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ C) $3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ D) $3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$
E) $3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$ F) $2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$ G) $4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ H) $4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$

Answer: B)

57. Let the position function of a particle be $\mathbf{r}(t) = \langle t^2, 1 - 2t, t \rangle$. Find the value of t at which its speed is smallest.

A) -2 B) $-\frac{3}{2}$ C) -1 D) $-\frac{1}{2}$
E) 0 F) $\frac{1}{2}$ G) 1 H) $\frac{3}{2}$

Answer: E)

58. Let the position function of a particle be $\mathbf{r}(t) = \langle t^2, 1 - 2t, t \rangle$. Find the smallest value of its speed.

A) 0 B) 1 C) $\sqrt{2}$ D) $\sqrt{3}$
E) 2 F) $\sqrt{5}$ G) $\sqrt{6}$ H) $\sqrt{7}$

Answer: F)

59. Let the position function of a particle be $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$. Find the tangential component of the acceleration vector when $t = 1$.

A) $\frac{1}{\sqrt{5}}$ B) $\frac{2}{\sqrt{5}}$ C) $\frac{3}{\sqrt{5}}$ D) $\frac{4}{\sqrt{5}}$
E) $\sqrt{5}$ F) $\frac{6}{\sqrt{5}}$ G) $\frac{7}{\sqrt{5}}$ H) $\frac{8}{\sqrt{5}}$

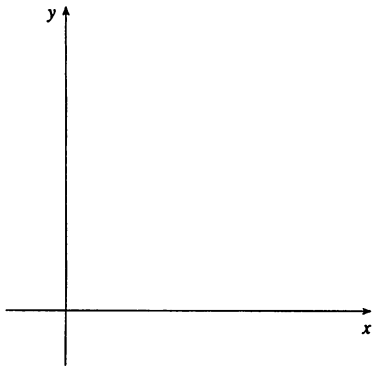
Answer: D)

60. Let the position function of a particle be $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$. Find the normal component of the acceleration vector when $t = 1$.

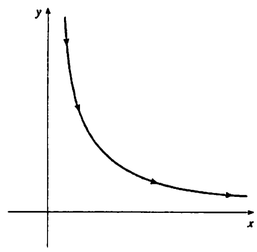
A) $\frac{1}{\sqrt{5}}$ B) $\frac{2}{\sqrt{5}}$ C) $\frac{3}{\sqrt{5}}$ D) $\frac{4}{\sqrt{5}}$
E) $\sqrt{5}$ F) $\frac{6}{\sqrt{5}}$ G) $\frac{7}{\sqrt{5}}$ H) $\frac{8}{\sqrt{5}}$

Answer: B)

61. Suppose a particle moves in the plane according to the vector-valued function $\mathbf{r}(t) = 2e^t\mathbf{i} + e^{-t}\mathbf{j}$, where t represents time. Find $\mathbf{v}(t)$, $|\mathbf{v}(t)|$, and $\mathbf{a}(t)$, and sketch a graph showing the path taken by the particle indicating the direction of motion.



Answer: $\mathbf{v}(t) = 2e^t\mathbf{i} - e^{-t}\mathbf{j}$, $|\mathbf{v}(t)| = \sqrt{4e^{2t} + e^{-2t}}$,
$\mathbf{a}(t) = 2e^t\mathbf{i} + e^{-t}\mathbf{j}$



62. An object's position vector at time t is given by the vector-valued function $\mathbf{r}(t) = \langle t^3 - t, 2t^2 + t\rangle$. At $t = 2$, find its velocity vector and its speed.
Answer: $\mathbf{v}(2) = \langle 11, 9\rangle$; speed is $\sqrt{202}$

63. Suppose a particle is moving in the xy-plane so that its position vector at time t is given by $\mathbf{r}(t) = \langle t^3 - t, t - t^2\rangle$. Find the velocity, speed, and acceleration of the particle at time $t = 2$.
Answer:$\mathbf{v}(2) = \langle 11, -3\rangle$; speed$= \sqrt{130}$; $\mathbf{a}(2) = \langle 12, -2\rangle$

64. Let $\mathbf{a}(t)$, $\mathbf{v}(t)$, and $\mathbf{r}(t)$ denote the acceleration, velocity, and position at time t of an object moving in the xy-plane. Find $\mathbf{r}(t)$, given that $\mathbf{a}(t) = \langle e^{2t} + 2t, e^{2t} - 3\rangle$, $\mathbf{v}(0) = \langle \frac{3}{2}, \frac{7}{2}\rangle$, and $\mathbf{r}(0) = \langle \frac{5}{4}, \frac{9}{4}\rangle$.
Answer: $\mathbf{r}(t) = \langle \frac{1}{4}e^{2t} + \frac{1}{3}t^3 + t + 1, \frac{1}{4}e^{2t} - \frac{3}{2}t^2 + 3t + 2\rangle$

65. A paper carrier is traveling 60 miles per hour down a straight road in the direction of the vector $\mathbf{i}$ when he throws a paper out the car window with a velocity (relative to the car) in the direction of $\mathbf{j}$ and of magnitude 10 miles per hour.
(a) Find the velocity of the paper relative to the ground when the paper carrier releases it.
(b) Find the speed of the paper at that time.
Answer: (a) $\mathbf{v} = 60\mathbf{i} + 10\mathbf{j}$; (b) $\sqrt{3700} \approx 60.83$ miles per hour

66. A particle has velocity given by $\mathbf{v}(t) = e^t\mathbf{i} + 2\mathbf{j}$. At time $t = 0$ it is at the origin. Find
(a) its speed
(b) its distance from the origin at time $t = 1$.
Answer: (a) $\sqrt{e^2 + 4}$; (b) $\sqrt{e^2 - 2e + 5}$

67. A cannon sits on top of a vertical tower 264 feet tall. It fires a cannonball at 80 ft/s. If the barrel of the cannon is elevated 30 degrees from the horizontal, find how far from the base of the tower the cannonball will land (assuming the ground around the tower is level).
Answer: $220\sqrt{3}$ ft

68. A person is standing 80 feet from a tall cliff. She throws a rock at 80 feet per second at an angle of 45° from the horizontal. Neglecting air resistance and discounting the height of the person, how far up the cliff does it hit?
Answer: 48 ft

69. If $\mathbf{r}(t) = (t^2+3)\,\mathbf{i} + (2t^2 - 3t + 5)\,\mathbf{j}$ describes the motion of a particle, find
(a) the particle's velocity when $t = 3$
(b) the particle's speed when $t = 3$
(c) the particle's acceleration when $t = 3$.
Answer: (a) $\mathbf{v}(3) = 6\mathbf{i} + 9\mathbf{j}$; (b) $3\sqrt{13}$; (c) $\mathbf{a}(3) = 2\mathbf{i} + 4\mathbf{j}$

70. For $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$, find $\mathbf{a}_T$ and $\mathbf{a}_N$, the tangential and normal components of acceleration.
Answer: $\mathbf{a}_T = \dfrac{4t}{\sqrt{4t^2+1}}$; $\mathbf{a}_N = \dfrac{-2}{\sqrt{4t^2+1}}$

71. Suppose the curve C is given by the following vector function: $\mathbf{r}(t) = \langle \sin t, \cos t, t^2 \rangle$, where t is between 0 and 1.
(a) Find the velocity vector at $t = \frac{\pi}{4}$.
(b) Find the acceleration vector at $t = \frac{\pi}{4}$.
Answer: (a) $\mathbf{v}\left(\frac{\pi}{4}\right) = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{\pi}{2} \right\rangle$; (b) $\mathbf{a}\left(\frac{\pi}{4}\right) = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 2 \right\rangle$

72. A particle is moving along the curve described by the parametric equations $x = 5t$, $y = 2t^3$, $z = \frac{3}{5}t^5$. Determine the velocity and acceleration vectors as well as the speed of the particle when $t = 3$.
Answer: $\mathbf{v}(3) = 5\mathbf{i} + 54\mathbf{j} + 243\mathbf{k}$; $\mathbf{a}(3) = 36\mathbf{j} + 324\mathbf{k}$; $|\mathbf{v}(3)| = \sqrt{61,990}$

73. Suppose that a dove is flying so that its acceleration vector at time t is given by $\mathbf{a}(t) = \langle t^4, t^3, t^2 \rangle$, its initial velocity is $\mathbf{v}(0) = \langle 1, 2, 3 \rangle$, and its initial displacement is $\mathbf{s}(0) = \langle 7, 6, 5 \rangle$. Find the position vector $\mathbf{s}(t)$ at time t.
Answer: $\mathbf{s}(t) = \left\langle \frac{1}{30}t^6 + t + 7, \frac{1}{20}t^5 + 2t + 6, \frac{1}{12}t^4 + 3t + 5 \right\rangle$

74. If $\mathbf{r}(t) = t^2\,\mathbf{i} + 3t\,\mathbf{j} + e^t\,\mathbf{k}$, find the acceleration vector and the tangential component of the acceleration vector.
Answer: $\mathbf{a}(t) = 2\,\mathbf{i} + e^t\,\mathbf{k}$; $a_T = \dfrac{4t + e^{2t}}{\sqrt{4t^2 + 9 + e^{2t}}}$

75. Let $\mathbf{r}(t) = \langle 5t, \sin 3t, \cos 3t \rangle$. Show that the velocity vector is perpendicular to the acceleration vector.
Answer: $\mathbf{v}(t) = \langle 5, 3\cos 3t, -3\sin 3t \rangle$; $\mathbf{r}(t) \cdot \mathbf{v}(t) = 0$

76. Let $\mathbf{r}(t) = \cos 2t\,\mathbf{i} + 2t\,\mathbf{j} + \sin 2t\,\mathbf{k}$. Show that the acceleration vector is parallel to the normal vector $\mathbf{N}(t)$.
Answer: $v(t) = |\mathbf{v}(t)| = \sqrt{4(\cos^2 2t + \sin^2 2t) + 2} = \sqrt{6}$, so $a_T = v'(t) = 0$ and thus $\mathbf{a}(t) = a_N\mathbf{N}$ is parallel to $\mathbf{N}$.

10.5 Parametric Surfaces

77. Find a parametric representation for the surface consisting of the upper half of the ellipsoid $x^2+5y^2+z^2=1$.

A) $x=x, y=y, z=\sqrt{1+x^2+5y^2}$
B) $x=x, y=y, z=\sqrt{1+x^2-5y^2}$
C) $x=x, y=y, z=\sqrt{1-x^2+5y^2}$
D) $x=x, y=y, z=\sqrt{1-x^2-5y^2}$
E) $x=x, y=y, z=\sqrt{x^2+5y^2-1}$
F) $x=x, y=y, z=\sqrt{x^2-5y^2-1}$
G) $x=x, y=y, z=\sqrt{-x^2+5y^2-1}$
H) $x=x, y=y, z=\sqrt{-x^2-5y^2-1}$

Answer: D)

78. Find a parametrization in cylindrical coordinates for the surface $z=x^2+y^2$.

A) $x=r\sin\theta, y=r\sin\theta, z=r$
B) $x=r\sin\theta, y=r\sin\theta, z=r^2$
C) $x=r\cos\theta, y=r\cos\theta, z=r$
D) $x=r\cos\theta, y=r\cos\theta, z=r^2$
E) $x=\cos\theta, y=\sin\theta, z=r$
F) $x=\cos\theta, y=\sin\theta, z=r^2$
G) $x=r\cos\theta, y=r\sin\theta, z=r$
H) $x=r\cos\theta, y=r\sin\theta, z=r^2$

Answer: H)

79. Find a parametric representation for the surface consisting of that part of the hyperboloid $-x^2-y^2+z^2=1$ that lies below the rectangle $[-1,1]\times[-3,3]$.

Answer: $x=x, y=y, z=-\sqrt{1+x^2+y^2}$ where $-1\le x\le 1$ and $-3\le y\le 3$.

80. Find a parametric representation for the surface consisting of that part of the elliptic paraboloid $x+y^2+2z^2=4$ that lies in front of the plane $x=0$.

Answer: $x=4-y^2-2z^2, y=y, z=z$ where $y^2+2z^2\le 4$ since $x\ge 0$.

81. Find a parametric representation for the surface consisting of that part of the cylinder $x^2+z^2=1$ that lies between the planes $y=-1$ and $y=3$.

Answer: $x=\sin\theta, y=y, z=\cos\theta, 0\le\theta\le 2\pi, -1\le y\le 3$.

82. Find a parametric representation for the surface consisting of that part of the plane $z=x+3$ that lies inside the cylinder $x^2+y^2=1$.

Answer: $x=x, y=y, z=x+3$, where $0\le x^2+y^2\le 1$.

83. Identify the surface with the vector equation $\mathbf{r}(u,v)=\cos u\sin v\,\mathbf{i}+\sin u\sin v\,\mathbf{j}+\cos v\,\mathbf{k}, 0\le u\le 2\pi, 0\le v\le\frac{\pi}{2}$. (*Hint:* First consider x^2+y^2.)

Answer: Top half of the unit sphere

84. Identify the surface with the vector equation $\mathbf{r}(u,v)=(1+2u+3v)\,\mathbf{i}+(5-u+4v)\,\mathbf{j}+(3+5u-7v)\,\mathbf{k}$.

Answer: Plane

85. Find a parametric representation for the surface consisting of that part of the hyperboloid $-x^2-y^2+z^2=1$ that lies below the disk $\{(x,y)\mid x^2+y^2\le 4\}$.

Answer: $x=r\cos\theta, y=r\sin\theta, z=-\sqrt{1+r^2}, 0\le\theta\le 2\pi, 0\le r\le 2$

11 Partial Derivatives

11.1 Functions of Several Variables

1. Let $f(x, y) = x \sin y$. Find $f\left(2, \frac{\pi}{3}\right)$.

A) $\sqrt{3}$ B) $\sqrt{2}$ C) $\frac{\sqrt{3}}{2}$ D) $\frac{\sqrt{2}}{2}$
E) $\frac{1}{2}$ F) $\frac{1}{3}$ G) 1 H) 0

Answer: A)

2. Let $f(x, y) = x^2 + 2xy + y^2$. If $x = 2$, find $f(x, 2x)$.

A) 12 B) 16 C) 24 D) 28
E) 32 F) 36 G) 42 H) 48

Answer: F)

3. Let $f(x, y) = x \sin y$. If $x = \pi$, find $f(x, x/2)$.

A) $\frac{\pi}{6}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{3}$ D) $\frac{\pi}{2}$
E) $\frac{2\pi}{3}$ F) $\frac{3\pi}{4}$ G) π H) 2π

Answer: G)

4. Let $f(x, y) = \left(x^2 + y\right)^3$. If $x = 1$, find $f(x, 2x)$.

A) 1 B) 2 C) 3 D) 4
E) 8 F) 9 G) 16 H) 27

Answer: H)

5. Find the domain of the function $f(x, y) = \sqrt{x - y^2}$.

A) All points on or to the left of $x = y^2$
B) All points on or to the right of $x = y^2$
C) All points to the left of $x = y^2$
D) All points to the right of $x = y^2$
E) All points on or to the left of $x = 0$
F) All points on or to the right of $x = 0$
G) All points to the left of $x = 0$
H) All points in the xy-plane

Answer: B)

6. Find the range of the function $f(x, y) = \sqrt{x - y^2}$.

A) $(0, \infty)$ B) $[0, \infty)$ C) $(-\infty, 0)$ D) $(-\infty, \infty)$
E) $(1, \infty)$ F) $[1, \infty)$ G) $(\sqrt{2}, \infty)$ H) $[\sqrt{2}, \infty)$

Answer: B)

7. Find the domain of the function $f(x, y) = e^{x - y^2}$.

A) All points on or to the left of $x = y^2$
B) All points on or to the right of $x = y^2$
C) All points to the left of $x = y^2$
D) All points to the right of $x = y^2$
E) All points on or to the left of $x = 0$
F) All points on or to the right of $x = 0$
G) All points to the left of $x = 0$
H) All points in the xy-plane

Answer: H)

8. Find the range of the function $f(x,y) = e^{x-y^2}$.

A) $(0,\infty)$ B) $[0,\infty)$ C) $(-\infty,\infty)$ D) $(-\infty,0)$
E) $(1,\infty)$ F) $[1,\infty)$ G) $(\sqrt{2},\infty)$ H) $[\sqrt{2},\infty)$
Answer: A)

9. Find the domain of the function $f(x,y) = \ln(x - y^2)$.

A) All points on or to the left of $x = y^2$
B) All points on or to the right of $x = y^2$
C) All points to the left of $x = y^2$
D) All points to the right of $x = y^2$
E) All points on or to the left of $x = 0$
F) All points on or to the right of $x = 0$
G) All points to the left of $x = 0$
H) All points in the x-y plane
Answer: D)

10. Find the range of the function $f(x,y) = \ln(x - y^2)$.

A) $(0,\infty)$ B) $[0,\infty)$ C) $(-\infty,\infty)$ D) $(-\infty,0)$
E) $(1,\infty)$ F) $[1,\infty)$ G) $(\sqrt{2},\infty)$ H) $[\sqrt{2},\infty)$
Answer: C)

11. Describe the level curves of the function $f(x,y) = x^2 + y^2 + 3x - 4y + 73$.

A) Concentric circles
B) Non-concentric circles
C) Concentric ellipses (not circles)
D) Non-concentric ellipses (not circles)
E) Parabolas with the same vertex
F) Parabolas with the same focus
G) Hyperbolas with the same vertices
H) Hyperbolas with the same foci
Answer: A)

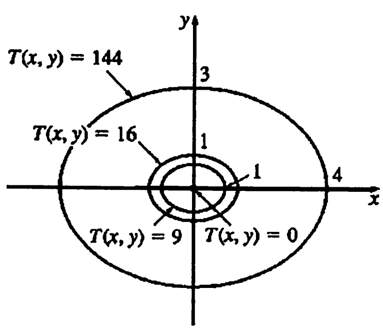
12. Describe the level curves of the function $f(x,y) = \sqrt{1 - x^2 - 2y^2}$.

A) Concentric circles
B) Non-concentric circles
C) Concentric ellipses (not circles)
D) Non-concentric ellipses (not circles)
E) Parabolas with the same vertex
F) Parabolas with the same focus
G) Hyperbolas with the same vertices
H) Hyperbolas with the same foci
Answer: C)

13. Identify the graph of the function $f(x,y) = 3 - x^2 - y^2$.

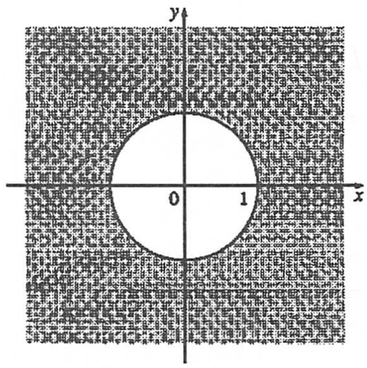
A) Cone
B) Paraboloid
C) Ellipsoid
D) Hyperboloid of one sheet
E) Hyperboloid of two sheets
F) Hyperbolic cylinder
G) Elliptic cylinder
H) Parabolic cylinder
Answer: B)

14. The temperature at a point (x,y) of a flat metal plate is $T(x,y) = 9x^2 + 16y^2$ where $T(x,y)$ is measured in degrees. Draw the isothermals for $T(x,y) = 0$, 9, 16, and 144 degrees.
Answer:



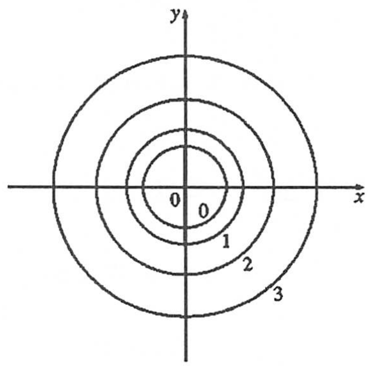
15. Sketch the domain of the function $z = \sqrt{x^2 + y^2 - 1}$.

Answer:

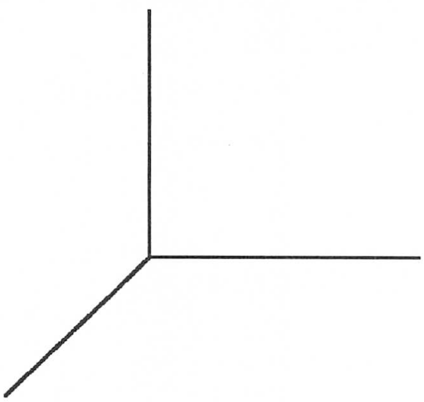


16. For the function $z = \sqrt{x^2 + y^2 - 1}$, sketch the level curves $z = k$ for $k = 0, 1, 2,$ and 3.

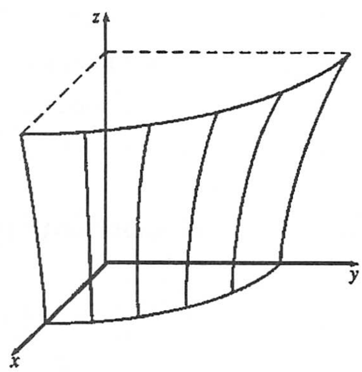
Answer:



17. For the function $z = \sqrt{x^2 + y^2 - 1}$, sketch the portion of the surface in the first octant.



Answer:



18. Suppose the point $(2, 3)$ is on a curve C which is a level curve of the surface $z = x^2 + 2y$. Can it be concluded that the point $(4, -3)$ is also on C? Explain.

Answer: Yes, given that C is a level curve of $z = x^2 + 2y$ means C has equation $x^2 + 2y = k$ for some constant k. The fact that $(2, 3)$ lies on C gives $k = 10$, so C has equation $x^2 + 2y = 10$ which does contain the point $(4, -3)$.

19. Describe the level surfaces of the function $f(x, y, z) = z - x^2 - y^2$.

Answer: Paraboloids with vertices along the z-axis

20. Describe the level surfaces of the function $f(x, y, z) = z - 2x - 3y$.

Answer: Parallel planes

21. Find the domain of the function $f(x, y, z) = \dfrac{1}{\sqrt{3x + y - z}}$.

A) All points above $3x + y - z = 0$
B) All points below $3x + y - z = 0$
C) All points above $3x + y - z = 1$
D) All points below $3x + y - z = 1$
E) All points below $3x + y - z = 0$ but not above the plane $3x + y - z = 1$
F) All points above $3x + y - z = 1$ but not above the plane $3x + y - z = 0$
G) All points below $3x + y - z = 1$ or between $3x + y - z = 0$ and $3x + y - z = 1$
H) All points in the xy-plane

Answer: E)

22. Describe the difference between the horizontal trace in $z = k$ for the function $z = f(x, y)$ and the contour curve $f(x, y) = k$.

