Math 21a First Midterm Review

Plan:

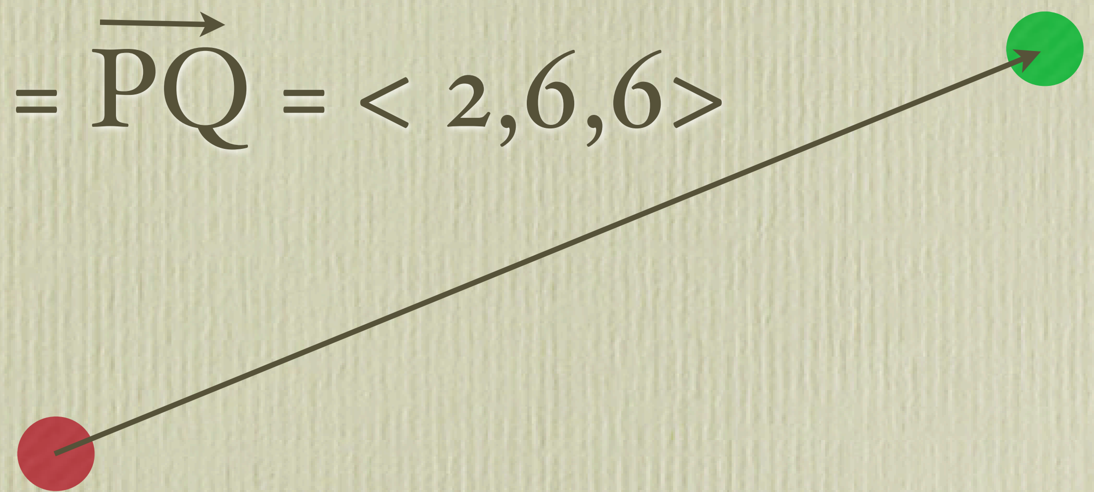
- Lines and Planes
- Distance formulas
- Parametrized curves
- Surfaces Quadrics
- Other coordinates
- Parametric Surfaces
- Continuity
- Partial Derivatives

Oliver Knill, October 4, 2011

Points and Vectors

$\vec{v} = \overrightarrow{PQ} = \langle 2,6,6 \rangle$

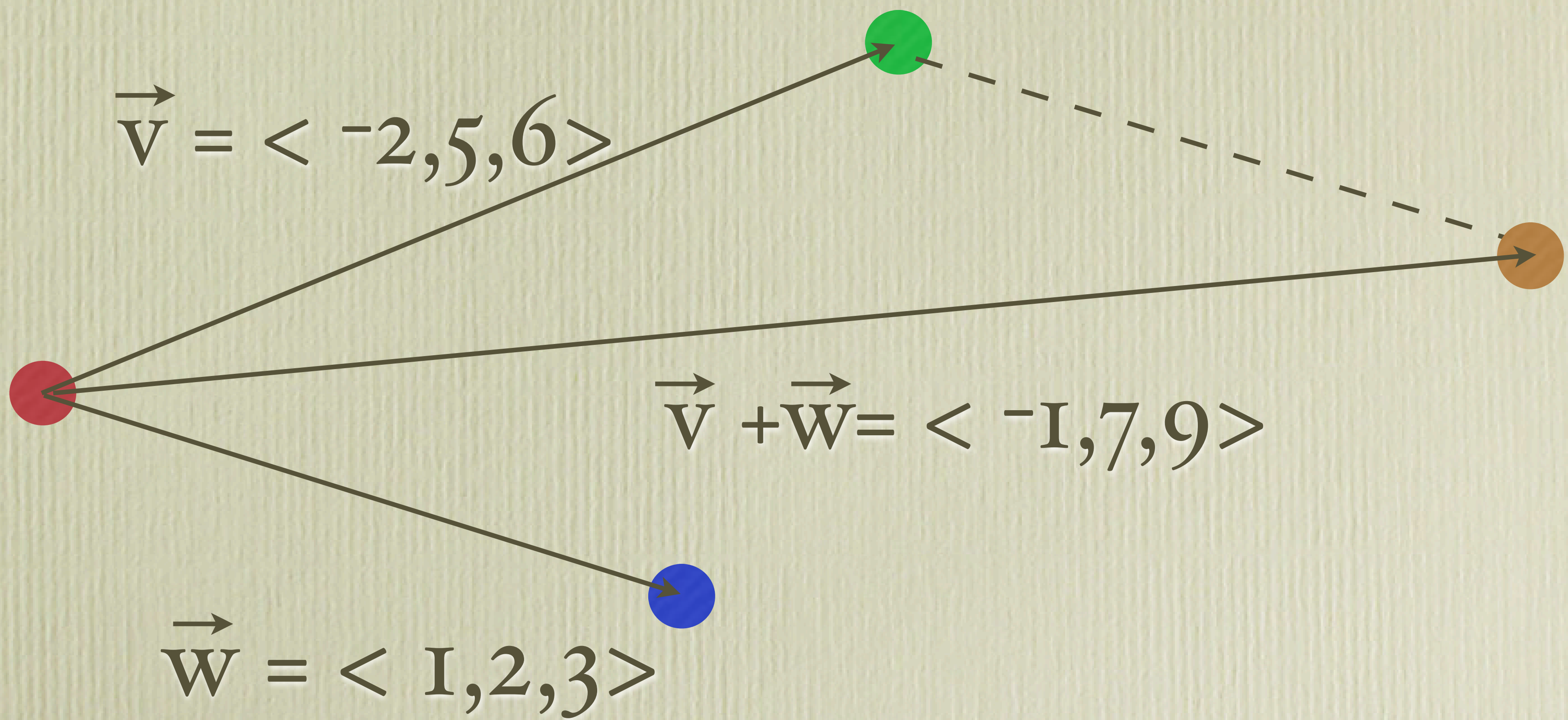
$Q = (5,6,7)$



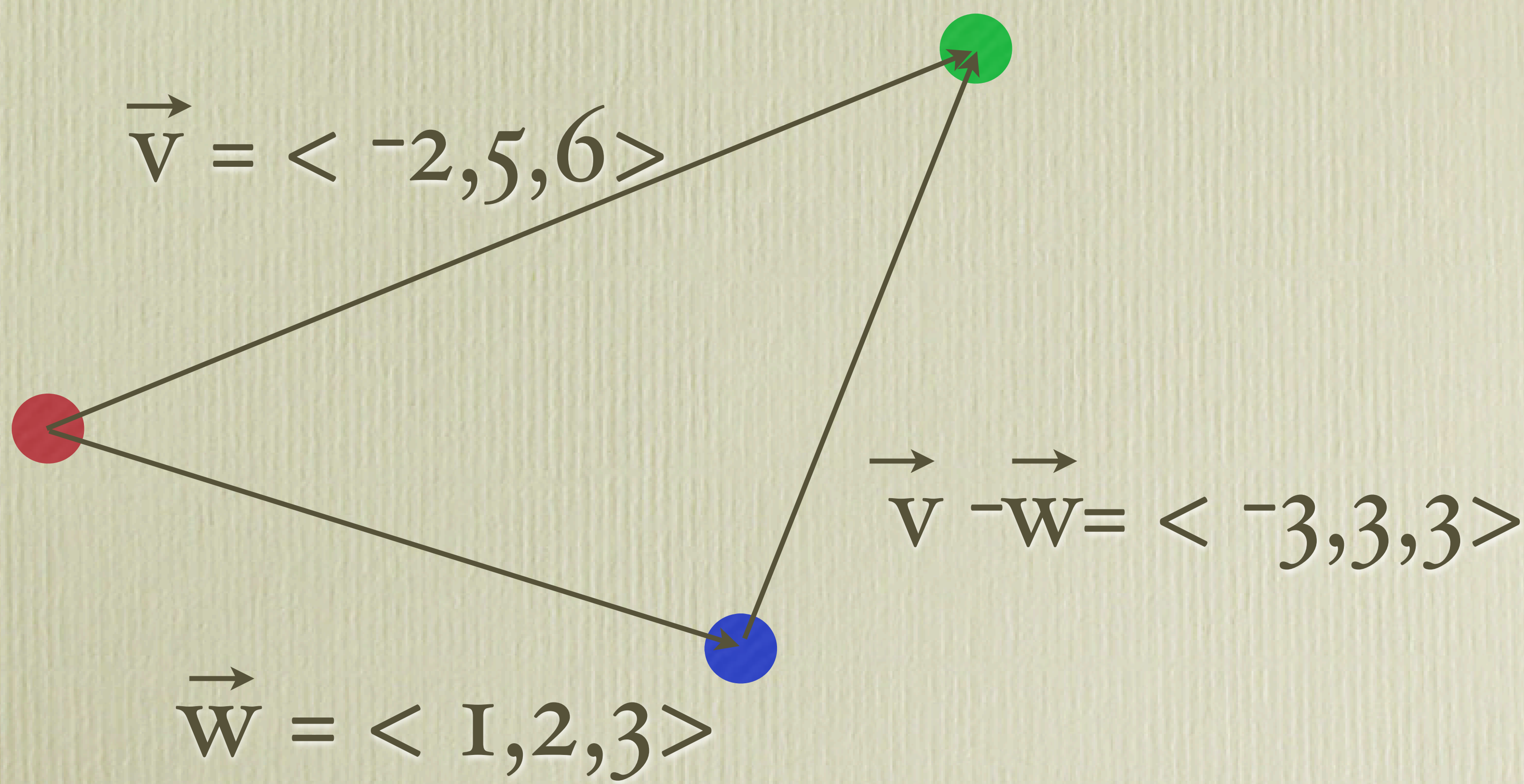
$P = (3,0,1)$

The components of $\vec{v}$
are the differences between
the coordinates of Q and P.

Addition

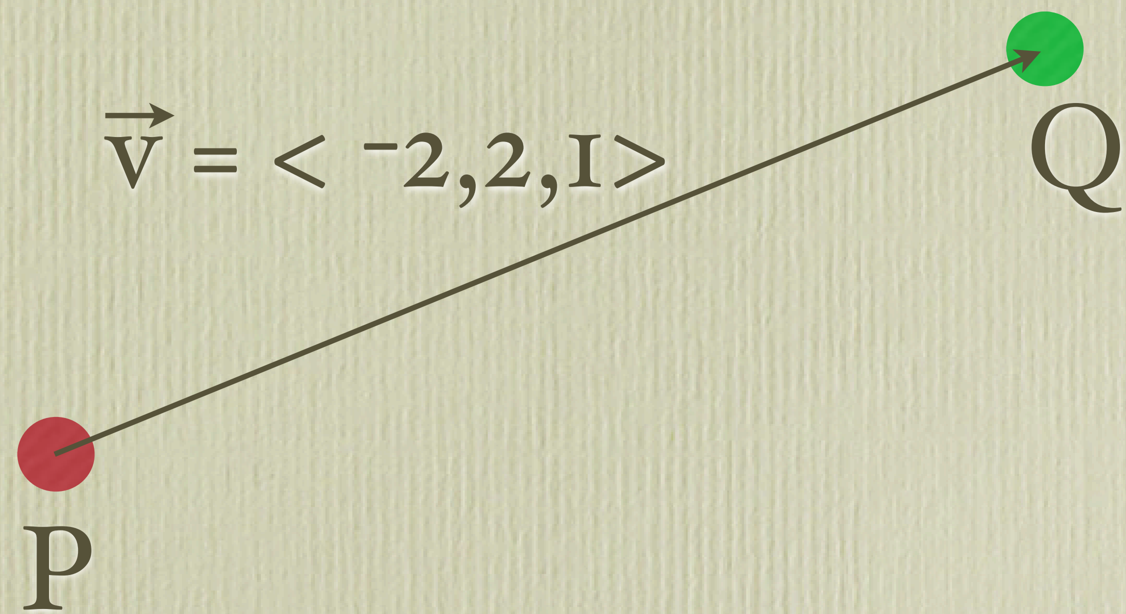


Subtraction



Distances

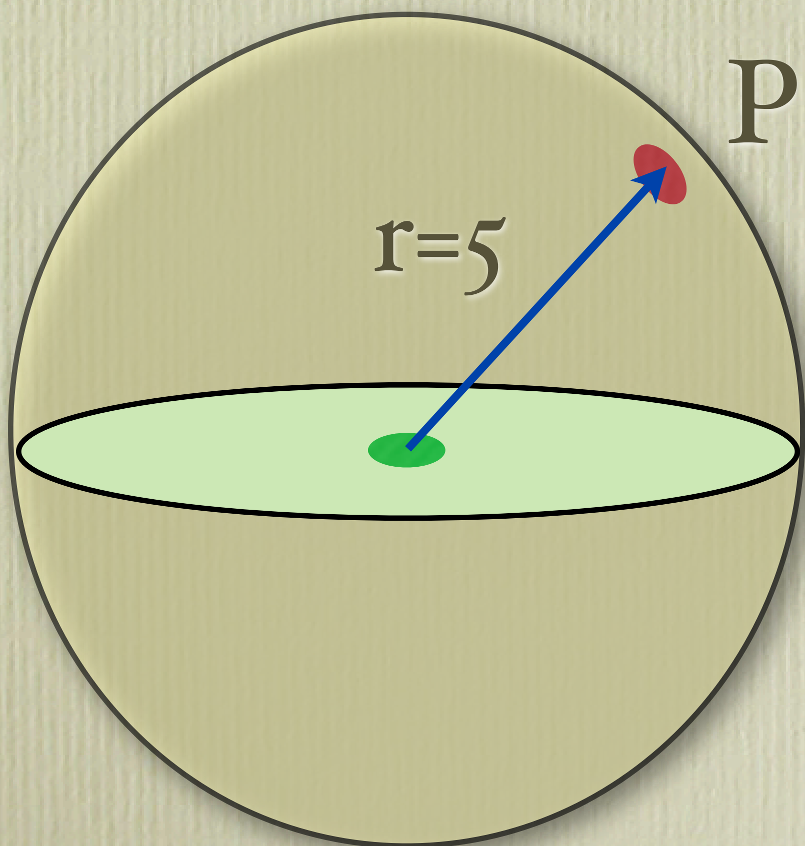
$\vec{v} = < {}^{-}2, 2, 1 >$



$$d(P,Q) = |\vec{v}|$$

Spheres

$$(x-3)^2 + (y+4)^2 + (z-2)^2 = 25$$

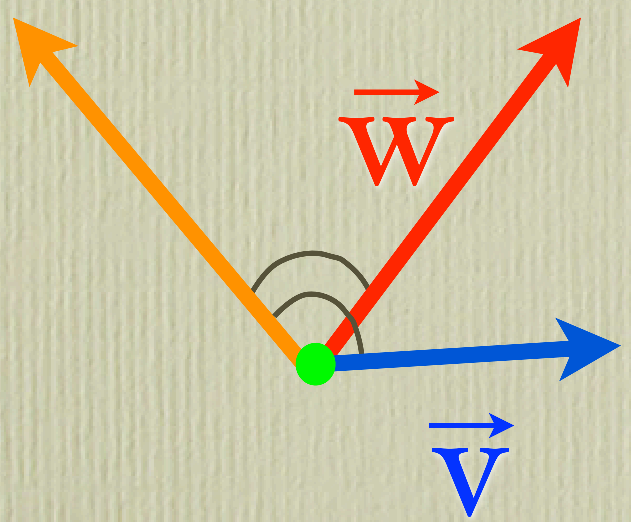


$Q=(3,-4,2)$

center

Dot and Cross product

$\vec{v} = \langle 3,4,1 \rangle$ $\quad\quad$ $\vec{w} = \langle 2,{}^{-}1,1 \rangle$



$$\vec{v} \cdot \vec{w} = 6 - 4 + 1 = 3$$

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 3 & 4 & 1 \\ 2 & -1 & 2 \end{vmatrix}$$

$$= \langle 9, {}^{-}4, {}^{-}11 \rangle$$

Two important formulas

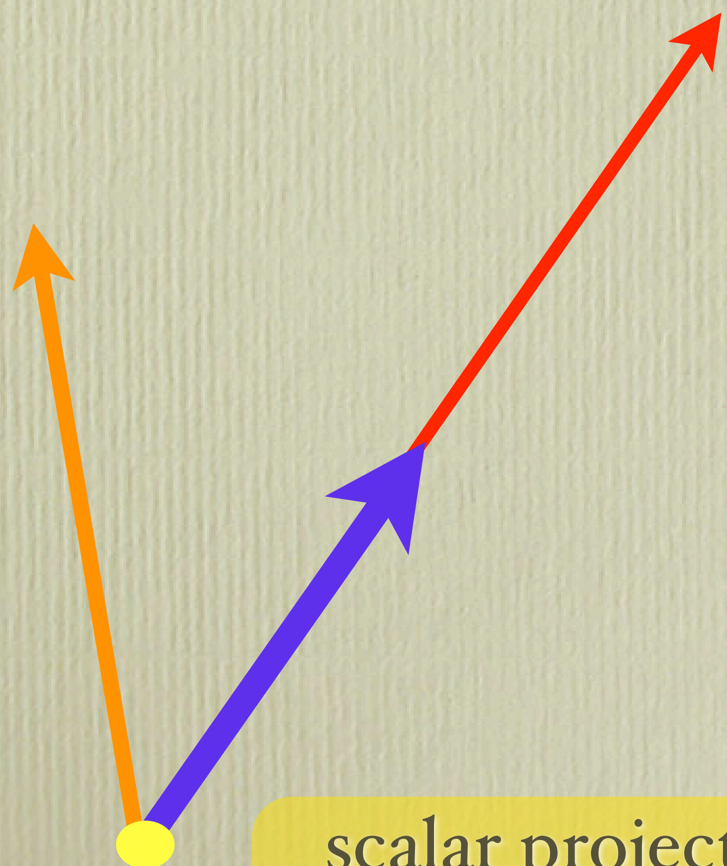
$$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}| \cos(\alpha)$$

$$|\vec{v} \times \vec{w}| = |\vec{v}|\,|\vec{w}| \sin(\alpha)$$

Projection

$\vec{v} = \langle 2, {}^{-}3, 4 \rangle$ $\quad\quad$ $\vec{w} = \langle 4, 0, 1 \rangle$

Project $\vec{v}$ onto $\vec{w}$:

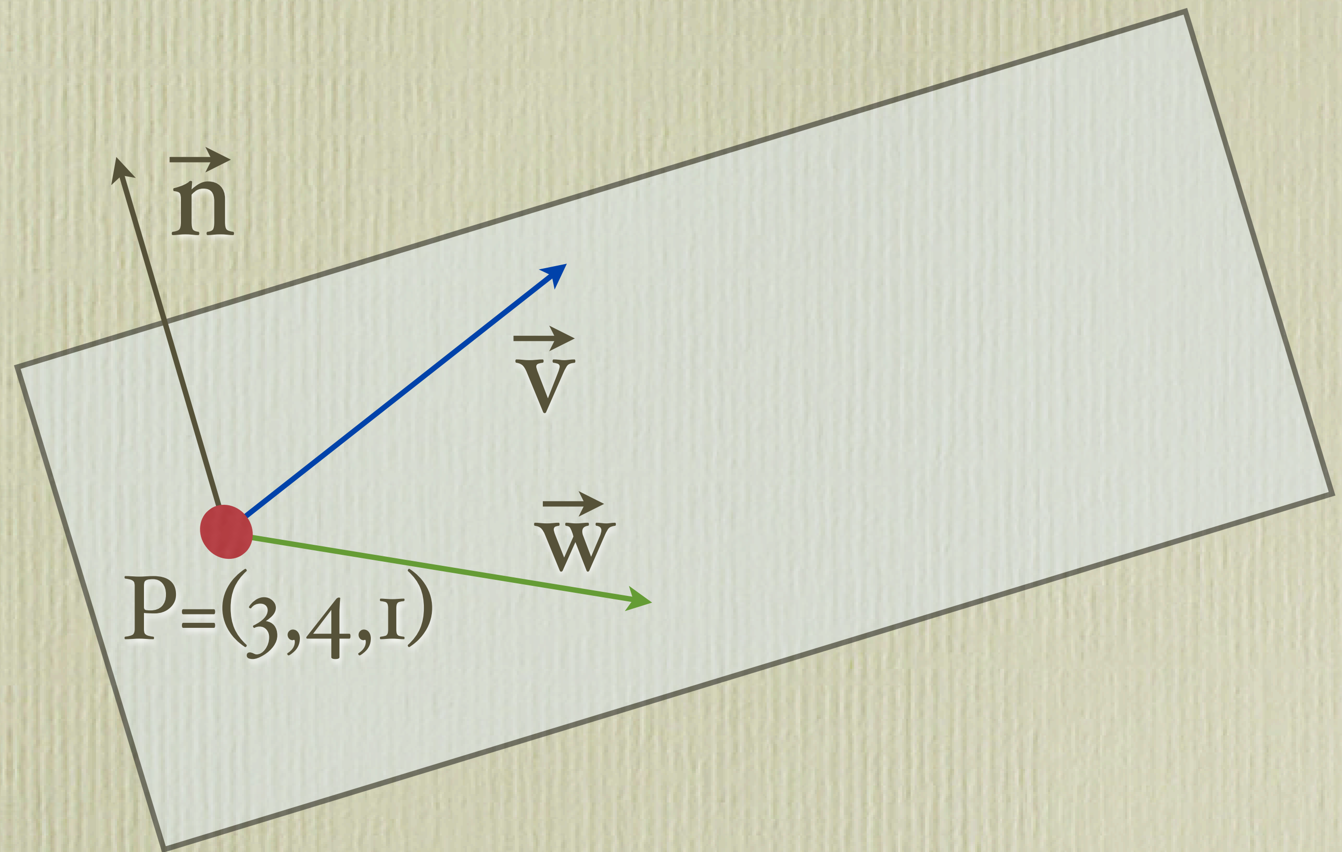


$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|} \frac{\vec{w}}{|\vec{w}|}$$

$$= \frac{12}{17} \langle 4, 0, 1 \rangle$$

scalar projection = component

Lines and Planes



$$\vec{OP} = \langle 3,4,1 \rangle$$

$$\vec{v} = \langle 1,1,-3 \rangle$$

$$\vec{w} = \langle 1,-2,1 \rangle$$

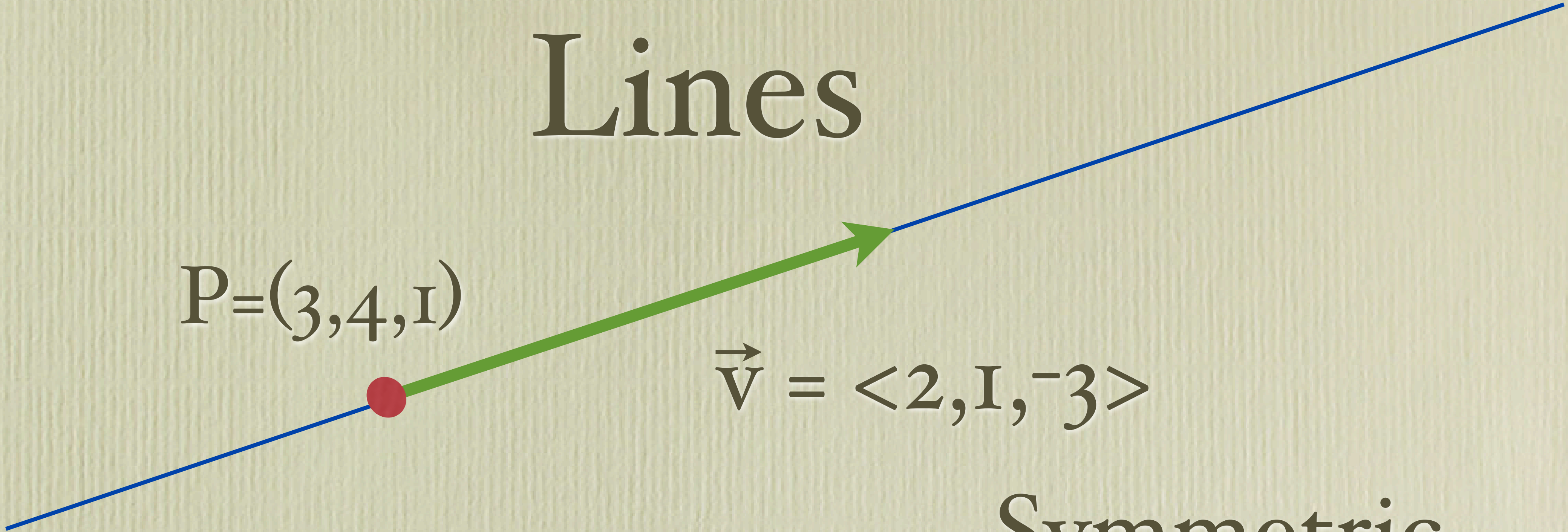
Parametrization

$$\vec{n} = \langle 7,-4,-3 \rangle$$

$$\vec{r}(t,s) = \langle\, 3+t+s, 4+t-2s, 1-3t+s \rangle$$

$$7x-4y-3z=2$$

Lines



Parametrization

$$\vec{r}(t) = \langle 3+2t, 4+t, 1-3t \rangle$$

$$= \langle x, y, z \rangle$$

Symmetric equations

$$\frac{x-3}{2} = \frac{y-4}{1} = \frac{z-1}{-3}$$

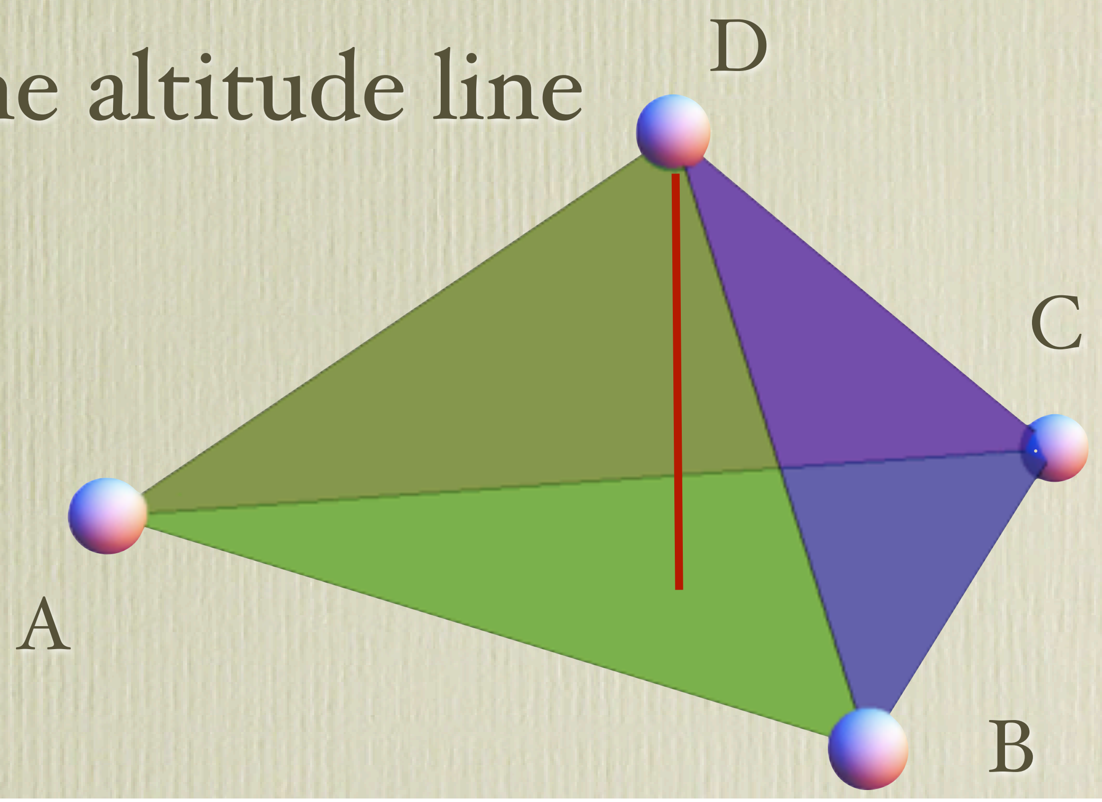
(solve for t)

Problem 1

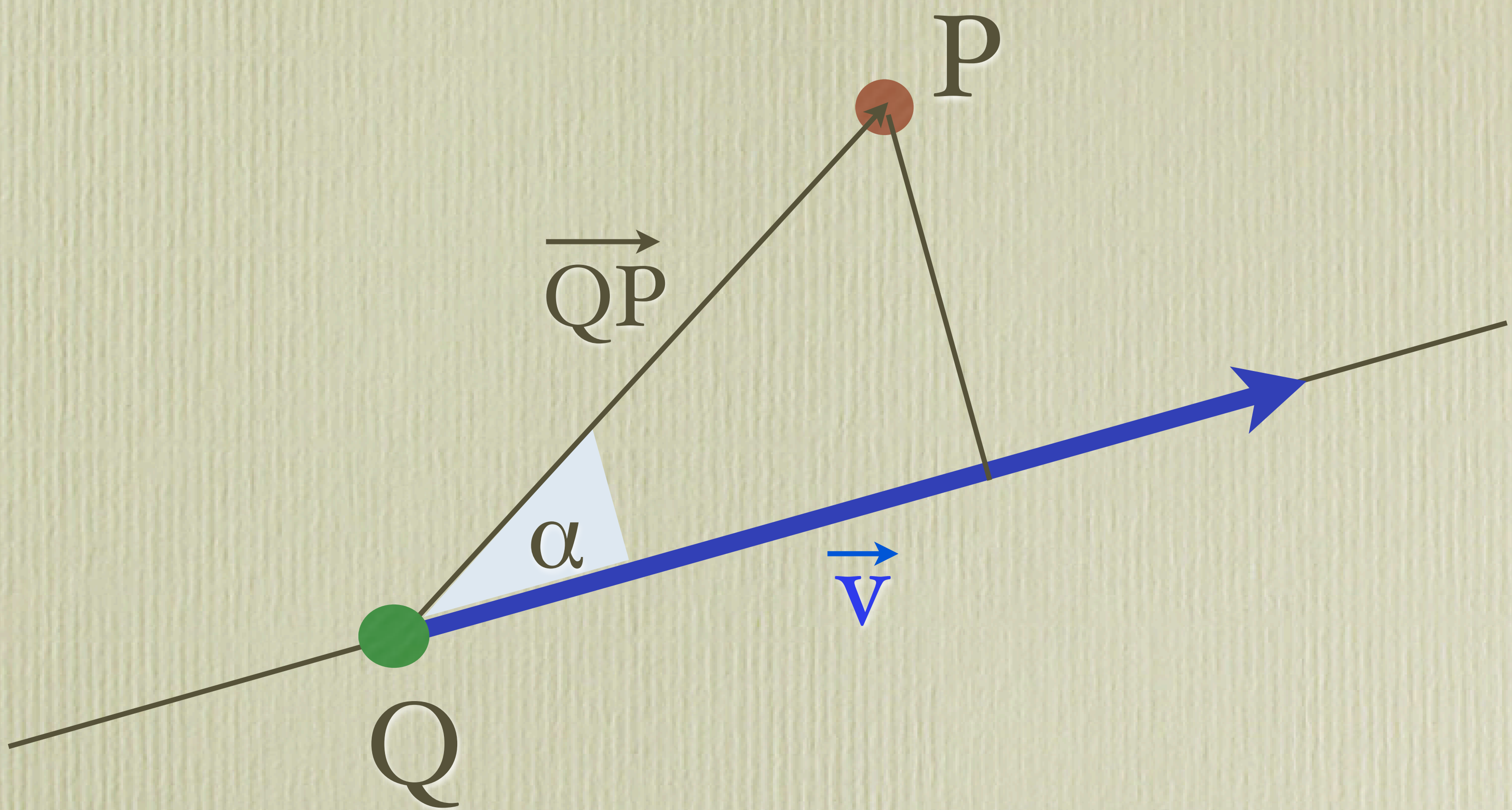
Given a Tetrahedron with vertices

A=(00,0),B=(1,1,0),C=(0,1,0), D=(2,2,3)

Find the equation of the altitude line through D!

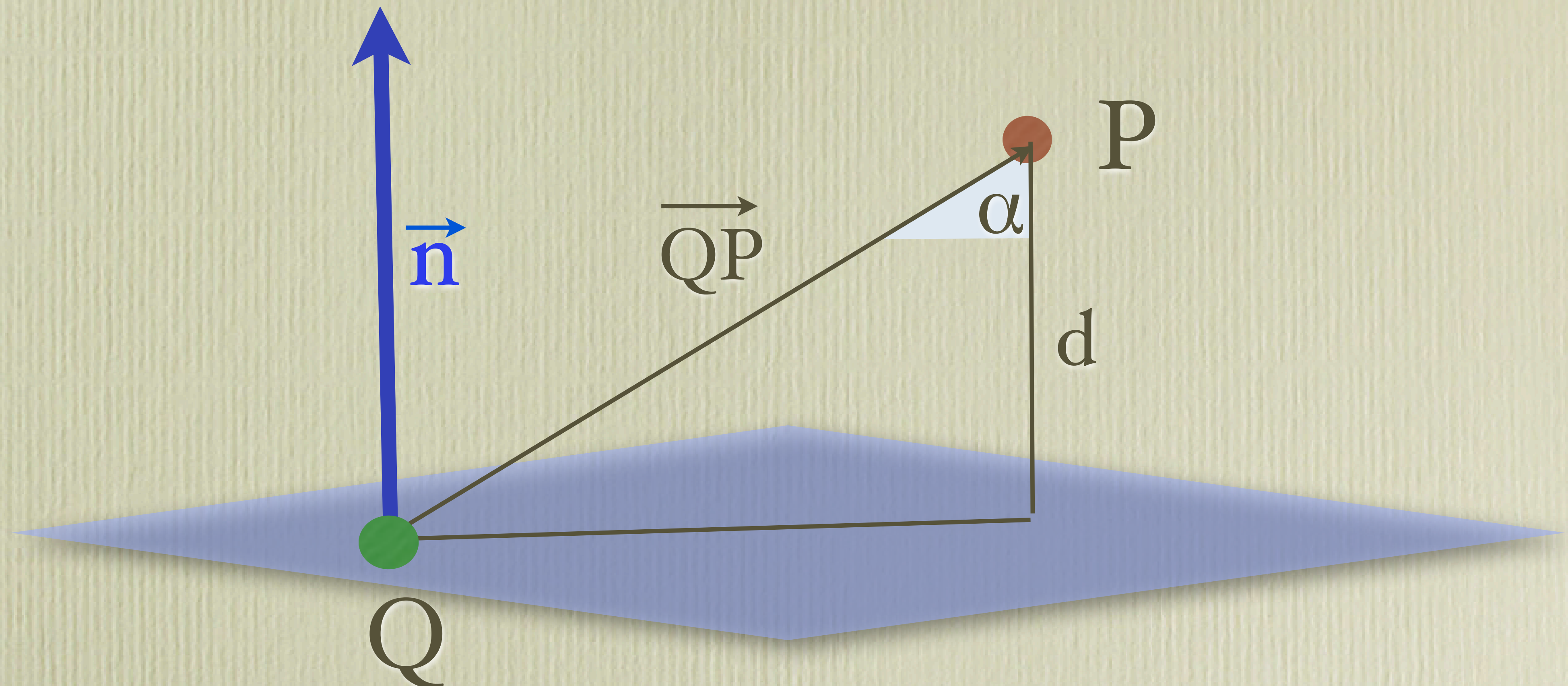


Distance Point/Line



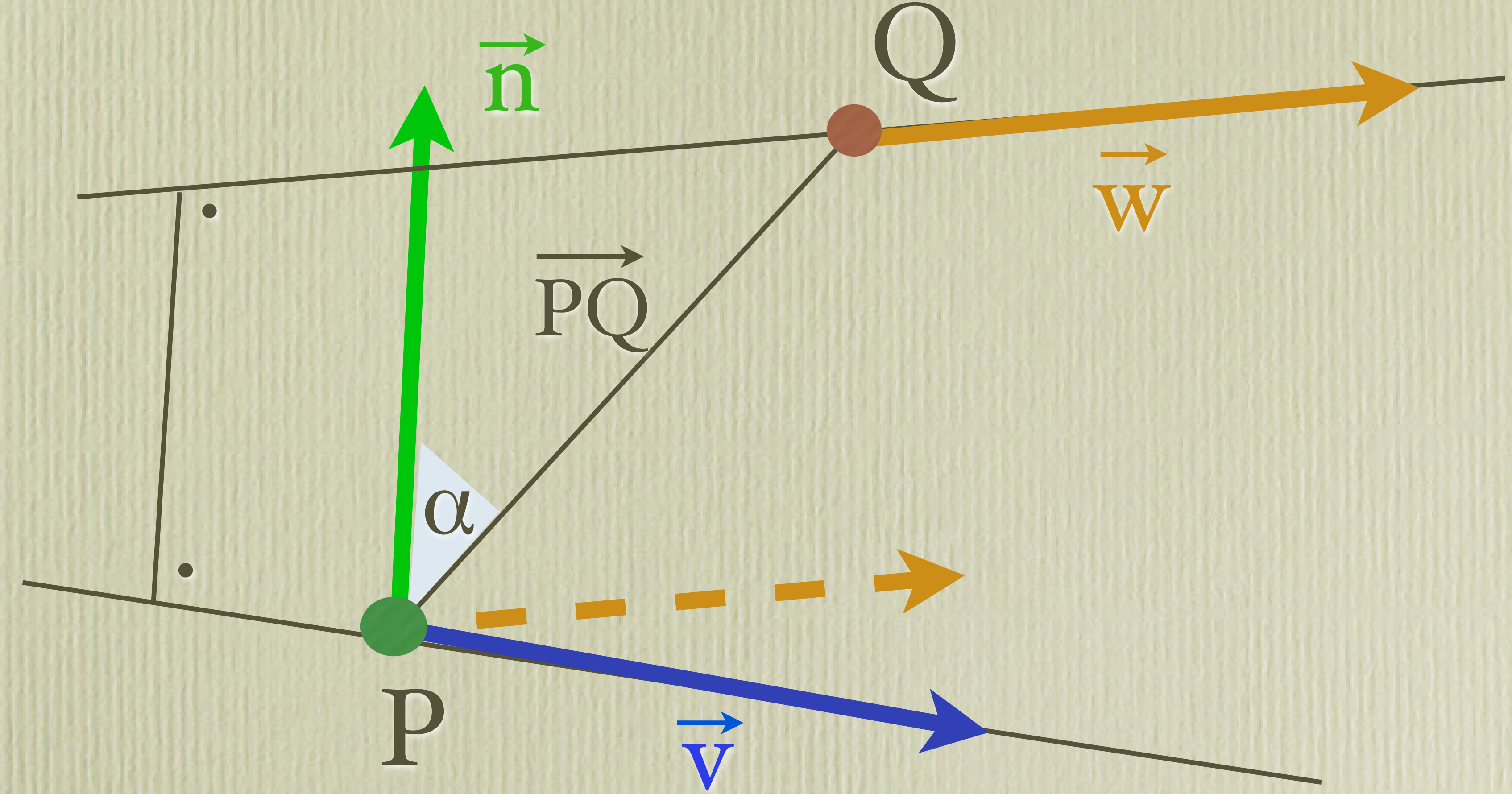
$$\frac{|\vec{QP}|\sin(\alpha)\,|\vec{v}|}{|\vec{v}|} = \frac{|\vec{QP} \times \vec{v}|}{|\vec{v}|}$$

Distance Point-Plane



$$d = \frac{|\vec{PQ}| \cos(\alpha)\, |\vec{n}|}{|\vec{n}|} = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

Distance Line/Line



$$\frac{|\vec{PQ}| \cos(\alpha)\ |\vec{n}|}{|\vec{n}|} = \frac{|\vec{QP} \cdot \vec{v} \times \vec{w}|}{|\vec{v} \times \vec{w}|}$$

Problem

We build two sacred pipes above Harvard yard. One from A=(1,1,2) to B=(2,6,2) and an other from C=(1,4,5) to D=(2,5,5) Find the distance.

