10/4/2011 FIRST HOURLY PRACTICE II **Math 21a, Fall 2011**

Name:

MWF 9 Chao Li
MWF 9 Thanos Papaïoannou
MWF 10 Emily Riehl
MWF 10 Jameel Al-Aidroos
MWF 11 Oliver Knill
MWF 11 Tatyana Kobylyatskaya
MWF 12 Tatyana Kobylyatskaya
MWF 12 Yu-Jong Tzeng
TTH 10 Junecue Suh
TTH 10 Pei-Yu Tsai
TTH 11:30 Paul Bourgade

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-2 and 8, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False (TF) questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) **T** F The surface $x^2 + y^2 + z^2 + 2z = 0$ is a sphere.

Solution:
Complete the square: add 1 on both sides, to get $x^2 + y^2 + (z+1)^2 = 1$ which is a sphere.

2) **T** F The length of the vector $\langle 1, 2, 2 \rangle$ is an integer.

Solution:
Indeed, $1 + 4 + 4 = 9$ is a perfect square.

3) **T** F The vector $\langle 3, 4 \rangle$ appears as a velocity vector of the curve $\vec{r}(t) = \langle \cos(5t), \sin(5t) \rangle$. Namely, there is a t such that $\vec{r}\,'(t) = \langle 3, 4 \rangle$.

Solution:
The velocity vector of that curve has length 5 and takes any possible direction, so also the direction of the vector $\langle 3, 4 \rangle$.

4) T **F** If $\vec{T}$ is the unit tangent vector, $\vec{N}$ is the unit normal vector, and $\vec{B}$ is the binormal vector, then $\vec{B} \times \vec{N} = \vec{T}$.

Solution:
The sign is wrong.

5) T **F** The curvature of a larger circle $r = 2$ is greater than the curvature of a smaller circle $r = 1/2$.

Solution:
False. The curvature of a circle of radius r is 1/r.

6) T **F** The surface $x^2 - y^2 - z^2 - 1 = 0$ is a one sheeted hyperboloid.

Solution:
It is a two sheeted one. Make a completion of a square

7) T **F** The function $f(x, y) = y^2 - x^2$ has a graph that is an elliptic paraboloid.

Solution:
It is a hyperbolic paraboloid.

8) **T** F Let $\vec{r}(t)$ be a parametrization of a curve. If $\vec{r}(t)$ is always parallel to the tangent vector $\vec{r}\,'(t)$, then the curve is part of a line through the origin.

Solution:
$\vec{T}\,'$ is zero at all times, so since $\vec{r}\,'(t)$ is parallel to the position vector $\vec{r}(t)$, the curve must lie on a line through the origin.

9) T **F** If $\text{proj}_{\vec{k}}(\vec{u})$ is perpendicular to $\vec{u}$, then $\vec{u}$ is the zero vector.

Solution:
Take $\vec{u} = \vec{i}$.

10) **T** F If $\text{proj}_{\vec{k}}(\vec{u})$ is perpendicular to $\vec{u}$, then $\text{proj}_{\vec{k}}(\vec{u})$ is the zero vector.

Solution:
Assume $\mathbf{proj}_{\vec{k}}(\vec{u})$ is a nonzero vector in the direction of $\vec{k}$. This means that $\vec{k}$ is perpendicular to $\vec{u}$, so $\text{proj}_{\vec{k}}(\vec{u}) = \vec{0}$ by the definition of the projection.

11) T **F** If $\vec{u} \times \vec{v} = \vec{0}$ then $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$.

Solution:
The two vectors can be parallel.

12) **T** F There are two vectors $\vec{a}$ and $\vec{b}$ such that the scalar projection of $\vec{a}$ onto $\vec{b}$ is 100 times the magnitude of $\vec{b}$.

Solution:
Take $\vec{b} = 100\vec{a}$.