Answer: The contour curve $f(x, y) = k$ is the horizontal trace of $z = f(x, y)$ in $z = k$ projected down to the xy-plane.

23. Describe the vertical traces $x = k$ and $y = k$ and the horizontal traces $z = k$ for the function $f(x, y) = x^2 - y^2$.

Answer: $x = k$: $z = k^2 - y^2$, parabola; $y = k$: $z = x^2 - k^2$, parabola; $z = k$: $x^2 - y^2 = k$, hyperbola

24. Describe the level surfaces $k = 1$, $k = 0$, $k = -1$ for the function $f(x, y, z) = 1 - x^2 - \frac{1}{2}y^2 - \frac{1}{3}z^2$.

Answer: $k = 1$: point $(0, 0, 0)$; $k = 0$: ellipsoid $x^2 + \frac{1}{2}y^2 + \frac{1}{3}z^2 = 1$; $k = -1$: ellipsoid $x^2 + \frac{1}{2}y^2 + \frac{1}{3}z^2 = 2$

25. Describe how the graph of $g(x, y) = 2f(x - 1, y - 1)$ is obtained from the graph of $z = f(x, y)$.

Answer: The graph of g is the graph of f shifted 1 unit in the positive x-direction, shifted 1 unit in the positive y-direction, and stretched vertically (that is, in the z-direction) by a factor of 2.

11.2 Limits and Continuity

26. Evaluate $\displaystyle\lim_{(x,y)\to(1,2)} (x^2 + 2xy)$.

A) 0	B) 1	C) 2	D) 3
E) 4	F) 5	G) 6	H) Does not exist

Answer: F)

27. Evaluate $\displaystyle\lim_{(x,y)\to(-1,1)} (x^2 + y^3)$.

A) 1	B) 2	C) 3	D) 4
E) 8	F) 9	G) 16	H) Does not exist

Answer: E)

28. Evaluate $\lim_{(x,y)\to(0,0)} x\sin y$.

A) 0 B) 1 C) 2 D) 3
E) 4 F) 5 G) 6 H) Does not exist

Answer: A)

29. Evaluate $\lim_{(x,y)\to(0,0)} \frac{x}{\sin y}$.

A) 0 B) 1 C) 2 D) 3
E) 4 F) 5 G) 6 H) Does not exist

Answer: H)

30. Evaluate $\lim_{(x,y)\to(-1,1)} \left(x^2 + x^3y^2 + y^4\right)$.

A) 0 B) 1 C) 2 D) 3
E) 4 F) 5 G) 6 H) Does not exist

Answer: B)

31. Evaluate $\lim_{(x,y)\to(1,-1)} \frac{x^2 - y^2}{x + y}$

A) 0 B) 1 C) 2 D) 3
E) 4 F) 5 G) 6 H) Does not exist

Answer: C)

32. Evaluate $\lim_{(x,y)\to(2,2)} \frac{\tan(x-y)}{x-y}$.

A) 0 B) 1 C) 2 D) 3
E) 4 F) 5 G) 6 H) Does not exist

Answer: B)

33. Evaluate $\lim_{(x,y)\to(1,1)} [(x+y)\cos(x-y)]$.

A) 0 B) 1 C) 2 D) 3
E) 4 F) 5 G) 6 H) Does not exist

Answer: C)

34. Evaluate $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + xy + y^2}$.

A) 0 B) 1 C) $\frac{1}{2}$ D) $\frac{1}{3}$
E) $\frac{1}{4}$ F) $\frac{1}{5}$ G) $\frac{1}{6}$ H) Does not exist

Answer: H)

35. Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$.

A) 0 B) 1 C) -1 D) $\frac{1}{2}$
E) $-\frac{1}{2}$ F) $\frac{1}{3}$ G) $-\frac{1}{3}$ H) Does not exist

Answer: Does not exist

36. Evaluate $\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2 + y^2}$.

A) 1 B) 0 C) $\frac{1}{2}$ D) ∞
E) 2 F) 4 G) 8 H) Does not exist

Answer: H)

37. Evaluate $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - 3xy + y^2}{x^2 + 2y^2}$.

A) 0 B) ∞ C) $-\infty$ D) 1
E) $\frac{1}{2}$ F) $\frac{1}{3}$ G) $\frac{1}{4}$ H) Does not exist

Answer: H)

38. Let $f(x,y) = \dfrac{xy}{x^2 + y^2}$ and let C_m be the curve with equation $y = mx$, where m is a constant. The value of the limit of $F(x,y)$ as (x,y) approaches $(0,0)$ along C_m is

A) 0 B) $\dfrac{1}{2}$ C) 1 D) $\dfrac{m}{1+m^2}$ E) $\dfrac{m^2}{1+m^2}$

Answer: D)

39. Let $f(x,y) = \begin{cases} \dfrac{x^4 + y^4}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ Does this function have a limit at the origin? If so, prove it. If not, demonstrate why not.

Answer: Approaching the origin along $y = 0$, the limit equals 1; approaching the origin along $y = x$, the limit equals $\frac{1}{2}$. Thus, this function does not have a limit at the origin.

40. Prove that the following limit does not exist: $\lim_{(x,y)\to(0,0)} f(x,y)$, where $f(x,y) = \dfrac{xy}{x^2 + y^4}$.

Answer: f is defined everywhere in $\mathbb{R}^2$ except at $(0,0)$. Let $S_1 = \{(0,y)\}$; then $(0,0) \in S_1$, and taking the limit of f as $y \to 0$ gives 0. Let $S_2 = \{(x,x)\}$; then $(0,0) \in S_2$ and taking the limit of f as $x \to 0$ gives 1. Since these limits are not equal, the limit does not exist.

41. Show that the limit $\lim_{(x,y)\to(0,0)} \dfrac{2x^2 - y^2}{x^2 + 2y^2}$ does not exist.

Answer: Approaching $(0,0)$ along the x-axis, we have $y = 0$, so the limit (if it exists) equals 2. Approaching $(0,0)$ along the y-axis we have $x = 0$, so the limit (if it exists) equals $-\frac{1}{2}$. Since these limits are different, the limit does not exist.

42. If $f(x,y) = \dfrac{xy}{x^2 + y^2}$ then $\lim_{(x,y)\to(0,0)} f(x,y)$

A) exists B) does not exist C) is equal to 0 D) is equal to $\frac{1}{2}$ E) is equal to 1

Answer: B)

43. Determine if $f(x,y) = \begin{cases} \dfrac{x^2 + xy}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$ is everywhere continuous, and if not, locate the point(s) of discontinuity.

Answer: f is continuous everywhere its denominator does not equal zero. The limit does not exist at $(0,0)$, and thus f is discontinuous at $(0,0)$.

44. Determine whether $f(x,y) = \begin{cases} \dfrac{3x^2 - 2y^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ is continuous at $(0,0)$.

Answer: The function is discontinuous at $(0,0)$

45. Consider $f(x,y) = \begin{cases} \dfrac{x^4 - y^4}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0.1 & \text{if } (x,y) = (0,0) \end{cases}$ Where is f continuous?

Answer: Continuous everywhere except at $(0,0)$

46. Find $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x - y^4}{x^3 - y^4}$ if it exists, or show that the limit does not exist.

Answer: Along $y = 1$, $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x - y^4}{x^3 - y^4} = \lim_{x\to 1} \frac{x-1}{x^3-1} = \frac{1}{3}$ (if the limit exists). Along $x = 1$, $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x - y^4}{x^3 - y^4} = \lim_{y\to 1} \frac{1-y^4}{1-y^4} = 1$ (if the limit exists). Because these limits are different, $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x - y^4}{x^3 - y^4}$ does not exist.

11.3 Partial Derivatives

47. Let $f(x,y) = x^2y^3$. Find the value of the partial derivative $f_x(1,1)$.

A) 0 B) 1 C) 2 D) 3
E) 4 F) 5 G) 6 H) 7

Answer: C)

48. Let $f(x,y) = x^3y^4$. Find the value of the partial derivative $f_y(1,1)$.

A) 0 B) 1 C) 2 D) 3
E) 4 F) 5 G) 6 H) 7

Answer: E)

49. Let $f(x,y) = (x+y)^2$. Find the value of the partial derivative $f_{xx}(3,1)$.

A) 0 B) 1 C) 2 D) 3
E) 4 F) 5 G) 6 H) 7

Answer: C)

50. Let $f(x,y) = (x^3 + x)\,y^2$. Find the value of the partial derivative $f_{xy}(0,1)$.

A) 0 B) 1 C) 2 D) 3
E) 4 F) 5 G) 6 H) 7

Answer: C)

51. Let $f(x,y) = x^3y^2$. Find the value of the partial derivative $f_{yxy}(1,7)$.

A) 0 B) 1 C) 2 D) 3
E) 4 F) 5 G) 6 H) 7

Answer: G)

52. Let $f(x,y) = \sin(2x + y)$. Find the value of the partial derivative $f_{xy}\left(\pi, \frac{\pi}{2}\right)$.

A) $\sqrt{2}$ B) $-\sqrt{2}$ C) $2\sqrt{2}$ D) $-2\sqrt{2}$
E) $\frac{\sqrt{2}}{2}$ F) -2 G) 2 H) 0

Answer: F)

53. Let $f(x,y) = e^{2x+y^2}$. Find the value of the partial derivative $f_{xyy}(0,0)$.

A) 1 B) 2 C) 4 D) $2e$
E) $4e$ F) e^2 G) $2e^2$ H) $4e^2$

Answer: 4

54. How many third-order partial derivatives does a function $f(x,y)$ have?

A) 3 B) 4 C) 5 D) 6
E) 7 F) 8 G) 9 H) 10

Answer: F)

55. If all the third-order partial derivatives of $f(x,y)$ are continuous, what is the largest number of them that can be distinct?

A) 3 B) 4 C) 5 D) 6
E) 7 F) 8 G) 9 H) 10

Answer: B)

56. Let $f(x,y) = (x^3 + y^4)^5$. Find the value of $f_{xy} - f_{yx}$ at the point $(1,2)$.

A) -32 B) -16 C) -8 D) -4
E) 0 F) 4 G) 8 H) 16

Answer: E)

57. Let $f(x,y) = xe^{y/x}$. Find the value of the partial derivative $f_x(2,4)$.

A) $-2e^2$ B) $-e^2$ C) 0 D) $e/2$
E) $e^2/2$ F) e^2 G) $2e^2$ H) $4e^2$

Answer: B)

58. Let $f(x,y) = \tan^{-1}(x/y)$. Find the value of the partial derivative $f_y(1,2)$.

A) -1 B) $-\frac{1}{2}$ C) $-\frac{1}{3}$ D) $-\frac{1}{5}$
E) $\frac{1}{5}$ F) $\frac{1}{3}$ G) $\frac{1}{2}$ H) 1

Answer: D)

59. Let $f(x,y,z) = z^{xy}$, $z > 0$. Find the value of the partial derivative $f_y(2,1,e)$.

A) 0 B) 1 C) e D) $2e$
E) $2e^2$ F) $\frac{1}{2}$ G) $\ln 2$ H) 2

Answer: E)

60. Let $f(x,y,z) = x^{yz}$, $x > 0$. Find the value of the partial derivative $f_z(2,3,0)$.

A) 0 B) 2 C) 3 D) 8
E) $\ln 2$ F) $\ln 3$ G) $2\ln 2$ H) $3\ln 2$

Answer: H)

61. If $f(r,\theta) = r\sin\theta + r^2\tan^2\theta$, find the partial derivative of f with respect to r and the partial derivative with respect to θ, both at the point $\left(-4, \frac{\pi}{6}\right)$.

Answer: $\left.\frac{\partial f}{\partial r}\right|_{(-4,\pi/6)} = \frac{-13}{6}$; $\left.\frac{\partial f}{\partial \theta}\right|_{(-4,\pi/6)} = \frac{110}{9}\sqrt{3}$

62. Let $f(x,y) == \begin{cases} \dfrac{x^2}{x+y} \text{if } (x,y) \neq (0,0) \\ 0 \quad \text{if } (x,y) = (0,0) \end{cases}$

(a) Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

(b) Find the values of the above derivatives at $(0,0)$, if they exist.

Answer: (a) $\dfrac{\partial f}{\partial x} = \dfrac{x^2+2xy}{(x+y)^2}$, $\dfrac{\partial f}{\partial y} = \dfrac{-x^2}{(x+y)^2}$; (b) $f_x(0,0) = 1$, $f_y(0,0) = 0$

63. Find both partial derivatives if $f(x,y) = \dfrac{2x-3y}{3x-2y}$.

Answer: $\dfrac{\partial f}{\partial x} = \dfrac{5y}{(3x-2y)^2}$; $\dfrac{\partial f}{\partial y} = \dfrac{-5x}{(3x-2y)^2}$

64. If $w = x^2z + xy^2 - yz^2$, find $\dfrac{\partial w}{\partial x}$, $\dfrac{\partial w}{\partial y}$, and $\dfrac{\partial w}{\partial z}$.

Answer: $\dfrac{\partial w}{\partial y} = 2zx + y^2$, $\dfrac{\partial w}{\partial y} = 2xy - z^2$, $\dfrac{\partial w}{\partial z} = x^2 - 2yz$

65. Given $f(x,y) = 2x^3y^2 - 3x^3 + 8y^4$, find f_x and f_y and evaluate each at $(1,2)$.

Answer: $f_x = 6x^2y^2 - 9x^2$, $f_x(1,2) = 15$; $f_y = 4x^3y + 32y^3$, $f_y(1,2) = 264$

66. Let $f(x,y,z) = x^2y^3 - \dfrac{x}{z} + e^x \ln y$. Find f_x, f_y, and f_z.

Answer: $f_x = 2xy^3 - \dfrac{1}{z} + e^x \ln y$, $f_y = 3x^2y^2 + \dfrac{e^x}{y}$, $f_z = \dfrac{x}{z^2}$

67. Given $f(x,y) = x^2 \sin(xy)$, find f_x, f_y, and f_{xy}.

Answer: $f_x = 2x\sin(xy) + x^2y\cos(xy)$; $f_y = x^3\cos(xy)$; $f_{xy} = 3x^2\cos(xy) - x^3y\sin(xy)$

68. Let $f(x,y,z) = (xyz)^2$. Find all second-order partial derivatives of f.

Answer: $f_{xy} = 4xyz^2$, $f_{yx} = 4xyz^2$, $f_{zx} = 4xy^2z$, $f_{xz} = 4xy^2z$, $f_{yz} = 4x^2yz$, $f_{zy} = 4x^2yz$, $f_{xx} = 2y^2z^2$, $f_{yy} = 2x^2z^2$, $f_{zz} = 2x^2y^2$

69. Find f_{xx}, f_{yy}, and f_{yx} if $f(x,y) = \sin x^2y$.

Answer: $f_{xx} = 2y\cos x^2y - 4x^2y^2\sin x^2y$; $f_{yy} = -x^4\sin x^2y$; $f_{xy} = 2x\cos x^2y - 2x^3y\sin x^2y$

70. If $f(x,y,z) = x\ln(yz^2)$, find f_{xy}, f_{xz}, and f_{yz}.

Answer: $f_{xy} = \dfrac{1}{y}$, $f_{xz} = \dfrac{2}{z}$, $f_{yz} = 0$

71. If $z = x^2\sin y + ye^x$, find $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$, $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial x \partial y}$, and $\dfrac{\partial^2 z}{\partial y^2}$.

Answer: $\dfrac{\partial z}{\partial x} = 2x\sin y + ye^x$, $\dfrac{\partial z}{\partial y} = x^2\cos y + e^x$, $\dfrac{\partial^2 z}{\partial x^2} = 2\sin y + ye^x$, $\dfrac{\partial^2 z}{\partial x \partial y} = 2x\cos y + e^x$, $\dfrac{\partial^2 z}{\partial y^2} = -x^2\sin y$

72. Let $f(xy) = x^3y - xy^2 + y^4 + x$. Find $\left.\dfrac{\partial^2 f}{\partial y \partial x}\right|_{(2,3)}$.

Answer: 6

73. Show that $f(x,y) = e^y \cos x - 2xy$ satisfies Laplace's Equation $f_{xx} + f_{yy} = 0$.

Answer: $f_x = -e^y \sin x - 2y$, $f_y = e^y \cos x - 2x \Rightarrow f_{xx} = -e^y \cos x$, $f_{yy} = e^y \cos x \Rightarrow f_{xx} + f_{yy} = 0$

74. If $g(x,y) = \begin{cases} \dfrac{x^4+y^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ find $\dfrac{\partial g}{\partial x}(1,0)$ and $\dfrac{\partial g}{\partial y}(1,0)$.

Answer: $\dfrac{\partial g}{\partial x}(1,0) = 2$, $\dfrac{\partial g}{\partial y}(1,0) = 0$

75. If g is a differentiable function and $f(x,y) = g(x^2+y^2)$, show that $yf_x - xf_y = 0$.

Answer: $f_x = g'(x^2+y^2)(2x)$, $f_y = g'(x^2+y^2)\,2y \Rightarrow yf_x - xf_y = 0$

11.4 Tangent Planes and Linear Approximations

76. Find an equation of the tangent plane to the surface $z = x^2$ at the point $(1,2,1)$.

A) $z = x$ B) $z = x + 2y - 4$ C) $z = 2x - 1$ D) $z = 2x + 2y - 5$
E) $z = x + y - 1$ F) $z = 2x + y - 3$ G) $z = 2x - y + 1$ H) $z = 2x - 2y + 3$

Answer: C)

77. Find an equation of the tangent plane to the surface $z = x^3$ at the point $(1,2,1)$.

A) $z = x$ B) $z = 3x - 2$ C) $z = 2x - 1$ D) $z = 3x + 2y - 5$
E) $z = x + y - 1$ F) $z = 3x + y - 4$ G) $z = 3x - y$ H) $z = 2x - 2y + 3$

Answer: B)

78. Find an equation of the tangent plane to the surface $z = x^2 + y^2$ at the point $(1,1,2)$.

A) $z = x + 1$ B) $z = x + 2y - 1$ C) $z = 2x$ D) $z = 2x + 2y - 2$
E) $z = x + y$ F) $z = 2x + y - 1$ G) $z = 2x - y + 1$ H) $z = 2x - 2y + 2$

Answer: D)

79. Find an equation of the tangent plane to the surface $z = x^2 + 2y^2$ at the point $(1,1,3)$.

A) $z = x + y + 1$ B) $z = x + 2y$ C) $z = x - y + 3$ D) $z = x - 2y + 4$
E) $z = 2x + 2y - 1$ F) $z = 2x + 4y - 3$ G) $z = 2x - 2y + 3$ H) $z = 2x - 4y + 5$

Answer: F)

80. Find an equation of the tangent plane to the surface $z = e^{x+y}$ at the point $(0,0,1)$.

A) $z = x + y + 1$ B) $z = ex + ey - 2e + 1$
C) $z = 2x + 2y + 1$ D) $z = 2ex + 2ey - 4e + 1$
E) $z = 4x + 4y - 7$ F) $z = e^2x + e^2y - 2e^2 + 1$
G) $z = 2e^2x + 2e^2y - 4e^2 + 1$ H) $z = 4ex + 4ey - 8e + 1$

Answer: A)

81. Find an equation of the tangent plane to the surface $z = e^{2x+2y}$ at the point $(0,0,1)$.

A) $z = x + y + 1$ B) $z = ex + ey - 2e + 1$
C) $z = 2x + 2y + 1$ D) $z = 2ex + 2ey - 4e + 1$
E) $z = 4x + 4y - 7$ F) $z = e^2x + e^2y - 2e^2 + 1$
G) $z = 2e^2x + 2e^2y - 4e^2 + 1$ H) $z = 4ex + 4ey - 8e + 1$

Answer: C)

82. Find the differential of $z = 3x + 2y$.

A) $3x\,dx + 2y\,dy$ B) $3x^2\,dx + 2y^2\,dy$ C) $2x\,dx + 3y\,dy$ D) $2x^2\,dx + 3y^2\,dy$
E) $3y\,dx + 2x\,dy$ F) $2y\,dx + 3x\,dy$ G) $2\,dx + 3\,dy$ H) $3\,dx + 2\,dy$

Answer: H)

83. Find the differential of $z = 3x + y^2$.

A) $3\,dx + 2y\,dy$ B) $3x\,dx + 2y^2\,dy$ C) $2\,dx + 3y\,dy$ D) $2x\,dx + 3y^2\,dy$
E) $3y\,dx + 2x\,dy$ F) $2y\,dx + 3x\,dy$ G) $2\,dx + 3\,dy$ H) $3\,dx + 2\,dy$

Answer: A)

84. Find the differential of $z = 2xy$.

A) $2x\,dx + 2y\,dy$ B) $2y\,dx + 2x\,dy$ C) $2x^2\,dx + 2y^2\,dy$ D) $2y^2\,dx + 2x^2\,dy$
E) $x\,dx + y\,dy$ F) $y\,dx + x\,dy$ G) $x^2\,dx + y^2\,dy$ H) $y^2\,dx + x^2\,dy$

Answer: B)

85. Find the differential of $z = 2x^2y$.

A) $2x\,dx + dy$ B) $2y\,dx + dy$ C) $4y\,dx + 2\,dy$ D) $2y^2\,dx + x\,dy$
E) $x\,dx + y\,dy$ F) $y\,dx + x\,dy$ G) $4xy\,dx + 2x^2\,dy$ H) $2xy\,dx + x^2\,dy$

Answer: G)

86. Find the differential of $z = \dfrac{1}{x + y^2}$ at the point $(x, y) = (1, 1)$.

A) $-dx - dy$ B) $-\frac{1}{4}\,dx - \frac{1}{2}\,dy$ C) $-dx - \frac{1}{2}\,dy$ D) $-\frac{1}{2}\,dx - dy$
E) $dx + dy$ F) $\frac{1}{4}\,dx + \frac{1}{2}\,dy$ G) $dx + \frac{1}{2}\,dy$ H) $\frac{1}{2}\,dx + dy$

Answer: B)

87. Find the differential of $w = xe^{y \sin z}$ at the point $(x, y, z) = (1, 1, 0)$.

A) dx B) dy C) dz D) $dy + dz$
E) $dx + dz$ F) $dx + dy$ G) $dx + dy + dz$ H) 0

Answer: E)

88. Use differentials to approximate $\sqrt{26}$.

A) 5.05 B) 5.06 C) 5.07 D) 5.08
E) 5.09 F) 5.10 G) 5.11 H) 5.12

Answer: F)

89. Use differentials to approximate $\sqrt{24} + \sqrt{5}$.

A) 6.85 B) 6.90 C) 6.95 D) 7.00
E) 7.05 F) 7.10 G) 7.15 H) 7.20

Answer: G)

90. Use differentials to approximate $\dfrac{\sqrt[3]{28}}{\sqrt{10}}$.

A) $\frac{154}{162}$ B) $\frac{155}{162}$ C) $\frac{156}{162}$ D) $\frac{157}{162}$
E) $\frac{158}{162}$ F) $\frac{159}{162}$ G) $\frac{160}{162}$ H) $\frac{161}{162}$

Answer: B)

91. A boundary stripe 3 inches wide is painted around a rectangle whose dimensions are 100 feet by 200 feet. Use differentials to approximate the number of square feet of paint in the stripe.