Area and Volume

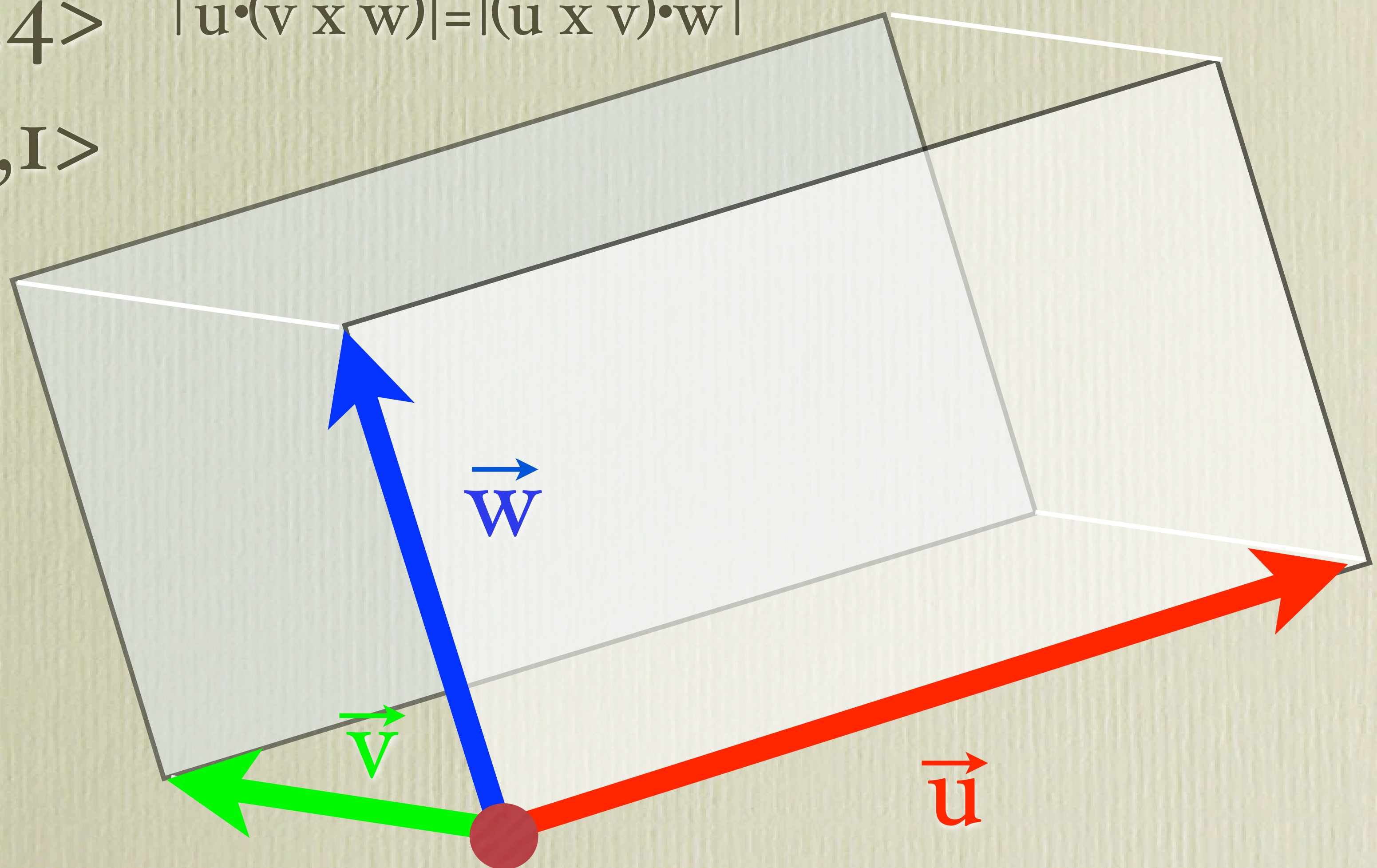
$\vec{u} = < 1,2,3>$

$\vec{v} = < 3,1,4>$

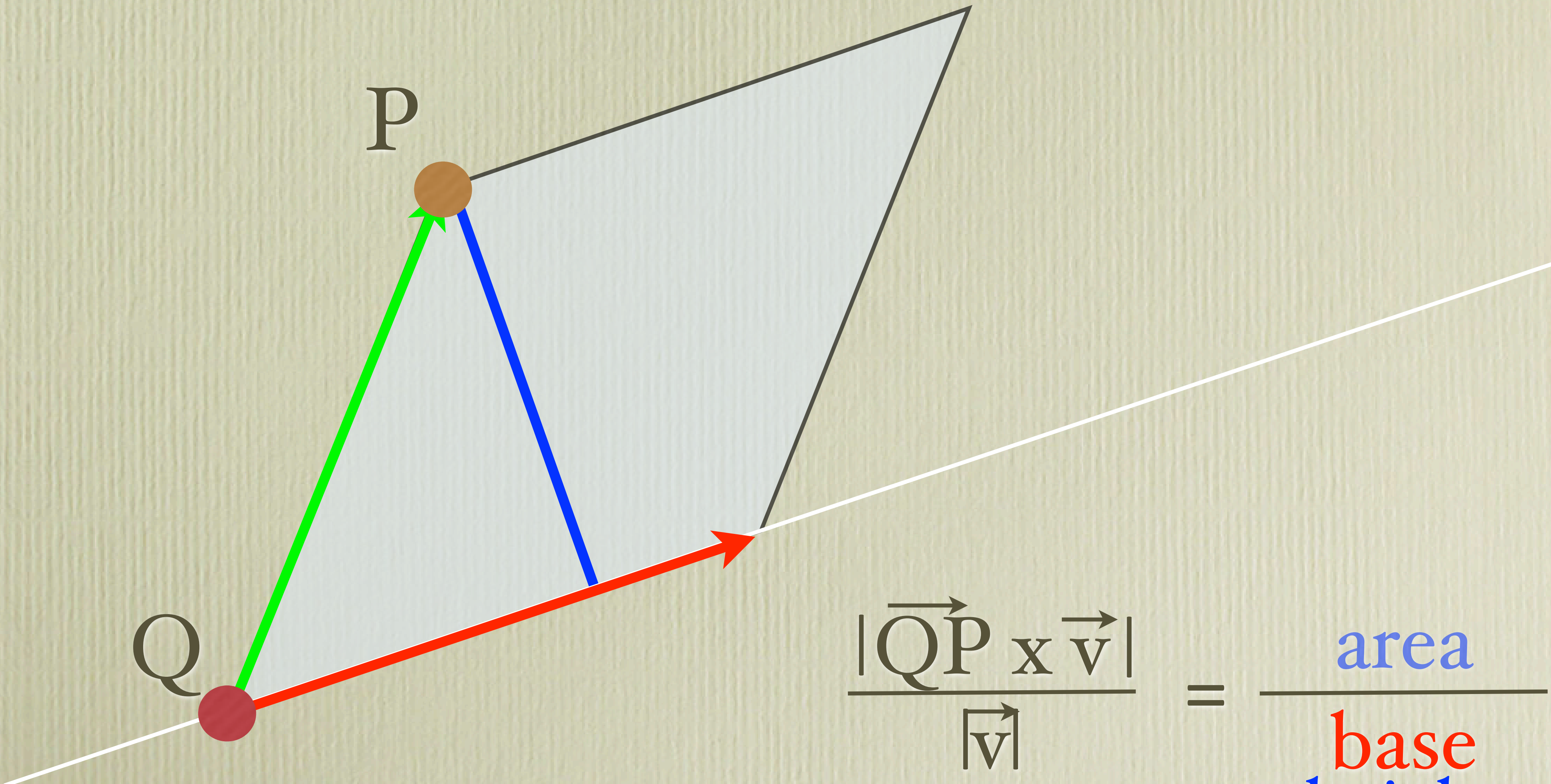
$\vec{w} = < 1,1,1>$

The volume of the parallelepiped spanned by $\vec{u},\vec{v},\vec{w}$ is

$|\vec{u}\cdot(\vec{v} \times \vec{w})| = |(\vec{u} \times \vec{v})\cdot\vec{w}|$

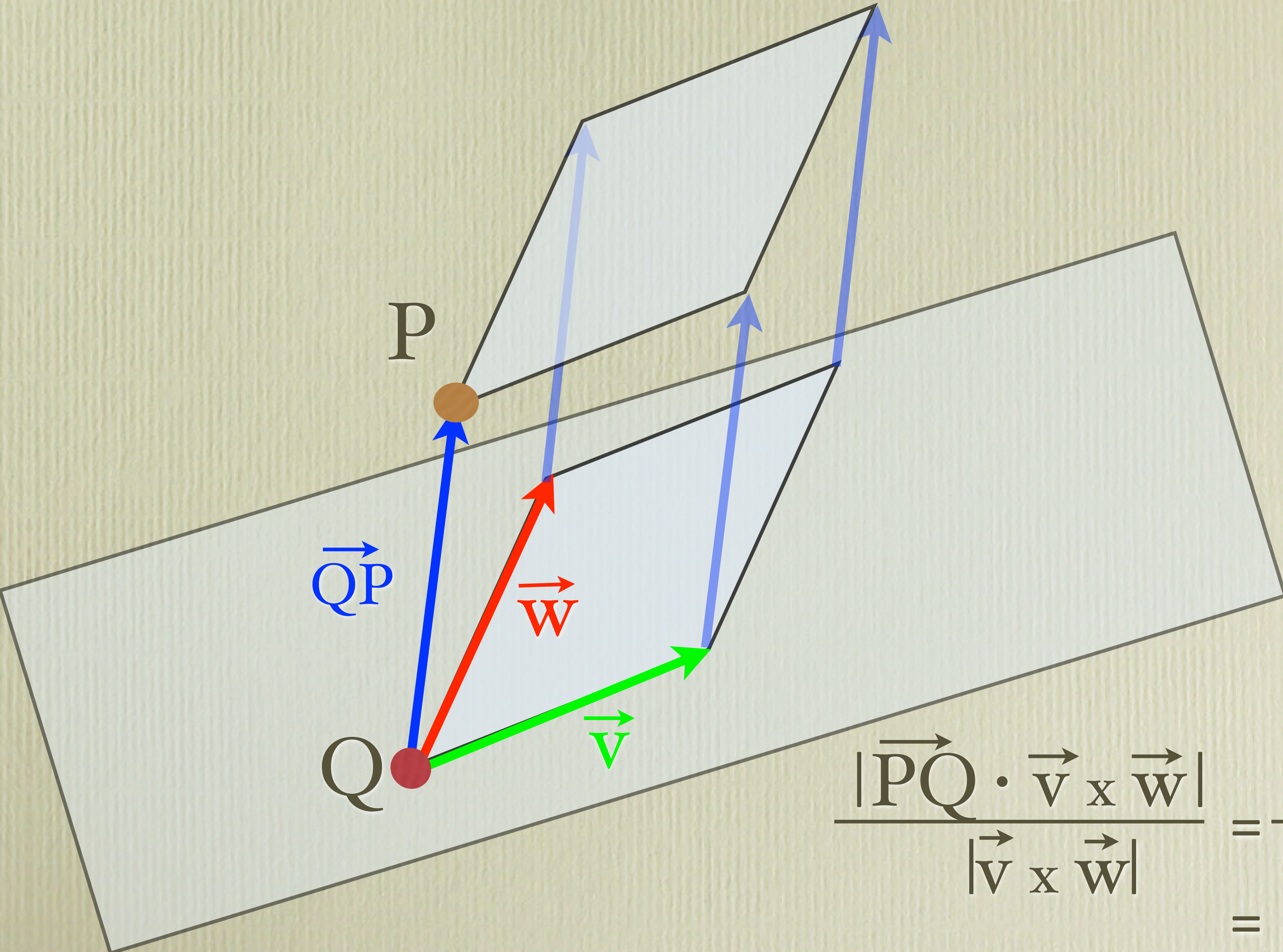


Distance Point-Line again



$$\frac{|\overrightarrow{QP} \times \vec{v}|}{|\vec{v}|} = \frac{\text{area}}{\text{base}} = \text{height}$$

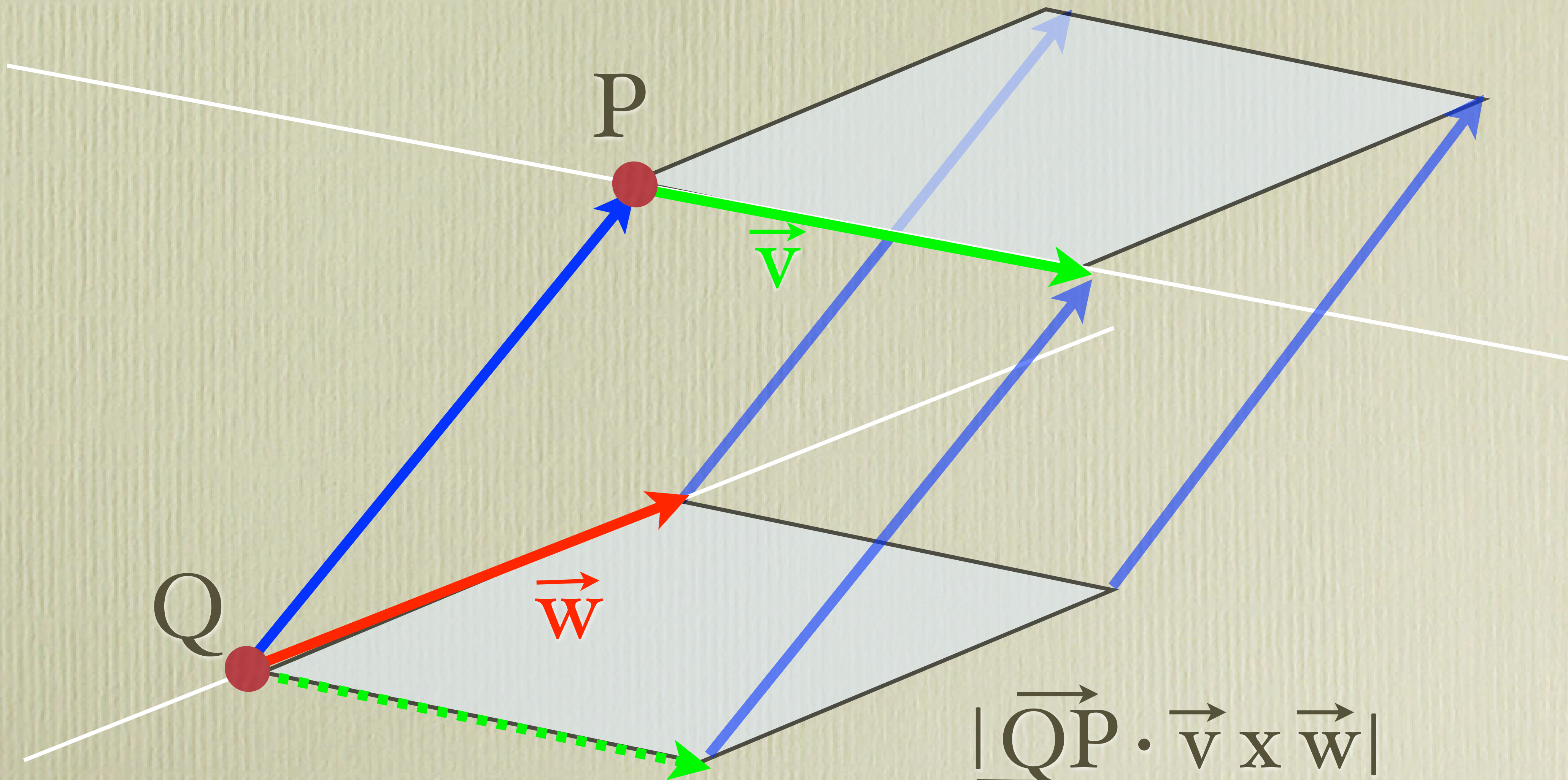
Distance Point-Plane again



$$\frac{|\overrightarrow{PQ} \cdot \vec{v} \times \vec{w}|}{|\vec{v} \times \vec{w}|} = \frac{\text{volume}}{\text{area}}$$

$$= \text{height}$$

Distance Line-Line



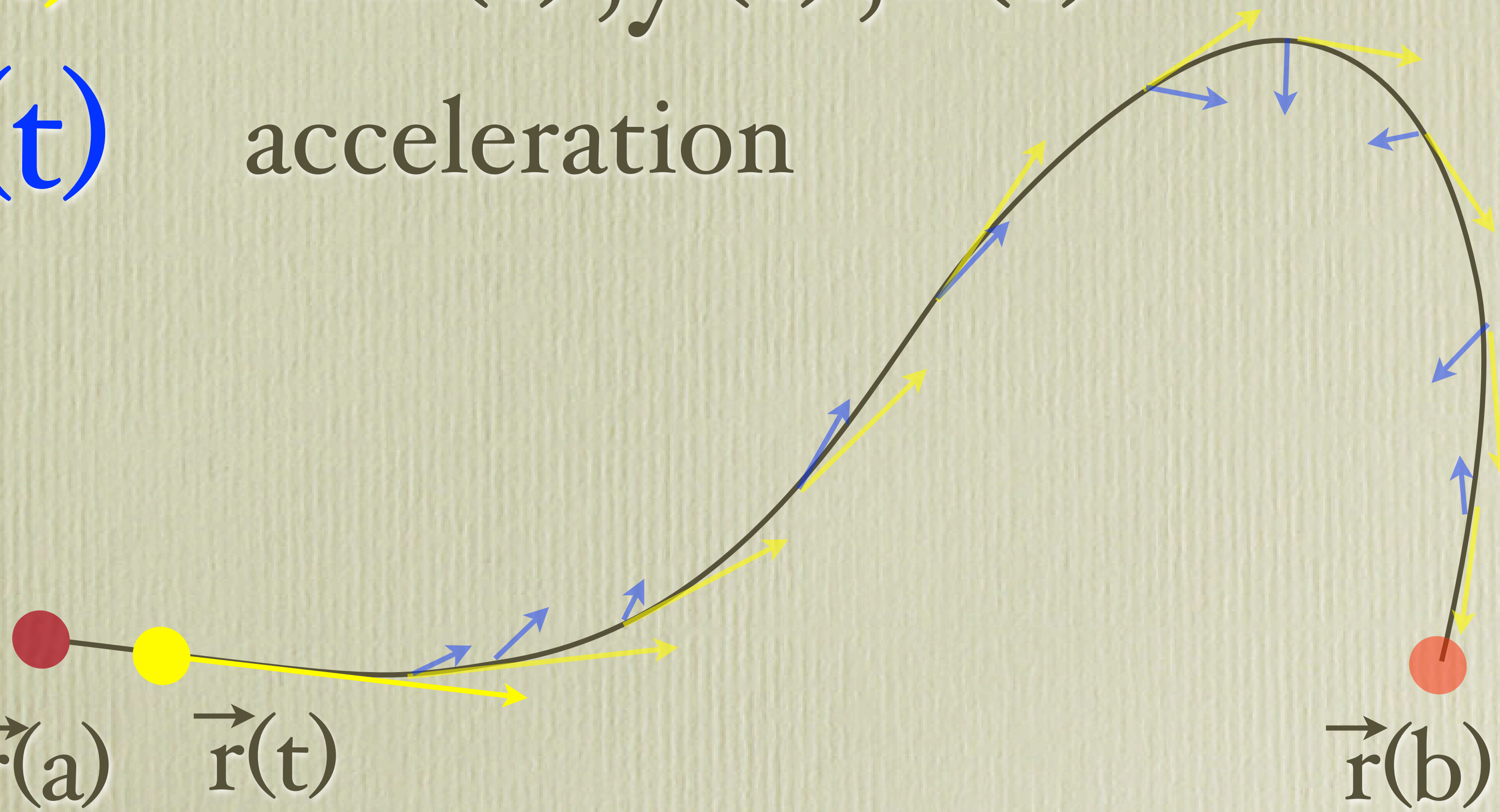
$$\frac{|\overrightarrow{QP} \cdot \vec{v} \times \vec{w}|}{|\vec{v} \times \vec{w}|}$$

$$= \frac{\text{Volume}}{\text{Area}}$$

$$\vec{r}(t) = < x(t), y(t), z(t) >$$

$$\vec{r}'(t) = < x'(t), y'(t), z'(t) >$$ velocity

$\vec{r}''(t)$ acceleration



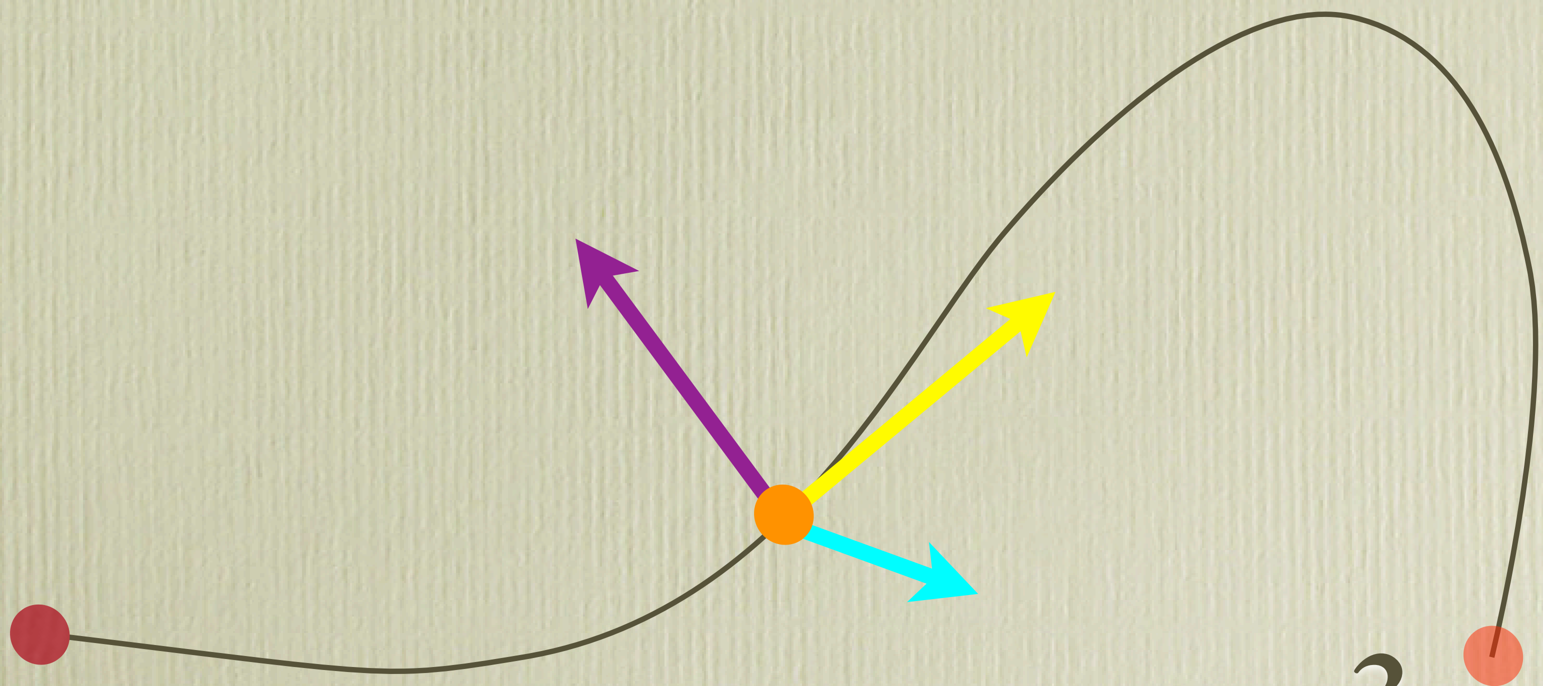
Problem.