13) **T** F The curve $\vec{r}(t) = \langle \cos(t), e^t + 10, t^2 \rangle, 2 \leq t \leq 6$ and the curve $\vec{r}(t) = \langle \cos(2t), e^{2t}, 4t^2 \rangle, 1 \leq t \leq 3$ have the same length.

Solution:
Make a change of variables.

14) T **F** The equation $\rho \sin(\phi) - 2\sin(\theta) = 0$ in spherical coordinates defines a two sheeted hyperboloid.

Solution:
The equation means $r = 2\sin(\theta)$ or $x^2 + y^2 = 2y$.

15) **T** F If triple scalar product of three vectors $\vec{u}, \vec{v}, \vec{w}$ is larger than $|\vec{u} \times \vec{v}|$ then $|\vec{w}| > 1$.

Solution:
Think of the triple scalar product as the volume. If that is larger than the base area, the height has to be bigger than than 1.

16) T **F** The distance between the x-axis and the line $x = y = 1$ is $\sqrt{2}$.

Solution:
The distance is 1. The distance between $x = y = 1$ and the z-axis would be $\sqrt{2}$.

17) **T** F The vector $\langle -1, 2, 3 \rangle$ is perpendicular to the plane $x - 2y - 3z = 9$.

Solution:
It is. Because $\langle 1, -2, -3 \rangle$ is perpendicular to the plane.

18) T F The function $f(x,y) = \begin{cases} (y^2+\sin(x^2))/x^3, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is continuous at $(0,0)$.

Solution:
We would have $\sin(x^2)/x^2$ continuous but $\sin(x^2)/x^3$ goes to infinity for $x \to 0$.

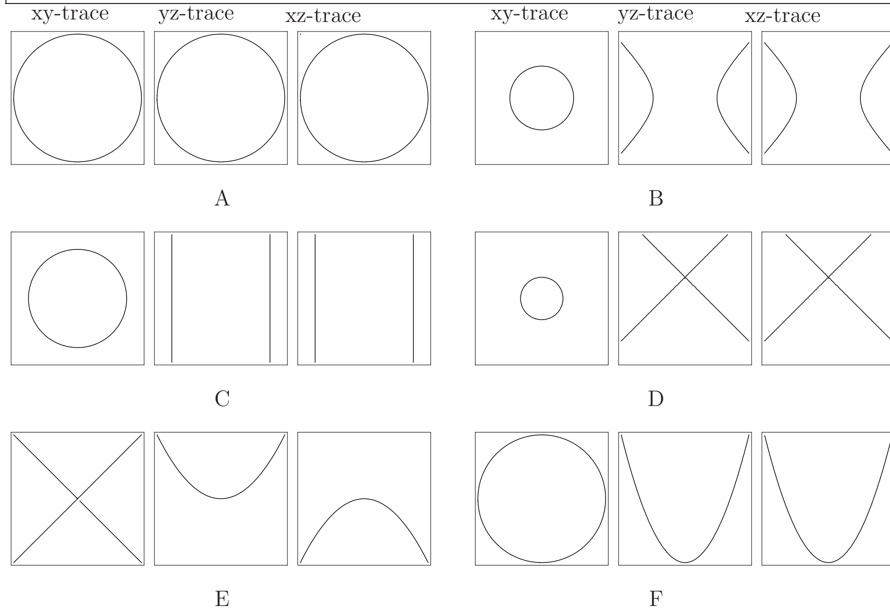
19) T F The point $(1,1,-\sqrt{3})$ is in spherical coordinates given by $(\rho,\theta,\phi) = (\sqrt{5},\pi/4,2\pi/3)$.

Solution:
Also the z coordinate is wrong.

20) T F If the cross product satisfies $(\vec{v}\times\vec{w})\times\vec{v} = \vec{0}$ then $\vec{v}$ and $\vec{w}$ are orthogonal.

Solution:
One can have $\vec{v}=\vec{w}$ in which case the two vectors are not orthogonal and still the product in question is zero.