A) 120 B) 130 C) 140 D) 150
E) 160 F) 170 G) 180 H) 190

Answer: D)

92. Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter if the metal in the wall is 0.05 cm thick and the metal in the top and bottom is 0.1 cm thick.

A) 2.0π B) 2.4π C) 2.8π D) 3.2π
E) 3.6π F) 4.0π G) 4.4π H) 4.8π

Answer: C)

93. Find an equation of the tangent plane to the surface $z = \ln(2x + y)$ at the point $(-1, 3, 0)$.

A) $2x + y - z = 1$ B) $x + 2y + z = 5$ C) $x - 2y + z = 7$ D) $3x + 2y + z = 3$
E) $2x + 3y + z = 7$ F) $x + 2y + 3z = 5$ G) $3x + y - z = 0$ H) $x + 3y + 2z = 8$

Answer: A)

94. Find an equation of the tangent plane to the surface $z = xy$ at the point $(-1, 2, -2)$.

A) $x + y + z + 1 = 0$ B) $2x - y - z + 2 = 0$
C) $x + 2y + z = 1$ D) $2x + y - z = 2$
E) $x + y - 2z = 5$ F) $3x + 2y + z + 1 = 0$
G) $2x + 3y + 2z = 0$ H) $x + y + 3z + 6 = 0$

Answer: B)

95. Find an equation of the tangent plane to the surface $z = \tan^{-1}(y/x)$ at the point $\left(-2, 2, -\frac{\pi}{4}\right)$.

A) $x + y + z = \pi$ B) $x + y + 4z + \pi = 0$
C) $4x + 4y + z + \pi = 0$ D) $2x - 2y + z = \frac{\pi}{4}$
E) $2x - 2y + 4z = \pi$ F) $4x - 2y - z + \pi = 0$
G) $x + 2y + 4z = 2 - \pi$ H) $x + 2y - 4z = 2 + \pi$

Answer: B)

96. Find an equation of the tangent plane to the surface $z = x^2 + 4y^2$ at the point $(2, 1, 8)$.

A) $4x + 8y - z = 8$ B) $4x + 8y + 2z = 32$
C) $6x + 2y + z = 22$ D) $2x + 6y - z = 2$
E) $4x - y + 2z = 23$ F) $6x + 4y - z = 8$
G) $4x + 2y - z = 2$ H) $8x + 2y + z = 26$

Answer: A)

97. A right triangle has leg A with length 4, leg B with length 3, and hypotenuse with length 5. Use a total differential to approximate the length of the hypotenuse if leg A had length 4.2 and leg B had length 2.9.

Answer: 5.10

98. Find the total differential of w if $w = f(x, y, z) = xy^2z^3 - e^{xz}y$.

Answer: $dw = \left(y^2z^3 - yze^{xz}\right)dx + \left(2xyz^3 - e^{xz}\right)dy + \left(3xy^2z^2 - xye^{xz}\right)dz$

99. Use the total differential to approximate $\sqrt[3]{25}\sqrt[4]{17}$.

Answer: 5.95

100. Find an equation of the plane tangent to the surface $xyz = 2$ at $(1, -1, -2)$.

Answer: $2x - 2y - z = 6$

101. If $f(x,y) = ye^{xy}$, find the values x_0 for which $f(x_0, 5) = 5$, and then find an equation of the plane tangent to the graph of f at $(x_0, 5, 5)$.

Answer: $x_0 = 0$; $z = 25x + y$

102. Find an equation of the tangent plane to the parametric surface $x = u$, $y = v$, $z = u^2 + v$ at the point $(1, 2, 3)$.

A) $x + y + z = 6$	B) $x + y - z = 0$	C) $x + y + 2z = 9$	D) $x + y - 2z = -3$
E) $x + 2y + z = 8$	F) $x - 2y + z = 0$	G) $2x + y + z = 7$	H) $2x + y - z = 1$

Answer: H)

103. Find an equation of the tangent plane to the parametric surface $x = u - v$, $y = u + v$, $z = u^2$ at the point $(0, 2, 1)$.

A) $x + y + z = 3$	B) $x + y - z = 1$	C) $x + y + 2z = 4$	D) $x + y - 2z = 0$
E) $x + 2y + z = 5$	F) $x - 2y + z = -3$	G) $2x + y + z = 3$	H) $2x + y - z = 1$

Answer: B)

104. Find an equation of the tangent plane to the parametric surface $x = u^2$, $y = u - v^2$, $z = v^2$ at the point $(1, 0, 1)$.

Answer: $x - 2y - 2z + 1 = 0$

105. Find an equation of the tangent plane to the surface given by $\mathbf{r}(u, v) = (u + v)\,\mathbf{i} + u\cos v\,\mathbf{j} + v\sin u\,\mathbf{k}$ at the point $(1, 1, 0)$.

Answer: $(\sin 1)\,x - (\sin 1)\,y - z = 0$

106. Find an equation of the tangent plane to the surface with parametric equations $x = \cos u \sin v$, $y = \sin u \sin v$, $z = \cos v$ at the point $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$.

Answer: $x + y + \sqrt{2}z = 2$

107. Find an equation of the tangent plane to the surface with parametric equations $x = 1 + 2u + 3v$, $y = 5 - u + 4v$, $z = 3 + 5u - 7v$ at the point $(3, 4, 8)$.

Answer: $-13x + 29y + 11z = 165$

108. Find a normal vector to the surface with parametric equations $x = 2uv$, $y = u^2 - v^2$, $z = u^2 + v^2$ at the point $(1, 0, 1)$.

Answer: $\left\langle \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right\rangle$

11.5 The Chain Rule

109. Let $z = xy$, and let x and y be functions of t with $x(1) = 1$, $y(1) = 2$, $x'(1) = 3$, and $y'(1) = 4$. Find dz/dt when $t = 1$.

A) 5	B) 6	C) 7	D) 8
E) 9	F) 10	G) 11	H) 12

Answer: F)

110. Let $z = x^2y$, and let x and y be functions of t with $x(1) = 1, y(1) = 2, x'(1) = 3$, and $y'(1) = 4$. Find dz/dt when $t = 1$.

A) 3 B) 4 C) 6 D) 8
E) 9 F) 10 G) 12 H) 16

Answer: H)

111. Let $z = \sin(xy)$, and let x and y be functions of t with $x(1) = 0, y(1) = 1, x'(1) = 2$, and $y'(1) = 3$. Find dz/dt when $t = 1$.

A) 0 B) 1 C) 2 D) 3
E) 4 F) 5 G) 6 H) 12

Answer: C)

112. Let $z = xy^2 + x^3y$, and let x and y be functions of t with $x(1) = 1, y(1) = 2, x'(1) = 3$, and $y'(1) = 4$. Find dz/dt when $t = 1$.

A) 38 B) 40 C) 42 D) 44
E) 46 F) 48 G) 50 H) 52

Answer: G)

113. Let $z = e^{x^2} \sin y$, and let x and y be functions of t with $x(1) = 0, y(1) = 0, x'(1) = 3$, and $y'(1) = 4$. Find dz/dt when $t = 1$.

A) 0 B) 1 C) 2 D) 3
E) 4 F) 5 G) 6 H) 7

Answer: E)

114. Let $z = x + y$, and let x and y be functions of s and t with $x(0,0) = 1, y(0,0) = 2, \partial x/\partial s = 3$, and $\partial y/\partial s = 4$ at $(s,t) = (0,0)$. Find $\partial z/\partial s$ when $(s,t) = (0,0)$.

A) 5 B) 6 C) 7 D) 8
E) 9 F) 10 G) 11 H) 12

Answer: C)

115. Let $z = xy + x^2y$, and let x and y be functions of s and t with $x(0,0) = 1, y(0,0) = 2, \partial x/\partial s = 3$, and $\partial y/\partial s = 4$ at $(s,t) = (0,0)$. Find $\partial z/\partial s$ when $(s,t) = (0,0)$.

A) 21 B) 22 C) 23 D) 24
E) 25 F) 26 G) 27 H) 28

Answer: F)

116. Let $z = e^x \sin y$, and let x and y be functions of s and t with $x(0,0) = 0, y(0,0) = 0, \partial x/\partial s = 3$, and $\partial y/\partial s = 4$ at $(s,t) = (0,0)$. Find $\partial z/\partial s$ when $(s,t) = (0,0)$.

A) 0 B) 1 C) 2 D) 3
E) 4 F) 5 G) 6 H) 7

Answer: E)

117. Let $x^3 + y^3 = x^2y^2$. Use implicit differentiation to find dy/dx when $(x,y) = (2,2)$.

A) $-\frac{1}{3}$ B) $\frac{1}{3}$ C) $-\frac{1}{2}$ D) $\frac{1}{2}$
E) -1 F) 1 G) -2 H) 2

Answer: E)

118. Let $e^x = 3\sin y$. Use implicit differentiation to find dy/dx when $(x,y) = (1,0)$.

A) $-\frac{1}{3}e$ B) $\frac{1}{3}e$ C) $-\frac{1}{2}e$ D) $\frac{1}{2}e$
E) $-e$ F) e G) $-2e$ H) $2e$

Answer: B)

119. Let $x + y + z - \sin(xyz) = 3$. Use implicit differentiation to find $\partial z/\partial y$ when $(x,y,z) = (1,0,2)$.

A) -3 B) -2 C) -1 D) 0
E) 1 F) 2 G) 3 H) 4

Answer: E)

120. Use implicit differentiation to find $\dfrac{\partial z}{\partial x}$ on the surface given by $x^3y + y^2z^2 + xz^3 = 3$.

Answer: $-\dfrac{3x^2y + z^3}{2y^2z + 3xz^2}$

121. If $z^3 + xz - y = 0$, find $\dfrac{\partial^2 z}{\partial x \partial y}$ in terms of x, y and z.

Answer: $\dfrac{3z^2 - x}{(3z^2 + x)^3}$

122. One side of a rectangle is increasing at 4 ft/min and another at 7 ft/min. At the time when the first side is 24 ft long and the second is 32 ft long, find

(a) how fast the area is changing.

(b) how fast the diagonal is changing.

Answer: (a) 296 ft^2/min (b) 8 ft/min

123. Suppose that $z = u^2 + uv + v^3$, and that $u = 2x^2 + 3xy$ and $v = 2x - 3y + 2$. Find $\dfrac{\partial z}{\partial x}$ at $(x,y) = (1,2)$.

Answer: 180

124. Given $f = x^2y^3$, $x = u^2 + v^2$, and $y = 2u + 3v$, the partial derivative of f with respect to v is

A) $\left(14u^2 + 29v^2 + 36uv\right)(2v + 3)$

B) $\left(u^2 + v^2\right)(2u + 3v)^2\left(8uv + 21v^2 + 9u^2\right)$

C) $2\left[\left(u^2 + v^2\right)(2u + 3v)^3 + v\right] + 3\left[\left(u^2 + v^2\right)^2(2u + 3v)^2 + 1\right]$

D) $2v(2u + 3v)^3 + 3\left(u^2 + v^2\right)^2$

E) $4\left(u^2 + v^2\right)(2u + 3v)^3 u + 6\left(u^2 + v^2\right)^2(2u + 3v)^2$

Answer: B)

125. If f is a function of x and y, and y is a function of x, then indirectly f depends only on x: $g(x) = f(x, y(x))$. Use the Chain Rule to write an expression for $\dfrac{dg}{dx}$ in terms of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

Answer: $\dfrac{\partial g}{\partial x} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$

126. If f is a function of x and y, and y is a function of x, then indirectly f depends only on x: $g(x) = f(x, y(x))$. If $f(x,y) = \sin x + \sqrt{1 - y^2}$ and $y(x) = \cos x$, calculate $\dfrac{dg}{dx}$.

Answer: $\dfrac{dg}{dx} = 2\cos x$

127. Suppose that $z = x - y$, $x = 4\left(t^3 - 1\right)$, and $y = \ln t$. Find $\dfrac{dz}{dt}$.

Answer: $12t^2 - \dfrac{1}{t}$

128. The radius of a right circular cylinder is increasing at a rate of 2 cm/min and its height is decreasing at 4 cm/min. At what rate is the volume changing at the instant when the radius is 4 cm and the height is 10 cm?

Answer: $96\pi\ \text{cm}^3/\text{min}$

129. Where does the equation $x^2 + \frac{1}{2}y^2 + \frac{1}{3}z^2 = 1$ define z as a function of x and y?

Answer: On or inside the ellipse $x^2 + \frac{1}{2}y^2 = 1$

130. Show that at $\left(\frac{1}{\sqrt{2}}, 0\right)$, the equation $x^2 + \frac{1}{2}y^2 + \frac{1}{3}z^2 = 1$ defines z implicitly as a function of x and y, and then compute $\dfrac{\partial z}{\partial x}\left(\frac{1}{\sqrt{2}}, 0, \sqrt{\frac{3}{2}}\right)$ and $\dfrac{\partial z}{\partial y}\left(\frac{1}{\sqrt{2}}, 0, \sqrt{\frac{3}{2}}\right)$.

Answer: $z = \sqrt{3\left(1 - x^2 - \frac{1}{2}y^2\right)}$ for $1 - x^2 - \frac{1}{2}y^2 \geq 0$, which is true at $\left(\frac{1}{\sqrt{2}}, 0\right)$. $\dfrac{\partial z}{\partial x} = -\sqrt{3}$, $\dfrac{\partial z}{\partial y} = 0$

131. Find $\partial z/\partial x$ and $\partial z/\partial y$ given that z is defined implicitly as a function of x and y by the equation $\sin xyz + \ln\left(x^2 + y^2 + z^2\right) = 0$.

Answer: $\dfrac{\partial z}{\partial x} = \dfrac{-\left(x^2yz + y^3z + yz^3\right)\cos xyz - 2x}{\left(x^3y + xy^3 + xyz^2\right)\cos xyz + 2z}$, $\dfrac{\partial z}{\partial y} = \dfrac{-\left(x^3z + xy^2z + xz^3\right)\cos xyz - 2y}{\left(x^3y + xy^3 + xyz^2\right)\cos xyz + 2z}$

11.6 Directional Derivatives and the Gradient Vector

132. Find the directional derivative of the function $f(x, y) = x^2 + y^2$ at the point $(1, 2)$ in the direction $\theta = \frac{\pi}{2}$.

A) 0 B) 1 C) 2 D) 3
E) 4 F) 5 G) 6 H) 7

Answer: E)

133. Find the directional derivative of the function $f(x, y) = x^2 + y^2$ at the point $(1, 1)$ in the direction $\theta = \frac{\pi}{4}$.

A) $\sqrt{2}$ B) $2\sqrt{2}$ C) $4\sqrt{2}$ D) $\frac{1}{\sqrt{2}}$
E) 1 F) 2 G) 4 H) $\frac{1}{2}$

Answer: B)

134. Find the directional derivative of the function $f(x, y) = x^2 + y^2$ at the point $\left(1, \sqrt{3}\right)$ in the direction $\theta = \frac{\pi}{6}$.

A) $\sqrt{3}$ B) $2\sqrt{3}$ C) $4\sqrt{3}$ D) $\frac{1}{\sqrt{3}}$
E) 1 F) 2 G) 4 H) $\frac{1}{2}$

Answer: B)

135. Let $f(x, y) = 3x + 2y$. Find the gradient vector ∇f.

A) $3x\mathbf{i} + 2y\mathbf{j}$ B) $3x^2\mathbf{i} + 2y^2\mathbf{j}$ C) $2x\mathbf{i} + 3y\mathbf{j}$ D) $2x^2\mathbf{i} + 3y^2\mathbf{j}$
E) $3y\mathbf{i} + 2x\mathbf{j}$ F) $2y\mathbf{i} + 3x\mathbf{j}$ G) $2\mathbf{i} + 3\mathbf{j}$ H) $3\mathbf{i} + 2\mathbf{j}$

Answer: H)

136. Let $f(x, y) = 2xy$. Find the gradient vector ∇f.

A) $2x\mathbf{i} + 2y\mathbf{j}$ B) $2y\mathbf{i} + 2x\mathbf{j}$ C) $2x^2\mathbf{i} + 2y^2\mathbf{j}$ D) $2y^2\mathbf{i} + 2x^2\mathbf{j}$
E) $x\mathbf{i} + y\mathbf{j}$ F) $y\mathbf{i} + x\mathbf{j}$ G) $x^2\mathbf{i} + y^2\mathbf{j}$ H) $y^2\mathbf{i} + x^2\mathbf{j}$

Answer: B)

137. Let $f(x, y) = \dfrac{1}{x + y^2}$. Find the gradient vector $\nabla f(1, 1)$ at the point $(x, y) = (1, 1)$.

A) $-\mathbf{i} - \mathbf{j}$ B) $-\frac{1}{4}\mathbf{i} - \frac{1}{2}\mathbf{j}$ C) $-\mathbf{i} - \frac{1}{2}\mathbf{j}$ D) $-\frac{1}{2}\mathbf{i} - \mathbf{j}$
E) $\mathbf{i} + \mathbf{j}$ F) $\frac{1}{4}\mathbf{i} + \frac{1}{2}\mathbf{j}$ G) $\mathbf{i} + \frac{1}{2}\mathbf{j}$ H) $\frac{1}{2}\mathbf{i} + \mathbf{j}$

Answer: B)

138. Let $f(x, y, z) = xe^{y \sin z}$. Find the gradient $\nabla f(1, 1, 0)$ at the point $(x, y, z) = (1, 1, 0)$.

A) $\mathbf{i}$ B) $\mathbf{j}$ C) $\mathbf{k}$ D) $\mathbf{j} + \mathbf{k}$
E) $\mathbf{i} + \mathbf{k}$ F) $\mathbf{i} + \mathbf{j}$ G) $\mathbf{i} + \mathbf{j} + \mathbf{k}$ H) 0

Answer: E)

139. Find the largest value of the directional derivative of the function $f(x, y) = xy + 2y^2$ at the point $(x, y) = (1, 2)$.

A) $\sqrt{53}$ B) $\sqrt{58}$ C) $\sqrt{63}$ D) $\sqrt{74}$
E) $\sqrt{85}$ F) $\sqrt{90}$ G) $\sqrt{97}$ H) $\sqrt{106}$

Answer: E)

140. Find the direction θ in which the directional derivative of the function $f(x, y) = xy + y^2$ at the point $(1, 1)$ is maximum.

A) $\cot^{-1} 2$ B) $\cot^{-1} 3$ C) $\cos^{-1} \frac{1}{2}$ D) $\cos^{-1} \frac{1}{3}$
E) $\sin^{-1} \frac{1}{2}$ F) $\sin^{-1} \frac{1}{3}$ G) $\tan^{-1} 2$ H) $\tan^{-1} 3$

Answer: H)

141. Find a normal vector to the surface $xyz = 8$ at the point $(1, 2, 4)$.

A) $2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ B) $\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$ C) $4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ D) $\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$
E) $4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ F) $2\mathbf{k} + \mathbf{j} + 4\mathbf{i}$ G) $\mathbf{i} + \mathbf{j} + \mathbf{k}$ H) $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

Answer: C)

142. Find an equation of the tangent plane to the surface $x^2 + y^2 + z^2 = 9$ at the point $(1, 2, 2)$.

A) $2x + y + z = 4$ B) $x + y + z = 5$
C) $x + 2y + 2z = 9$ D) $x + 4y + 4z = 17$
E) $2x^2 + y^2 + z^2 = 10$ F) $x^2 + y^2 + z^2 = 9$
G) $x^2 + 2y^2 + 2z^2 = 17$ H) $x^2 + 4y^2 + 4z^2 = 33$

Answer: C)

143. Find the directional derivative of the function $f(x, y, z) = \sqrt{xyz}$ at the point $(2, 4, 2)$ in the direction of the vector $\langle 4, 2, -4 \rangle$.

A) $-\frac{1}{6}$ B) $-\frac{1}{4}$ C) $-\frac{1}{2}$ D) 0
E) $\frac{1}{2}$ F) $\frac{1}{4}$ G) $\frac{1}{6}$ H) $\frac{1}{8}$

Answer: G)

144. Find the directional derivative of the function $f(x,y,z) = xe^{xy/z}$ in the direction $\mathbf{u} = -\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$ at the point $(3,0,1)$.