We get kicked with initial velocity $\langle 1,0,10 \rangle$ at the point $\langle 0,2,0 \rangle$ and feel an acceleration of $\langle 0,0,-10+\exp(-t) \rangle$. Where do we hit the ground?

$$\vec{T}(t) = \vec{r}'(t)/|\vec{r}'(t)|$$ Unit Tangent

$$\vec{N}(t) = \vec{T}'(t)/|\vec{T}'(t)|$$ Normal

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$ Binormal



curvature $\varkappa = |r' \times r''|/|r'|^3$

Problem:

We fly in the plane

$$4x + 2y - 4z = 12$$

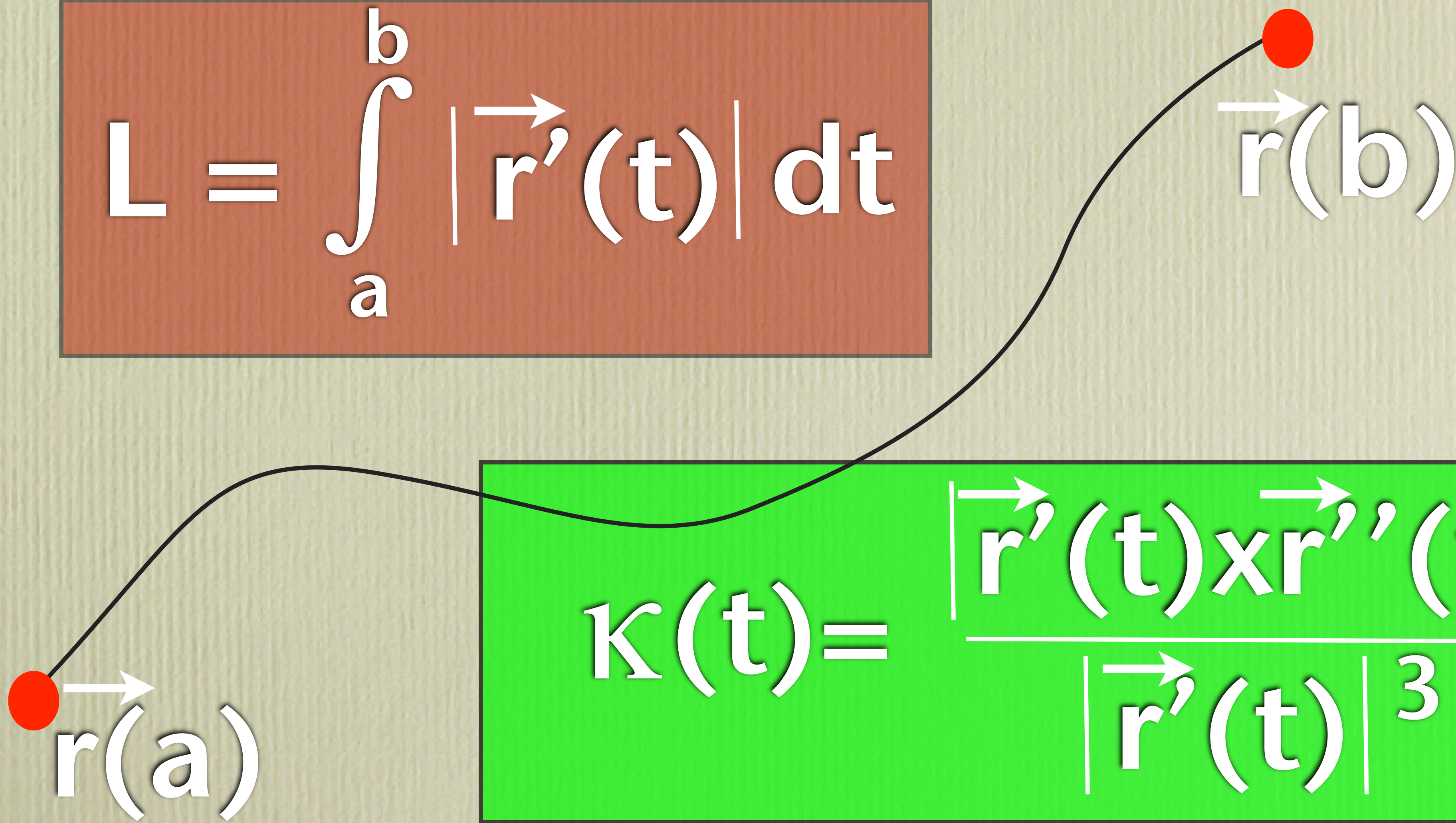
as well as in the surface

$$x^2/9 + z^2/25 = 1$$

Parametrize the path

Arc Length and Curvature

$$L = \int_a^b \left|\vec{r}'(t)\right| dt$$



$$\kappa(t) = \frac{\left|\vec{r}'(t) \times \vec{r}''(t)\right|}{\left|\vec{r}'(t)\right|^3}$$

Independent of parametrization

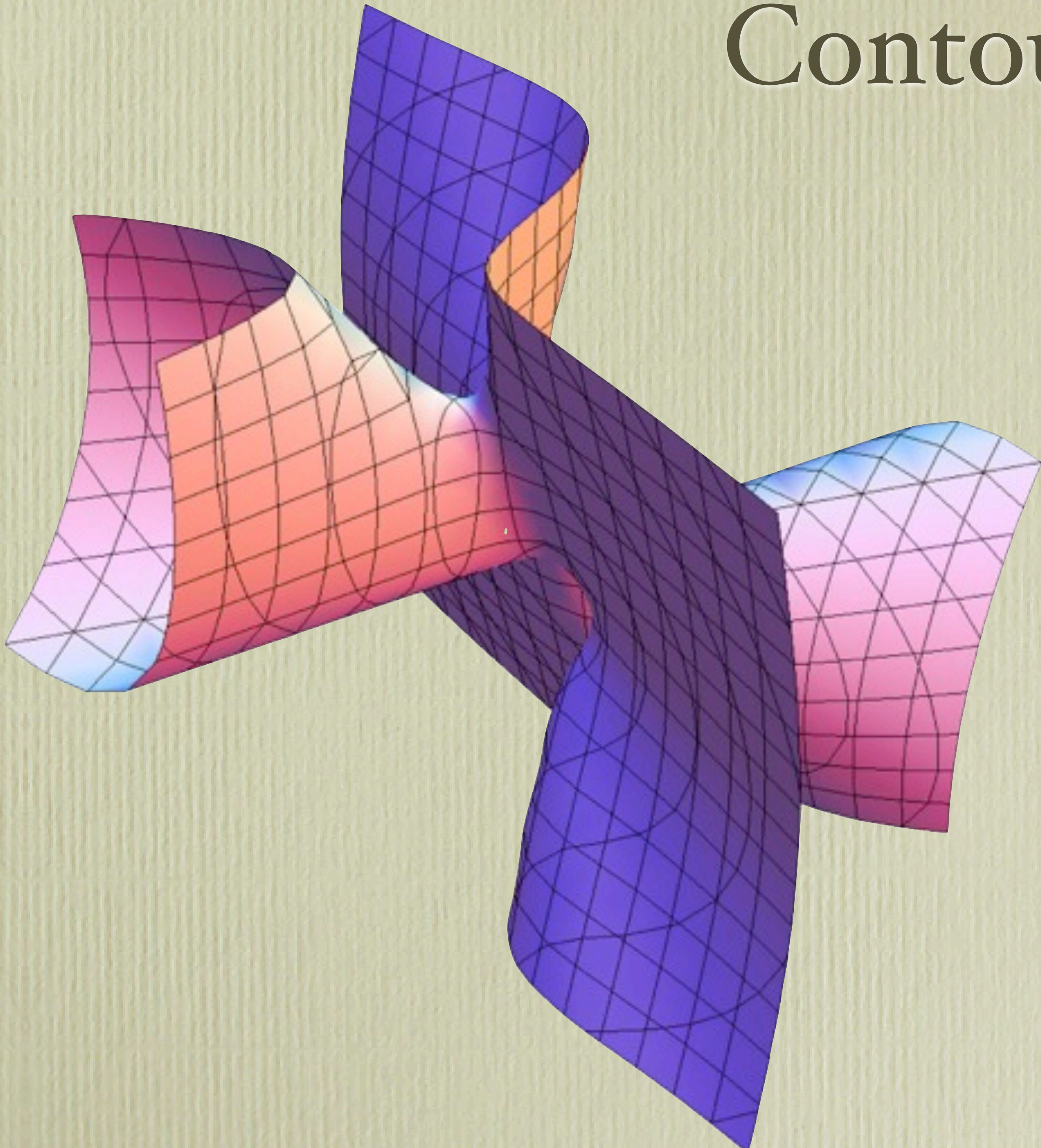
Problem 3

A dragon flies along the curve

$$\vec{r}(t) = \langle t^2/2, 1, -t^3/3 \rangle$$

From t=0 to t=1. Find the arc length of that path.

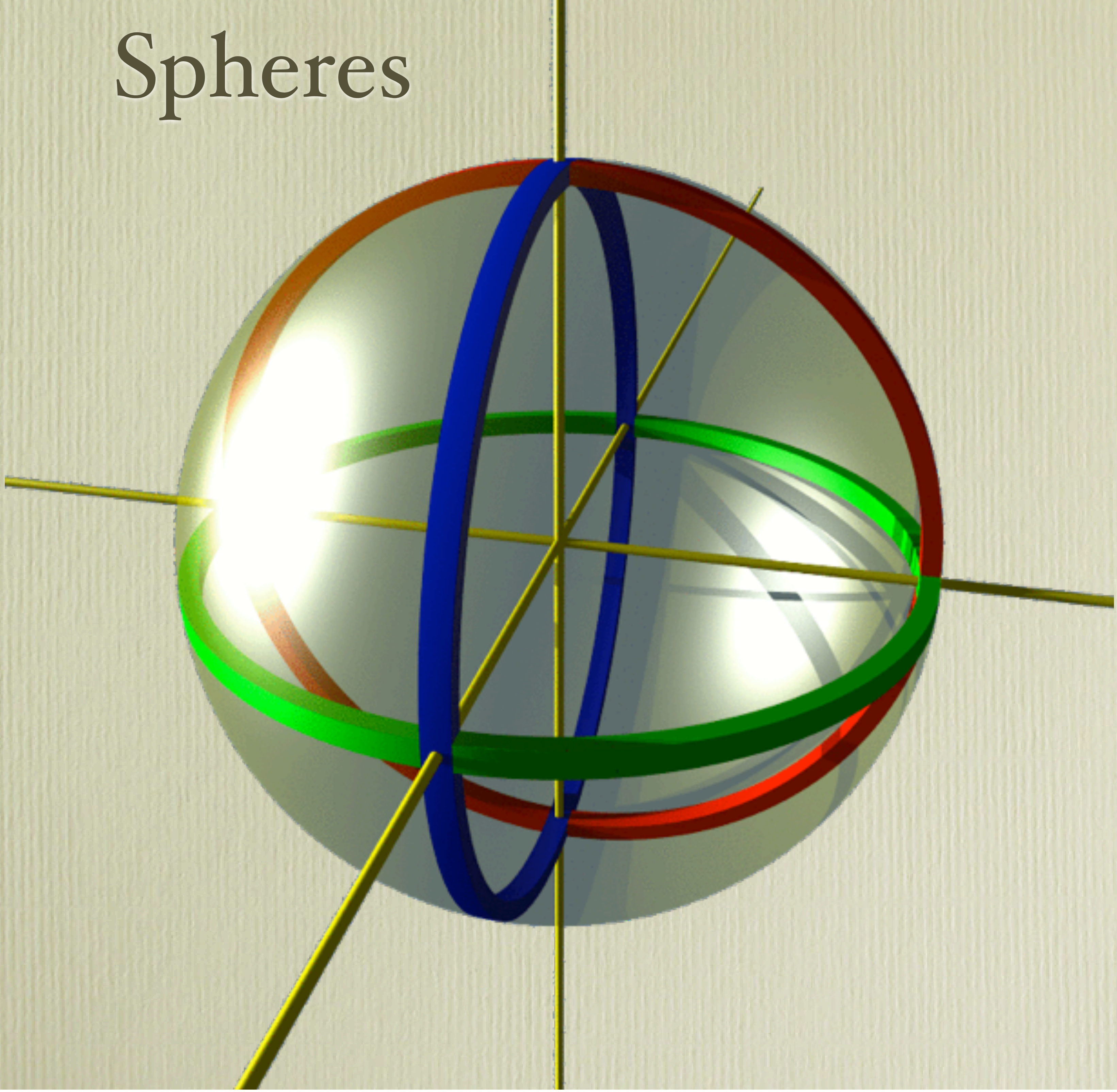
Contour Surfaces



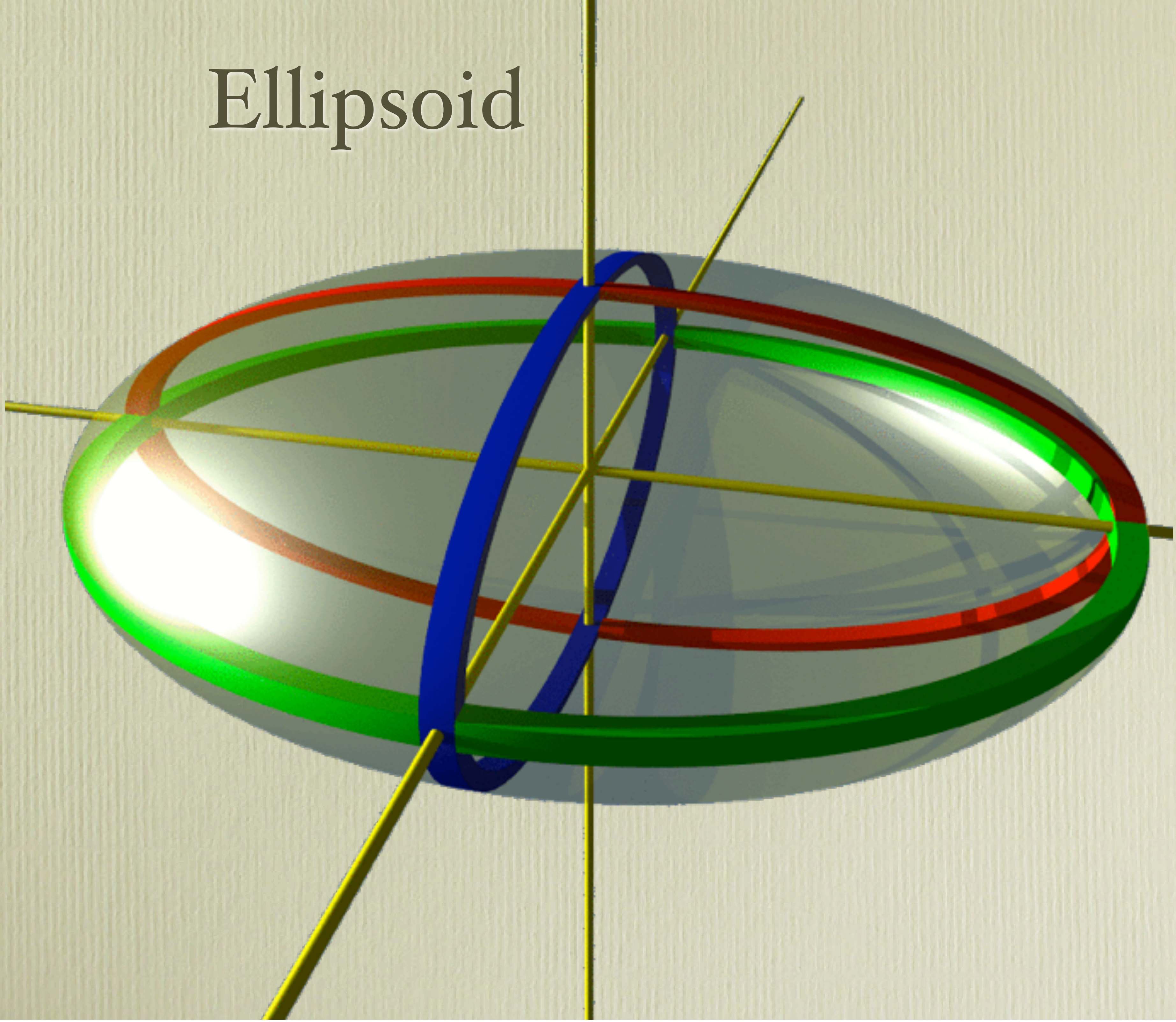
$g(x,y,z) = c$

ContourPlot3D[x^3+y^3+x z^2-2x y^2+z+x^2 z+x y z-3, {x,-6,6}, {y,-6, 6}, {z,-6, 6}]

