MATH 21A SECOND MIDTERM REVIEW

Math 21a, Fall 2011

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TOPICS

Partial differential equations

Linearization and estimation

Tangent lines and tangent spaces

Chain rule and implicit differentiation

Directional derivative

Extrema with or without constraints

Double Integration

Polar integration

Surface area

PDE'S

An equation for an unknown function which involves derivatives with respect to at least two variables

$$u_t = u_x \quad \text{Transport}$$

$$u_t = u_{xx} \quad \text{Heat}$$

$$u_{tt} = u_{xx} \quad \text{Wave}$$

$$u_t + u\, u_x = u_{xx} \quad \text{Burgers}$$

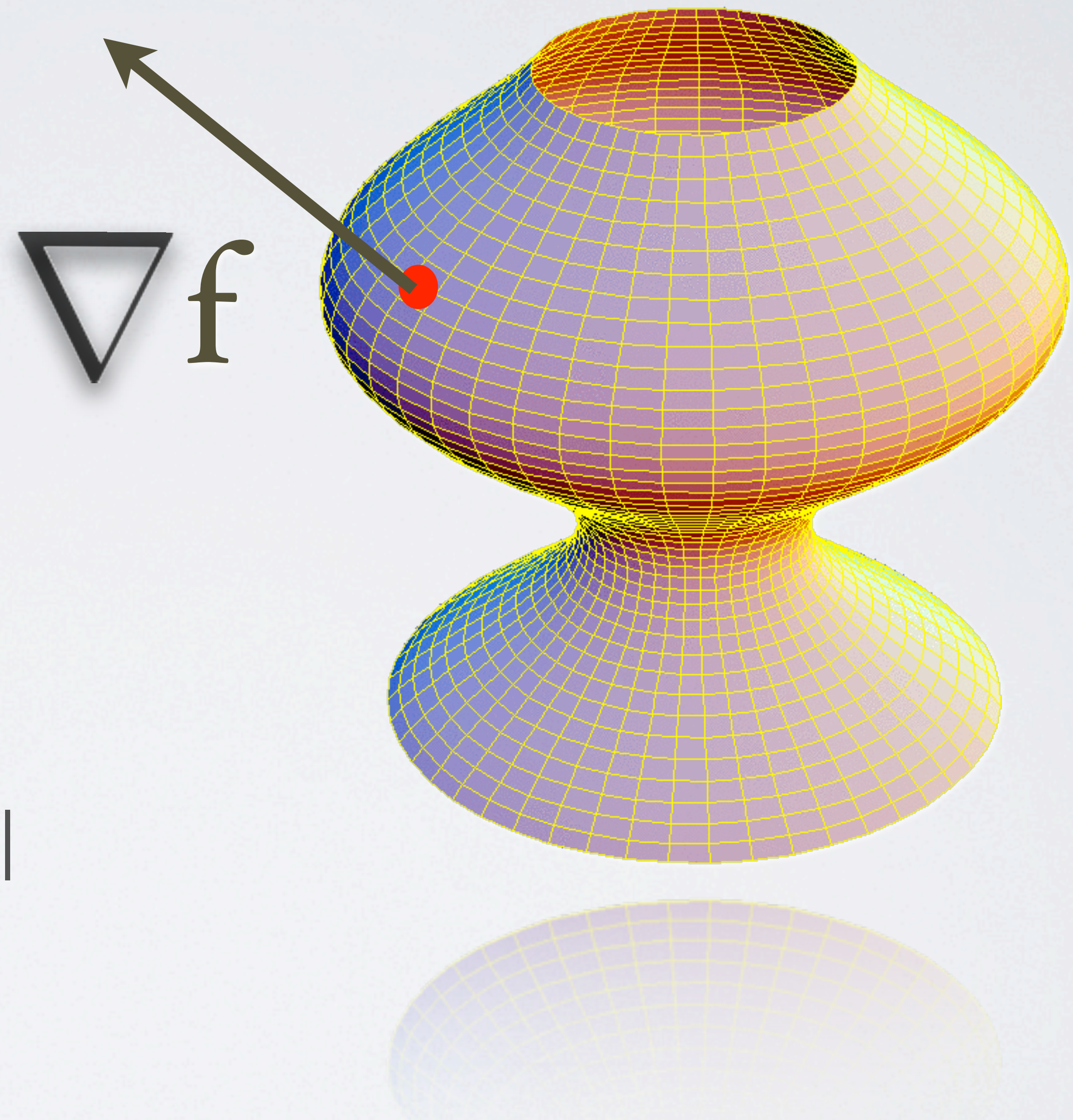
$$u_{xy} = u_{yx} \quad \text{Clairot}$$