A) $\frac{17}{3}$ B) 6 C) $\frac{19}{3}$ D) $\frac{20}{3}$
E) 7 F) $\frac{22}{3}$ G) $\frac{23}{3}$ H) 8

Answer: A)

145. Find the direction of maximum increase of the function $f(x,y) = xe^{-y} + 3y$ at the point $(1,0)$.

A) $\langle 1,1\rangle$ B) $\langle 1,2\rangle$ C) $\langle 2,1\rangle$ D) $\langle 2,2\rangle$
E) $\langle 1,3\rangle$ F) $\langle 3,1\rangle$ G) $\langle 2,3\rangle$ H) $\langle 3,2\rangle$

Answer: B)

146. Find the direction of maximum increase of the function $f(x,y,z) = x^2 - 2xy + z^2$ at the point $(1,1,2)$.

A) $\langle 1,0,-2\rangle$ B) $\langle 4,1,2\rangle$ C) $\langle 2,1,-4\rangle$ D) $\langle 0,-2,4\rangle$
E) $\langle 2,0,3\rangle$ F) $\langle 3,1,2\rangle$ G) $\langle 0,2,1\rangle$ H) $\langle 5,-1,6\rangle$

Answer: D)

147. Find an equation of the tangent plane to the hyperboloid $x^2 + y^2 - z^2 - 2xy + 4xz = 4$ at the point $(1,0,1)$.

A) $3x - y + z = 4$ B) $2x - 4y + z = 3$ C) $x + 2y + 3z = 4$ D) $2x + y - z = 1$
E) $3x + 2y - z = 2$ F) $4x - y + 2z = 6$ G) $x + y - z = 0$ H) $5x + 3y - 2z = 3$

Answer: A)

148. Find an equation of the tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 4$ at the point $(4,1,1)$.

A) $2x + y - z = 1$ B) $x + 2y + 3z = 9$ C) $x - 2y + 4z = 0$ D) $4x - y + 6z = 1$
E) $x + 2y + 2z = 8$ F) $3x + 2y + z = 1$ G) $x + y + z = 6$ H) $2x + y + z = 10$

Answer: E)

149. Find the directional derivative of $f(x,y) = x^2 - 3xy + 2y^2$ at $(-1,2)$ in the direction of $\mathbf{i} + \sqrt{3}\mathbf{j}$.

Answer: $\frac{11\sqrt{3}-8}{2}$

150. Consider $f(x,y,z) = x^2y + y^3z + xz^3$ at the point $P = (2,1,-1)$. Find the directional derivative at P in the direction of $\langle 1,2,3\rangle$.

Answer: $\frac{26}{\sqrt{14}}$

151. Consider $f(x,y,z) = x^2y + y^3z + xz^3$ at the point $P = (2,1,-1)$. Find a vector in the direction in which f increases most rapidly at P.

Answer: $\langle 3,1,7\rangle$

152. Consider $f(x,y,z) = x^2y + y^3z + xz^3$ at the point $P = (2,1,-1)$. Find the rate of change of f in the direction in which f increases most rapidly at P.

Answer: $\sqrt{59}$

153. For $f(x,y) = x^2y^3$, $\mathbf{u} = \langle \frac{3}{5}, -\frac{4}{5}\rangle$, the directional derivative of f in the direction u at the point (x,y) is

A) $\dfrac{6xy^3 - 12x^2y^2}{5}$ B) $\dfrac{3x^2 - 4y^3}{5}$ C) $\dfrac{6x - 12y^2}{5}$ D) $2xy^3 + 3x^2y^2$ E) $\sqrt{4x^2y^6 + 9x^4y^4}$

Answer: A)

154. Find the directional derivative of $f(x,y,z) = x^2 + y^2 - z$ at the point $(1,3,5)$ in the direction of $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$.

Answer: $\frac{-2\sqrt{21}}{7}$

155. Let the temperature in a flat plate be given by the function $T(x, y) = 3x^2 + 2xy$. What is the value of the directional derivative of this function at the point $(3, -6)$ in the direction $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$? In what direction is the plate cooling most rapidly at $(3, -6)$?
Answer: $\frac{6}{5}$; the plate is cooling most rapidly in the direction $-\mathbf{i} - \mathbf{j}$

156. Find the directional derivative of $f(x, y) = x^3 + xy^2$ at the point $(1, -2)$ in the direction toward the origin.
Answer: $-3\sqrt{5}$

157. Find the directional derivative of $f(x, y) = 3x^2 + xy - y^3$ in the direction $\theta = \frac{\pi}{3}$.
Answer: $\left(\frac{6+\sqrt{3}}{2}\right)x + \frac{1}{2}y - \frac{3\sqrt{3}}{2}y^2$

158. Let $f(x, y, z) = x^2 + y^2 + xz$. Find the directional derivative of f at $(1, 2, 0)$ in the direction of the vector $\mathbf{v} = \langle 1, -1, 1 \rangle$.
Answer: $-\frac{1}{\sqrt{3}}$

159. Find the directional derivative of $f(x, y, z) = x^3 + y^2 - z$ at the point $(1, -1, 2)$ in the direction of the vector $\mathbf{v} = \langle -1, 2, 2 \rangle$.
Answer: -3

160. Calculate the directional derivative of the function $f(x, y) = 4 - x^3 - y^3$ at the point $(1, 1)$ in the direction given by the vector $\mathbf{v} = \langle 1, 1 \rangle$.
Answer: $-3\sqrt{2}$

161. Find the directional derivative of the function $f(x, y) = x^2y^5 + x^3$ at the point $(-1, 2)$ in the direction toward $(2, 3)$.
Answer: $-\frac{103}{\sqrt{10}}$

162. Find the directional derivative of $f(x, y) = x^2y + \ln y$ $(y > 0)$ at the point $(1, 1)$ in the direction of the origin.
Answer: $-2\sqrt{2}$

163. A bug is crawling on the surface $z = x^2 + xy + 2y^2$. When he reaches the point $(2, 1, 8)$ he wants to avoid vertical change. In which direction should he head? (He wants the directional derivative in the z-direction to be zero.)
Answer: $\mathbf{u} = \pm\frac{1}{\sqrt{61}}\langle -6, 5 \rangle$

164. Given $f(x, y) = x^2y$, $P_0 = (3, 2)$, let $\mathbf{u}$ be the unit vector for which the directional derivative $D_{\mathbf{u}}f(P_0)$ has maximum value. This maximum value is

A) $\frac{288}{5}$ B) 1 C) 6 D) $12\sqrt{3}$ E) 15

Answer: E)

165. Let $f(x, y) = 3x^3 + y^2 - 9x + 4y$. Find the direction at the origin in which $f(x, y)$ is decreasing the fastest.
Answer: $9\mathbf{i} - 4\mathbf{j}$

166. Given $f(x, y) = \frac{1}{4}x^4y^3$, at the point $(-1, 2)$. Find (a) the maximum value of the directional derivative and (b) the unit vector in the direction in which the directional derivative takes on its maximum value.
Answer: (a) $\sqrt{73}$; (b) $\left\langle -\frac{8}{\sqrt{73}}, \frac{3}{\sqrt{73}} \right\rangle$

167. If $f(x,y) = 2x^2 + 4y^2 - xy$, find the gradient at the point $(2,1)$. Also find the rate of change of $f(x,y)$ in the direction $\theta = \frac{\pi}{3}$ at $(2,1)$.

Answer: $\nabla f(2,1) = \langle 7,6 \rangle$; $D_{\mathbf{u}} f(2,1) = \frac{7+6\sqrt{3}}{2}$

168. The surface of a certain lake is represented by a region in the xy-plane such that the depth under the point corresponding to (x,y) is $f(x,y) = 300 - 2x^2 - 3y^2$. Zeke the dog is at the point $(3,2)$.

(a) In what direction should Zeke swim in order for the depth to decrease most rapidly?

(b) In what direction would the depth remain the same?

Answer: (a) $\mathbf{i} + \mathbf{j}$; (b) $\mathbf{i} - \mathbf{j}$ and $-\mathbf{i} + \mathbf{j}$

169. Find an equation of the tangent plane to the surface $4x^2 - y^2 + 3z^2 = 10$ at the point $(2,-3,1)$.

Answer: $16x + 6y + 6z = 20$

170. Find an equation of the tangent plane to the surface $z = f(x,y) = x^3y^4$ at the point $(-1,2,-16)$.

Answer: $48x - 32y - z + 96 = 0$

171. Given $f(x,y) = xe^{-2y}$, at the point $(4,0)$ find the equation of the tangent plane to the graph of f. Also, find a normal vector to the graph of f.

Answer: $z = x - 8y$; two normal vectors are $\langle 1,-8,-1 \rangle$ and $\langle -1,8,1 \rangle$

172. Consider the surface given by $z = xy^3 - x^2y$. Find an equation for the tangent plane to the surface at the point $(3,2,6)$. Also, find parametric equations for the normal line to the surface at the point $(3,2,6)$.

Answer: tangent plane: $4x - 27y + z = -36$; normal line: $x = 3 - 4t$, $y = 2 + 27t$, $z = 6 - t$

173. Find the equation of the tangent plane to the surface $xy^2 - 3xz = -3$ at the point $(-3,-2,1)$.

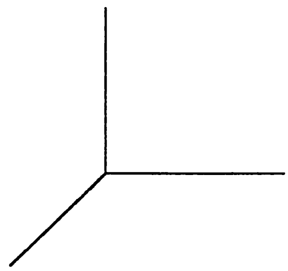
Answer: $x + 12y + 9z + 18 = 0$

174. Find an equation of the plane tangent to the surface $z = x^2y - xy + 2$ at the point $P(2,1,4)$.

Answer: $-3(x-2) - 2(y-1) + (z-4) = 0$

175. Consider the equation $x^2 + y^2 + z^2 = 49$.

(a) Sketch this surface.

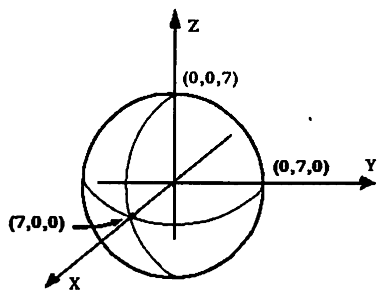


(b) Find an equation of the tangent plane to the surface at the point $(6, 2, 3)$.

(c) Find a symmetric equation of the line perpendicular to the tangent plane at the point $(6, 2, 3)$.

Answer:

(a)



The surface is a sphere with radius 7, centered at the origin.

(b) $6x + 2y + 3z = 49$

(c) $\dfrac{x-6}{6} = \dfrac{y-2}{2} = \dfrac{z-3}{3}$

176. Let S be the surface $x^2y + 4xz^3 - yz = 0$. An equation of the tangent plane to S at $(1, 2, -1)$ is

A) $y + 5z = -3$ B) $2x - 3y + z = 3$ C) $x - 3z = 4$
D) $x - y + z = 1$ E) $2x + y + 5z = -1$

Answer: A)

177. For the surface given by $2x^2 - 3y^2 + xz - 6 = 0$, find the equation of the tangent plane at $(2, -2, 5)$.

Answer: $13x + 12y + 2z - 12 = 0$

178. Find the unit vectors $\mathbf{u}$ and $\mathbf{v}$ for $f(x,y) = x^3 + 3y^2$ which describe the direction of maximal and minimal increase at $(2, 1)$ on the level curve $f(x,y) = 11$.

Answer: $\mathbf{u} = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$, $\mathbf{v} = \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle$

179. In what direction does the directional derivative for $f(x,y) = x^2y$ at $(1, 1)$ have value 1? Value -2? Is there a direction for which the value is 4?

Answer: Value 1 in direction of $\mathbf{u} = \langle 0, 1\rangle$ and $\mathbf{u} = \langle \frac{4}{5}, -\frac{3}{5}\rangle$; value -2 in direction of $\mathbf{u} = \langle -1, 0\rangle$ and $\mathbf{u} = \langle -\frac{3}{5}, \frac{4}{5}\rangle$; no

180. What is the direction of maximal decrease for $f(x,y,z) = xy + yz + xz$ at $(1, 1, 1)$?

Answer: Any vector $\mathbf{u}$ perpendicular to $\langle 2, 2, 2\rangle$, for example, $\langle -1, -1, 2\rangle$

181. Find the directional derivative of $f(x,y) = x^2 + xy + y^2$ at $(2,1)$ in the direction pointing toward $(1,2)$.

Answer: $\frac{13}{\sqrt{5}}$

11.7 Maximum and Minimum Values

182. The function $f(x,y) = x^2 - 2y^2$ has one critical point. Determine its location and type.

A) $(2,1)$, saddle point
B) $(2,1)$, minimum point
C) $(2,1)$, maximum point
D) $(\sqrt{2},1)$, saddle point
E) $(\sqrt{2},1)$, minimum point
F) $(\sqrt{2},1)$, maximum point
G) $(0,0)$, saddle point
H) $(0,0)$, minimum point

Answer: G)

183. The function $f(x,y) = x^2 + y^2 + xy$ has one critical point. Determine its location and type.

A) $(2,1)$, saddle point
B) $(2,1)$, minimum point
C) $(2,1)$, maximum point
D) $(\sqrt{2},1)$, saddle point
E) $(\sqrt{2},1)$, minimum point
F) $(\sqrt{2},1)$, maximum point
G) $(0,0)$, saddle point
H) $(0,0)$, minimum point

Answer: H)

184. The function $f(x,y) = x^2 + y^2 + 3xy$ has one critical point. Determine its location and type.

A) $(2,1)$, saddle point
B) $(2,1)$, minimum point
C) $(2,1)$, maximum point
D) $(\sqrt{2},1)$, saddle point
E) $(\sqrt{2},1)$, minimum point
F) $(\sqrt{2},1)$, maximum point
G) $(0,0)$, saddle point
H) $(0,0)$, minimum point

Answer: G)

185. The function $f(x,y) = x^2 + y^2 + xy + x$ has one critical point. Determine its location and type.

A) $\left(-\frac{2}{3},\frac{1}{3}\right)$, saddle point
B) $\left(-\frac{2}{3},\frac{1}{3}\right)$, minimum point
C) $\left(-\frac{2}{3},\frac{1}{3}\right)$, maximum point
D) $\left(\frac{1}{3},\frac{2}{3}\right)$, saddle point
E) $\left(\frac{1}{3},\frac{2}{3}\right)$, minimum point
F) $\left(\frac{1}{3},\frac{2}{3}\right)$, maximum point
G) $\left(\frac{1}{3},-\frac{2}{3}\right)$, saddle point
H) $\left(\frac{1}{3},-\frac{2}{3}\right)$, minimum point

Answer: B)

186. Determine how many critical points the function $f(x,y) = x^2 + y^2 + 2x^2y + 3$ has.

A) 0
B) 1
C) 2
D) 3
E) 4
F) 5
G) 6
H) 7

Answer: D)

187. Find the minimum value of the function $f(x,y) = 2x^2 + 3y^2$.

A) 0
B) 1
C) 2
D) 3
E) $-\sqrt{\frac{2}{3}}$
F) $-\sqrt{\frac{3}{2}}$
G) $-\frac{2}{3}$
H) $-\frac{3}{2}$

Answer: A)

188. Find the minimum value of the function $f(x,y) = x^2 + y^2 + 2x - 2y$.

A) -2
B) -1
C) -0
D) 1
E) 2
F) 3
G) 4
H) 5

Answer: A)

189. Find the shortest distance from the origin to the surface $z^2 = 2xy + 2$.

A) $\frac{1}{2}$ B) $\frac{1}{\sqrt{2}}$ C) 1 D) $\sqrt{2}$
E) 2 F) $2\sqrt{2}$ G) 4 H) $4\sqrt{2}$

Answer: D)

190. A cardboard box without a lid is to have volume 100 cubic inches, with total area of cardboard as small as possible. Find its height in inches.

A) 2 B) 5/2 C) 4 D) 5
E) $25^{1/3}$ F) $25^{2/3}$ G) $200^{1/3}$ H) $200^{2/3}$

Answer: E)

191. Find the point at which the function $f(x,y) = xy - x^2y - xy^2$ has a local maximum.

A) $(0,0)$ B) $(1,1)$ C) $(0,2)$ D) $(2,0)$
E) $(-1,0)$ F) $(0,-1)$ G) $\left(\frac{1}{3},\frac{1}{3}\right)$ H) $\left(\frac{1}{2},\frac{1}{2}\right)$

Answer: G)

192. Find the local maximum and minimum values and saddle points of the function $f(x,y) = 3x^3 + y^2 - 9x + 4y$.

Answer: $f(1,-2) = -10$ is a local minimum; $(-1,-2)$ is a saddle point.

193. Find the local maximum and minimum values and saddle points of the function $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$.

Answer: $f(0,0) = 4$ is a local maximum; $f(2,0) = 0$ is a local minimum; $(1,1)$ and $(1,-1)$ are saddle points.

194. The function $z = xy + (x+y)(120 - x - y)$ has a maximum. Find the values of x and y at which it occurs.

Answer: $x = y = 40$

195. Find the local maximum and minimum values and saddle points of the function $f(x,y) = x^3 - 3xy - y^3$.

Answer: $(0,0)$ is a saddle point; $f(-1,1) = 1$ is a local maximum

196. Find the local maximum and minimum values and saddle points of the function $f(x,y) = 3x^2 - 6xy + 6y^2 + 3x - 9y + 10$.

Answer: $f\left(\frac{1}{2},1\right) = \frac{25}{4}$ is a local minimum

197. Find the local maximum and minimum values and saddle points of the function $f(x,y) = x^2 + y^2 - xy + x - 5y$.

Answer: $f(1,3) = -7$ is a local minimum

198. Find the local maximum and minimum values and saddle points of the function $f(x,y) = -x^4 + 4xy - 2y^2 + 1$.

Answer: $f(-1,-1) = 2$ is a local maximum; $(0,0)$ is a saddle point; $f(1,1) = 2$ is a local maximum

199. Find the local maximum and minimum values and saddle points of the function $f(x,y) = 2x^3 + 4y^3 + 3x^2 - 12x - 192y + 5$.

Answer: $f(-2,-4) = 537$ is a local maximum; $(-2,4)$ is a saddle point; $(1,-4)$ is a saddle point; $f(1,4) = -514$ is a local minimum

200. Find the local maximum and minimum values and saddle points of the function $f(x,y) = x^3 + y^3 - 3xy + 5$.

Answer: $(0,0)$ is a saddle point; $f(1,1) = 4$ is a local minimum

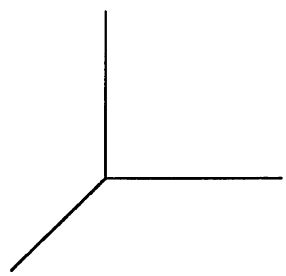
201. Find the local maximum and minimum values and saddle points of the function $f(x,y) = 4xy - x^4 - y^4 + \frac{1}{16}$.

Answer: $(0,0)$ is a saddle point; $f(1,1) = \frac{33}{16}$ is a local maximum; $f(-1,-1) = \frac{33}{16}$ is a local maximum

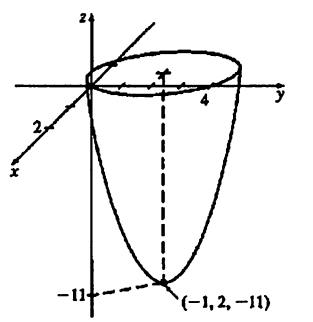
202. Examine the function $f(x,y) = x^2 + y^2 - 4x + 6y + 25$ for maximum or minimum points. At such points, give the maximum or minimum function values.

Answer: $f(2,-3) = 12$ is a minimum

203. Compute the minimum value of z and sketch a portion of the graph of $z = 3x^2 + 6x + 2y^2 - 8y$ near its lowest point.



Answer:



$z(-1,2) = -11$ is the absolute minimum.

204. For each of the following functions, find the critical point, if there is one, and determine if it is a local maximum, local minimum, saddle point, or otherwise.

(a) $f(x,y) = 4x^2 + 3x - 5y^2 - 8y + 7$

(b) $f(x,y) = -2x^2 + 7x - 7y^2 + 4y + 9$

(c) $f(x,y) = 3x^2 - 5y + 4y^2 + 12x - 11$

(d) $f(x,y) = 2y^2 + 7x + 17y + 12$

Answer: (a) $f\left(-\frac{3}{8}, -\frac{4}{5}\right) = \frac{771}{80}$ is a saddle point.
(b) $f\left(\frac{7}{4}, \frac{2}{7}\right) = \frac{879}{56}$ is a local maximum.
(c) $f\left(-2, \frac{5}{8}\right) = -\frac{393}{16}$ is a local minimum.
(d) No critical point

205. Find the critical points (if any) for $f(x,y) = \dfrac{1}{x} + \dfrac{1}{y} + xy$ and determine if each is a local extreme value or a saddle point.

Answer: $f(1,1) = 3$ is a local minimum.

11.8 Lagrange Multipliers

206. In using Lagrange multipliers to minimize the function $f(x,y) = x^2 + y^2$ subject to the constraint that $x + y = 3$, what is the value of the multiplier λ?

A) $\frac{1}{2}$ B) 1 C) $\frac{3}{2}$ D) 2
E) $\frac{5}{2}$ F) 3 G) $\frac{7}{2}$ H) 4

Answer: F)

207. In using Lagrange multipliers to minimize the function $f(x,y) = x^2 + y^2$ subject to the constraint that $xy = 2$, what is the value of the multiplier λ?

A) $\frac{1}{2}$ B) 1 C) $\frac{3}{2}$ D) 2
E) $\frac{5}{2}$ F) 3 G) $\frac{7}{2}$ H) 4

Answer: D)

208. Find the minimum value of the function $f(x,y) = x^2 + y^2$ subject to the constraint that $xy = 2$.

A) $\frac{1}{2}$ B) 1 C) $\frac{3}{2}$ D) 2
E) $\frac{5}{2}$ F) 3 G) $\frac{7}{2}$ H) 4

Answer: H)

209. Find the minimum value of the function $f(x,y) = 2x^2 + y^2$ subject to the constraint that $xy = 2$.

A) $\frac{1}{2}$ B) $\frac{1}{\sqrt{2}}$ C) 1 D) $\sqrt{2}$
E) 2 F) $2\sqrt{2}$ G) 4 H) $4\sqrt{2}$

Answer: H)

210. Find the minimum value of the function $f(x,y) = xy$ subject to the constraint that $x^2 + y^2 = 2$.

A) $\frac{1}{2}$ B) 1 C) $\frac{3}{2}$ D) 2
E) $-\frac{1}{2}$ F) -1 G) $-\frac{3}{2}$ H) -2

Answer: F)

211. Find the maximum value of the function $f(x,y) = xy$ subject to the constraint that $x^2 + y^2 = 2$.

A) $\frac{1}{2}$ B) 1 C) $\frac{3}{2}$ D) 2
E) $-\frac{1}{2}$ F) -1 G) $-\frac{3}{2}$ H) -2

Answer: B)

212. Solve completely, using Lagrange multipliers: Find the dimensions of a box with volume 1000 which minimizes the total length of the 12 edges.