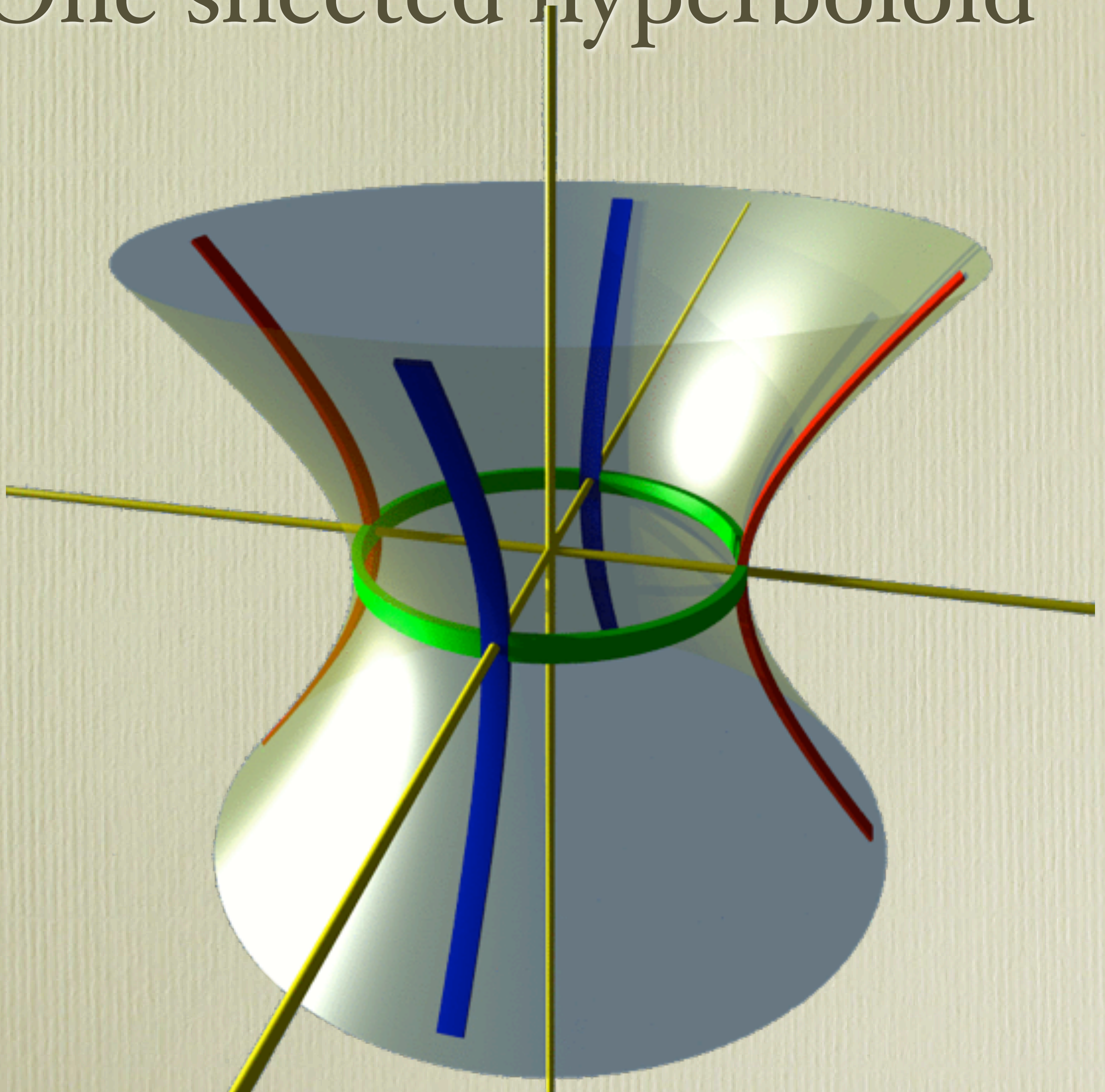
Spheres



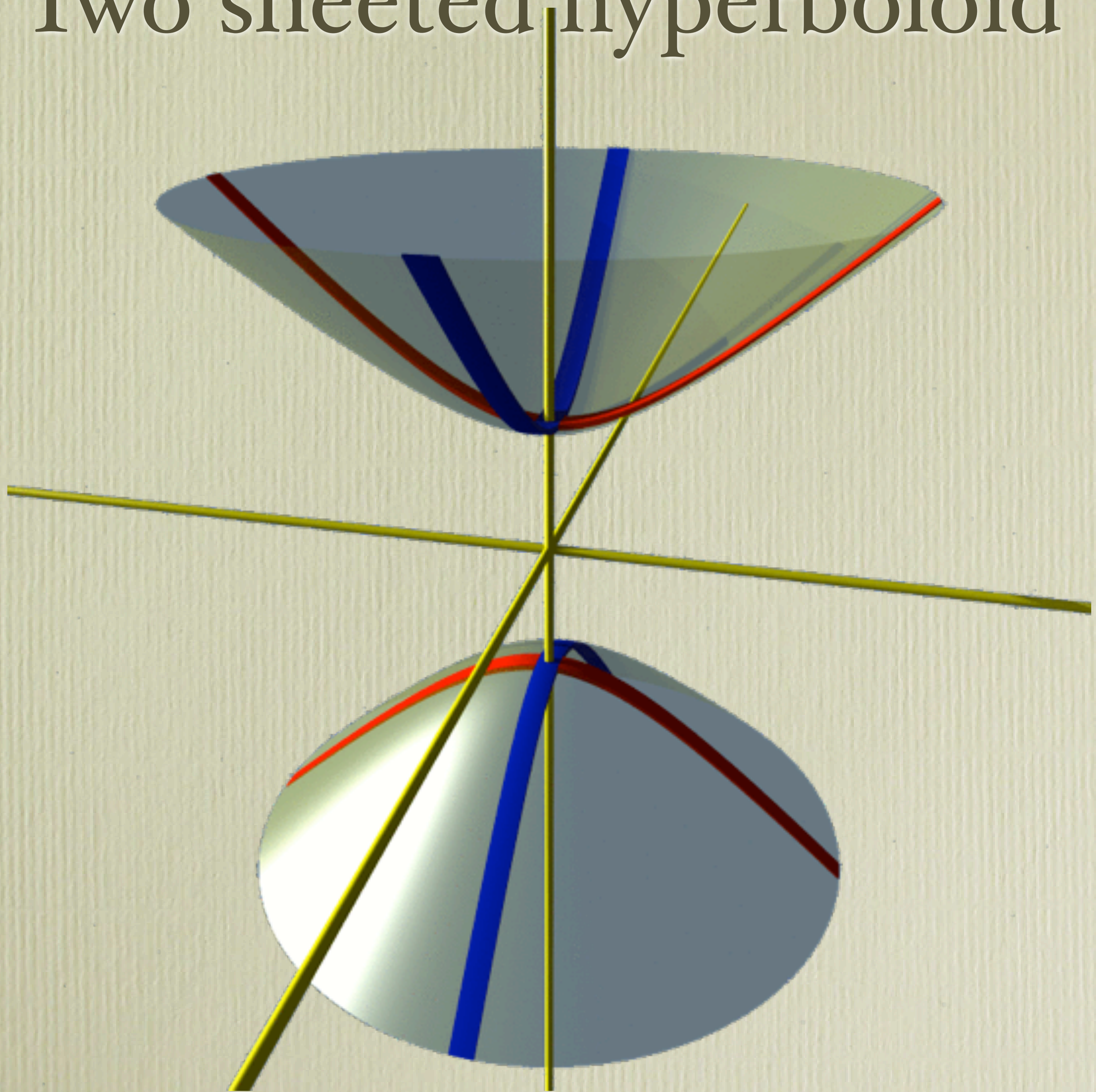
Ellipsoid



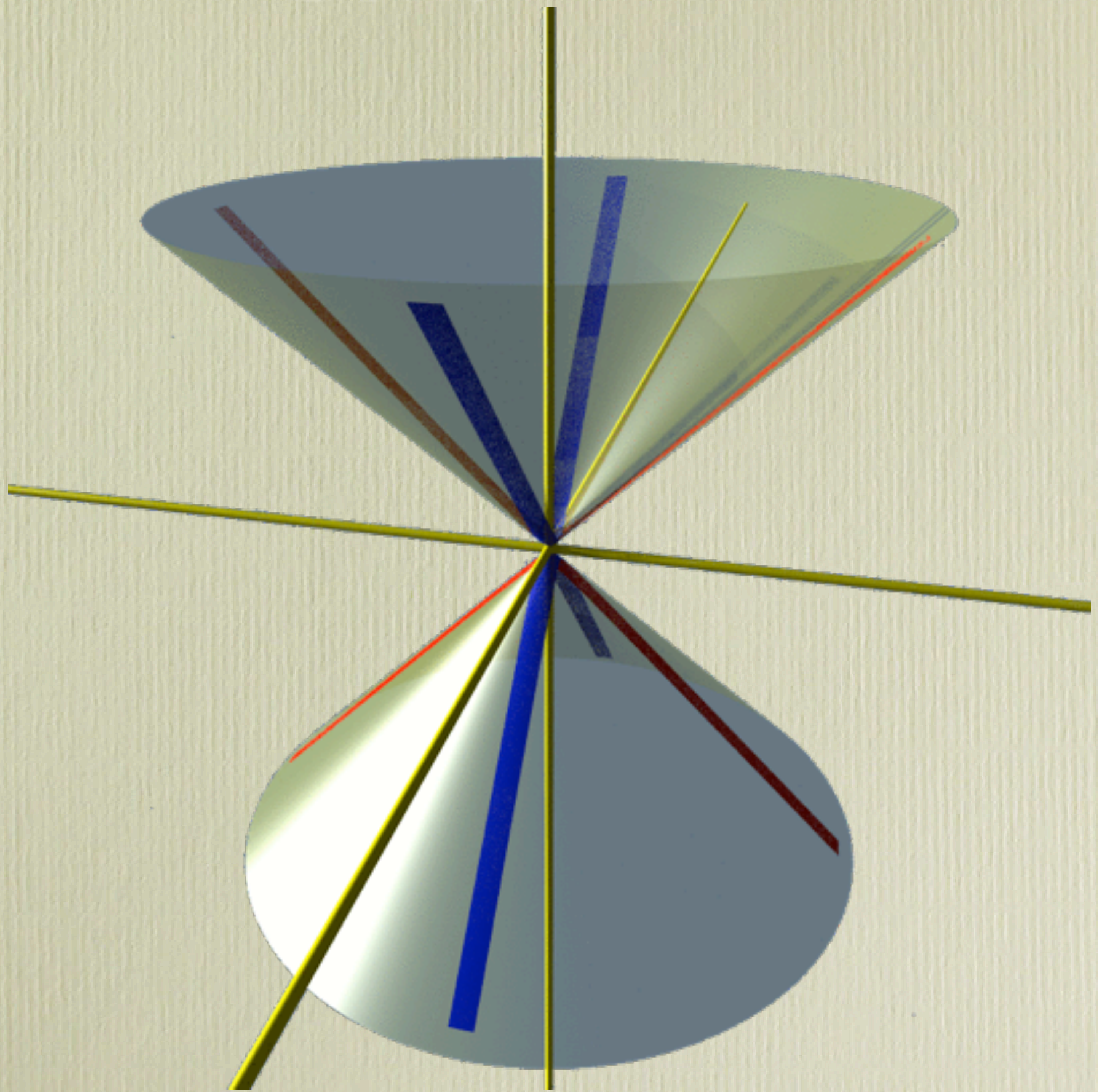
One sheeted hyperboloid



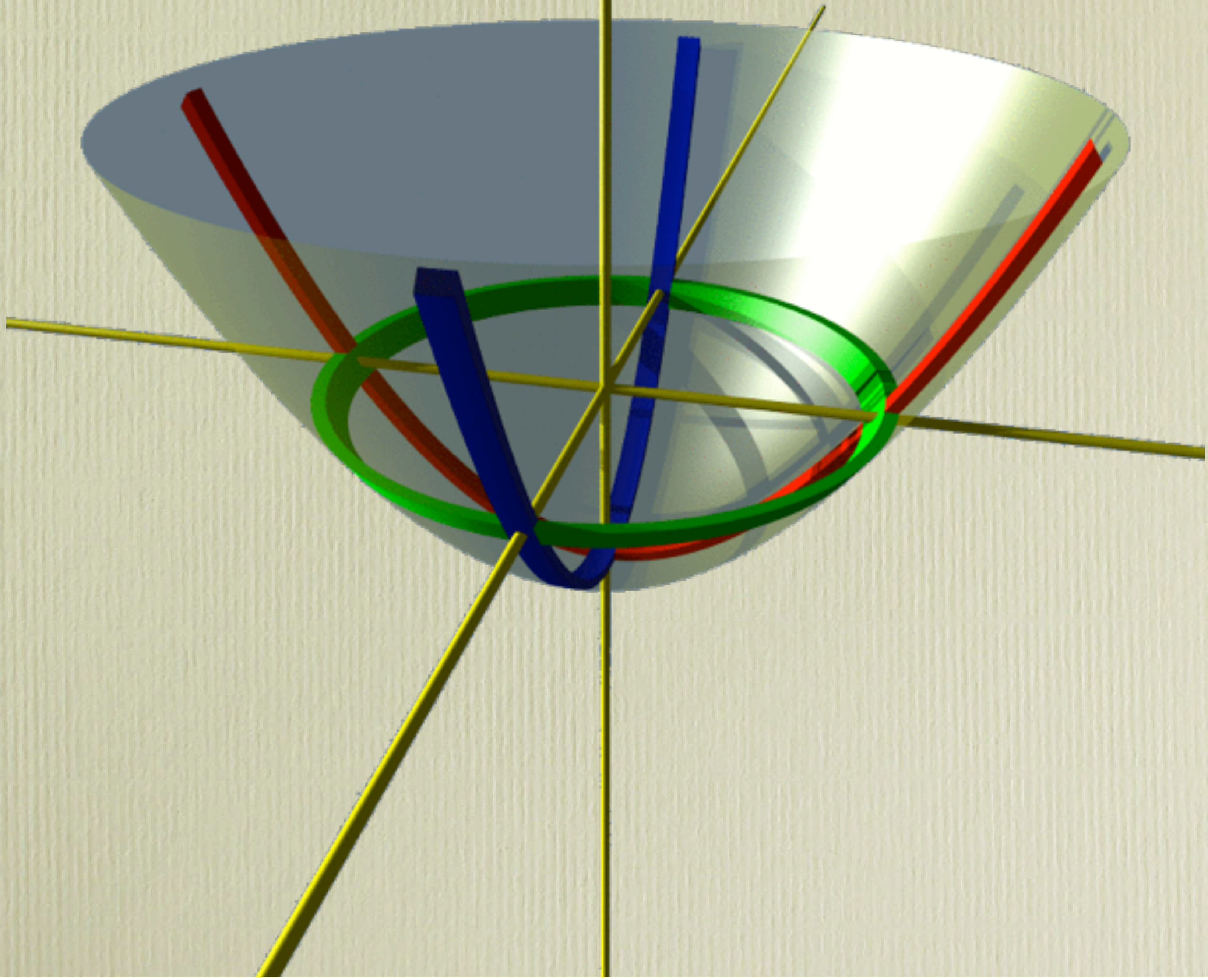
Two sheeted hyperboloid



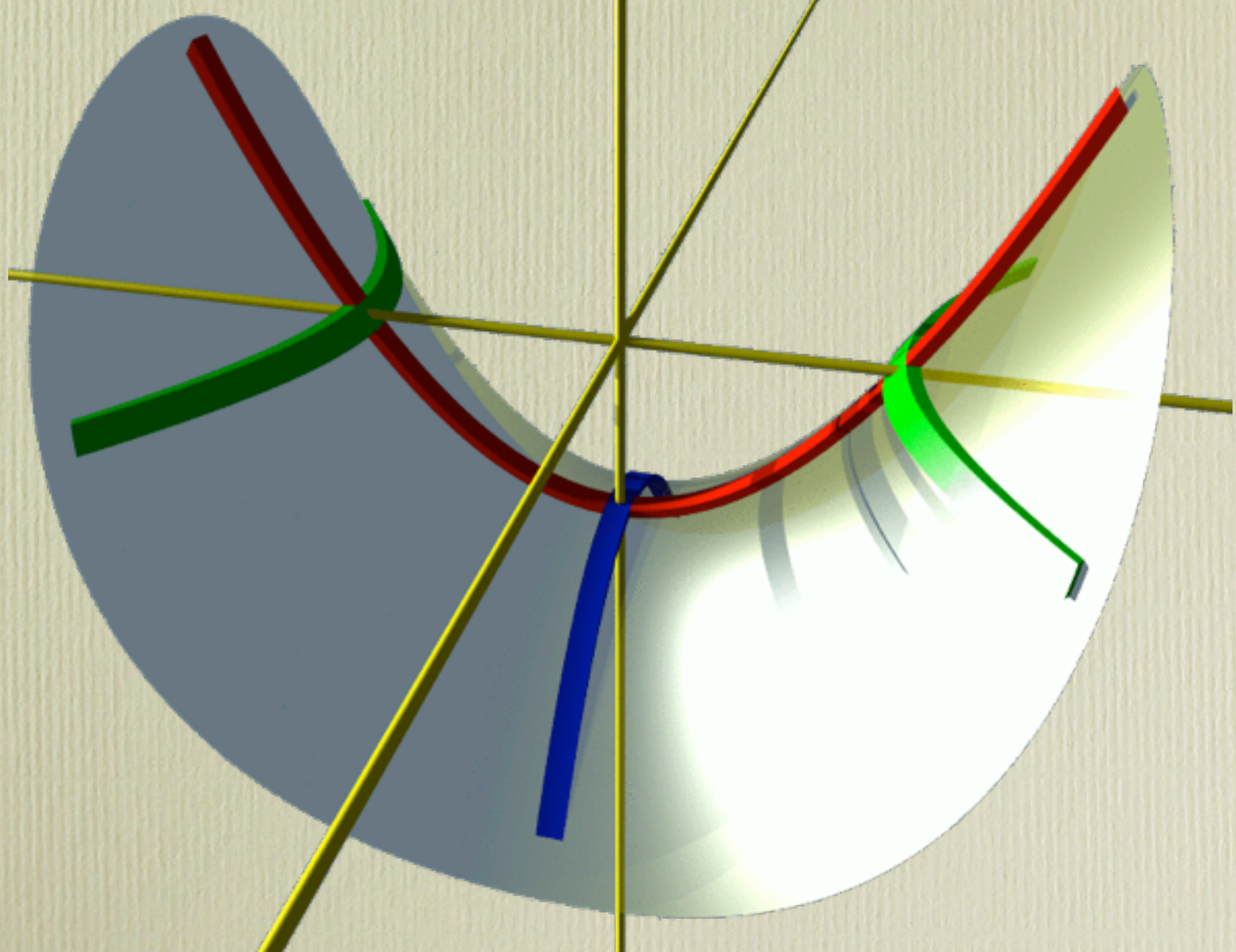
Cone



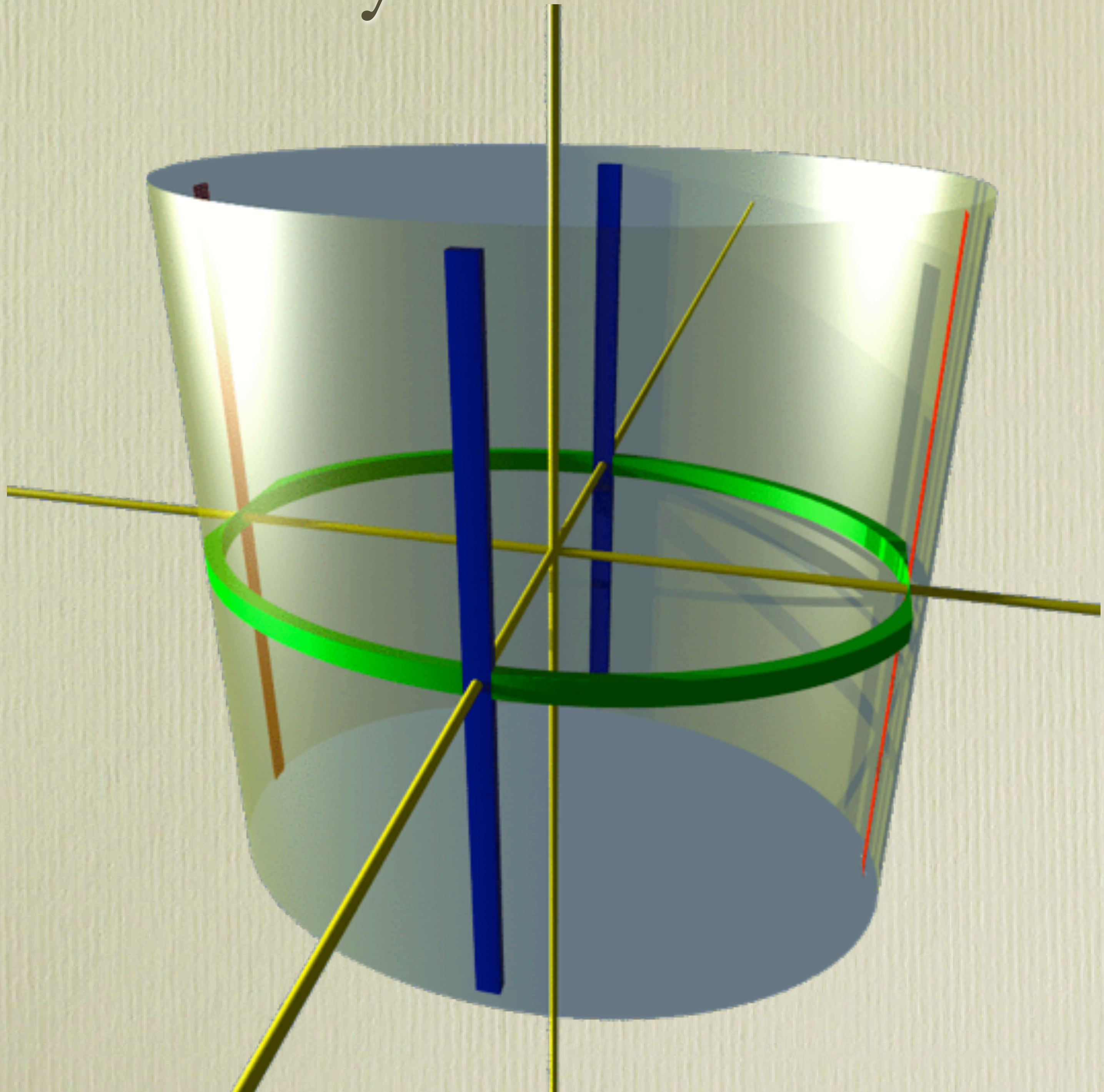
Elliptic Paraboloid



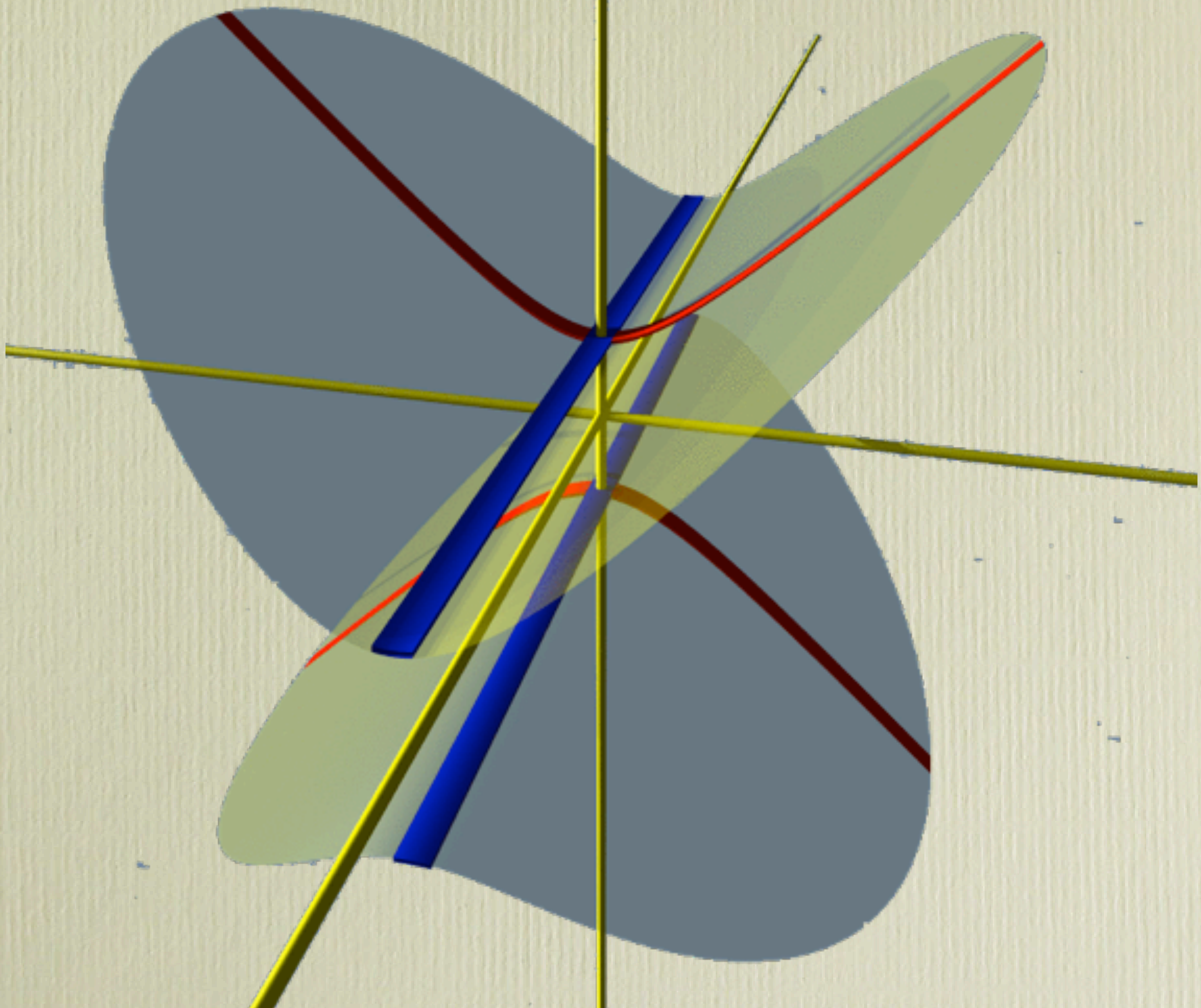
Hyperbolic Paraboloid



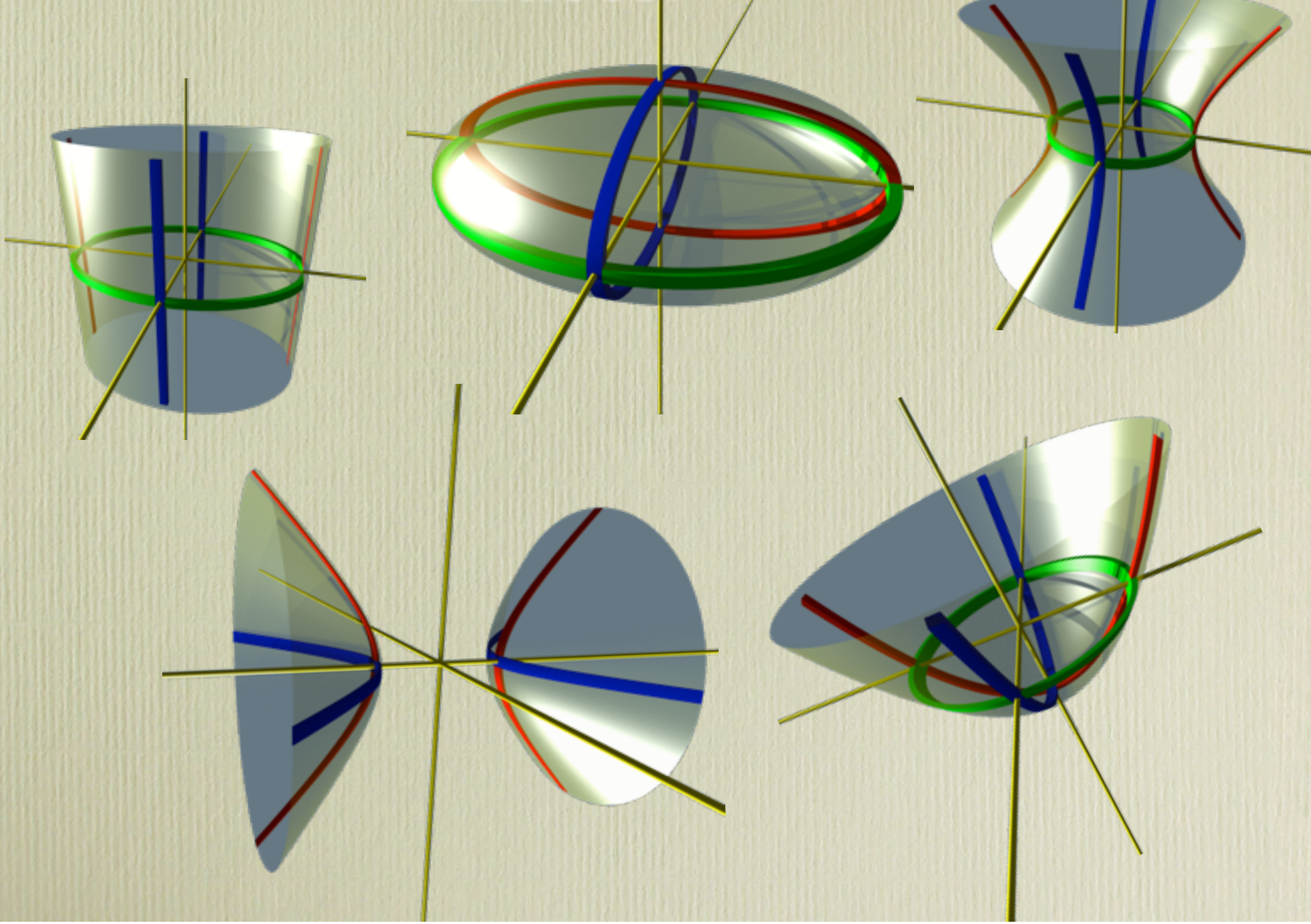
Cylinder



Cylindrical hyperboloid

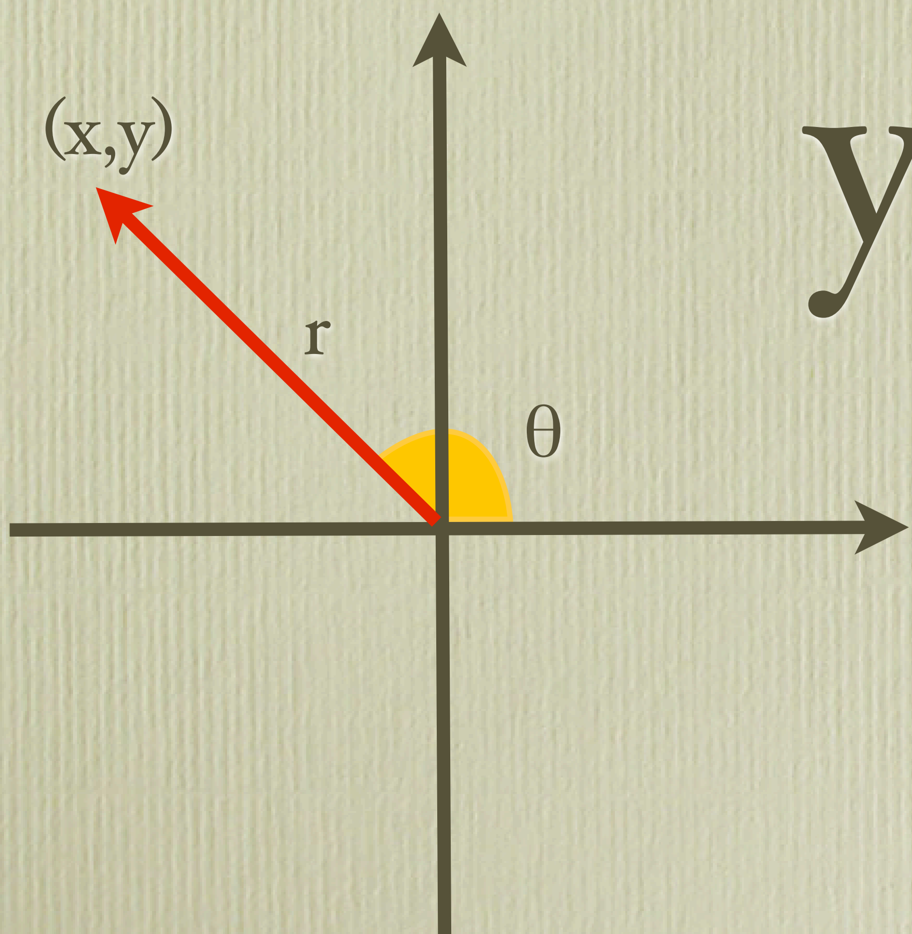


Traces!



Polar coordinates

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$



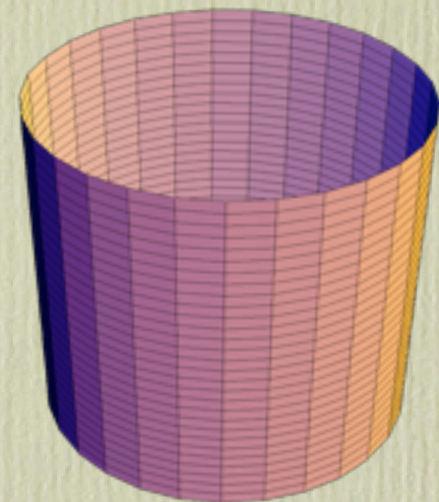
Cylindrical coordinates

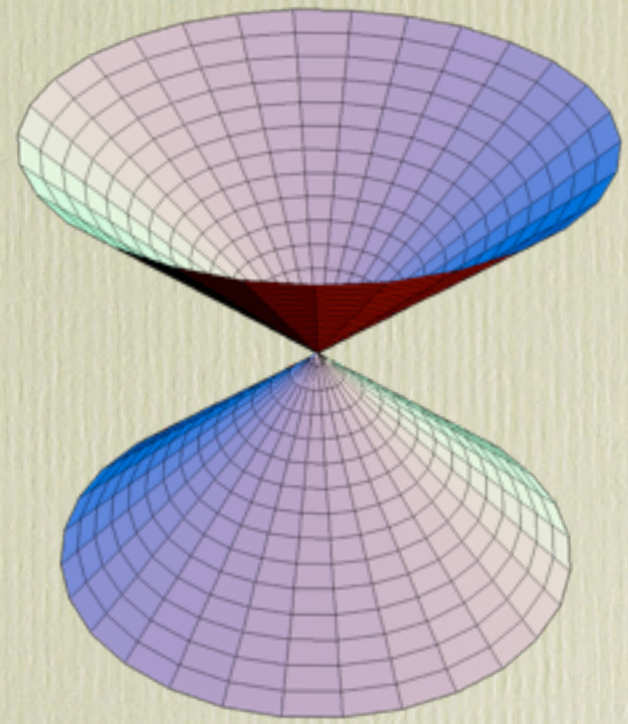
