

CONTINUITY

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CONTINUITY: A function $f(x, y)$ with domain R is **continuous at a point** $(a, b) \in R$ if $f(x, y) \rightarrow f(a, b)$ whenever $(x, y) \rightarrow (a, b)$.

The function f is **continuous on R** , if f is continuous for every point (a, b) on R .

EXAMPLES.

a) $f(x, y) = x^2 + y^4 + xy + \sin(y + \sin \sin \sin \sin(x)^2)$ is continuous on the entire plane. It is built up from functions which are continuous using addition, multiplication and composition.

b) $f(x, y) = 1/x^2 + y^2$ is continuous everywhere except at the origin, where it is not defined.

c) $f(x, y) = y + \sin(x)/|x|$ is continuous away from $x = 0$. At every point $(0, y)$ it is discontinuous. $f(1/n, y) \rightarrow y + 1$ and $f(1/n, y) \rightarrow y - 1$ for $n \rightarrow \infty$.

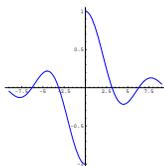
d) $f(x, y) = \sin(1/(x + y))$ is continuous except on the line $x + y = 0$.

e) $f(x, y) = xy^2/(x^2 + y^4)$ an example, where f is continuous on every line through the origin but not continuous. Choose paths $y = mx$ or $x = y^2$.

THREE SOURCES FOR DISCONTINUITY. Roughly speaking there are three reasons, why a function can be discontinuous: it can **jump**, it can **diverge to infinity**, or it can **oscillate**. The prototypes in one dimensions are

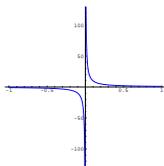
JUMP

$f(x) = \sin(x)/|x|$



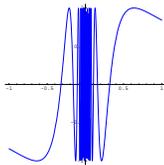
DIVERGE

$g(x) = 1/x$



OSCILLATE

$h(x) = \sin(1/x)$



One can have mixtures of these three things of course like the function $f(x) + h(x)$, which jumps and a oscillatory problem at $x = 0$.

HOW DO WE CHECK FOR CONTINUITY? There are two handy tools to check whether a function $f(x, y)$ is continuous at a point:

- 1) Use polar coordinates with coordinate center at the point to analyze the function.
- 2) Restrict the function to one dimensional curves and check continuity there. This can show discontinuity.

EXAMPLE I. $f(x, y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$.

EXAMPLE II. $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$.

EXAMPLE III. $f(x, y) = \frac{x^2y}{x^4+y^2}$.

EXAMPLE IV. $f(x, y) = \frac{xy^2+y^3}{x^2+y^2}$.

EXAMPLE V. $f(x, y) = \frac{\sin(1/(x^2+y^2))}{\sqrt{x^2+y^2}}$.