Answer: $x = y = z = 10$

213. Find the greatest product three numbers can have if the sum of their squares must be 48.

Answer: 64

214. Optimize $f(x,y) = x^2 + y^2 + 2$ subject to $xy = 4$.

Answer: $f(2,2) = 10$ is a local minimum; $f(-2,-2) = 10$ is also a local minimum

215. Find the point on the plane $x - 2y + z = 3$ where $x^2 + 4y^2 + 2z^2$ is minimum.

Answer: $\left(\frac{6}{5}, -\frac{3}{5}, \frac{3}{5}\right)$

216. Maximize $3x - y + 1$ on the ellipse $3x^2 + y^2 = 16$.

Answer: $f(2,-2) = 9$

217. Compute the minimum value of $f(x,y,z) = x^2+y+z^2$ subject to the condition that $g(x) = 2x+y+4z = 6$.

Answer: $f(1,-4,2) = 1$

218. Find the maximum and minimum values of the function $f(x,y) = xy$ on the ellipse given by the equation $x^2 + \frac{y^2}{4} = 1$.

Answer: Maximum value is 1, minimum value is -1

219. Find two positive numbers whose sum is eighteen and whose product is a maximum, using the method of Lagrange multipliers.

Answer: 9 and 9

220. Use the method of Lagrange multipliers to find points on the surface $x^2+y^2+z^2 = 3$ where the function $f(x,y,z) = x+y+z$ has (a) a minimum, (b) a maximum.

Answer: (a) $(-1,-1,-1)$; (b) $(1,1,1)$

221. Find the extreme values of $f(x,y) = xy+2y^2+x^4-y^4$ on the circle $x^2+y^2 = 1$.

Answer: Maximum value is $f\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right) = f\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right) = \frac{3}{2}$, minimum value is $f\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right) = f\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right) = \frac{1}{2}$

222. What is the shortest distance from the origin to the surface $xyz^2 = 2$?

Answer: 2

223. What point on the surface $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 1$, $x > 0$, $y > 0$, $z > 0$ is closest to the origin?

Answer: $(3,3,3)$

224. Find the maximum value of $z = 2x+3y+4$ on the circle $x^2+y^2 = 1$.

Answer: $f\left(\frac{2}{\sqrt{13}},\frac{3}{\sqrt{13}}\right) = \sqrt{13}+4$

12 Multiple Integrals

12.1 Double Integrals over Rectangles

1. Let $f(x,y) = x^2y$, and let $R = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1\}$. Let R be its own partition, and let (x_1^*, y_1^*) be the center of R. Calculate the double Riemann sum of f.

A) $\frac{3}{2}$ B) $\frac{3}{4}$ C) $\frac{3}{8}$ D) $\frac{3}{16}$
E) $\frac{1}{2}$ F) $\frac{1}{4}$ G) $\frac{1}{8}$ H) $\frac{1}{16}$

Answer: G)

2. Let $f(x,y) = x$, and let $R = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1\}$. Let R be partitioned into two subrectangles by the line $x = \frac{1}{4}$, and let $\left(x_i^*, y_j^*\right)$ be the center of R_{ij}. Calculate the double Riemann sum of f.

A) $\frac{3}{16}$ B) $\frac{1}{4}$ C) $\frac{5}{16}$ D) $\frac{3}{8}$
E) $\frac{7}{16}$ F) $\frac{1}{2}$ G) $\frac{9}{16}$ H) $\frac{3}{4}$

Answer: F)

3. Let $f(x,y) = x^2$, and let $R = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1\}$. Let R be partitioned into two subrectangles by the line $x = \frac{1}{2}$, and let $\left(x_i^*, y_j^*\right)$ be the center of R_{ij}. Calculate the double Riemann sum of f.

A) $\frac{3}{16}$ B) $\frac{1}{4}$ C) $\frac{5}{16}$ D) $\frac{3}{8}$
E) $\frac{7}{16}$ F) $\frac{1}{2}$ G) $\frac{9}{16}$ H) $\frac{3}{4}$

Answer: C)

4. Let $f(x,y) = xy$, and let $R = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1\}$. Let R be partitioned into four subrectangles by the lines $x = \frac{1}{2}$ and $y = \frac{1}{2}$, and let $\left(x_i^*, y_j^*\right)$ be the upper right corner of R_{ij}. Calculate the double Riemann sum of f.

A) $\frac{3}{16}$ B) $\frac{1}{4}$ C) $\frac{5}{16}$ D) $\frac{3}{8}$
E) $\frac{7}{16}$ F) $\frac{1}{2}$ G) $\frac{9}{16}$ H) $\frac{5}{8}$

Answer: G)

5. Let $f(x,y) = xy$, and let $R = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1\}$. Let R be partitioned into four subrectangles by the lines $x = \frac{1}{2}$ and $y = \frac{1}{2}$, and let $\left(x_i^*, y_j^*\right)$ be the upper left corner of R_{ij}. Calculate the double Riemann sum of f.

A) $\frac{3}{16}$ B) $\frac{1}{4}$ C) $\frac{5}{16}$ D) $\frac{3}{8}$
E) $\frac{7}{16}$ F) $\frac{1}{2}$ G) $\frac{9}{16}$ H) $\frac{5}{8}$

Answer: A)

6. Calculate the double Riemann sum of f for the partition of R given by the indicated lines and the given choice of $\left(x_{ij}^*, y_{ij}^*\right)$. $f(x,y) = x^2 + 4y$, $R = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 3\}$, $x = 1$, $y = 1$, $y = 2$; $\left(x_{ij}^*, y_{ij}^*\right)$ = center of R_{ij}.

Answer: $\frac{87}{2}$

7. Calculate the double Riemann sum of f for the partition of R given by the indicated lines and the given choice of $\left(x_{ij}^*, y_{ij}^*\right)$. $f(x,y) = 2x + x^2y$, $R = \{(x,y) \mid -2 \le x \le 2,\ -1 \le y \le 1\}$, $x = -1$, $x = 0$, $x = 1$, $y = -\frac{1}{2}$, $y = 0$, $y = \frac{1}{2}$; $\left(x_{ij}^*, y_{ij}^*\right)$ = lower left corner of R_{ij}.

Answer: -11

8. Use the Midpoint Rule to estimate $\iint_R \left(x^2 + y^2\right) dA$ over $R = \{(x,y) \mid 0 \le x \le 2, 0 \le y \le 2\}$ partitioned by the lines $x = 1$ and $y = 1$. Then estimate the average value of $f(x,y) = x^2 + y^2$ over R.

Answer: $\iint_R \left(x^2 + y^2\right) dA \approx 10$; average value $\approx \frac{10}{4}$

9. Give an example of a non-constant function $f(x,y)$ such that the average value of f over $R = \{(x,y) \mid -1 \le x \le 1, -1 \le y \le 1\}$ is 0.

Answer: Possible functions include $f(x,y) = x$ and $f(x,y) = y$. Any linear function of x and y with no constant term will work, as will many other functions.

10. Use the Midpoint Rule to estimate $\iint_R (7 - 2x - y)\, dA$ over $R = \{(x,y) \mid -1 \le x \le 2, -1 \le y \le 2\}$ partitioned by the lines $x = 0$, $x = 1$, $y = 0$, and $y = 1$ into nine subrectangles.

Answer: 49.5

12.2 Iterated Integrals

11. Evaluate the iterated integral $\int_0^1 \int_0^1 x^2 y\, dx\, dy$.

A) $\frac{1}{2}$ B) $\frac{1}{3}$ C) $\frac{1}{4}$ D) $\frac{3}{4}$
E) $\frac{2}{3}$ F) $\frac{1}{6}$ G) $\frac{5}{6}$ H) 1

Answer: F)

12. Evaluate the iterated integral $\int_0^1 \int_0^2 (x + y)\, dx\, dy$.

A) $\frac{1}{2}$ B) 2 C) $\frac{3}{2}$ D) 2
E) $\frac{5}{2}$ F) 3 G) $\frac{7}{2}$ H) 4

Answer: F)

13. Evaluate the iterated integral $\int_0^1 \int_0^1 \left(x^2 + y\right) dx\, dy$.

A) $\frac{1}{2}$ B) $\frac{1}{3}$ C) $\frac{1}{4}$ D) $\frac{3}{4}$
E) $\frac{2}{3}$ F) $\frac{1}{6}$ G) $\frac{5}{6}$ H) 1

Answer: G)

14. Evaluate the iterated integral $\int_0^\pi \int_0^1 x \sin y \, dx \, dy$.

A) $\frac{1}{2}$ B) $-\frac{1}{2}$ C) $\frac{1}{4}$ D) $-\frac{1}{4}$
E) 2 F) -2 G) 1 H) -1

Answer: G)

15. Evaluate the iterated integral $\int_0^2 \int_0^3 (x^2 - y) \, dx \, dy$.

A) 2 B) 3 C) 4 D) 6
E) 9 F) 12 G) 18 H) 24

Answer: F)

16. Evaluate the double integral $\iint_R (x + 2y) \, dA$, where $R = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$.

A) $\frac{1}{4}$ B) $\frac{1}{2}$ C) $\frac{3}{4}$ D) 1
E) $\frac{5}{4}$ F) $\frac{3}{2}$ G) $\frac{7}{4}$ H) 2

Answer: F)

17. Evaluate the double integral $\iint_R (x^2 + y) \, dA$, where $R = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$.

A) $\frac{1}{2}$ B) $\frac{1}{3}$ C) $\frac{1}{4}$ D) $\frac{3}{4}$
E) $\frac{2}{3}$ F) $\frac{1}{6}$ G) $\frac{5}{6}$ H) 1

Answer: G)

18. Evaluate the double integral $\iint_R xy \, dA$, where $R = \{(x,y) \mid 0 \le x \le 1, 1 \le y \le 2\}$.

A) $\frac{1}{4}$ B) $\frac{1}{2}$ C) $\frac{3}{4}$ D) 1
E) $\frac{5}{4}$ F) $\frac{3}{2}$ G) $\frac{7}{4}$ H) 2

Answer: C)

19. Evaluate the double integral $\iint_R (x + \sin y) \, dA$, where $R = \{(x,y) \mid 0 \le x \le 2, 0 \le y \le \pi\}$.

A) $2 + \pi$ B) $4 + \pi$ C) $(2 + \pi)/2$ D) $4 + 2\pi$
E) $-2 - \pi$ F) $-4 - \pi$ G) $-(2 + \pi)/2$ H) $-(4 + \pi)/2$

Answer: D)

20. Evaluate the double integral $\iint_R e^{x-y} \, dA$, where $R = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$.

A) $e^2 + e^{-2} + 2$ B) $e^2 - e^{-2} + 2$ C) $e^2 + e^{-2} - 2$ D) $e^2 - e^{-2} - 2$
E) $e + e^{-1} + 2$ F) $e - e^{-1} + 2$ G) $e + e^{-1} - 2$ H) $e - e^{-1} - 2$

Answer: G)

21. Evaluate the double integral $\iint_R y \cos(xy) \, dA$, where $R = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le \pi\}$.

A) -3 B) -2 C) -1 D) 0
E) 1 F) 2 G) 3 H) 4

Answer: F)

22. Find $\int_0^2 f(x,y) \, dy$ and $\int_0^1 f(x,y) \, dx$ for $f(x,y) = 2xy - 3x^2$.

Answer: $\int_0^2 f(x,y) \, dy = 4x - 6x^2$, $\int_0^1 f(x,y) \, dx = y - 1$

23. Calculate the iterated integral $\int_0^2 \int_0^3 e^{x-y} \, dy \, dx$.

Answer: $e^2 - e^{-1} - 1 + e^{-3}$

24. Calculate the iterated integral $\int_0^1 \int_1^2 (x^4 - y^2) \, dx \, dy$.

Answer: $\frac{88}{15}$

25. Calculate the iterated integral $\int_0^{\pi/2} \int_0^{\pi/2} \sin(x + y) \, dy \, dx$

Answer: 2

26. Calculate the double integral $\iint_R \left(xy^2 + \frac{y}{x}\right) dA$, where $R = \{(x, y) \mid 2 \leq x \leq 3,\ -1 \leq y \leq 0\}$.

Answer: $\frac{5}{6} + \ln\sqrt{\frac{2}{3}}$

27. Calculate the double integral $\iint_R \frac{1+x}{1+y}\, dA$, where $R = \{(x, y) \mid -1 \leq x \leq 2,\ 0 \leq y \leq 1\}$.

Answer: $\frac{9}{2}\ln 2$

28. Compute the average value of $f(x, y) = x^2 + y^2$ over $R = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 2\}$.

Answer: $\frac{8}{3}$

29. Show that the average value of $f(x, y) = ax + by$ over $R = \{(x, y) \mid -1 \leq x \leq 1, -1 \leq y \leq 1\}$ is 0 for all values of a and b.

Answer: Average value $= \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}(ax + by)\, dy\, dx = \frac{1}{4}\int_{-1}^{1}\left[axy + \frac{1}{2}by^2\right]_{-1}^{1} dx$
$= \frac{1}{4}\int_{-1}^{1} 2ax\, dx = \frac{1}{4}\left[ax^2\right]_{-1}^{1} = 0$

30. Compute $\int_0^2 \int_{-1}^{1} \frac{ye^{y^2}}{1+x^2}\, dx\, dy$.

Answer: $\frac{1}{4}\pi\left(e^4 - 1\right)$

31. Compute $\int_0^1 \int_0^1 xy^2 e^{xy^3}\, dx\, dy$.

Answer: $\frac{1}{3}(e - 2)$

32. Describe the shape of the solid whose volume is given by the integral $\int_0^3 \int_{-2}^{2} \sqrt{9 - y^2}\, dx\, dy$.

Answer: Quarter-cylinder of radius 3 and height 4

12.3 Double Integrals over General Regions

33. Evaluate the iterated integral $\int_0^1 \int_0^y dx\, dy$.

A) $\frac{1}{2}$ B) $\frac{1}{3}$ C) $\frac{1}{4}$ D) $\frac{3}{4}$
E) $\frac{2}{3}$ F) $\frac{1}{6}$ G) $\frac{5}{6}$ H) 1

Answer: A)

34. Evaluate the iterated integral $\int_0^1 \int_x^1 dy\, dx$.

A) $\frac{1}{2}$ B) $\frac{1}{3}$ C) $\frac{1}{4}$ D) $\frac{3}{4}$
E) $\frac{2}{3}$ F) $\frac{1}{6}$ G) $\frac{5}{6}$ H) 1

Answer: A)

35. Evaluate the iterated integral $\int_0^1 \int_0^y x\, dx\, dy$.

A) $\frac{1}{2}$ B) $\frac{1}{3}$ C) $\frac{1}{4}$ D) $\frac{3}{4}$
E) $\frac{2}{3}$ F) $\frac{1}{6}$ G) $\frac{5}{6}$ H) 1

Answer: F)

36. Evaluate the iterated integral $\int_0^1 \int_x^1 y\, dy\, dx$.

A) $\frac{1}{2}$ B) $\frac{1}{3}$ C) $\frac{1}{4}$ D) $\frac{3}{4}$
E) $\frac{2}{3}$ F) $\frac{1}{6}$ G) $\frac{5}{6}$ H) 1

Answer: B)

37. Evaluate the iterated integral $\int_0^1 \int_{y^2}^1 x\,dx\,dy$.

A) 0.1 B) 0.2 C) 0.3 D) 0.4
E) 0.5 F) 0.6 G) 0.7 H) 0.8

Answer: D)

38. Evaluate the iterated integral $\int_0^1 \int_0^{y^2} y\,dx\,dy$.

A) $\frac{1}{2}$ B) $\frac{1}{3}$ C) $\frac{1}{4}$ D) $\frac{3}{4}$
E) $\frac{2}{3}$ F) $\frac{1}{6}$ G) $\frac{5}{6}$ H) 1

Answer: C)

39. Evaluate the iterated integral $\int_0^1 \int_0^y (x+y)\,dx\,dy$.

A) $\frac{1}{2}$ B) $\frac{1}{3}$ C) $\frac{1}{4}$ D) $\frac{3}{4}$
E) $\frac{2}{3}$ F) $\frac{1}{6}$ G) $\frac{5}{6}$ H) 1

Answer: A)

40. Evaluate the iterated integral $\int_0^1 \int_0^y e^{y^2}\,dx\,dy$.

A) $e-1$ B) $(e-1)/2$ C) $(e-1)/3$ D) $(e-1)/4$
E) $e-2$ F) $(e-2)/2$ G) $(e-2)/3$ H) $(e-2)/4$

Answer: B)

41. Evaluate the double integral $\iint_R (x+2y)\,dA$, where $R = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x\}$.

A) $\frac{1}{4}$ B) $\frac{1}{2}$ C) $\frac{3}{4}$ D) 1
E) $\frac{1}{6}$ F) $\frac{1}{3}$ G) $\frac{2}{3}$ H) $\frac{5}{6}$

Answer: G)

42. Evaluate the double integral $\iint_R xy\,dA$, where $R = \{(x,y) \mid 0 \le x \le y,\ 0 \le y \le 2\}$.

A) $\frac{1}{4}$ B) $\frac{1}{2}$ C) $\frac{3}{4}$ D) 1
E) $\frac{5}{4}$ F) $\frac{3}{2}$ G) $\frac{7}{4}$ H) 2

Answer: H)

43. Evaluate the double integral $\iint_R y\,dA$, where $R = \{(x,y) \mid y \le x \le 2,\ 0 \le y \le 2\}$.

A) $\frac{2}{3}$ B) $\frac{4}{3}$ C) 2 D) $\frac{8}{3}$
E) $\frac{10}{3}$ F) 4 G) $\frac{14}{3}$ H) $\frac{16}{3}$

Answer: B)

44. Find the volume under the paraboloid $z = x^2 + y^2$ above the region $R = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1\}$.

A) $\frac{1}{6}$ B) $\frac{1}{4}$ C) $\frac{1}{3}$ D) $\frac{1}{2}$
E) $\frac{2}{3}$ F) $\frac{3}{4}$ G) $\frac{5}{6}$ H) 1

Answer: E)

45. Find the volume under the paraboloid $z = x^2 + y^2$ above the region bounded by the x-axis, the y-axis, and the line $x + y = 1$.

A) $\frac{1}{6}$ B) $\frac{1}{4}$ C) $\frac{1}{3}$ D) $\frac{1}{2}$
E) $\frac{2}{3}$ F) $\frac{3}{4}$ G) $\frac{5}{6}$ H) 1

Answer: A)

46. Find the volume under the paraboloid $z = 4x^2 + y^2$ above the triangle with vertices $(0,0,0)$, $(3,0,0)$, and $(3,1,0)$.

A) $\frac{109}{4}$ B) $\frac{55}{2}$ C) $\frac{111}{4}$ D) 28
E) $\frac{113}{4}$ F) $\frac{57}{2}$ G) $\frac{115}{4}$ H) 29

Answer: A)

47. Express the integral $\int_0^2 \int_{x^2}^4 (xy^2 + x)\, dy\, dx$ as an equivalent integral with the order of integration reversed.

Answer: $\int_0^4 \int_0^{\sqrt{y}} (xy^2 + x)\, dx\, dy$

48. Evaluate $\int_0^1 \int_0^{x^3} (x^2 + y - y^2)\, dy\, dx$ and describe the region of integration.

Answer: $\frac{129}{630}$; R is bounded below by $y = 0$, above by $y = x^3$, on the left by $x = 0$ and on the right by $x = 1$.

49. Change the order of integration in the following integral and evaluate: $\int_0^9 \int_{\sqrt{y}}^3 \sin(\pi x^3)\, dx\, dy$

Answer: $\int_0^3 \int_0^{x^2} \sin(\pi x^3)\, dy\, dx = \frac{2}{3\pi}$

50. Evaluate the double integral of ye^{y^4} over the region bounded by $y = \sqrt{x}$, $y = 2$, and $x = 0$.

Answer: $\dfrac{e^{16} - 1}{4}$

51. Evaluate the double integral $\int_0^1 \int_x^1 \sin y^2\, dy\, dx$.

Answer: $\frac{1}{2}(1 - \cos 1)$

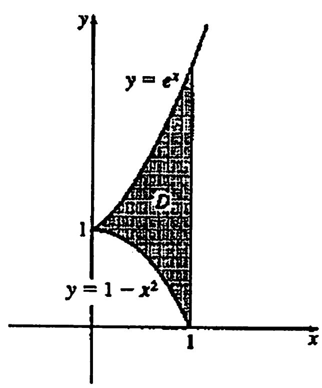
52. Evaluate the following double integral: $\int_0^a \int_0^{\sqrt{a^2 - x^2}} (x + y)\, dy\, dx$, where $a > 0$

Answer: $\dfrac{2a^3}{3}$

53. Write $\int_0^{16} \int_0^{\sqrt{x}} f(x, y)\, dy\, dx$ with the order of integration reversed.

Answer: $\int_0^4 \int_{y^2}^{16} f(x, y)\, dx\, dy$

54. The region D in $\mathbb{R}^2$ shown below is bounded by $x = 1$, $y = e^x$, and $y = 1 - x^2$.



(a) Compute $\iint_R x\, dA$ by finding $\int_0^1 \int_{1-x^2}^{e^x} x\, dy\, dx$.

(b) Write down the integral or integrals needed to compute $\iint_R x\, dA$ with the order of integration reversed.

Answer: (a) $\frac{3}{4}$; (b) $\int_0^1 \int_{\sqrt{1-y}}^1 x\, dx\, dy + \int_1^e \int_{\ln y}^1 x\, dx\, dy$

55. Evaluate $\displaystyle\int_1^2 \int_1^{x^2} \frac{x}{y}\, dy\, dx$.

Answer: $2 \ln 4 - \frac{3}{2}$

56. Evaluate $\int_0^2 \int_0^y x^2 y^4\, dx\, dy$.

Answer: $\frac{32}{3}$

57. Find the missing limits of integration for $\int_0^9 \int_{\sqrt{y}}^3 f(x,y)\, dx\, dy = \int\int f(x,y)\, dy\, dx$.

Answer: $\int_0^3 \int_0^{x^2} f(x,y)\, dy\, dx$

58. Find the volume under the surface $f(x,y) = xy$ over the region bounded by $y = 2x$, $x = 2y$, $x = 0$, and $x = 1$.

Answer: $\frac{15}{32}$

59. Use a double integral to find the volume of the solid bounded above by $y = 9 - x^2$, below by $z = 0$, and laterally by $y^2 = 3x$.

Answer: $\frac{432}{7}$

60. Find the volume of the solid in the first octant that is bounded by the plane $y + z = 4$, the cylinder $y = x^2$, and the xy- and yz-planes.