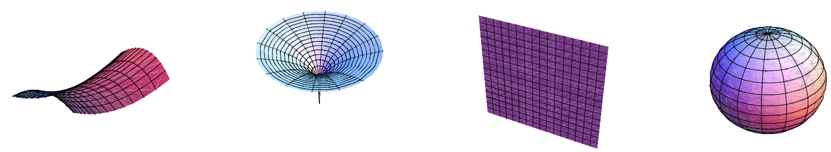
Problem 2a) (6 points)



The figures above show the xy-trace, (the intersection of the surface with the xy-plane), the yz-trace (the intersection of the surface with the yz-plane), and the xz-trace (the intersection of the surface with the xz-plane). Match the following equations with the traces. No justifications required.

Enter A,B,C,D,E,F here	Equation
	$x^2+y^2-(z-1/3)^2=0$
	$x^2+y^2+z^2-1=0$
	$x^2-y^2-z=0$
	$x^2+y^2-1=0$
	$x^2+y^2-z^2-1=0$
	$x^2+y^2-z=1$

Problem 2b) (4 points)



I II III IV

Match the parametric surfaces with their parameterization. No justifications are needed.

Enter I,II,III,IV here	Parameterization
	$\vec{r}(u,v)=\langle u^2,v^2,u^4-v^4\rangle$
	$\vec{r}(u,v)=\langle\cos(u)\sin(v),1+\sin(u)\sin(v),\cos(v)\rangle$
	$\vec{r}(u,v)=\langle v\cos(u),v\sin(u),v^{1/4}\rangle$
	$\vec{r}(u,v)=\langle u,3,v\rangle$

Solution:
a) DAECBF.
b) I,IV,II,III

Problem 3) (10 points)

Find the distance of the point $P = (3, 4, 5)$ to the line

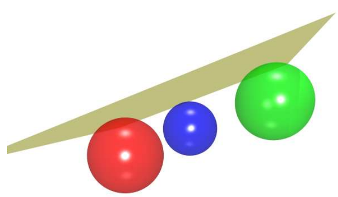
$$\frac{x-1}{4} = \frac{y-2}{5} = \frac{z-3}{6} .$$

Solution:
First parametrize the line as $\vec{r}(t) = \langle 1, 2, 3\rangle + t\langle 4, 56\rangle = \langle 1, 2, 3\rangle + t\vec{v}$. We also need the vector $\vec{QP} = \langle 2, 2, 2\rangle$ connecting a point $Q = (1, 2, 3)$ on the line with the point $P = (3, 4, 5)$. Use the distance formula

$$d = |\langle 4, 5, 6\rangle \times \langle 2, 2, 2\rangle|/|\langle 4, 5, 6\rangle| = |\langle -2, 4, -2\rangle|/|\langle 4, 5, 6\rangle|$$

$\boxed{= \sqrt{24/77}}$.

Problem 4) (10 points)



Given three spheres of radius 1 centered at $A = (1, 2, 0), B = (4, 5, 0), C = (1, 3, 2)$. Find a plane $ax + by + cz = d$ which touches each of three spheres from the same side.

Solution:
The normal vector to the plane is $\vec{n} = \langle 3, 3, 0\rangle \times \langle 0, 1, 2\rangle = \langle 6, -6, 3\rangle$. The plane touching the three spheres has the equation $6x - 6y + 3z = d$, where d is a constant still to be determined. To find this constant, we have to find a point P on the plane. We do that by going from the point A by 1 unit in the direction of the normal vector. The point $P = A + \vec{n}/|\vec{n}| = (1, 2, 0) + \langle 6/9, -6/9, 3/9\rangle = (5, 4, 1)/3$ is on the plane. Plug in this point into the equation gives $d = 3$. The equation of the plane is $\boxed{6x - 6y + 3z - 3 = 0}$.