$$u_{xx} + u_{yy} = 0$$

Laplace

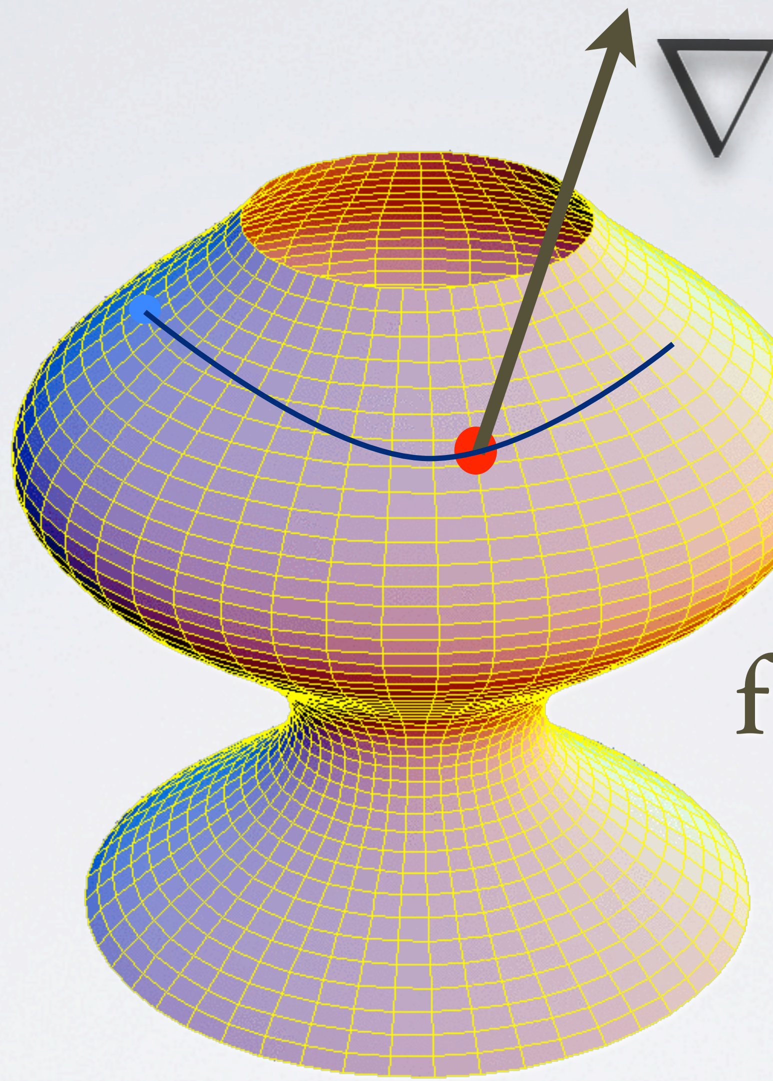
Important Fact

$f(x,y,z) = c$



The gradient is perpendicular to level sets

Proof

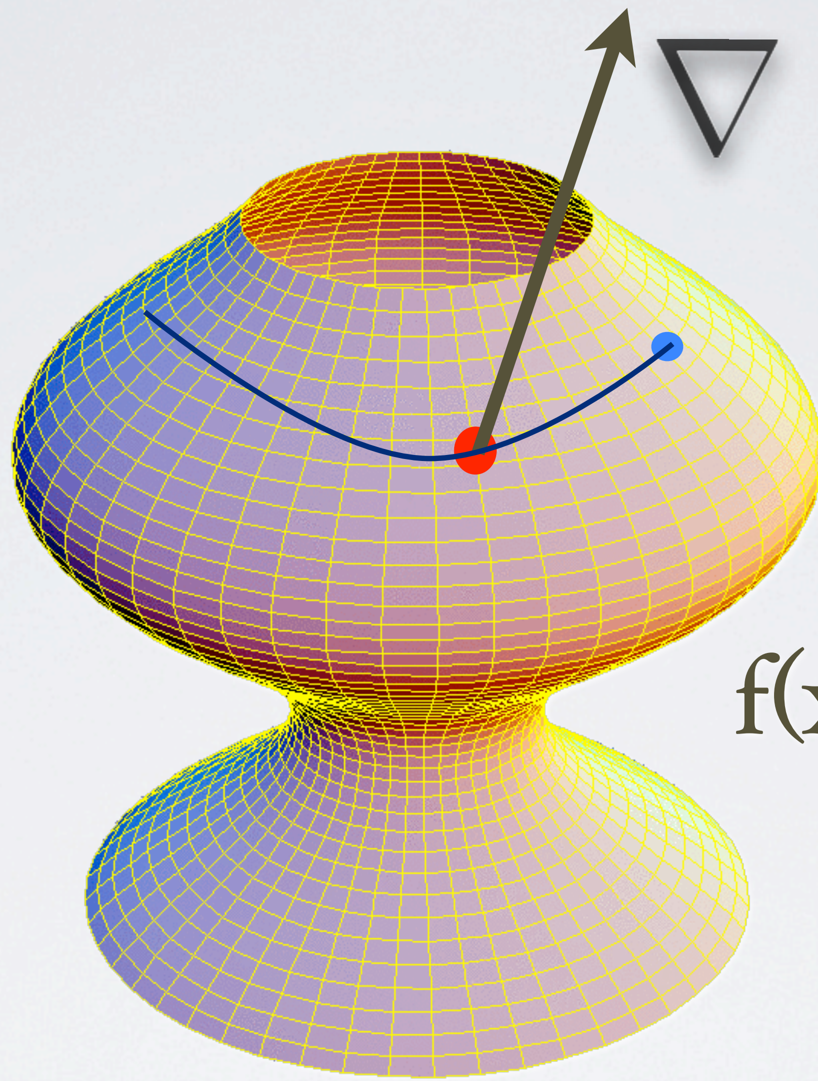


$f(x,y,z)=c$

$f(r(t))=\text{const}$

$$0 = \frac{d}{dt} f(r(t)) = \nabla f(r(t)) \,.\, r'(t)$$

Proof



$f(r(t))=\text{const}$

$$0 = \frac{d}{dt} f(r(t)) = \nabla f(r(t)) \cdot r'(t)$$

LINEARIZATION

$$L(x) = f(a) + f'(a)\,(x-a)$$

$$L(x,y) = f(a,b) + f_x(a,b)\,(x-a) + f_y(a,b)\,(y-b)$$

$$L(x,y,z) = f(a,b,c) + f_x(a,b,c)\,(x-a) + f_y(a,b,c)\,(y-b) + f_z(a,b,c)\,(z-c)$$

$$L(x) = f(a) + \quad f(a)\cdot(x-a)$$

LINEARIZATION

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$$L(x,y) = f(a,b) + f_x(a,b)\,(x-a) + f_y(a,b)\,(y-b)$$