$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

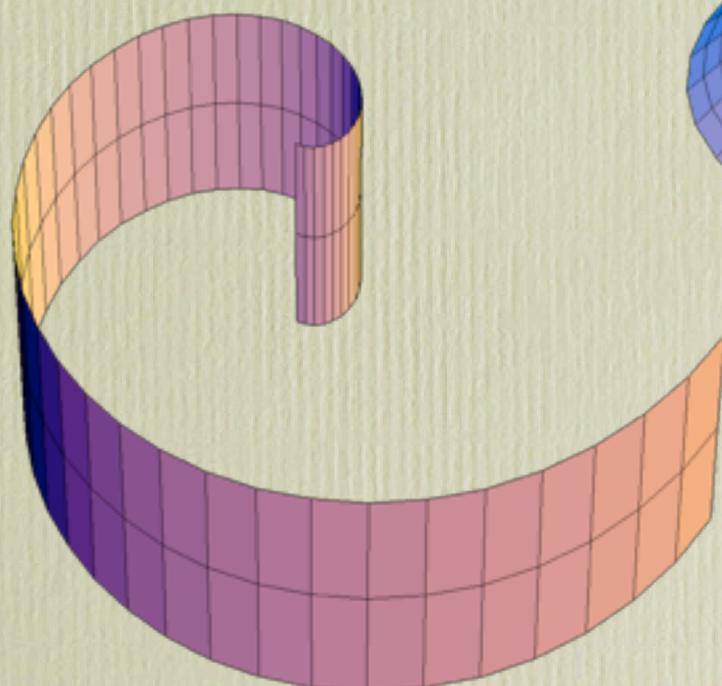
$$z = z$$

Surfaces in Cylindrical coordinates





- $r=1$
- $r=z$
- $r = \theta$



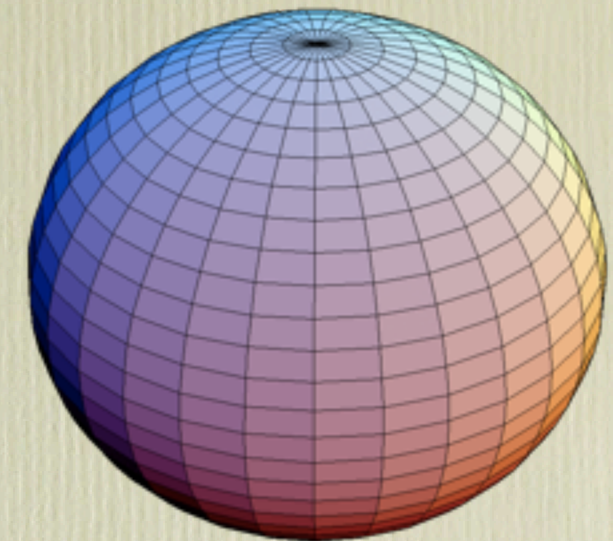
Spherical coordinates

$$x = \rho \sin(\phi) \cos(\theta)$$

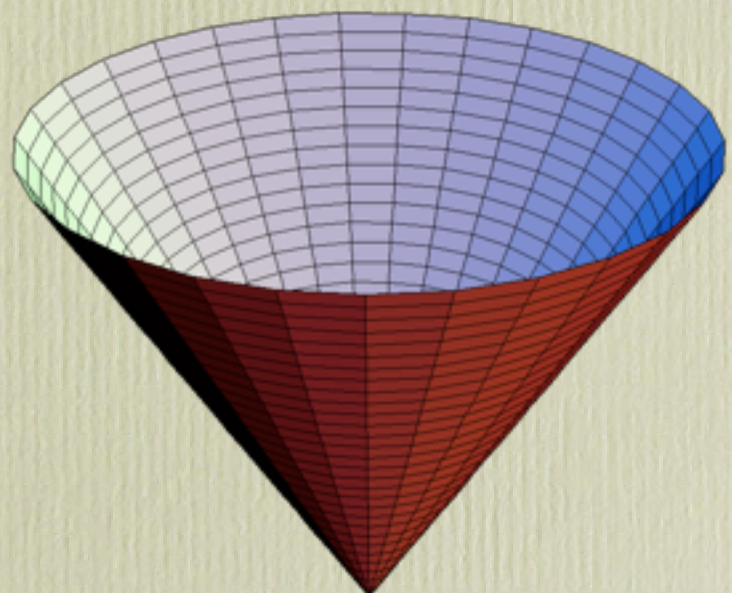
$$y = \rho \sin(\phi) \sin(\theta)$$

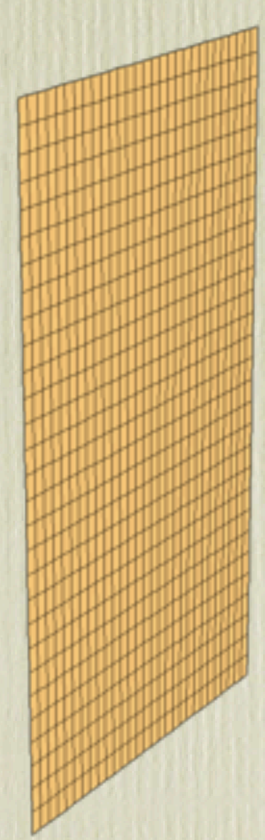
$$z = \rho \cos(\phi)$$

Surfaces in Spherical coordinates



- $\rho = 1$
- $\phi = 1$
- $\theta = 1$



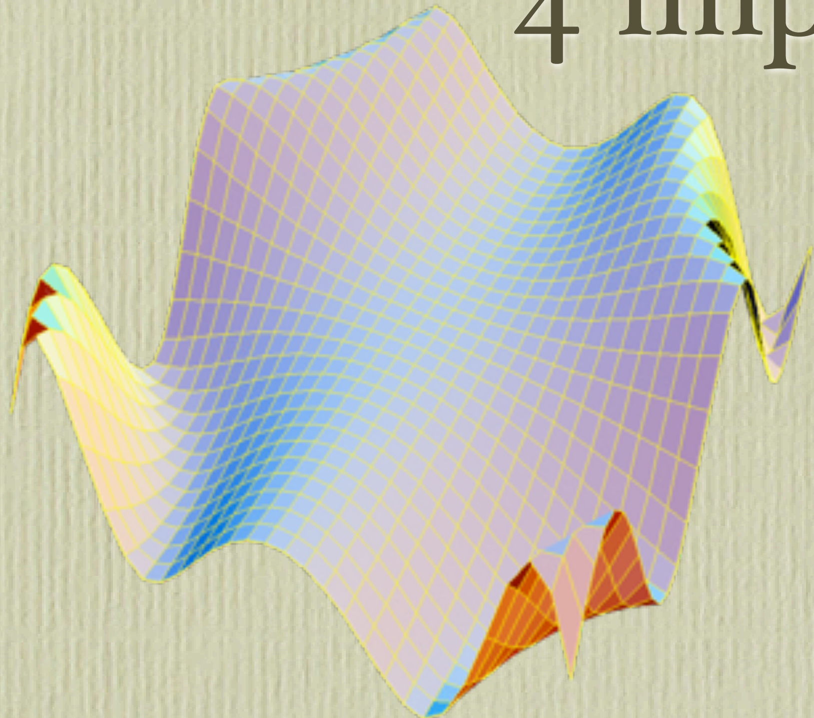


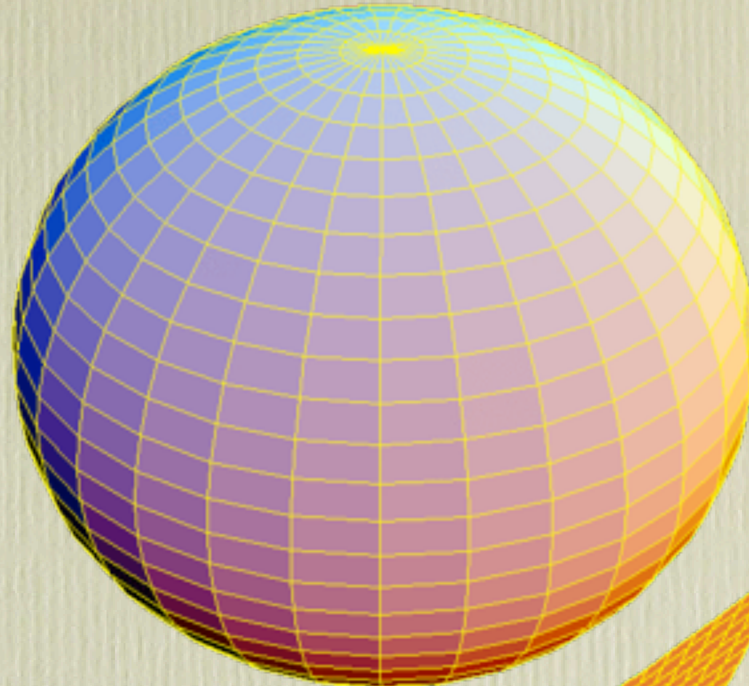
Problem:

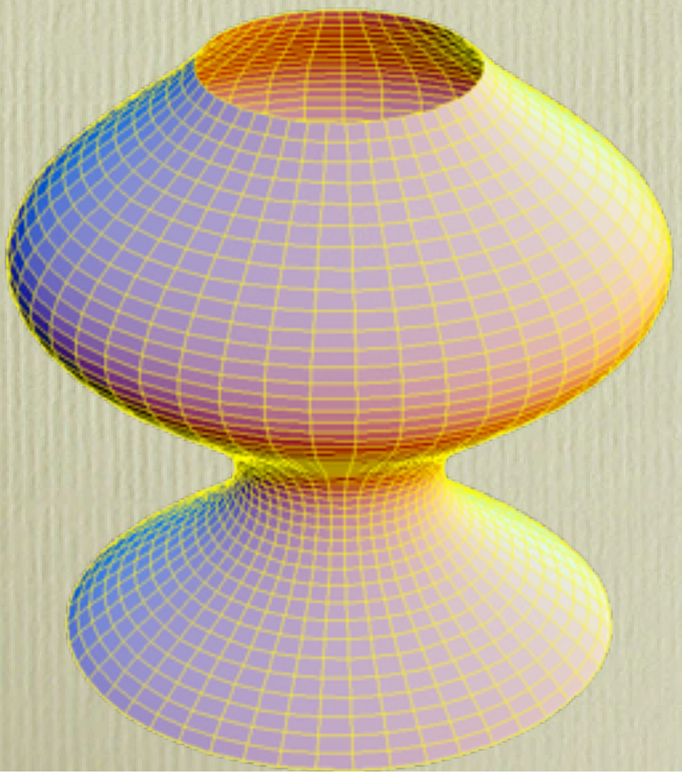
What surface is this:

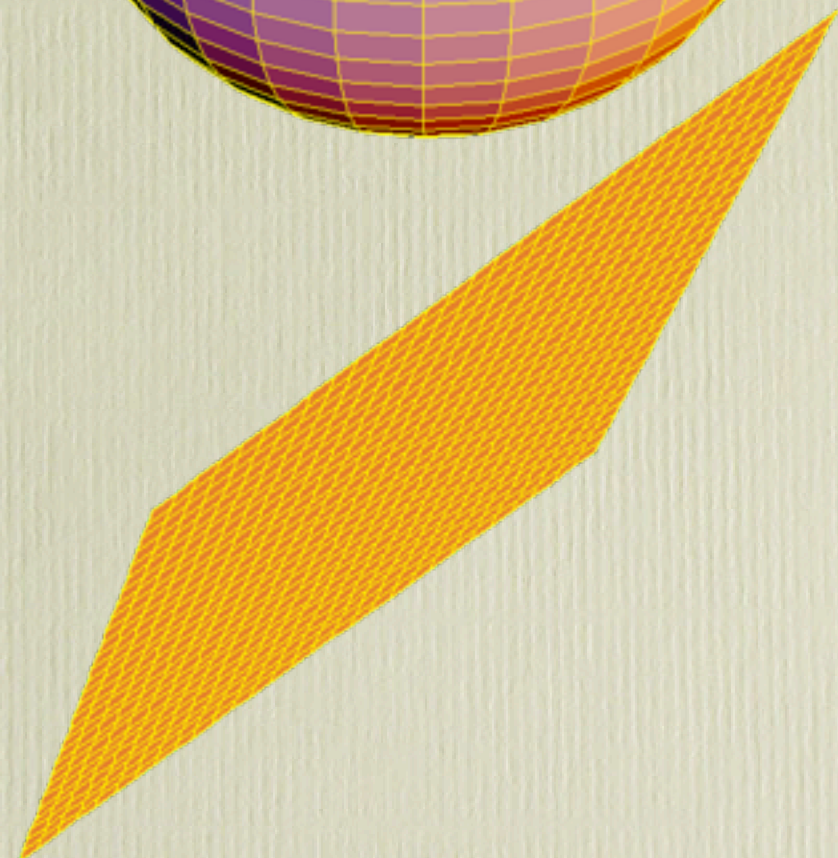
$$r \cos(\theta) = z$$

4 important classes:



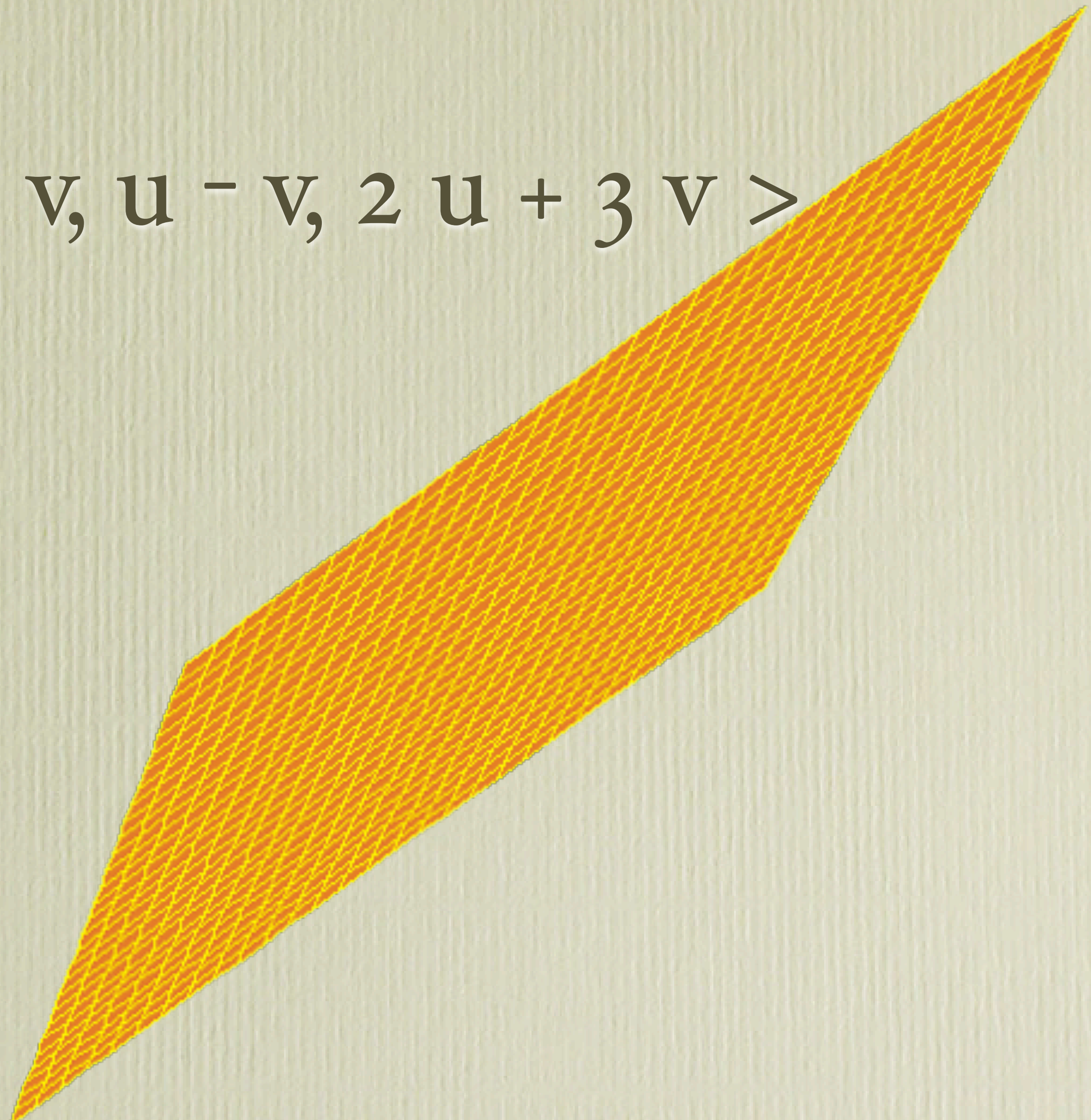




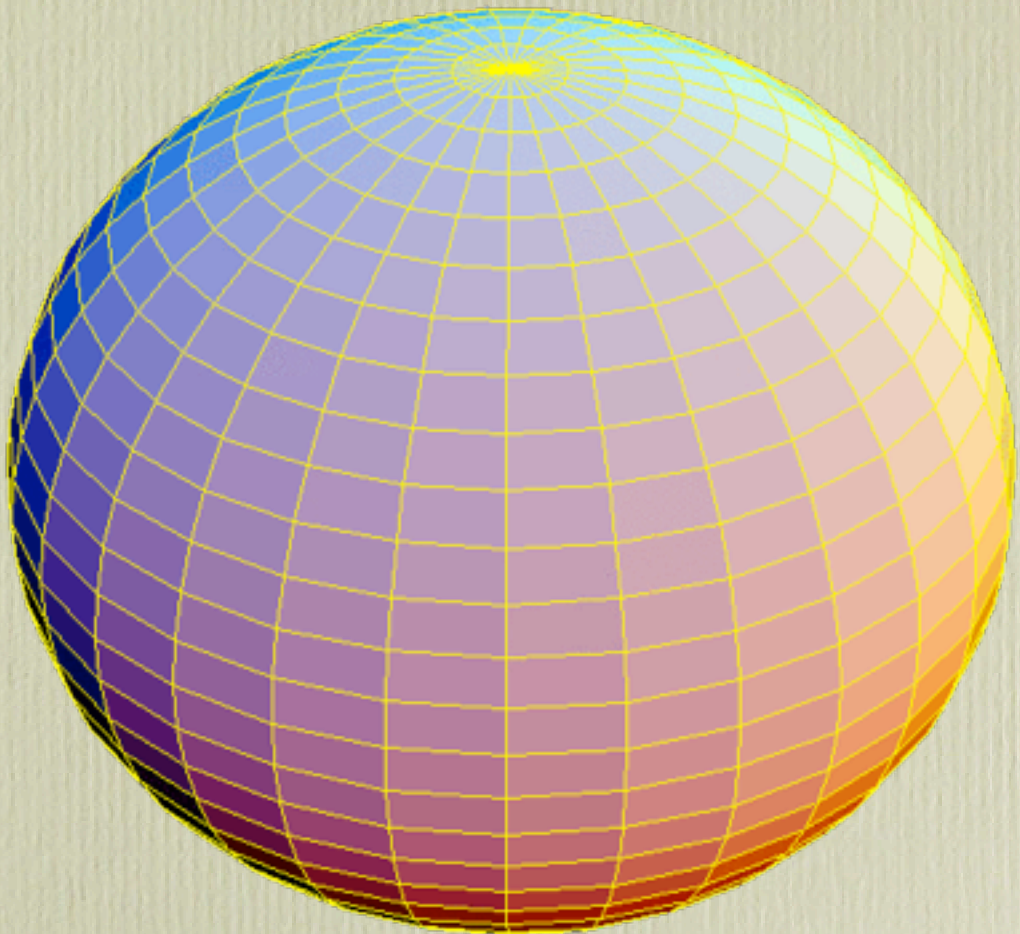


Planes

$$\vec{r}\,(u,v) = \langle 1 + u + v,\ u - v,\ 2u + 3v \rangle$$

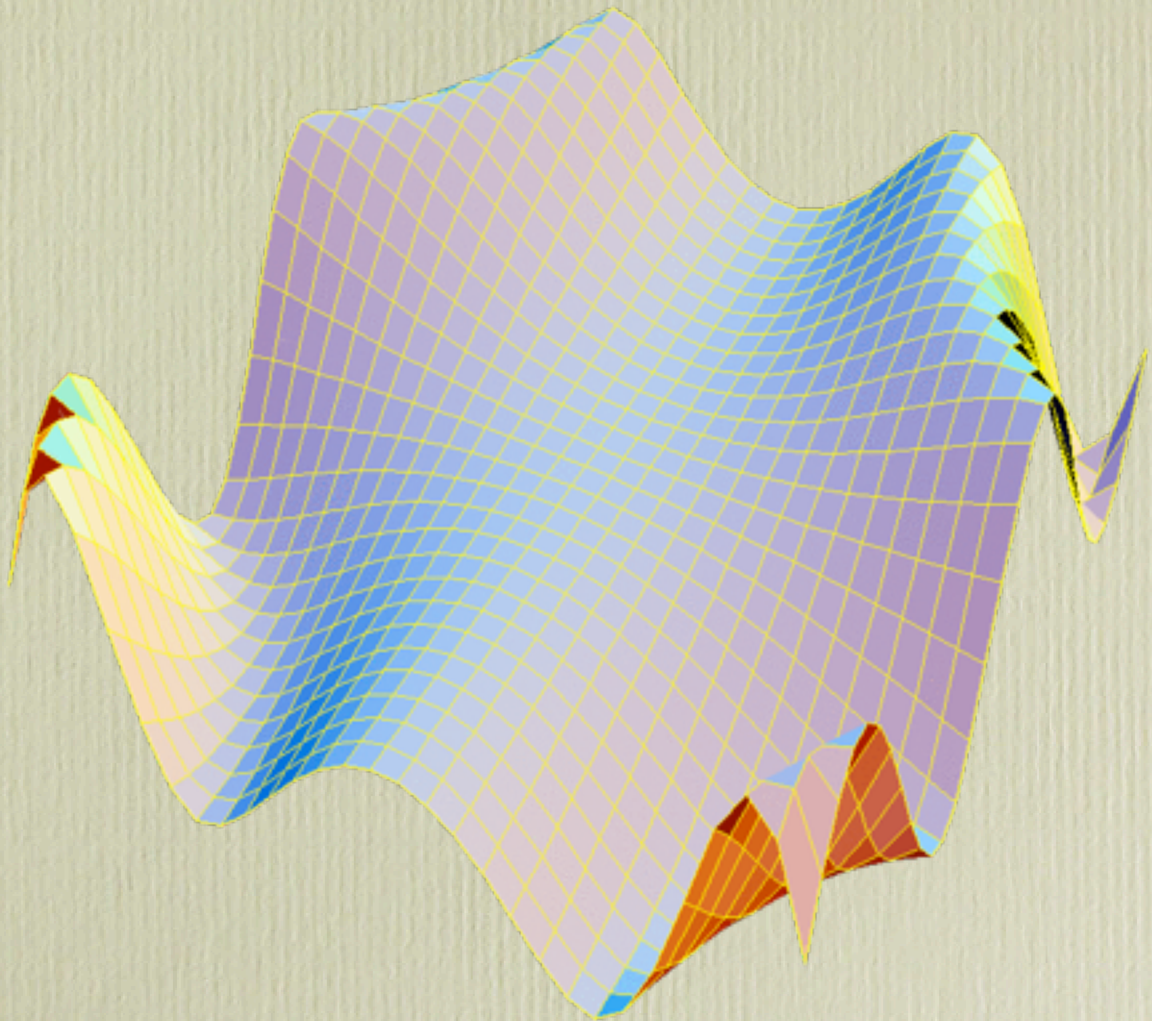


Spheres



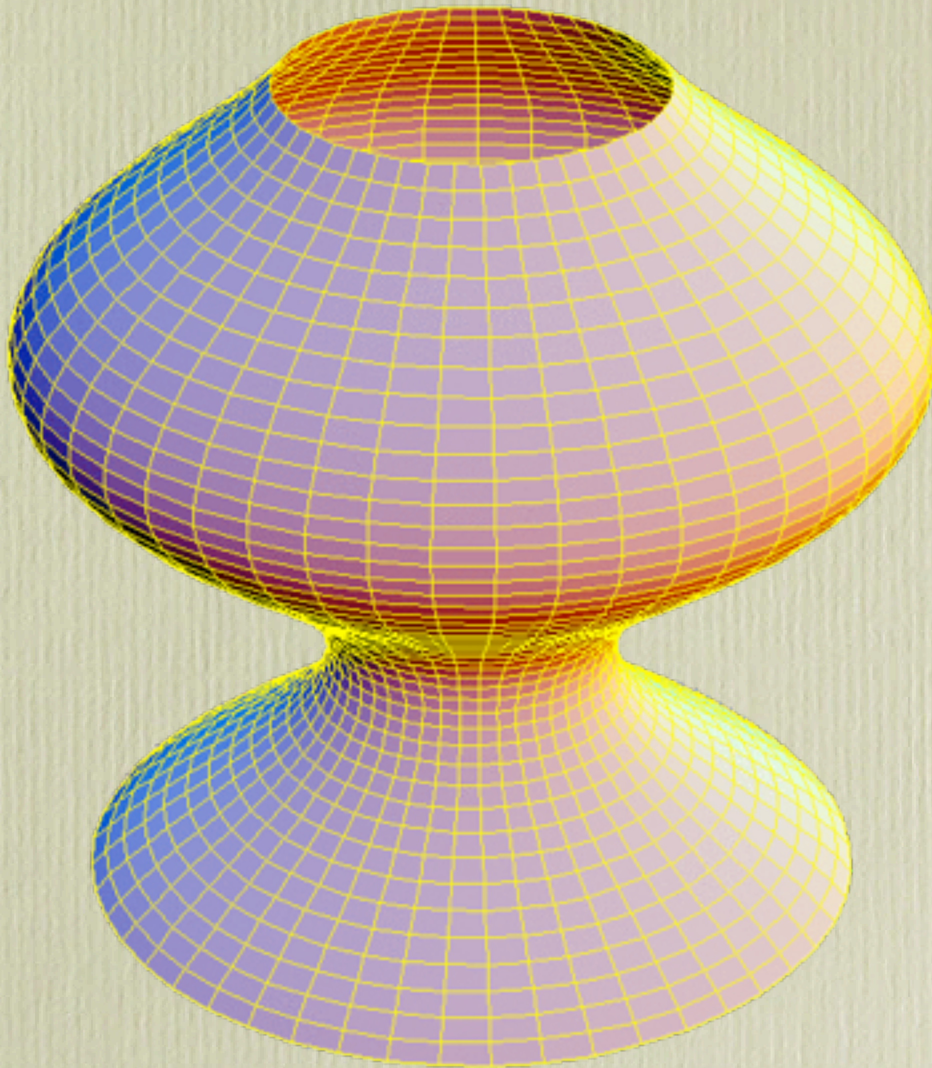
$\vec{r}(u,v) = \langle \cos(u)\sin(v), \sin(u)\sin(v), \cos(v) \rangle$

Graphs

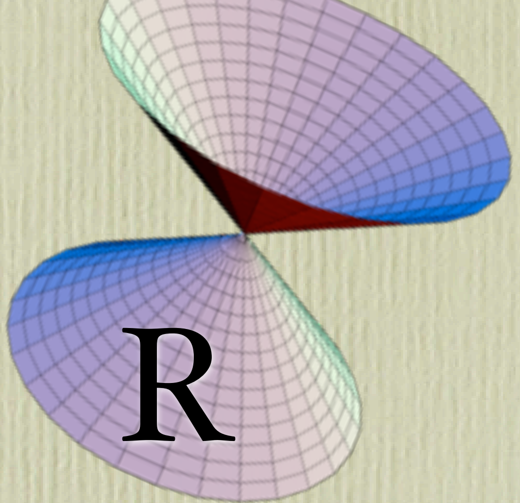


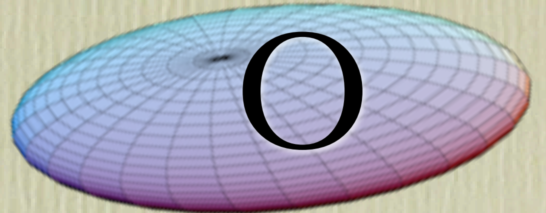
$$\vec{r}\,(u,v) = \langle u, v, f(u,v) \rangle$$

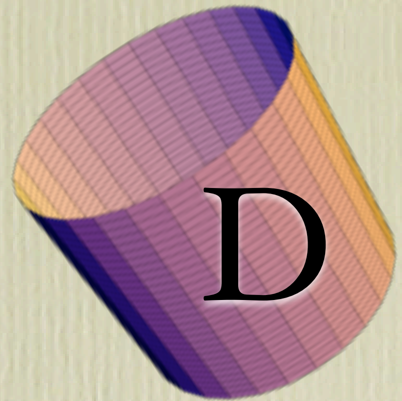
Surfaces of revolution



$$\vec{r}(u,v) = <r(v)\cos(u), r(v)\sin(u), v>$$







$$< \cos(u), \sin(u), v >$$

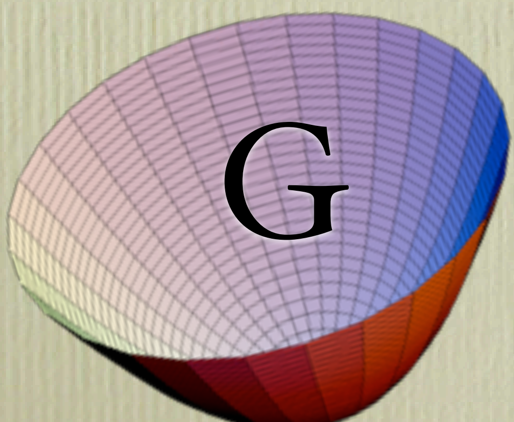
$$< v\cos(u), v\sin(u), v >$$

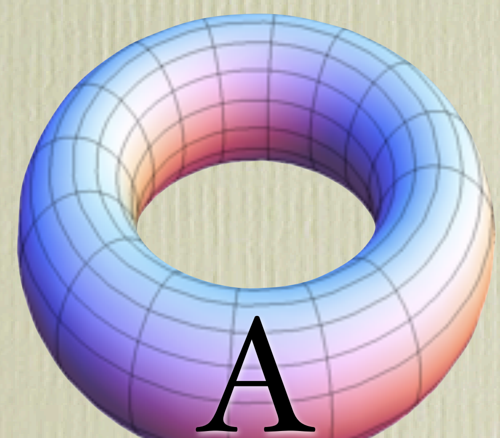
$$< (2+\cos(v))\cos(u), (2+\cos(v))\sin(u), \sin(v) >$$