Answer: $\frac{128}{15}$

61. Find the volume of the solid bounded by $z = 4 - x^2$, $y^2 = 4x$, and the xy-plane.

Answer: $\frac{256}{21}\sqrt{2}$

62. Find the volume of the solid under the surface $z = x^2 + y^2$ and lying above the region $\{(x,y) \mid 0 \le x \le 1, x^2 \le y \le \sqrt{x}\}$.

Answer: $\frac{6}{35}$

63. Evaluate the iterated integral $\int_1^2 \int_0^{\pi/x} x^2 \sin xy\, dy\, dx$.

Answer: 3

64. Evaluate the iterated integral $\int_0^2 \int_{x^2}^4 \left(2x^2 y + 7\right) dy\, dx$.

Answer: $\frac{432}{7}$

65. Evaluate the iterated integral $\int_0^{\pi/2} \int_0^{\sin y} e^x \cos y\, dx\, dy$.

Answer: $e - 2$

66. $\int_0^1 \int_0^x F(x,y)\, dy\, dx$ is equivalent to

A) $\int_0^1 \int_y^1 F(x,y)\, dx\, dy$ B) $\int_0^1 \int_1^y F(x,y)\, dx\, dy$ C) $\int_0^1 \int_0^1 F(x,y)\, dx\, dy$
D) $\int_0^1 \int_{-y}^y F(x,y)\, dx\, dy$ E) $\int_0^1 \int_0^y F(x,y)\, dx\, dy$

Answer: A)

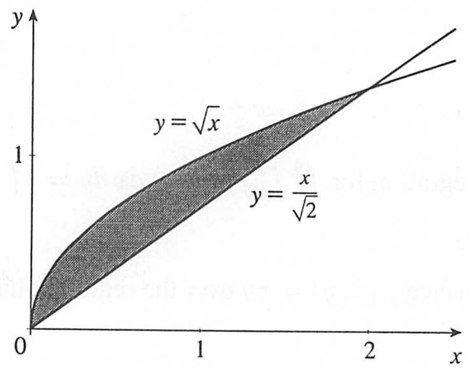
67. Let S be the surface defined by $z = 1 - y - x^2$. Let V be the volume of the 3-dimensional region in the first octant bounded by S and the coordinate planes. Set up (but do not evaluate) the iterated integrals for V in two ways:

(a) Integrate first with respect to x and then with respect to y.

(b) Integrate first with respect to y and then with respect to x.

Answer: (a) $\int_0^1 \int_0^{\sqrt{1-y}} \left(1 - y - x^2\right) dx\, dy$; (b) $\int_0^1 \int_0^{1-x^2} \left(1 - y - x^2\right) dy\, dx$

68. Describe the region sketched below both as a type I and as a type II region.



Answer: Type I: $0 \le x \le 2$, $\frac{1}{\sqrt{2}}x \le y \le \sqrt{x}$; type II: $0 \le y \le \sqrt{2}$, $y^2 \le x \le \sqrt{2}y$

69. Give good estimates of lower and upper bounds for $\int_1^2 \int_0^2 xy\sqrt{x^2+y^2}\, dx\, dy$.

Answer: Lower bound: 0; upper bound: $16\sqrt{2}$

70. Evaluate the iterated integral $\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$.

Answer: $\frac{1}{2}(e-1)$

12.4 Double Integrals in Polar Coordinates

71. Evaluate the integral $\iint_R (x^2+y^2)\, dA$, where R is the disk with center the origin and radius 1.

A) $\frac{\pi}{6}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{3}$ D) $\frac{\pi}{2}$
E) $\frac{2\pi}{3}$ F) $\frac{3\pi}{4}$ G) $\frac{5\pi}{6}$ H) π

Answer: D)

72. Find the volume beneath the cone $z = r$ above the disk with center at the polar coordinates $(1, 0)$ and radius 1.

A) $\frac{5\pi}{2}$ B) $\frac{11\pi}{3}$ C) $\frac{32\pi}{9}$ D) $\frac{46\pi}{27}$
E) $\frac{5}{2}$ F) $\frac{11}{3}$ G) $\frac{32}{9}$ H) $\frac{46}{27}$

Answer: G)

73. Find the volume of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

A) 4π B) 8π C) 12π D) 16π
E) 24π F) 32π G) 48π H) 64π

Answer: B)

74. Find the volume of the solid obtained by intersecting the two paraboloids $z = x^2+y^2$ and $z = 1-x^2-y^2$.

A) $\frac{\pi}{6}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{3}$ D) $\frac{\pi}{2}$
E) $\frac{2\pi}{3}$ F) $\frac{3\pi}{4}$ G) $\frac{5\pi}{6}$ H) π

Answer: B)

75. Find the volume of the solid obtained by intersecting the two cones $z = \sqrt{x^2+y^2}$ and $z = 1-\sqrt{x^2+y^2}$.

A) $\frac{\pi}{12}$ B) $\frac{\pi}{6}$ C) $\frac{\pi}{4}$ D) $\frac{\pi}{3}$
E) $\frac{\pi}{2}$ F) $\frac{3\pi}{4}$ G) $\frac{5\pi}{6}$ H) π

Answer: A)

76. Find the volume of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$.

A) $\frac{\pi}{6}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{3}$ D) $\frac{\pi}{2}$
E) $\frac{2\pi}{3}$ F) $\frac{3\pi}{4}$ G) $\frac{5\pi}{6}$ H) π

Answer: D)

77. Evaluate the integral $\iint_R \sqrt{x^2+y^2}\,dA$, where R is the disk with center the origin and radius 2.

A) $\frac{\pi}{3}$ B) $\frac{2\pi}{3}$ C) $\frac{4\pi}{3}$ D) $\frac{8\pi}{3}$
E) $\frac{16\pi}{3}$ F) $\frac{32\pi}{3}$ G) $\frac{64\pi}{3}$ H) $\frac{128\pi}{3}$

Answer: E)

78. If R is the region inside the circle $x^2 + y^2 = 4$, then $\iint_R x\sqrt{x^2+y^2}\,dA$ is equal to

A) $\int_0^2 \int_0^{2\pi} r^2\,d\theta\,dr$ B) $\int_0^2 \int_0^{\pi/2} r^2 \cos\theta\,d\theta\,dr$ C) $4\int_0^2 \int_0^{2\pi} r^3 \sin\theta\,d\theta\,dr$
D) $\int_0^2 \int_0^{2\pi} r^3 \cos\theta\,d\theta\,dr$ E) $\int_{-2}^2 \int_{-2}^2 x\sqrt{x^2+y^2}\,dy\,dx$

Answer: D)

79. Find the area inside the circle $r = 2\cos\theta$ and outside the circle $r = 1$.

Answer $\frac{\pi}{3} + \frac{\sqrt{3}}{2}$

80. Use polar coordinates to evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{-(x^2+y^2)}\,dy\,dx$.

Answer: $\dfrac{e-1}{4e}\pi$

81. Convert the integral $\displaystyle\int_0^2 \int_{-\sqrt{4-x^2}}^0 \frac{xy}{\sqrt{x^2+y^2}}\,dy\,dx$ to polar coordinates and evaluate it.

Answer: $-\frac{4}{3}$

82. Evaluate $\int_{-1}^0 \int_{-\sqrt{1-y^2}}^0 \cos\left(x^2+y^2\right)\,dx\,dy$.

Answer: $\dfrac{(\sin 1)\,\pi}{4}$

83. A solid is bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the xy-plane. Compute the volume of this solid using polar coordinates.

Answer: $V = \int_0^{2\pi} \int_0^1 \left(1-r^2\right) r\,dr\,d\theta = \frac{\pi}{2}$

84. Evaluate $\int_{-2}^2 \int_0^{\sqrt{4-x^2}} \sqrt{9-x^2-y^2}\,dy\,dx$.

Answer: $\left(9 - \dfrac{5^{3/2}}{3}\right)\pi$

85. Use polar coordinates to find the area inside the circle $x^2 + y^2 = 4$ and to the right of the line $x = 1$.

Answer: $\frac{4}{3}\pi - \frac{\sqrt{3}}{2}$

86. Find the area common to the two disks bounded by $r = \sin\theta$ and $r = \cos\theta$.

Answer: $\frac{\pi}{4}$

87. Evaluate $\iint_D \cos\left(x^2+y^2\right)\,dA$, where $D = \left\{(x,y) \mid x^2+y^2 \le 1\right\}$, the unit disk.

Answer $\pi \sin 1$

12.5 Applications of Double Integrals

88. Find the mass of the lamina that occupies the region $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$ and has density function $\rho(x, y) = x$.

A) $\frac{1}{6}$ B) $\frac{1}{4}$ C) $\frac{1}{3}$ D) $\frac{1}{2}$
E) $\frac{2}{3}$ F) $\frac{3}{4}$ G) $\frac{5}{6}$ H) 1

Answer: D)

89. Find the mass of the lamina that occupies the region $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$ and has density function $\rho(x, y) = xy$.

A) $\frac{1}{6}$ B) $\frac{1}{4}$ C) $\frac{1}{3}$ D) $\frac{1}{2}$
E) $\frac{2}{3}$ F) $\frac{3}{4}$ G) $\frac{5}{6}$ H) 1

Answer: B)

90. Find the mass of the lamina that occupies the region $D = \{(x, y) \mid 0 \le x \le 1, x^2 \le y \le 1\}$ and has density function $\rho(x, y) = x + y$.

A) 0.35 B) 0.40 C) 0.45 D) 0.50
E) 0.55 F) 0.60 G) 0.65 H) 0.70

Answer: G)

91. Find the y-coordinate of the center of mass of the lamina that occupies the region $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$ and has density function $\rho(x, y) = x$.

A) $\frac{1}{6}$ B) $\frac{1}{4}$ C) $\frac{1}{3}$ D) $\frac{1}{2}$
E) $\frac{2}{3}$ F) $\frac{3}{4}$ G) $\frac{5}{6}$ H) 1

Answer: $\frac{1}{2}$

92. Find the x-coordinate of the center of mass of the lamina that occupies the region $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$ and has density function $\rho(x, y) = xy$.

A) $\frac{1}{6}$ B) $\frac{1}{4}$ C) $\frac{1}{3}$ D) $\frac{1}{2}$
E) $\frac{2}{3}$ F) $\frac{3}{4}$ G) $\frac{5}{6}$ H) 1

Answer: E)

93. Find the x-coordinate of the center of mass of the lamina that occupies the region $D = \{(x, y) \mid 0 \le x \le 1, x^2 \le y \le 1\}$ and has density function $\rho(x, y) = x + y$.

A) $\frac{6}{13}$ B) $\frac{7}{13}$ C) $\frac{8}{13}$ D) $\frac{9}{13}$
E) $\frac{6}{11}$ F) $\frac{7}{11}$ G) $\frac{8}{11}$ H) $\frac{9}{11}$

Answer: A)

94. Find the moment of inertia I_y of the lamina that occupies the region $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$ and has density function $\rho(x, y) = x$.

A) $\frac{1}{24}$ B) $\frac{1}{18}$ C) $\frac{1}{12}$ D) $\frac{1}{9}$
E) $\frac{1}{6}$ F) $\frac{1}{4}$ G) $\frac{1}{3}$ H) $\frac{1}{2}$

Answer: F)

95. Find the moment of inertia I_x of the lamina that occupies the region $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$ and has density function $\rho(x,y) = x$.

A) $\frac{1}{24}$ B) $\frac{1}{18}$ C) $\frac{1}{12}$ D) $\frac{1}{9}$
E) $\frac{1}{6}$ F) $\frac{1}{4}$ G) $\frac{1}{3}$ H) $\frac{1}{2}$

Answer: E)

96. Find the polar moment of inertia I_0 of the lamina that occupies the region $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$ and has density function $\rho(x,y) = 1$.

A) $\frac{1}{6}$ B) $\frac{1}{4}$ C) $\frac{1}{3}$ D) $\frac{1}{2}$
E) $\frac{2}{3}$ F) $\frac{3}{4}$ G) $\frac{5}{6}$ H) 1

Answer: E)

97. A phonograph turntable is made in the shape of a circular disk of radius 6 inches with density function $\rho(x,y) = \sqrt{x^2+y^2}$. Find the mass of the disk.

A) 60π B) 96π C) 108π D) 120π
E) 144π F) 180π G) 192π H) 240π

Answer: E)

98. A phonograph turntable is made in the shape of a circular disk of radius 6 inches with density function $\rho(x,y) = \sqrt{x^2+y^2}$. Find the polar moment of inertia I_0 of the disk.

A) 2945.4π B) 3110.4π C) 3245.4π D) 3362.4π
E) 3495.4π F) 3560.4π G) 3632.4π H) 3775.4π

Answer: B)

99. Let R be the region bounded by $y = x^2$, $y = 0$, and $x = 1$. Find the center of mass of a lamina in the shape of R with density function $\rho(x,y) = xy$.

Answer: $\left(\frac{6}{7}, \frac{1}{2}\right)$

100. Find the moment of inertia I_x about the x-axis and the moment of inertia I_y about the y-axis for the region in the first quadrant bounded by $y = x$ and $y^2 = x^3$, assuming $\rho = 1$.

Answer: $I_x = \frac{1}{44}$; $I_y = \frac{1}{36}$

101. Electric charge is distributed over the unit disk $x^2 + y^2 \le 1$ so that the charge density at (x,y) is $\sigma(x,y) = 1 + x^2 + y^2$ (measured in coulombs per square meter). Find the total charge on the disk.

Answer: $\frac{3\pi}{2}$ C

102. Find the mass and center of mass of the lamina that occupies the region $D = \{(x,y) \mid 0 \le x \le 2, 0 \le y \le 3\}$ and has density function $\rho(x,y) = y$.

Answer: $m = 9$, $(\overline{x}, \overline{y}) = (1, 2)$

103. Find the mass and center of mass of the lamina that occupies the triangular region with vertices $(0,0)$, $(1,1)$, and $(4,0)$, and has density function $\rho(x,y) = x$.

Answer: $m = \frac{10}{3}$; $(\overline{x}, \overline{y}) = (2.1, 0.3)$

104. Find the center of mass of the lamina that occupies the part of the disk $x^2 + y^2 \le 1$ in the first quadrant if the density at any point is proportional to the square of its distance from the origin.

Answer: $(\overline{x}, \overline{y}) = \left(\frac{8}{5\pi}, \frac{8}{5\pi}\right)$

105. Find the moments of inertia I_x, I_y, and I_0 for the lamina that occupies the region given by $D = \{(x,y) \mid 0 \le x \le 2, 0 \le y \le 3\}$.
Answer: $I_x = 18$, $I_y = 8$, $I_0 = 26$

106. The average value of $f(x,y)$ over a region D in the plane with area $A(D)$ is $\frac{1}{A(D)} \iint_D f(x,y)\,dA$.
Find the average value of $f(x,y) = xy$ over the region $D = \{(x,y) \mid x^2 + y^2 \le 1, xy \ge 0\}$.
Answer: $\frac{1}{2\pi}$

107. The centroid $(\overline{x}, \overline{y})$ of a planar region D is the center of mass with the density function $\rho(x,y) = 1$. Find the centroid of $D = \{(x,y) \mid x^2 + y^2 \le 4, x \ge 0, y \ge 0\}$.
Answer $\left(\frac{8}{3\pi}, \frac{8}{3\pi}\right)$

108. Find the center of mass of $D = \{(x,y) \mid x^2 + y^2 \le 4, x \ge 0, y \ge 0\}$ if $\rho(x,y) = \sqrt{x^2 + y^2}$.
Answer: $\left(\frac{3}{\pi}, \frac{3}{\pi}\right)$

109. Find the centroid of $D = \left\{(x,y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1, y \ge 0\right\}$, the region enclosed by the upper half of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
Answer: $\left(0, \frac{8b}{3\pi}\right)$

12.6 Surface Area

110. Find the surface area of the surface $z = xy$ inside the cylinder $x^2 + y^2 = 1$.
Answer: $\frac{2\pi}{3}\left(2\sqrt{2} - 1\right)$

111. Compute the area of that part of the graph of $3z = 5 + 2x^{3/2} + 4y^{3/2}$ which lies above the rectangular region in the first quadrant of the xy-plane bounded by the lines $x = 0$, $x = 3$, $y = 0$, and $y = 6$.
Answer: $\frac{4}{15}\left(392\sqrt{7} - 789\right)$

112. Find the surface area of the part of the cone $z = \left(x^2 + y^2\right)^{1/2}$ lying inside the cylinder $x^2 - 2x + y^2 = 0$.
Answer: $\sqrt{2}\pi$

113. Set up, but do not evaluate, the integral to find the surface area of the portion of the sphere $x^2 + y^2 + z^2 = 16$ between the planes $z = 1$ and $z = 2$.
Answer: $\int_0^{2\pi} \int_{\sqrt{12}}^{\sqrt{15}} \sqrt{\frac{16}{16 - r^2}}\, r\, dr\, d\theta$

114. Find the area of the surface cut from the cone $z = 1 - \sqrt{x^2 + y^2}$ by the cylinder $x^2 + y^2 = y$.
Answer: $\frac{\pi\sqrt{2}}{4}$

115. Find the area of the portion of the surface $z = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 4$.
Answer: $\frac{\pi}{6}\left(17\sqrt{17} - 1\right)$

116. Find the area of that part of the plane $2x + 3y - z + 1 = 0$ that lies above the rectangle $[1,4] \times [2,4]$.
Answer: $6\sqrt{14}$

117. Find the area of that part of the surface $z = x + y^2$ that lies above the triangle with vertices $(0,0)$, $(1,1)$, and $(0,1)$.
Answer: $\frac{1}{2}\sqrt{6} - \frac{1}{6}\sqrt{2}$

118. Find the area of that part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the xy-plane.
Answer: $\frac{\pi}{6}\left(17\sqrt{17} - 1\right)$

119. Find the area of that part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = 1$.
Answer: 4π

120. Find the area of that part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies inside the paraboloid $z = x^2 + y^2$.
Answer: 4π

121. Find the surface area for the part of the plane $5z = 3x - 4y$ that lies inside the elliptic cylinder $x^2 + 2y^2 = 2$.
Answer: 2π

122. Find the surface area of $f(x,y) = (2xy)^{1/2}$ over the unit box $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$.
Answer: $\frac{4\sqrt{2}}{3}$

123. Show that the surface areas for the functions $f(x,y) = 2x^2 + 2y^2$ and $g(x,y) = 4xy$ over the disk $D = \left\{(x,y) \mid x^2 + y^2 \le 1\right\}$ are equal.
Answer: $\dfrac{\partial f}{\partial x} = \dfrac{\partial g}{\partial y} = 4x$ and $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x} = 4y$, so the surface area integrals $\iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA$ and $\iint_D \sqrt{g_x^2 + g_y^2 + 1}\, dA$ are equal.

124. Set up the formula for the surface area of the part of the hyperbolic paraboloid $z = x^2 - y^2$ above the xy-plane and inside the disk of radius 2.
Answer: $4\int_0^{\pi/4}\int_0^2 r\sqrt{4r^2 + 1}\, dr\, d\theta$

125. Find the area of the part of the plane $x + z = 4$ that lies above the square with vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$.

A) 1 B) $\sqrt{2}$ C) $\sqrt{3}$ D) 2
E) $\sqrt{5}$ F) $\sqrt{6}$ G) $\frac{\sqrt{5}}{2}$ H) $\frac{\sqrt{6}}{2}$

Answer: B)

126. Find the area of the part of the plane $x + y + z = 6$ that lies above the square with vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$.

A) 1 B) $\sqrt{2}$ C) $\sqrt{3}$ D) 2
E) $\sqrt{5}$ F) $\sqrt{6}$ G) $\frac{\sqrt{5}}{2}$ H) $\frac{\sqrt{6}}{2}$

Answer: C)

127. Find the area of the hemisphere $z = \sqrt{4 - x^2 - y^2}$.

A) $\left(4 - 2\sqrt{2}\right)\pi$ B) $\left(6 - 3\sqrt{2}\right)\pi$ C) $\left(8 - 4\sqrt{2}\right)\pi$ D) $\left(6 - 3\sqrt{3}\right)\pi$
E) $\left(8 - 4\sqrt{3}\right)\pi$ F) 6π G) 8π H) 16π

Answer: G)

128. Find the area of the part of the hemisphere $z = \sqrt{4 - x^2 - y^2}$ that lies above the area inside the circle $x^2 + y^2 = 1$.

A) $(4 - 2\sqrt{2})\,\pi$ B) $(6 - 3\sqrt{2})\,\pi$ C) $(8 - 4\sqrt{2})\,\pi$ D) $(6 - 3\sqrt{3})\,\pi$
E) $(8 - 4\sqrt{3})\,\pi$ F) 6π G) 8π H) 16π

Answer: E)

129. Find the area of the surface with parametric equations $x = uv$, $y = u + v$, $z = u - v$, $u^2 + v^2 \leq 1$.

A) $\frac{\sqrt{2}}{3}\pi$ B) $\frac{\sqrt{6}}{3}\pi$ C) $\frac{\sqrt{2}-1}{3}\pi$ D) $\frac{\sqrt{6}-1}{3}\pi$
E) $\frac{2\sqrt{2}-1}{3}\pi$ F) $\frac{6\sqrt{6}-8}{3}\pi$ G) $\frac{10\sqrt{2}-1}{3}\pi$ H) $\frac{10\sqrt{6}-9}{3}\pi$

Answer: F)

130. Find the area of the part of the surface $z = x + y^2$ that lies above the triangle with vertices $(0, 0)$, $(1, 1)$, and $(0, 1)$.