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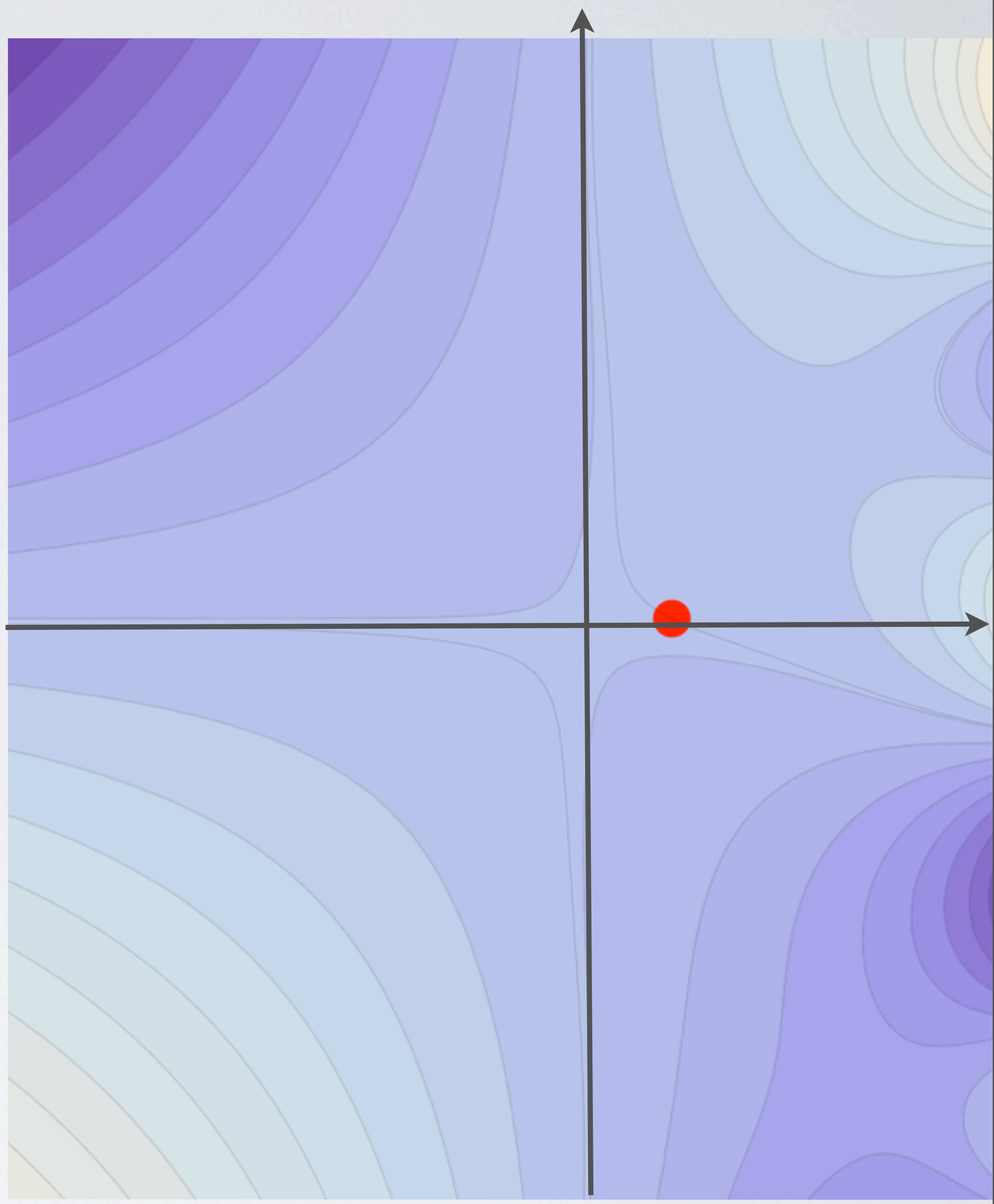
$$L(x) = f(a) + \nabla f(a)\cdot(x-a)$$

PROBLEM

Find the tangent plane to

$$2xy + e^{x-1}\cos(y) = 1$$

at (1,0)