Problem 5) (10 points)

Find the arc length of the curve

$$\vec{r}(t) = \langle t^3/3, t^4/2, 2t^5/5\rangle$$

from $0 \leq t \leq 1$.

Solution:
We have $\vec{r}\,'(t) = \langle t^2, 2t^3, 2t^4\rangle$ and $|\vec{r}\,'(t)| = \sqrt{t^4 + 4t^6 + 4t^8} = t^2 + 2t^4$. The arc length is

$$\int_0^1 t^2 + 2t^4 \, dt = 1/3 + 2/5 = 11/15 .$$

The arc length is $\boxed{11/15}$.

Problem 6) (10 points)

An apple at position $(0, 0, 20)$ rests 20 meters above Newton's head, the tip of whose nose is at $(1, 0, 0)$. The apple falls with constant acceleration $\vec{r}\,''(t) = \langle a, 0, -10\rangle$ (where $\langle 0, 0, -10\rangle$ is caused by gravity and $\langle a, 0, 0\rangle$ by the wind) precisely onto the nose of Newton. Find the wind force $\langle a, 0, 0\rangle$ which achieves this. Give a parametrization for the path along which the apple falls.

Solution:
From

$$\vec{r}\,''(t) = \langle a, 0, -10 \rangle$$

we get by integration

$$\vec{r}\,'(t) = \langle at, 0, -10t \rangle + \langle 0, 0, 0 \rangle$$

and

$$\vec{r}\,(t) = \langle at^2/2, 0, -5t^2 \rangle + \langle 0, 0, 20 \rangle \; = \langle at^2/2, 0, -5t^2 + 20 \rangle \; .$$

Now, in order that we reach the nose $(1, 0, 0)$, we have to get the time t such the apple is at the ground $5t^2 = 20$ gives $t = 2$. In order that $at^2/2 = 1$ we have $a = 1/2$. The wind force is $\langle 1/2, 0, 0 \rangle$. The path is

$$\vec{r}\,(t) = \langle t^2/4, 0, 20 - 5t^2 \rangle \; .$$

Problem 7) (10 points)

a) (5 points) A red maple leaf falls to the ground $z = 0$. It falls along the curve $\vec{r}(t) = \langle 3\sqrt{3}\cos(t), 3\sqrt{3}\sin(t), 5 - t - 4t^2 \rangle$. At which angle does it hit the xy-plane?

b) (5 points) Find the tangent line to the curve at the impact point.

Solution:
a) The leaf hits the ground at the point $\vec{r}(1) = \langle 3\sqrt{3}\cos(1), 3\sqrt{3}\sin(1), 0 \rangle$ at time $t = 1$. We compute the velocity vector at that time

$$\vec{r}\,'(1) = \langle -3\sqrt{3}\sin(1), 3\sqrt{3}\cos(1), -9 \rangle \; .$$

In order to compute the impact angle, we compute the angle with the normal vector $\langle 0, 0, 1 \rangle$ which is $\cos(\alpha) = -9/(6\sqrt{3}) = -\sqrt{3}/2$ so that $\alpha = 5\pi/6$. The angle between the plane and the velocity vector is $5\pi/6 - \pi/2 = \pi/3$. The result is $\boxed{\pi/3}$.
b) the tangent line has the parametrization $\vec{r}(t) = \vec{r}(1) + t\vec{r}\,'(1)$ which is

$$\vec{r}(t) = \langle 3\sqrt{3}\cos(1), 3\sqrt{3}\sin(1), 0 \rangle + t\langle -3\sqrt{3}\sin(1), 3\sqrt{3}\cos(1), -9 \rangle \; .$$

Problem 8) (10 points)

a) (5 points) The surface

$$\vec{r}(t, s) = \langle 1 + t + s, 1 - t - 2s, 1 + t - s \rangle$$

with $0 \leq t \leq 1, 0 \leq s \leq 1$ is a parallelogram in space. Find the area of this parallelogram.

b) (5 points) Another surface is given in spherical coordinates by $\rho = 2\sin(\phi)\cos(\theta)$. Write down the equation of this surface in rectangular coordinates as well as in cylindrical coordinates.