Answer: $\frac{3}{\sqrt{6}} - \frac{1}{3\sqrt{2}}$

131. Find the area of the part of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 9$.

Answer: $\frac{\pi}{6}\left(37\sqrt{37} - 1\right)$

12.7 Triple Integrals

132. Evaluate the iterated integral $\int_0^1 \int_0^z \int_0^y dx\,dy\,dz$.

A) $\frac{1}{120}$ B) $\frac{1}{90}$ C) $\frac{1}{60}$ D) $\frac{1}{48}$
E) $\frac{1}{36}$ F) $\frac{1}{24}$ G) $\frac{1}{12}$ H) $\frac{1}{6}$

Answer: H)

133. Evaluate the iterated integral $\int_0^1 \int_0^z \int_0^y y\,dx\,dy\,dz$.

A) $\frac{1}{120}$ B) $\frac{1}{90}$ C) $\frac{1}{60}$ D) $\frac{1}{48}$
E) $\frac{1}{36}$ F) $\frac{1}{24}$ G) $\frac{1}{12}$ H) $\frac{1}{6}$

Answer: G)

134. Evaluate the iterated integral $\int_0^1 \int_0^z \int_0^{y^2} dx\,dy\,dz$.

A) $\frac{1}{120}$ B) $\frac{1}{90}$ C) $\frac{1}{60}$ D) $\frac{1}{48}$
E) $\frac{1}{36}$ F) $\frac{1}{24}$ G) $\frac{1}{12}$ H) $\frac{1}{6}$

Answer: G)

135. Evaluate the iterated integral $\int_0^1 \int_0^z \int_0^{y^2} x\,dx\,dy\,dz$.

A) $\frac{1}{120}$ B) $\frac{1}{90}$ C) $\frac{1}{60}$ D) $\frac{1}{48}$
E) $\frac{1}{36}$ F) $\frac{1}{24}$ G) $\frac{1}{12}$ H) $\frac{1}{6}$

Answer: C)

136. Evaluate the triple integral $\iiint_E (x + 2y)\,dV$, where $E = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$.

A) $\frac{1}{4}$ B) $\frac{1}{2}$ C) $\frac{3}{4}$ D) 1
E) $\frac{5}{4}$ F) $\frac{3}{2}$ G) $\frac{7}{4}$ H) 2

Answer: F)

137. Evaluate the triple integral $\iiint_E (x^2 + 2y)\,dV$, where $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$.

A) $\frac{1}{4}$ B) $\frac{1}{3}$ C) $\frac{1}{2}$ D) $\frac{2}{3}$
E) $\frac{3}{4}$ F) 1 G) $\frac{5}{4}$ H) $\frac{4}{3}$

Answer: H)

138. Evaluate the triple integral $\iiint_E x\,dV$, where $E = \{(x,y,z) \mid 0 \le x \le y, 0 \le y \le 1, 0 \le z \le 1\}$.

A) $\frac{1}{6}$ B) $\frac{1}{4}$ C) $\frac{1}{3}$ D) $\frac{1}{2}$
E) $\frac{2}{3}$ F) $\frac{1}{12}$ G) $\frac{5}{6}$ H) 1

Answer: A)

139. Evaluate the triple integral $\iiint_E x^2\,dV$, where $E = \{(x,y,z) \mid 0 \le x \le y, 0 \le y \le 1, 0 \le z \le 1\}$.

A) $\frac{1}{6}$ B) $\frac{1}{4}$ C) $\frac{1}{3}$ D) $\frac{1}{2}$
E) $\frac{2}{3}$ F) $\frac{1}{12}$ G) $\frac{5}{6}$ H) 1

Answer: F)

140. Find the mass of the solid that occupies the region $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$ and has density function $\rho(x,y,z) = x$.

A) $\frac{1}{6}$ B) $\frac{1}{4}$ C) $\frac{1}{3}$ D) $\frac{1}{2}$
E) $\frac{2}{3}$ F) $\frac{3}{4}$ G) $\frac{5}{6}$ H) 1

Answer: D)

141. Find $\iiint_R 2x\,dV$, where R is the region in the first octant bounded by the cylinders $z = 1 - y^2$ and $z = 1 - x^2$.

Answer: $\frac{2}{5}$

142. Find the volume, using triple integrals, of the region in the first octant beneath the plane $x + 2y + 3z = 6$.

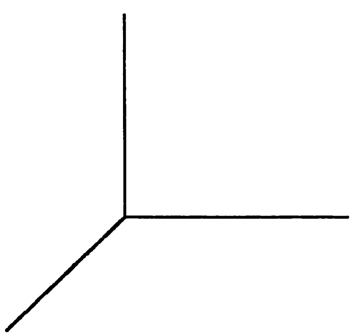
Answer: 6

143. Find $\iiint_S x^2 y\,dV$, where S is the solid bounded by the cylinder $y = x^2$ and the planes $z = 0$, $y = 1$, and $z = y$.

Answer: $\frac{4}{27}$

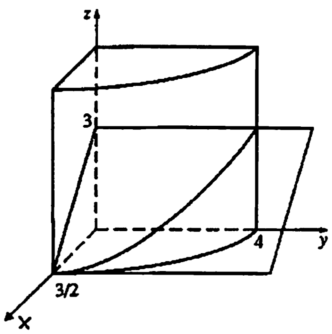
144. Suppose the volume of a solid is given by $V = \int_0^3 \int_0^{(3-z)/2} \int_0^{4-x^2} dy\,dx\,dz$.

(a) Sketch the solid whose volume is given by V.



(b) Evaluate the integral to find the volume of the solid.

Answer: (a)



(b) $\frac{261}{32}$

145. Find the volume of the solid formed by the intersection of the cylinder $y = x^2$ and the two planes given by $z = 0$ and $y + z = 4$.

Answer: $\frac{256}{15}$

146. Evaluate $\int_0^1 \int_z^{2z} \int_{x+z}^{2x+2z} (x-1)\, dy\, dx\, dz$.

Answer: $\frac{1}{8}$

147. Use the method of iterated integration in order to evaluate the triple integral $\iiint_N x\, dV$ where N is the region cut off from the first octant by the plane defined by $x + y + z = 3$.

Answer: $\frac{27}{8}$

148. Evaluate $\int_0^2 \int_0^{2z} \int_0^{xz} xyz^2\, dy\, dx\, dz$.

Answer: $\frac{1024}{9}$

149. Evaluate $\int_0^1 \int_{x^2}^1 \int_0^{3y} (y + 2x^2 z)\, dz\, dy\, dx$.

Answer: $\frac{32}{21}$

150. Set up the triple integral for $f(x,y,z) = \pi x^3$ over the solid E with vertices $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(0,1,1)$, $(1,1,0)$, and $(1,1,1)$.

Answer: $\int_0^1 \int_0^1 \int_0^y \pi x^2\, dx\, dy\, dz$

151. The centroid $(\overline{x}, \overline{y}, \overline{z})$ of a region E in $\mathbb{R}^3$ is the center of mass with the density function $\rho(x,y,z) = 1$. Find the centroid of the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$.

Answer: $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$

152. The average value of $f(x,y,z)$ over a region E in space with volume $V(E)$ is $\frac{1}{V(E)} \iiint_E f(x,y,z)\, dV$. Find the average value of $f(x,y,z) = xy$ over the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$.

Answer: $\frac{1}{20}$

153. Set up the volume integral for the solid cylinder $x^2 + y^2 \le 1$, bounded above by $z = x^3 + y^2 = 9$ and below by $z = 2x + 3y + 3$.

Answer: $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^3 + y^4 - 2x - 3y + 6)\, dy\, dx$

12.8 Triple Integrals in Cylindrical and Spherical Coordinates

154. Evaluate the triple integral $\iiint_E 1\,dV$ in cylindrical coordinates, where $E = \{(r,\theta,z) \mid 0 \le r \le 1, 0 \le \theta \le \pi, 0 \le z \le 1\}$.

A) $\frac{\pi}{4}$ B) $\frac{\pi}{3}$ C) $\frac{\pi}{2}$ D) $\frac{2\pi}{3}$
E) $\frac{3\pi}{4}$ F) π G) $\frac{4\pi}{3}$ H) 2π

Answer: C)

155. Evaluate the triple integral $\iiint_E r\,dV$ in cylindrical coordinates, where $E = \{(r,\theta,z) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi, 0 \le z \le 1\}$.

A) $\frac{\pi}{4}$ B) $\frac{\pi}{3}$ C) $\frac{\pi}{2}$ D) $\frac{2\pi}{3}$
E) $\frac{3\pi}{4}$ F) π G) $\frac{4\pi}{3}$ H) 2π

Answer: D)

156. Evaluate the triple integral $\iiint_E 1\,dV$ in spherical coordinates, where $E = \{(\rho,\theta,\phi) \mid 0 \le \rho \le 1, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$.

A) $\frac{\pi}{4}$ B) $\frac{\pi}{3}$ C) $\frac{\pi}{2}$ D) $\frac{2\pi}{3}$
E) $\frac{3\pi}{4}$ F) π G) $\frac{4\pi}{3}$ H) 2π

Answer: G)

157. Evaluate the triple integral $\iiint_E 1\,dV$ in spherical coordinates, where $E = \{(\rho,\theta,\phi) \mid 0 \le \rho \le 1, 0 \le \theta \le \pi, 0 \le \phi \le \pi\}$.

A) $\frac{\pi}{4}$ B) $\frac{\pi}{3}$ C) $\frac{\pi}{2}$ D) $\frac{2\pi}{3}$
E) $\frac{3\pi}{4}$ F) π G) $\frac{4\pi}{3}$ H) 2π

Answer: D)

158. Evaluate the triple integral $\iiint_E \rho\,dV$ in spherical coordinates, where $E = \left\{(\rho,\theta,\phi) \mid 0 \le \rho \le 1, 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{2}\right\}$.

A) $\frac{\pi}{4}$ B) $\frac{\pi}{3}$ C) $\frac{\pi}{2}$ D) $\frac{2\pi}{3}$
E) $\frac{3\pi}{4}$ F) π G) $\frac{4\pi}{3}$ H) 2π

Answer: C)

159. Evaluate the triple integral $\iiint_E \rho\,dV$ in spherical coordinates, where $E = \left\{(\rho,\theta,\phi) \mid 0 \le \rho \le 1, 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{3}\right\}$.

A) $\frac{\pi}{4}$ B) $\frac{\pi}{3}$ C) $\frac{\pi}{2}$ D) $\frac{2\pi}{3}$
E) $\frac{3\pi}{4}$ F) π G) $\frac{4\pi}{3}$ H) 2π

Answer: A)

160. Evaluate the triple integral $\iiint_E \rho\,dV$ in spherical coordinates, where $E = \{(\rho,\theta,\phi) \mid 0 \le \rho \le 1, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$.

A) $\frac{\pi}{4}$ B) $\frac{\pi}{3}$ C) $\frac{\pi}{2}$ D) $\frac{2\pi}{3}$
E) $\frac{3\pi}{4}$ F) π G) $\frac{4\pi}{3}$ H) 2π

Answer: F)

161. Find the mass of the solid that occupies the cylindrical coordinate region $E = \{(r,\theta,z) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi, 0 \le z \le 1\}$ and has density function $\rho(r,\theta,z) = z$.

A) $\frac{\pi}{4}$ B) $\frac{\pi}{3}$ C) $\frac{\pi}{2}$ D) $\frac{2\pi}{3}$
E) $\frac{3\pi}{4}$ F) π G) $\frac{4\pi}{3}$ H) 2π

Answer: C)

162. Find the mass of the solid that occupies the cylindrical coordinate region $E = \{(r,\theta,z) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi, 0 \le z \le 1\}$ and has density function $\rho(r,\theta,z) = r$.

A) $\frac{\pi}{4}$ B) $\frac{\pi}{3}$ C) $\frac{\pi}{2}$ D) $\frac{2\pi}{3}$
E) $\frac{3\pi}{4}$ F) π G) $\frac{4\pi}{3}$ H) 2π

Answer: D)

163. Evaluate the triple integral $\iiint_E (x^2+y^2+z^2)\, dV$, where $E = \{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$.

A) $\frac{2\pi}{3}$ B) $\frac{2\pi}{5}$ C) $\frac{4\pi}{3}$ D) $\frac{4\pi}{5}$
E) 2π F) $\frac{6\pi}{5}$ G) $\frac{8\pi}{3}$ H) $\frac{8\pi}{5}$

Answer: D)

164. Let E be the solid that lies below the sphere $x^2+y^2+z^2 = a^2$ and above the cone $\phi = \beta$, where $0 < \beta < \frac{\pi}{2}$. Find the value of the triple integral $\iiint_E z\, dV$.

A) $\pi a^2 \sin\beta$ B) $\frac{1}{2}\pi a^2 \sin\beta$ C) $\pi a^2 \sin^2\beta$ D) $\frac{1}{2}\pi a^2 \sin^2\beta$
E) $\frac{1}{4}\pi a^2 \sin^2\beta$ F) $\frac{1}{2}\pi a^4 \sin^2\beta$ G) $\frac{1}{4}\pi a^4 \sin^2\beta$ H) $\pi a^4 \sin\beta$

Answer: G)

165. Use cylindrical coordinates to find $\iiint_R z\, dV$, where R is the region bounded by $z = \sqrt{x^2+y^2}$ and $z = x^2+y^2$.

Answer: $\frac{\pi}{12}$

166. Find the volume bounded above by the surface $z = x^2 - y^2$, $x \ge 0$, below by the xy-plane, and laterally by the cylinder $x^2+y^2 = 1$.

Answer: $\frac{1}{2}$

167. Find the volume of the region inside the cylinder $x^2+y^2 = 7$ which is bounded below by the xy-plane and above by the sphere $x^2+y^2+z^2 = 16$.

Answer: $\frac{74\pi}{3}$

168. Let R be the three-dimensional region in the first octant that is outside the cylinder $r = 1$ and inside the sphere $r^2+z^2 = 4$ (in cylindrical coordinates). Set up, but do not integrate, the iterated triple integral for the volume of R, integrating with respect to z, r, and θ, in that order.

Answer: $\int_0^{\pi/2} \int_1^2 \int_0^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta$

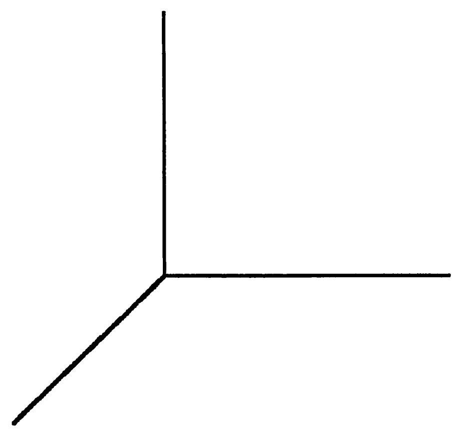
169. Find the volume of the region above the paraboloid $z = x^2+y^2$ and below the hemisphere $z - 9 = \sqrt{9-x^2-y^2}$.

Answer: $\frac{117\pi}{2}$

170. Use spherical coordinates to evaluate $\iiint_S \left(x^2+y^2+z^2\right)^2\,dV$, where S is the solid in the first octant bounded by the sphere $x^2+y^2+z^2=4$.
Answer: $\frac{64\pi}{7}$

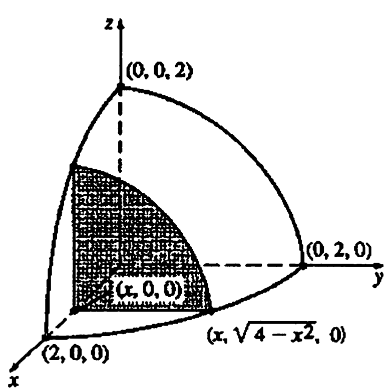
171. Evaluate the following integral by changing to spherical coordinates:
$$\int_{-5}^{5}\int_{0}^{\sqrt{25-x^2}}\int_{0}^{\sqrt{25-x^2-y^2}} \left(x^2+y^2+z^2\right)^{1/3}\,dz\,dy\,dx$$
Answer: $\frac{375}{11}\left(5^{2/3}\right)\pi$

172. A region W in $\mathbb{R}^3$ is described completely by $x \geq 0$, $y \geq 0$, $z \geq 0$, and $x^2+y^2+z^2 \leq 4$.
(a) Describe or sketch this region.



(b) Write an integral in rectangular coordinates which gives the volume of W. Do not work out this integral.
(c) Write an integral in spherical coordinates which gives the volume of W. Find the volume of W using this integral.

Answer: (a)



(b) $\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} dz\,dy\,dx$
(c) $\int_0^2\int_0^{\pi/2}\int_0^{\pi/2} \rho^2\sin\phi\,d\phi\,d\theta\,d\rho = \frac{4\pi}{3}$

173. Use a triple integral in spherical coordinates to find the volume of that part of the sphere $x^2+y^2+z^2=9$ which lies inside the cone $z=\sqrt{x^2+y^2}$.
Answer: $18\pi\left(1-\frac{1}{\sqrt{2}}\right)$

174. Find the mass of that portion of the solid bounded above by the sphere $x^2+y^2+z^2=3$ which lies in the first octant, if the density varies as the distance from the center of the sphere.

Answer: $\dfrac{9k\pi}{8}$

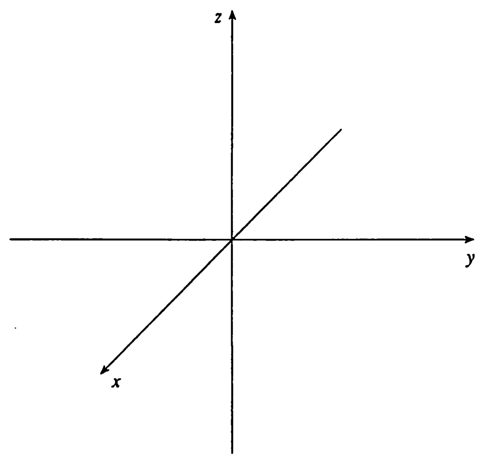
175. A sphere of radius k has a volume of $\frac{4}{3}\pi k^3$. Set up the iterated integrals in rectangular, cylindrical, and spherical coordinates to show this.

Answer: Rectangular coordinates: $V = \int_{-k}^{k}\int_{-\sqrt{k^2-x^2}}^{\sqrt{k^2-x^2}}\int_{-\sqrt{k^2-x^2-y^2}}^{\sqrt{k^2-x^2-y^2}} dz\,dy\,dx$; cylindrical coordinates: $V = \int_0^{2\pi}\int_0^k\int_{-\sqrt{k^2-r^2}}^{\sqrt{k^2-r^2}} r\,dz\,dr\,d\theta$; spherical coordinates: $V = \int_0^{2\pi}\int_0^{\pi}\int_0^k \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

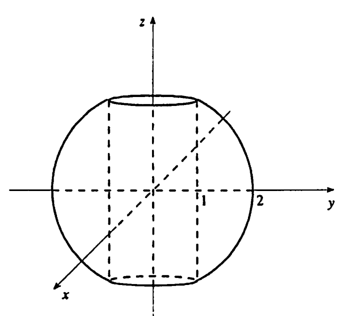
176. Evaluate $\iiint_E z\left(x^2+y^2\right)dV$, where E is the solid bounded by the cylinder $x^2+y^2=4$, above by $z=3$ and below by $z=0$.

Answer: 36π

177. Sketch the region E whose volume is given by the integral $\int_0^{2\pi}\int_{\pi/6}^{5\pi/6}\int_{1/\sin\phi}^{2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$.



Answer:



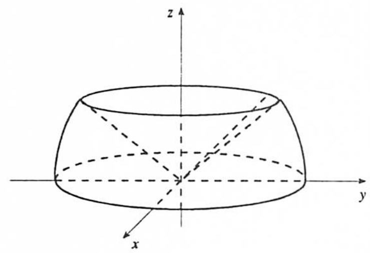
A sphere of radius 2 with a hole of radius 1 drilled through the center

178. Find the mass of a solid ball of radius 2 if the density at each point (x,y,z) is $\dfrac{3}{1+\sqrt{x^2+y^2+z^2}}$.

Answer: $12\pi\ln 3$

179. Give a geometric description of the solid S whose volume in spherical coordinates is given by $V = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$.

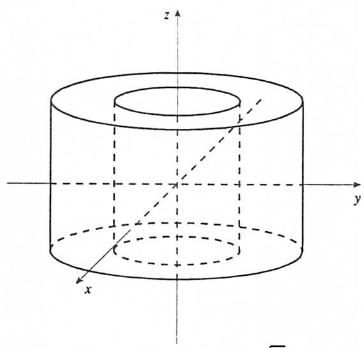
Answer:



A hemisphere with a cone cut out

180. Give a geometric description of the solid S whose volume in cylindrical coordinates is given by $V = \int_0^{2\pi} \int_1^2 \int_{-\sqrt{3}}^{\sqrt{3}} r \, dz \, dr \, d\theta$.

Answer:



A cylinder of radius 2 and height $2\sqrt{3}$ with a hole of radius 1 cut out in the center along the axis of the cylinder.

12.9 Change of Variables in Multiple Integrals

181. Find the Jacobian of the transformation $x = u + v$, $y = 2u - v$.

A) 1	B) 2	C) 3	D) 4
E) −1	F) −2	G) −3	H) −4

Answer: G)

182. Find the Jacobian of the transformation $x = u$, $y = 2v$, $z = 3w$.

A) 1	B) 2	C) 3	D) 6
E) −1	F) −2	G) −3	H) −6

Answer: D)

183. Find the Jacobian of the transformation $x = u^2$, $y = v^3$.

A) uv^2	B) $2uv^2$	C) $3uv^2$	D) $6uv^2$
E) $-uv^2$	F) $-2uv^2$	G) $-3uv^2$	H) $-6uv^2$

Answer: D)

184. Find the Jacobian of the transformation $x = u \sin v$, $y = u \cos v$.

A) $-u$ B) $-u^2$ C) u D) u^2
E) $-u^2 \sin v \cos v$ F) $u^2 \sin v \cos v$ G) $-2u^2 \sin v \cos v$ H) $2u^2 \sin v \cos v$

Answer: A)

185. Find the Jacobian of the transformation $x = u + v$, $y = 2u - v$ when $u = 1$ and $v = 2$.

A) 1 B) 2 C) 3 D) 4
E) -1 F) -2 G) -3 H) -4

Answer: G)

186. Find the Jacobian of the transformation $x = u$, $y = 2v$, $z = 3w$ when $u = 1$, $v = -1$, and $w = \frac{1}{2}$.

A) 1 B) 2 C) 3 D) 6
E) -1 F) -2 G) -3 H) -6

Answer: D)

187. Find the Jacobian of the transformation $x = u^2$, $y = v^3$ when $u = \frac{1}{2}$ and $v = 1$.

A) 1 B) 2 C) 3 D) 6
E) -1 F) -2 G) -3 H) -6

Answer: C)

188. Find the Jacobian of the transformation $x = u \sin v$, $y = u \cos v$ when $u = 3$ and $v = 5$.

A) 3 B) 5 C) 7.5 D) 15
E) -3 F) -5 G) -7.5 H) -15

Answer: E)

189. Under the transformation $x = u + v$, $y = v - 2u$, the image of the circle $x^2 + y^2 \leq 1$ is an ellipse. What is the area of that ellipse?