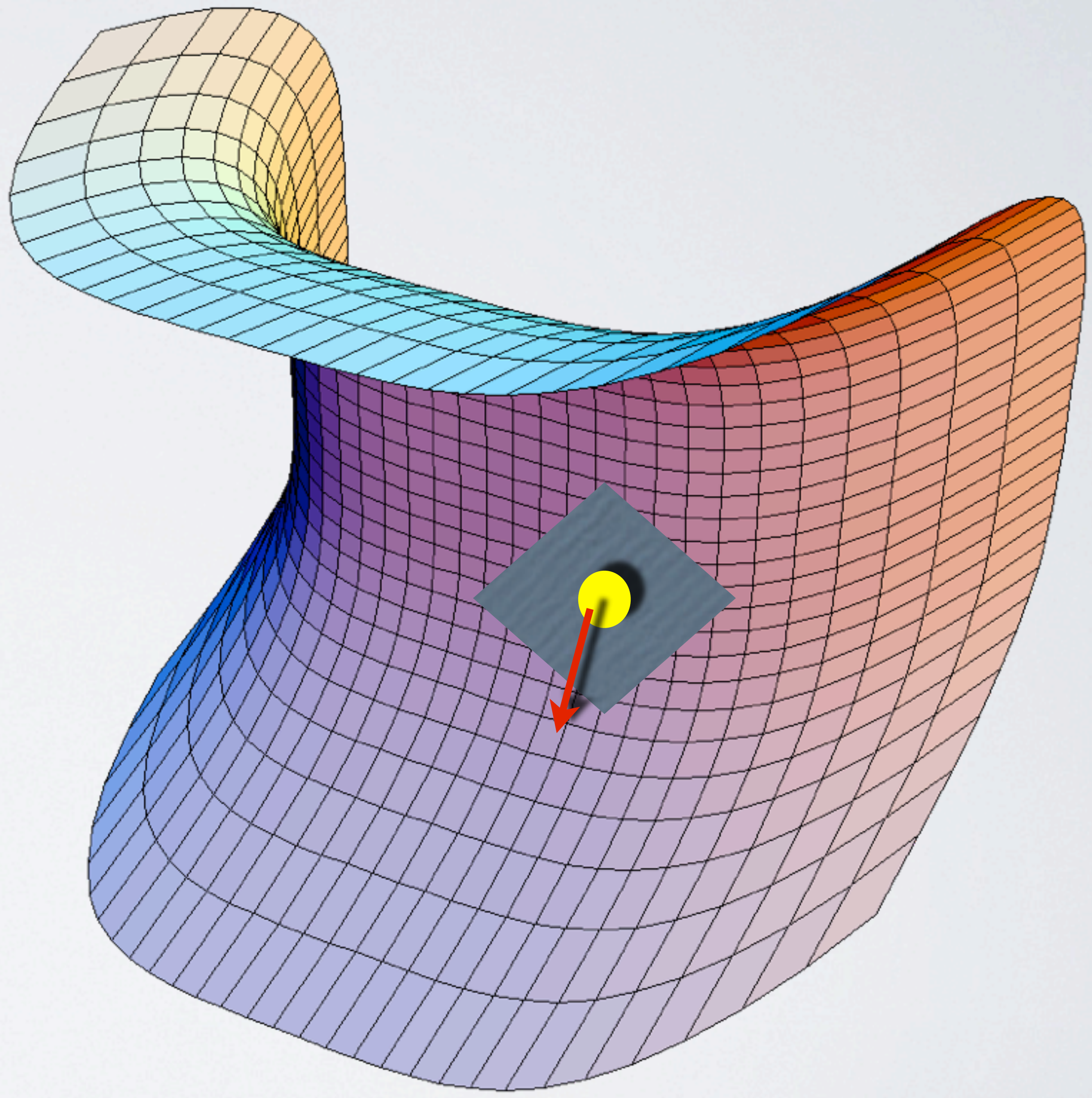