Solution:
a) The parallelogram is spanned by $\langle 1, -1, 1 \rangle$ and $\langle 1, -2, -1 \rangle$. The area is the length of the cross product $\langle 3, 2, -1 \rangle$ which is $\boxed{\sqrt{14}}$.
b) To get the equation in rectangular coordinates, multiply both sides of the equation with ρ, this gives

$$x^2 + y^2 + z^2 = \rho^2 = 2\rho\sin(\phi)\cos(\theta) = 2x \; .$$

Complete the square to get $\boxed{(x-1)^2 + y^2 + z^2 = 1}$.
To get from rectangular to cylindrical coordinates, just replace $x^2 + y^2$ with r^2 and x with $r\cos(\theta)$ and leave z as it is. In cylindrical coordinates the surface is given by the equation $\boxed{r^2 - 2r\cos(\theta) + z^2 = 0}$.

Problem 9) (10 points)

a) (5 points) Parametrize the curve obtained by intersecting the surface $z - x^2 + y^3 = 0$ with the cylindrical surface $x^2/4 + 9y^2 = 1$.

b) (5 points) Find the unit tangent vector $\vec{T}$ and the normal vector $\vec{N}(t) = \vec{T}\,'(t)/|\vec{T}\,'(t)|$ to the curve

$$\vec{r}(t) = \langle 3, t^2, t \rangle$$

at the point $(3, 0, 0)$. What is the binormal vector $\vec{B} = \vec{T} \times \vec{N}$?

Solution:
a) First parametrize the first two coordinates: $\vec{r}(t) = \langle 2\cos(t), 1/3\sin(t), ...\rangle$, then fill in the third coordinate $z = x^2 - y^3$ to get

$$\vec{r}(t) = \langle 2\cos(t), 1/3\sin(t), 4\cos^2(t) - \sin^3(t)/27\rangle$$

b) $\vec{T}(t) = \langle 0, 2t, 1\rangle/\sqrt{1+4t^2}$. At $t = 0$ we have $\boxed{\vec{T} = \langle 0,0,1\rangle}$. Now $\vec{T}\,'(t) = \langle 0,2,0\rangle/\sqrt{1+4t^2} - 4t\langle 0,2t,1\rangle/(1+4t^2)^{3/2}$. Which is at $t = 0$ equal to $\langle 0,2,0\rangle$. Normalized, we get $\boxed{\vec{N} = \langle 0,1,0\rangle}$. The third vector is just the cross product of the first two $\boxed{\vec{B} = \langle -1,0,0\rangle}$.

Problem 10) (10 points)

a) (4 points) Give a parametrization of the hyperboloid $x^2 + y^2 = z^2 + 1$.
b) (3 points) Give a parametrization of the plane $x + y = 1$.
c) (3 points) Give a parametrization of the ellipsoid $x^2 + y^2 + z^2/4 = 1$.

Solution:
a) Since the hyperboloid is in cylindrical coordinates given as $r^2 = z^2 + 1$, we have

$$\vec{r}(z,\theta) = \langle \sqrt{z^2+1}\cos(\theta), \sqrt{z^2+1}\sin(\theta), z\rangle\ .$$

b) Since the points $(1,0,0), (0,1,0), (1,0,1)$ are in the plane, we have two vectors $\vec{v} = \langle -1,1,0\rangle, \vec{w} = \langle 0,0,1\rangle$ parallel to the plane and can parametrize

$$\vec{r}(s,t) = \langle 1,0,0\rangle + t\langle -1,1,0\rangle + s\langle 0,0,1\rangle\ .$$

c) The ellipsoid is a deformation of the sphere. We just have to make it higher:

$$\vec{r}(\theta,\phi) = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), 2\cos(\phi)\rangle\ .$$