A) $\frac{\pi}{4}$ B) $\frac{\pi}{3}$ C) $\frac{\pi}{2}$ D) $\frac{2\pi}{3}$
E) π F) $\frac{3\pi}{2}$ G) 2π H) 3π

Answer: B)

190. Find the area of the region whose image under the transformation $x = u + v$, $y = v - 2u$ is $D = \{(x,y) \mid -1 \leq x \leq 1, 0 \leq y \leq 1 - x^2\}$.

A) $\frac{1}{9}$ B) $\frac{2}{9}$ C) $\frac{1}{3}$ D) $\frac{4}{9}$
E) $\frac{5}{9}$ F) $\frac{2}{3}$ G) $\frac{7}{9}$ H) $\frac{8}{9}$

Answer: D)

191. Evaluate the integral $\iint_R \frac{x+y}{x-y}\, dA$, where R is the triangular region with vertices $(1,0)$, $(0,-1)$, and $(0,0)$.

A) 0 B) $\frac{1}{4}$ C) $\frac{1}{2}$ D) $\frac{3}{4}$
E) 1 F) $\frac{3}{2}$ G) 2 H) -2

Answer: A)

192. Evaluate the integral $\iint_R e^{x^2+y^2}\, dA$, where R is the circle with center $(0,0)$ and radius 1.

A) $\frac{\pi}{2}$ B) π C) $\frac{3\pi}{2}$ D) 2π
E) πe F) $2\pi e$ G) $\pi(e-1)$ H) $2\pi(e-1)$

Answer: G)

193. Use the change of variables $u = 2x - y$, $v = x + y$ to evaluate $\iint_R (6x - 3y)\, dA$ where R is the region bounded by $2x - y = 1$, $2x - y = 3$, $x + y = 1$, and $x + y = 2$.

Answer: 4

194. Find the Jacobian of the transformation $x = u - v^2$, $y = u + v^2$.

Answer: $4v$

195. Find the Jacobian of the transformation $x = se^t$, $y = se^{-t}$.

Answer: $-2s$

196. Find the Jacobian of the transformation $x = 2u$, $y = 3v^2$, $z = 4w^3$.

Answer: $144vw^2$

197. Use the change of variables $x = 2u + 3v$, $y = 3u - 2v$ to evaluate $\iint_R (x + y)\, dA$, where R is the square with vertices $(0, 0)$, $(2, 3)$, $(5, 1)$, and $(3, -2)$.

Answer: 39

198. Use the change of variables $x = \sqrt{2}u - \sqrt{\frac{2}{3}}v$, $y = \sqrt{2}u + \sqrt{\frac{2}{3}}v$ to evaluate $\iint_R (x^2 - xy + y^2)\, dA$, where R is the region bounded by the ellipse $x^2 - xy + y^2 = 2$.

Answer: $\frac{4\pi}{\sqrt{3}}$

199. Use the change of variables $u = xy$, $v = xy^2$ to evaluate $\iint_R y^2\, dA$, where R is the region bounded by the curves $xy = 1$, $xy = 2$, $xy^2 = 1$, and $xy^2 = 2$.

Answer: $\frac{3}{4}$

200. Use the change of variables $x = au$, $y = bv$, $z = cw$ to evaluate $\iiint_E y\, dV$, where E is the solid enclosed by the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

Answer: 0

201. Evaluate $\iint_R \sin(9x^2 + 4y^2)\, dA$ by making an appropriate change of variables, where R is the region in the first quadrant bounded by the ellipse $9x^2 + 4y^2 = 1$.

Answer: $\frac{\pi}{24}(1 - \cos 1)$

202. Compute the Jacobian of the transformation T given by $x = \frac{1}{\sqrt{2}}(u - v)$, $y = \frac{1}{\sqrt{2}}(u + v)$, and find the image of $S = \{(u, v) \mid 0 \le u \le 1, 0 \le v \le 1\}$ under T.

Answer: $J = 1$, image of $S = \left\{(x, y) \mid -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}, |x| \le y \le \sqrt{2} - |x|\right\}$

203. Compute the Jacobian of the transformation T given by $x = \sqrt{2}u - \frac{1}{\sqrt{2}}v$, $y = \frac{1}{\sqrt{2}}u + \sqrt{2}v$. Compute the area of the image of $S = \{(u, v) \mid 0 \le u \le 1, 0 \le v \le 1\}$ and compare it to the area of S.

Answer: $J = \frac{5}{2}$, area of image $= \frac{5}{2}$

204. Compute the Jacobian of the transformation T given by $x = v\cos 2\pi u$, $y = v\sin 2\pi u$. Describe the image of $S = \{(u, v) \mid 0 \le u \le 1, 0 \le v \le 1\}$, and compute its area.

Answer: $J = \pi$, image of $S = \{(x, y) \mid x^2 + y^2 \le 1\}$, area of image $= \pi$

205. Describe the image R of the set $S = \{(u, v) \mid 0 \le u \le 1, 0 \le v \le 1\}$ under the transformation $x = 3u + v$, $y = u + 2v$, and then compute $\iint_R (xy + y^2)\, dA$.

Answer: R is a parallelogram with vertices $(0, 0)$, $(3, 1)$, $(4, 3)$, and $(1, 2)$; $\iint_R (xy + y^2)\, dA = \frac{365}{12}$

13 Vector Calculus

13.1 Vector Fields

1. Find the gradient vector field of the function $f(x, y) = xy^2$.

A) $\mathbf{i} + 2\mathbf{j}$ B) $\mathbf{i} - 2\mathbf{j}$ C) $x\mathbf{i} + y^2\mathbf{j}$ D) $x\mathbf{i} - y^2\mathbf{j}$
E) $\mathbf{i} + 2y\mathbf{j}$ F) $\mathbf{i} - 2y\mathbf{j}$ G) $y^2\mathbf{i} + 2xy\mathbf{j}$ H) $y^2\mathbf{i} - 2xy\mathbf{j}$

Answer: G)

2. Find the gradient vector field of the function $f(x, y) = x + y^2$.

A) $2\mathbf{j}$ B) $\mathbf{i} + 2\mathbf{j}$ C) $\mathbf{i} + 2y\mathbf{j}$ D) $x\mathbf{i} + 2y\mathbf{j}$
E) $-2\mathbf{j}$ F) $-\mathbf{i} - 2\mathbf{j}$ G) $-\mathbf{i} - 2y\mathbf{j}$ H) $-x\mathbf{i} - 2y\mathbf{j}$

Answer: C)

3. Find the value of the gradient vector field of the function $f(x, y) = e^{xy^2}$ at the point $(1, 1)$.

A) $e\mathbf{i}$ B) $e\mathbf{j}$ C) $2e\mathbf{i}$ D) $2e\mathbf{j}$
E) $e\mathbf{i} + e\mathbf{j}$ F) $e\mathbf{i} + 2e\mathbf{j}$ G) $2e\mathbf{i} + 2e\mathbf{j}$ H) $2e\mathbf{i} + e\mathbf{j}$

Answer: F)

4. Find the gradient vector field of the function $f(x, y) = x \sin y$.

A) $\sin y\,\mathbf{i} + \cos y\,\mathbf{j}$ B) $\sin y\,\mathbf{i} + x\cos y\,\mathbf{j}$ C) $x\sin y\,\mathbf{i} + \cos y\,\mathbf{j}$ D) $x\sin y\,\mathbf{i} + x\cos y\,\mathbf{j}$
E) $\sin y\,\mathbf{i} - \cos y\,\mathbf{j}$ F) $\sin y\,\mathbf{i} - x\cos y\,\mathbf{j}$ G) $x\sin y\,\mathbf{i} - \cos y\,\mathbf{j}$ H) $x\sin y\,\mathbf{i} - x\cos y\,\mathbf{j}$

Answer: B)

5. Find the value of the gradient vector field of the function $f(x, y) = x^2y^3$ at the point $(1, 1)$.

A) $2\mathbf{i} - 3\mathbf{j}$ B) $2\mathbf{i} + 3\mathbf{j}$ C) $3\mathbf{i} - 2\mathbf{j}$ D) $3\mathbf{i} + 2\mathbf{j}$
E) $6\mathbf{i} - 6\mathbf{j}$ F) $6\mathbf{i} + 6\mathbf{j}$ G) $\mathbf{i} - \mathbf{j}$ H) $\mathbf{i} + \mathbf{j}$

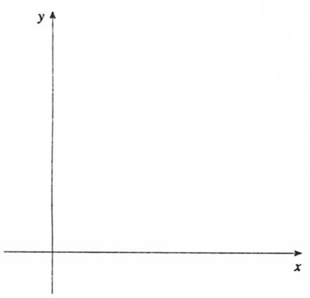
Answer: B)

6. Find the value of the gradient vector field of the function $f(x, y) = \dfrac{1}{x^2 + y^2}$ at the point $(1, 2)$.

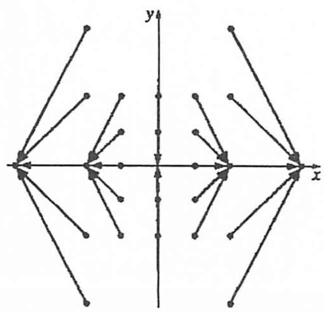
A) $0.08\mathbf{i} + 0.16\mathbf{j}$ B) $-0.08\mathbf{i} - 0.16\mathbf{j}$ C) $0.04\mathbf{i} + 0.08\mathbf{j}$ D) $-0.04\mathbf{i} - 0.08\mathbf{j}$
E) $0.02\mathbf{i} + 0.04\mathbf{j}$ F) $-0.02 - 0.04\mathbf{j}$ G) $0.01\mathbf{i} + 0.02\mathbf{j}$ H) $-0.01\mathbf{i} - 0.02\mathbf{j}$

Answer: B)

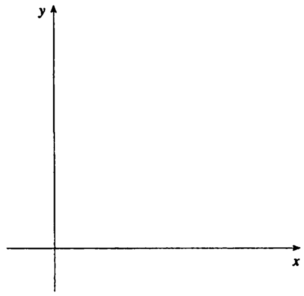
7. Sketch the vector field $\mathbf{F}$ where $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$.



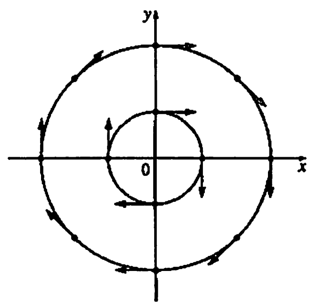
Answer:



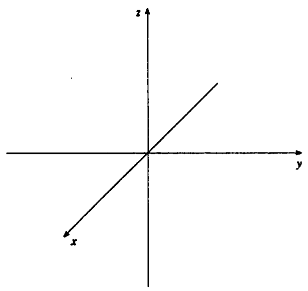
8. Sketch the vector field $\mathbf{F}$ where $\mathbf{F}(x,y) = \dfrac{y\mathbf{i} - x\mathbf{j}}{\sqrt{x^2+y^2}}$.



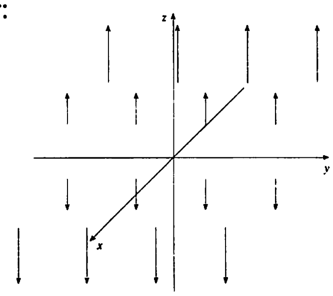
Answer:



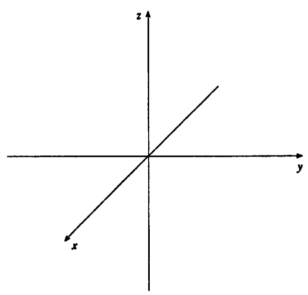
9. Sketch the vector field $\mathbf{F}$ where $\mathbf{F}(x,y,z) = z\mathbf{k}$.



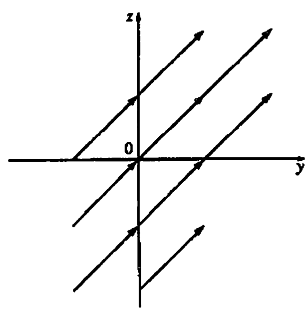
Answer:



10. Sketch the vector field $\mathbf{F}$ where $\mathbf{F}(x,y,z) = \mathbf{j} + \mathbf{k}$.



Answer:

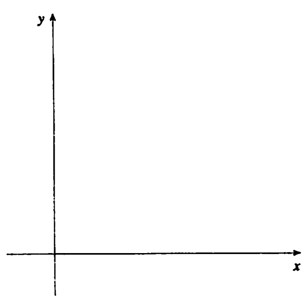


11. Find the gradient vector field of $f(x,y) = \sin(2x+3y)$.
Answer: $\nabla f = 2\cos(2x+3y)\,\mathbf{i} + 3\cos(2x+3y)\,\mathbf{j}$

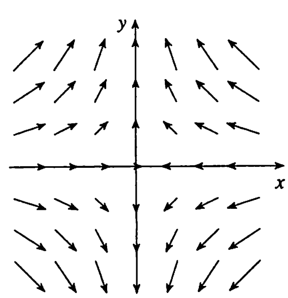
12. Find the gradient vector field of $f(x,y,z) = xyz$.
Answer: $\nabla f = \langle yz, xz, xy \rangle$

13. Find the gradient vector field of $f(x,y,z) = x\ln(y-z)$.
Answer: $\nabla f = \langle \ln(y-z), x/(y-z), -x/(y-z) \rangle$

14. Sketch the vector field $\mathbf{F}(x,y) = -x\mathbf{i} + y\mathbf{j}$.



Answer:



142. Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = z^2\mathbf{i} + y^2\mathbf{j} + xy\mathbf{k}$ and C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 2)$.
Answer: $\frac{4}{3}$

143. Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = x^2z\mathbf{i} + xy^2\mathbf{j} + z^2\mathbf{k}$ and C is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$.
Answer: $\frac{81\pi}{2}$

144. Let $\mathbf{F}(x, y, z) = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve of intersection of the paraboloid $x^2 + y^2 = 2z$ and the cylinder $x^2 + y^2 = 2x$.

A) 1	B) -1	C) $\sqrt{2}$	D) $-\sqrt{2}$
E) $\sqrt{3}$	F) $-\sqrt{3}$	G) 2	H) 0

Answer: H)

145. Let S be the parametric surface $x = r\cos\theta$, $y = r\sin\theta$, $z = \theta$, $0 \le r \le 1$, $0 \le \theta \le \frac{\pi}{2}$. Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$.
Answer: $\frac{\pi}{2}$

146. Consider the surfaces S_1: $\frac{1}{9}x^2 + \frac{1}{9}y^2 + \frac{1}{4}z^2 s = 1$, $z \ge 0$, and S_2: $4z = 9 - x^2 - y^2$, $z \ge 0$, and let $\mathbf{F}$ be a vector field with continuous partial derivatives everywhere. Why do we know that $\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$?
Answer: The boundary curves C_1 and C_2 for S_1 and S_2 are the same

147. Parametrize the boundary curve C of the surface S: $\frac{1}{4}x^2 + \frac{1}{9}y^2 + \frac{1}{16}z^2 = 1$, $z \ge 0$, so that it has positive orientation with S.
Answer: $x = 2\cos\theta$, $y = 3\sin\theta$, $0 \le \theta \le 2\pi$

13.8 The Divergence Theorem

148. Let $\mathbf{F}(x, y, z) = \mathbf{i}$ and let S be the boundary surface of the solid $E = \{(x, y, z) \mid 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

A) 0	B) $\frac{1}{4}$	C) $\frac{1}{3}$	D) $\frac{1}{2}$
E) $\frac{2}{3}$	F) $\frac{3}{4}$	G) 1	H) $\frac{3}{2}$

Answer: A)

149. Let $\mathbf{F}(x, y, z) = x\mathbf{i}$ and let S be the boundary surface of the solid $E = \{(x, y, z) \mid 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

A) 0	B) $\frac{1}{4}$	C) $\frac{1}{3}$	D) $\frac{1}{2}$
E) $\frac{2}{3}$	F) $\frac{3}{4}$	G) 1	H) $\frac{3}{2}$

Answer: G)

150. Let $\mathbf{F}(x,y,z) = x^2\mathbf{i}$ and let S be the boundary surface of the solid $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

A) 0 B) $\frac{1}{4}$ C) $\frac{1}{3}$ D) $\frac{1}{2}$
E) $\frac{2}{3}$ F) $\frac{3}{4}$ G) 1 H) $\frac{3}{2}$

Answer: G)

151. Let $\mathbf{F}(x,y,z) = xy\mathbf{i}$ and let S be the boundary surface of the solid $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

A) 0 B) $\frac{1}{4}$ C) $\frac{1}{3}$ D) $\frac{1}{2}$
E) $\frac{2}{3}$ F) $\frac{3}{4}$ G) 1 H) $\frac{3}{2}$

Answer: D)

152. Let $\mathbf{F}(x,y,z) = xy^2\mathbf{i}$ and let S be the boundary surface of the solid $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

A) 0 B) $\frac{1}{4}$ C) $\frac{1}{3}$ D) $\frac{1}{2}$
E) $\frac{2}{3}$ F) $\frac{3}{4}$ G) 1 H) $\frac{3}{2}$

Answer: C)

153. Let $\mathbf{F}(x,y,z) = \mathbf{i}$ and let S be the boundary surface of the solid $E = \{(x,y,z) \mid x^2 + y^2 + z^2 \le 1\}$. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

A) 0 B) $\frac{\pi}{4}$ C) $\frac{\pi}{3}$ D) $\frac{\pi}{2}$
E) $\frac{2\pi}{3}$ F) $\frac{3\pi}{4}$ G) π H) $\frac{3\pi}{2}$

Answer: A)

154. Let $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and let S be the boundary surface of the solid $E = \{(x,y,z) \mid x^2 + y^2 + z^2 \le 1\}$. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

A) 0 B) $\frac{\pi}{2}$ C) π D) $\frac{4\pi}{3}$
E) $\frac{2\pi}{3}$ F) $\frac{8\pi}{3}$ G) 2π H) 4π

Answer: H)

155. Let $\mathbf{F}(x,y,z) = x\mathbf{i} + z\mathbf{j} + z\mathbf{k}$ and let S be the boundary surface of the solid $E = \{(x,y,z) \mid x^2 + y^2 + z^2 \le 1\}$. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

A) 0 B) $\frac{\pi}{2}$ C) π D) $\frac{4\pi}{3}$
E) $\frac{2\pi}{3}$ F) $\frac{8\pi}{3}$ G) 2π H) 4π

Answer: F)

156. Evaluate the flux integral $\iint_S (2x\mathbf{i} - y\mathbf{j} + 3z\mathbf{k}) \cdot \mathbf{n}\, dS$ over the boundary of the ball $x^2 + y^2 + z^2 \le 9$.

Answer: 144π

157. Evaluate $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, where S is the cube bounded by the planes $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$, $\mathbf{F} = x^2y\mathbf{i} + xy\mathbf{j} + y^2z^3\mathbf{k}$, and $\mathbf{n}$ is the outward normal.

Answer: $\frac{8}{3}$

158. Define the vector function $\mathbf{F}$ by $\mathbf{F}(x,y,z) = (2x - z)\,\mathbf{i} + x^2y\mathbf{j} + xz^2\mathbf{k}$. Use the Divergence Theorem to evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the surface enclosing the unit cube with $\mathbf{n}$ the outward-pointing unit normal.
Answer: $\frac{17}{6}$

159. Let $\mathbf{F}(x,y,z) = x^2y\mathbf{i} - x^2z\mathbf{j} + z^2y\mathbf{k}$ and let S be the surface of the rectangular box bounded by the planes $x = 0$, $x = 3$, $y = 0$, $y = 2$, $z = 0$, and $z = 1$. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$.
Answer: 24

160. Let $\mathbf{F}(x,y,z) = 3xy\mathbf{i} + y^2\mathbf{j} - x^2y^4\mathbf{k}$ and let S be the surface of the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$.
Answer: $\frac{5}{24}$

161. Let $\mathbf{F}(x,y,z) = \left(x + e^{y \tan z}\right)\mathbf{i} + 3xe^{xz}\mathbf{j} + (\cos y - z)\,\mathbf{k}$ and let S be the surface with equation $x^4 + y^4 + z^4 = 1$. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$.
Answer: 0

162. Let $\mathbf{F}(x,y,z) = x^3\mathbf{i} + 2xz^2\mathbf{j} + 3y^2z\mathbf{k}$ and let S be the surface of the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$.
Answer: 32π

163. Let $\mathbf{F}(x,y,z) = \left(x^3 + yz\right)\mathbf{i} + x^2y\mathbf{j} + xz^2\mathbf{k}$ and let S be the surface of the solid bounded by the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$.
Answer: $\frac{3376\pi}{15}$

164. Let $\mathbf{F}(x,y,z) = \left(x^2 + ye^z\right)\mathbf{i} + \left(y^2 + ze^x\right)\mathbf{j} + \left(z^2 + xe^y\right)\mathbf{k}$ and let S be the boundary surface of the solid $E = \left\{(x,y,z) \mid x^2 + y^2 \le 1, 0 \le z \le x + 2\right\}$. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$.
Answer: $\frac{19\pi}{4}$

165. Let $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$ and let S be the boundary surface of the solid $E = \left\{(x,y,z) \mid x^2 + y^2 \le z \le 4\right\}$. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$.
Answer: 2π

166. Let $\mathbf{F}(x,y,z) = \dfrac{z\mathbf{i} + x\mathbf{j} + y\mathbf{k}}{\left(x^2 + y^2 + z^2\right)^{3/2}}$ and let S be the boundary surface of the solid $E = \left\{(x,y,z) \mid 1 \le x^2 + y^2 + z^2 \le 4\right\}$. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$.
Answer: 0

167. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the boundary surface of the solid sphere E: $x^2 + y^2 + z^2 \le 4$ and $\mathbf{F} = 3x\mathbf{i} + 4y\mathbf{j} + 5z\mathbf{k}$.
Answer: 96π

168. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the boundary surface of the region outside the sphere $x^2 + y^2 + z^2 = 1$ and inside the ball $x^2 + y^2 + z^2 \le 4$ and $\mathbf{F} = e^{y^2}\mathbf{i} + \left(5y + e^{x^2}\right)\mathbf{j} + (-3z + 2x)\,\mathbf{k}$.
Answer $\frac{56\pi}{3}$