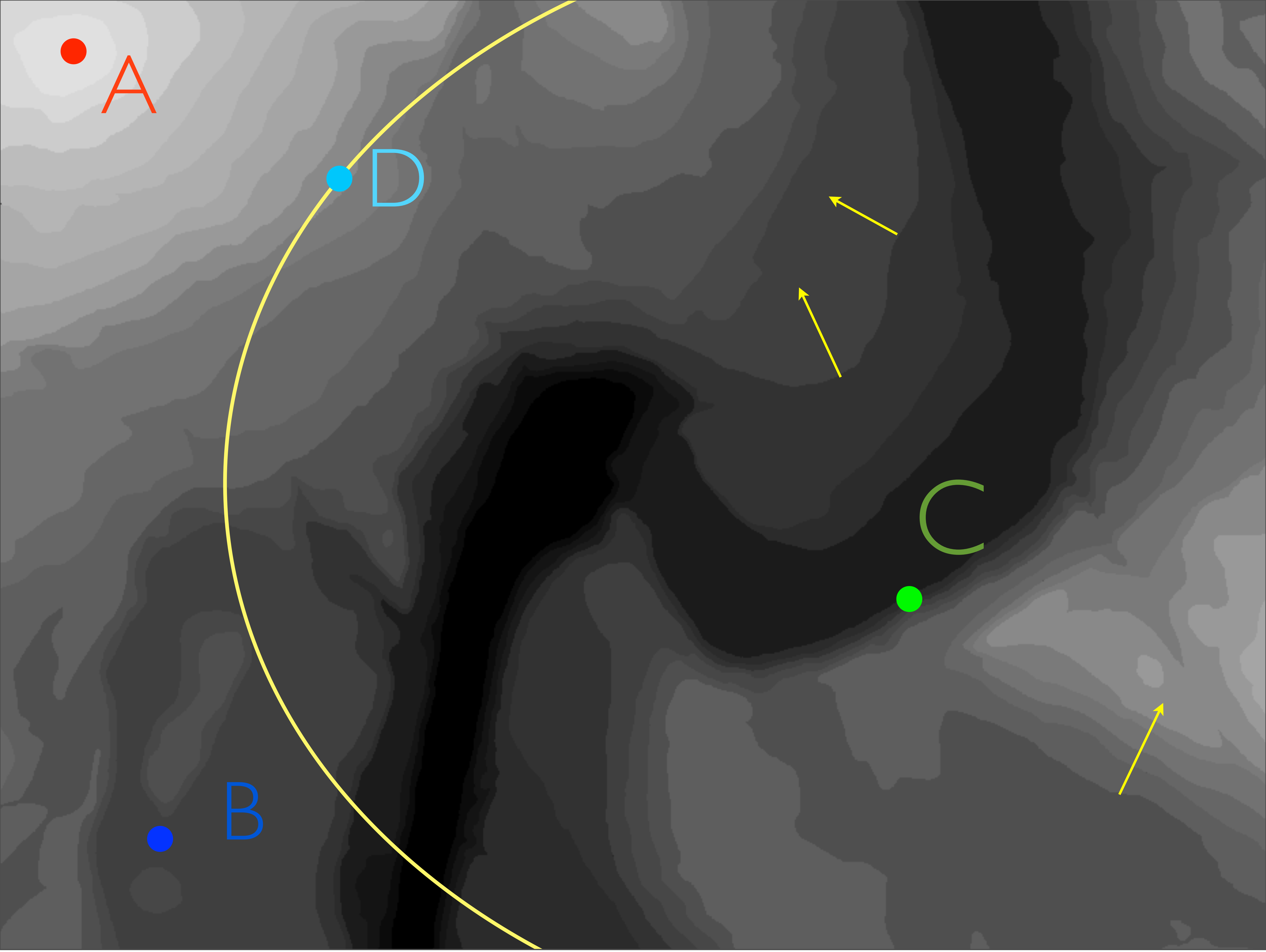
Problem

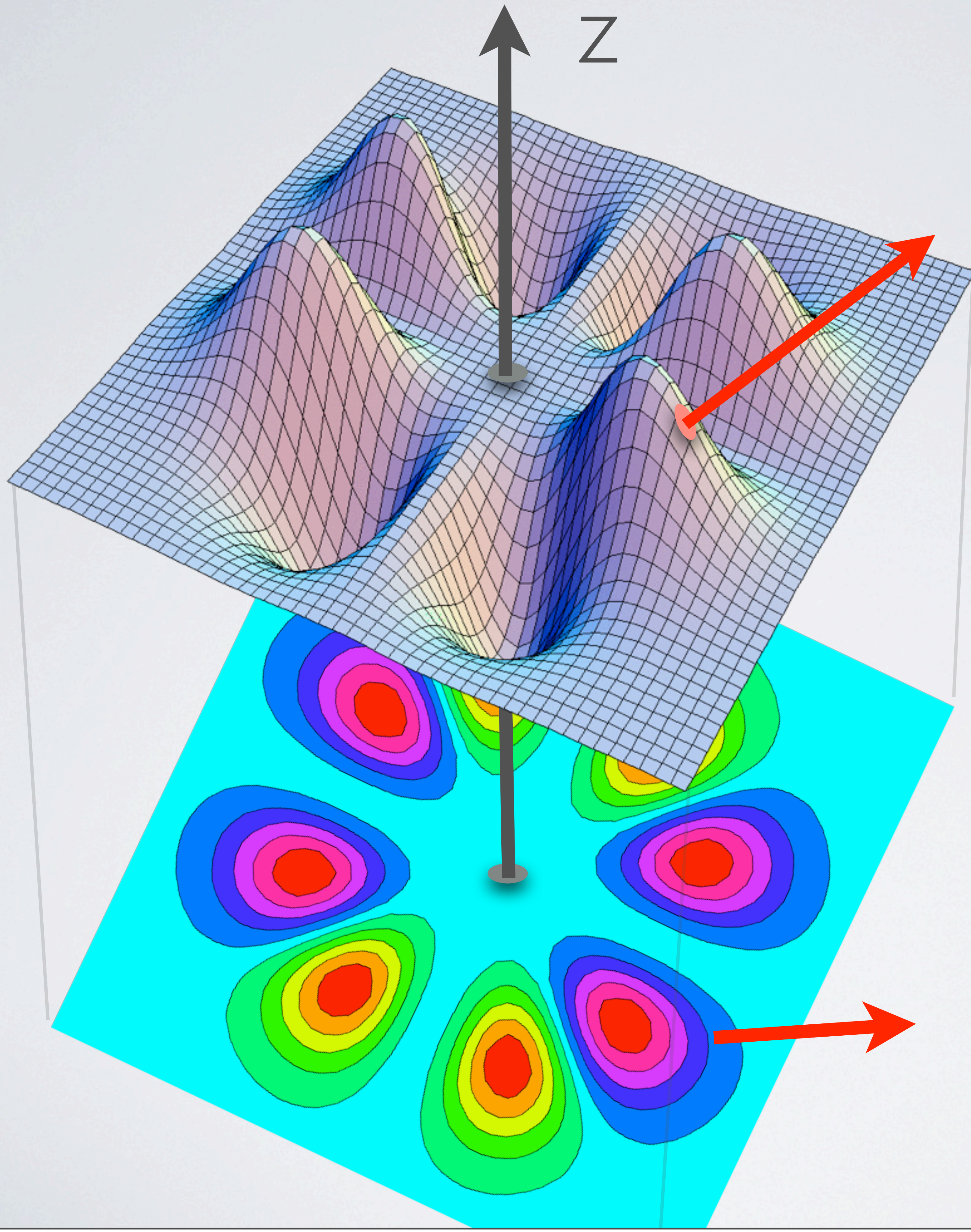
Find the **tangent plane** to the surface