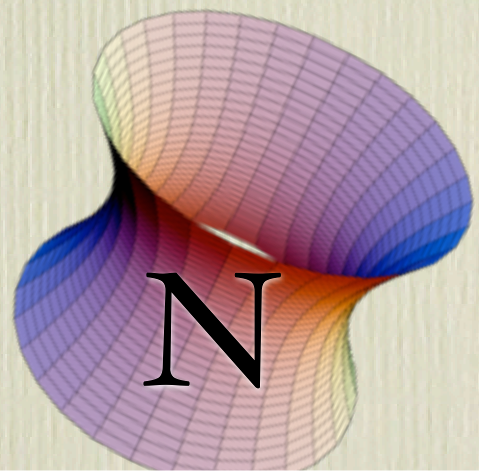
$$< v^{1/2}\cos(u), v^{1/2}\sin(u), v >$$

$$< \cos(u)\sin(v), \sin(u)\sin(v), \cos(v)/3 >$$

$$< v\sin(u), v\sin(u), (-1+v^{2})^{1/2} >$$

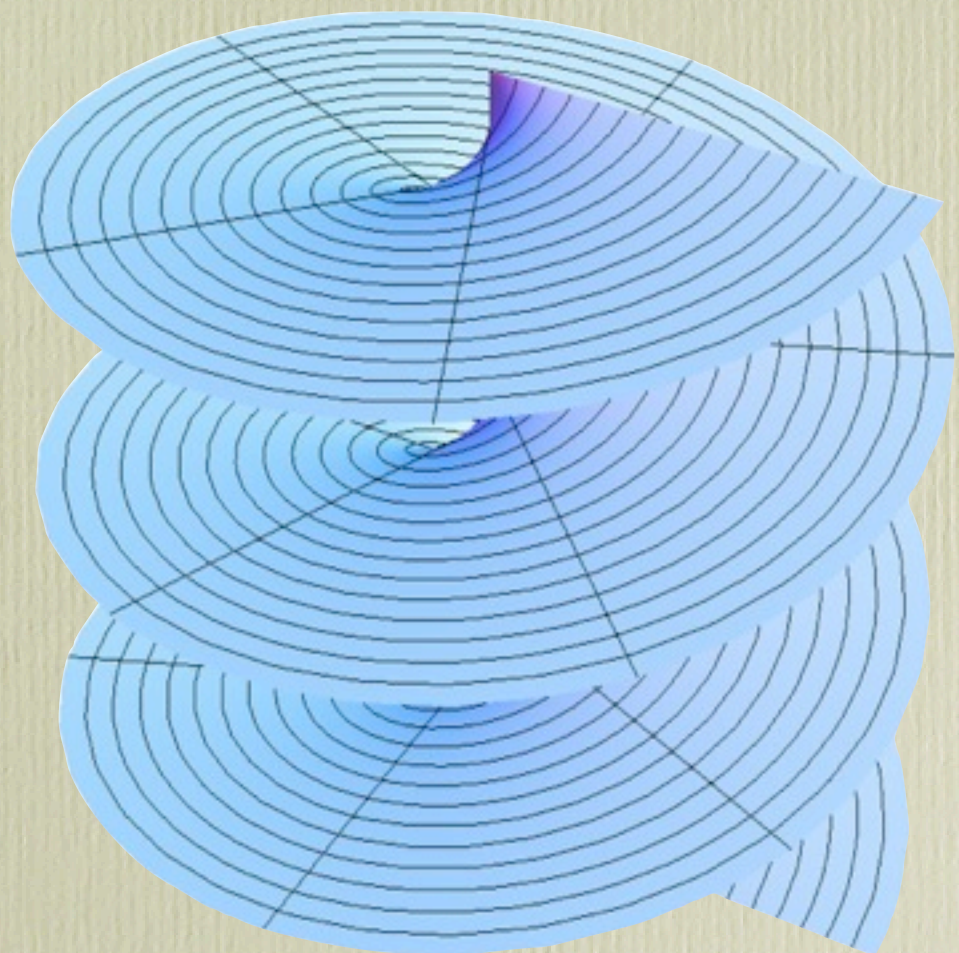




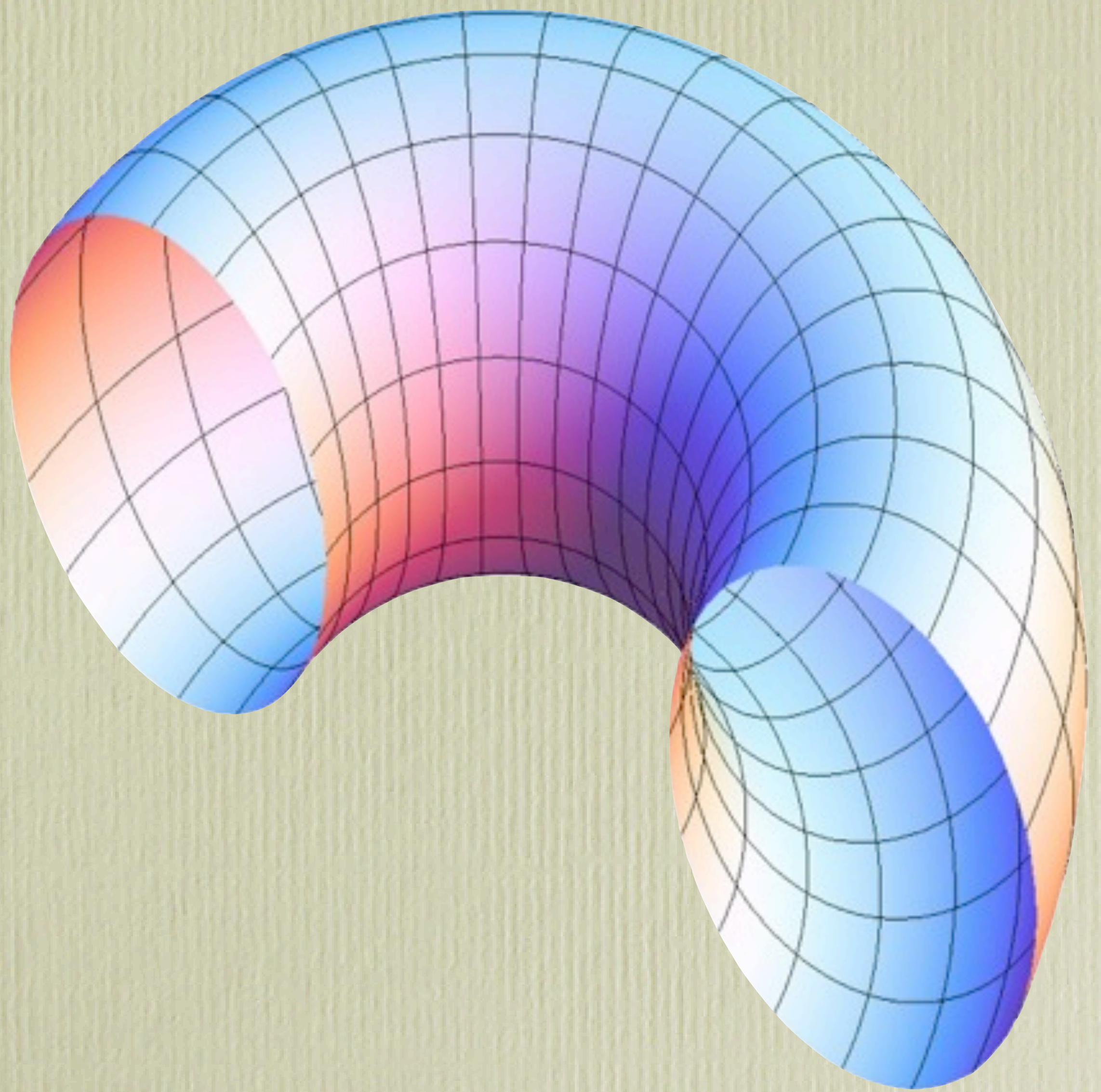


Problem: What is this?

$\vec{r}(u,v) = \langle v\cos(u), v\sin(u), u \rangle$

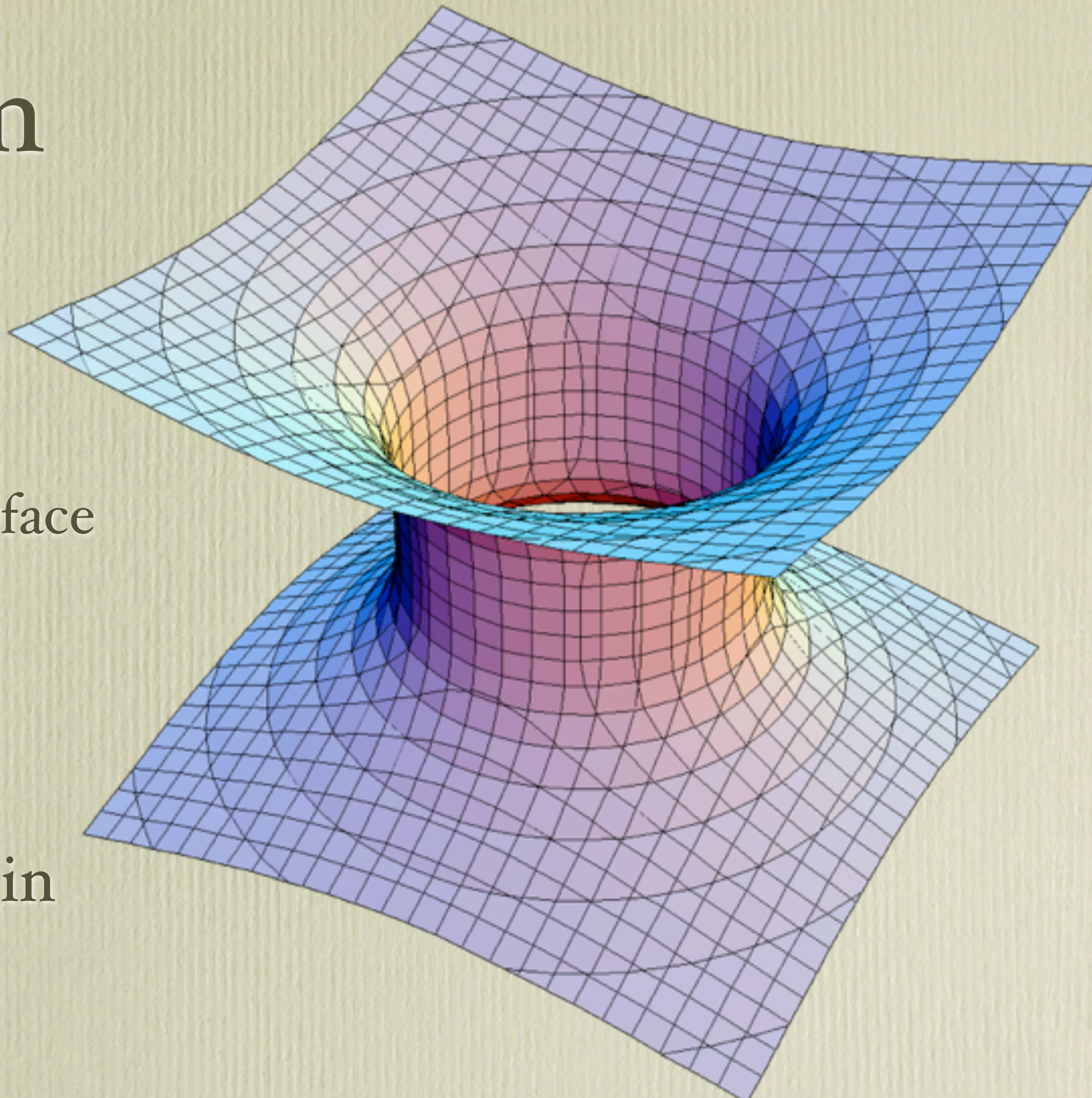


Example:



$\vec{r}\,(u,v) = <(2+\cos(v))\cos(u), (2+\cos(v))\sin(u), \sin(v)>$

Problem

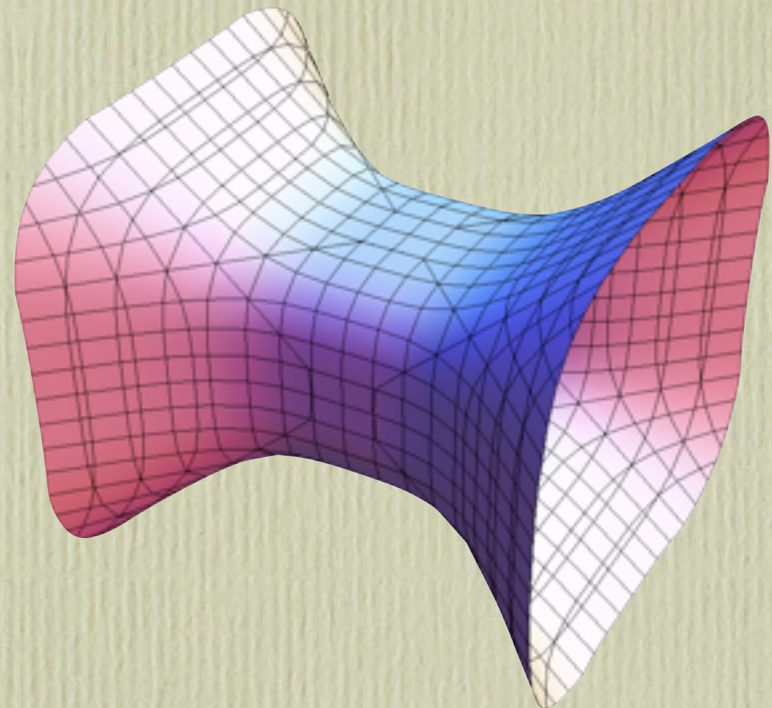


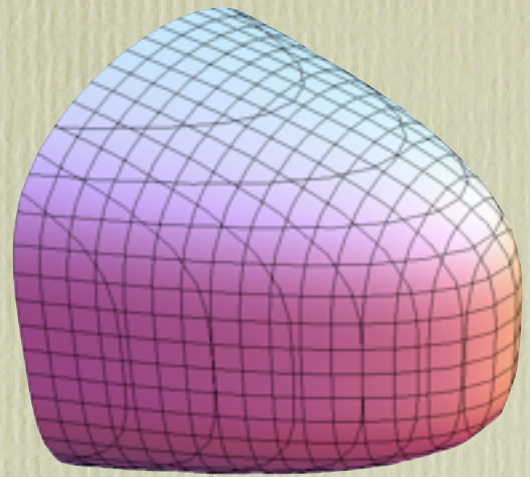
Parametrize the surface

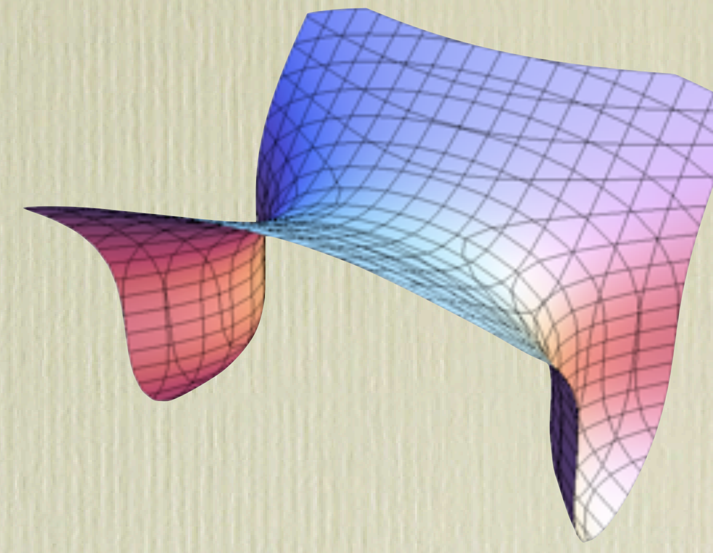
$$x^2 + y^2 - z^4 = 1$$

and describe it in cylindrical coordinates.

Matching Surfaces





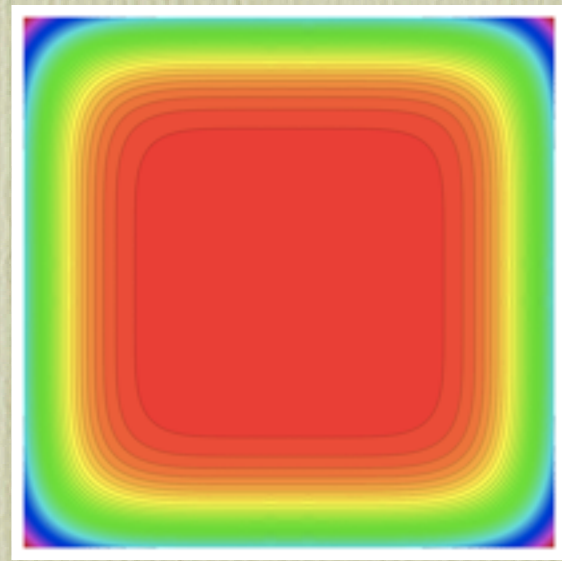


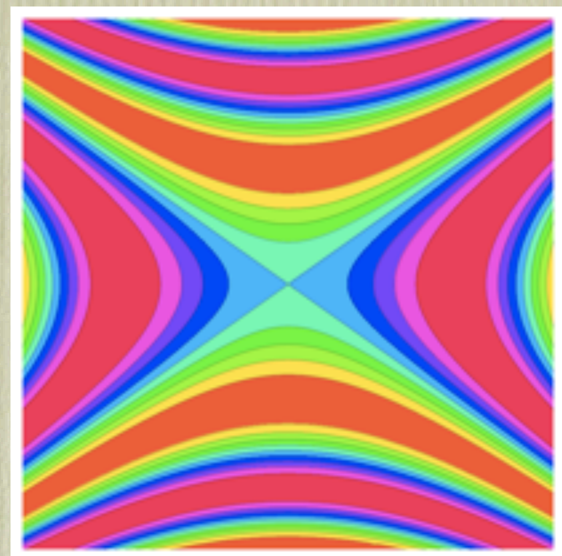
$$x^2 - y^2 + z^3 = 1$$

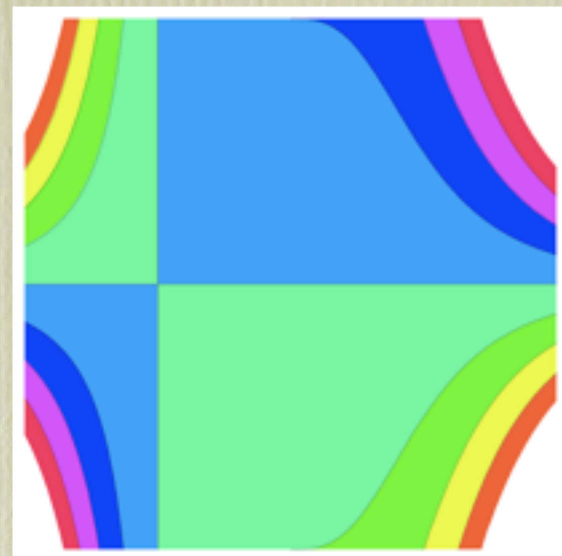
$$x + y^2 + z^4 = 1$$

$$-x^4 + y^4 + z^4 = 1$$

Functions







$$f(x,y) = \sin(x^2 - y^2)$$

$$f(x,y) = (y + y^3)x$$

$$f(x,y) = x^6 + y^6$$

Problem

a) Is the function

$$f(x,y)=x/(x+y)$$

continuous at the origin?

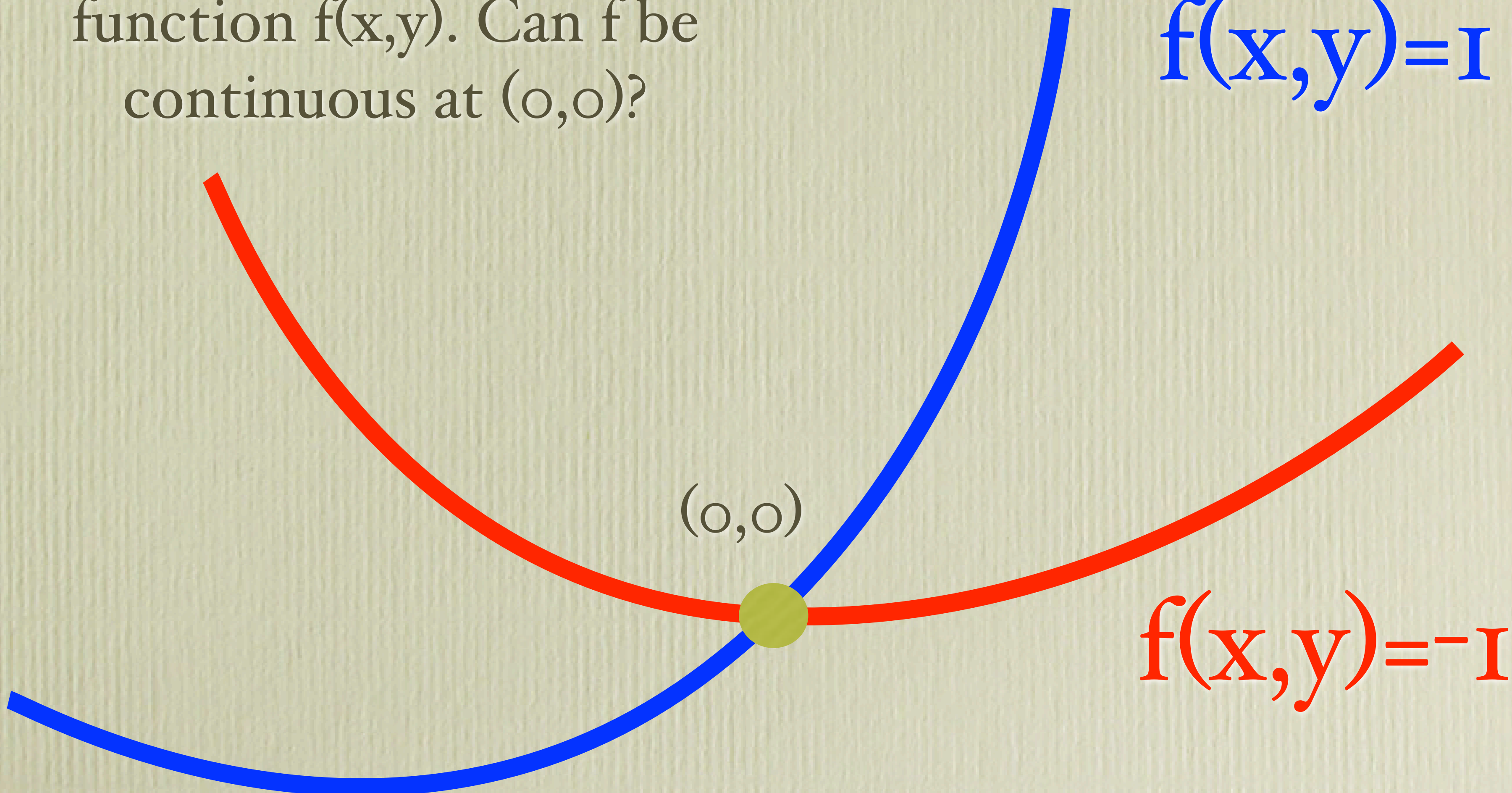
b) Is the function

$$f(x,y)=x^3/(x^2+y^2)$$

continuous at the origin?

Problem

You see two level curves of a function f(x,y). Can f be continuous at (0,0)?



Partial derivatives

$$f(x,y) = \sin(x\,y + x^2)$$

Find $f_x(x,y)$