$f(x,y,z) = x^4 - 2z^4 - y = 0$

at the point (1,0,1)




A
D
C
B



$\nabla g(x,y,z)$

$g(x,y,z)=z-f(x,y)$

$\nabla f(x,y)$

THE CHAIN RULE

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$$

PROBLEM

A helicopter flies along a curve r(t) = < t, sin(t), 4t+1 > .

The pressure at position (x,y,z) is f(x,y,z) =sin(x+y z). Find the rate of change of f(r(t)) at time t=0.

PROBLEM

The mountain has the height

$f(x,y)$

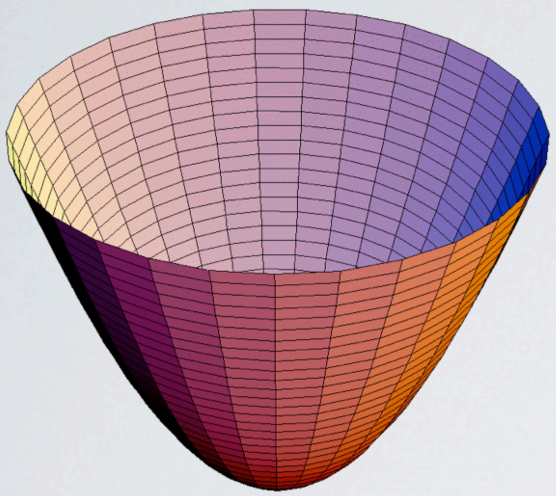
Jean-Yves drives along two paths

$\vec{r}(t) = \langle 1-4t, -3\sin(t)\rangle/5.$

$\vec{r}(t) = \langle 1+t, \cos(t)-1\rangle.$

and measures slope 2 and 3. Find the gradient of f!

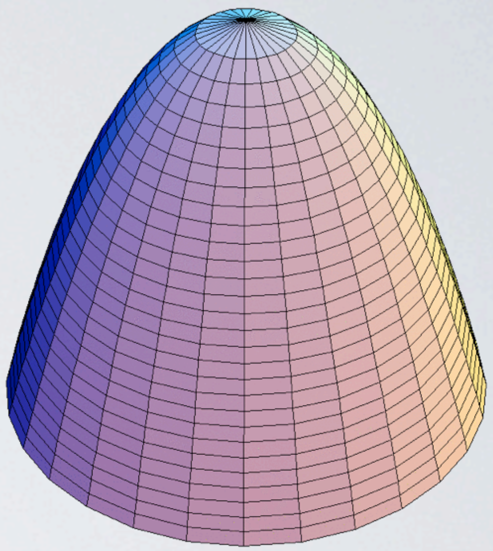
SECOND DERIVATIVE TEST



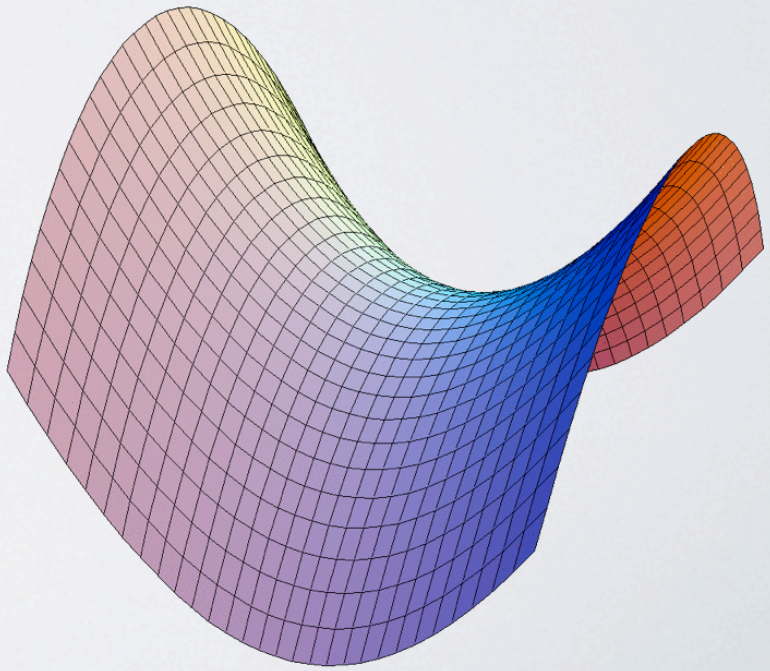
$D > 0$, $f_{xx} > 0$ Min

$D > 0$, $f_{xx} < 0$ Max

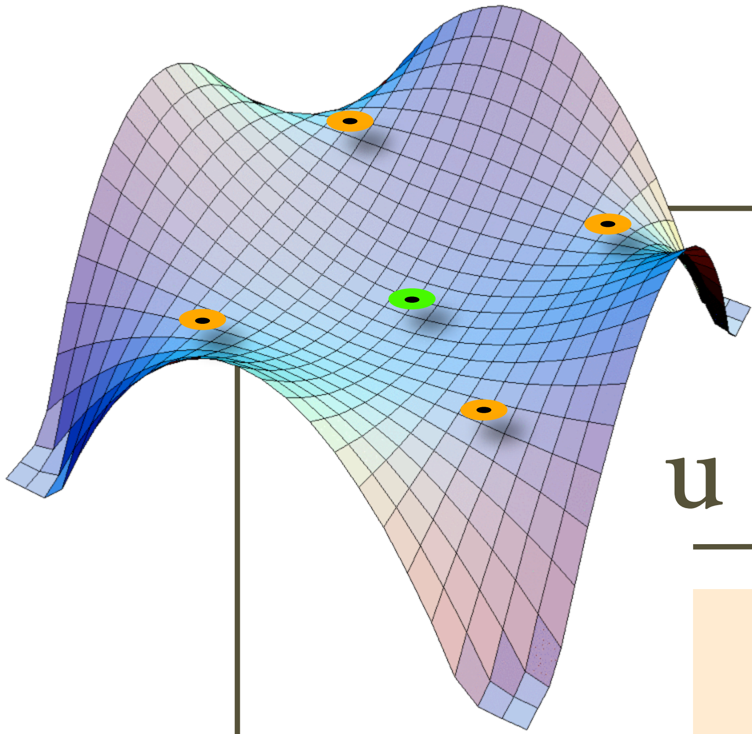
$D < 0$, Saddle



$$D = f_{xx} \cdot f_{yy} - f_{xy}^{2}$$



D Discriminant at the critical point (x,y)



u	v	D	f_{xx}	Type	f
-1	-1	-16	0	saddle	1
-1	1	-16	0	saddle	1
0	0	4	2	min	0
1	-1	-16	0	saddle	1
1	1	-16	0	saddle	1

PROBLEM

Classify the extrema of

$$f(x,y) = x^2 + y^2 - x^2 y^2$$

y= gory

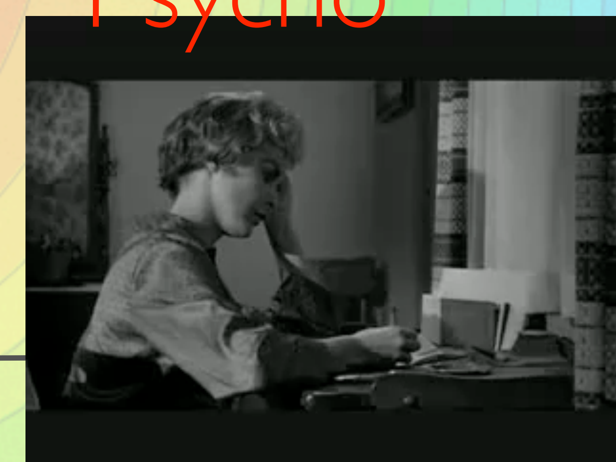
x= scary

Devils Pray

Sigaw

Poltergeist

Psycho



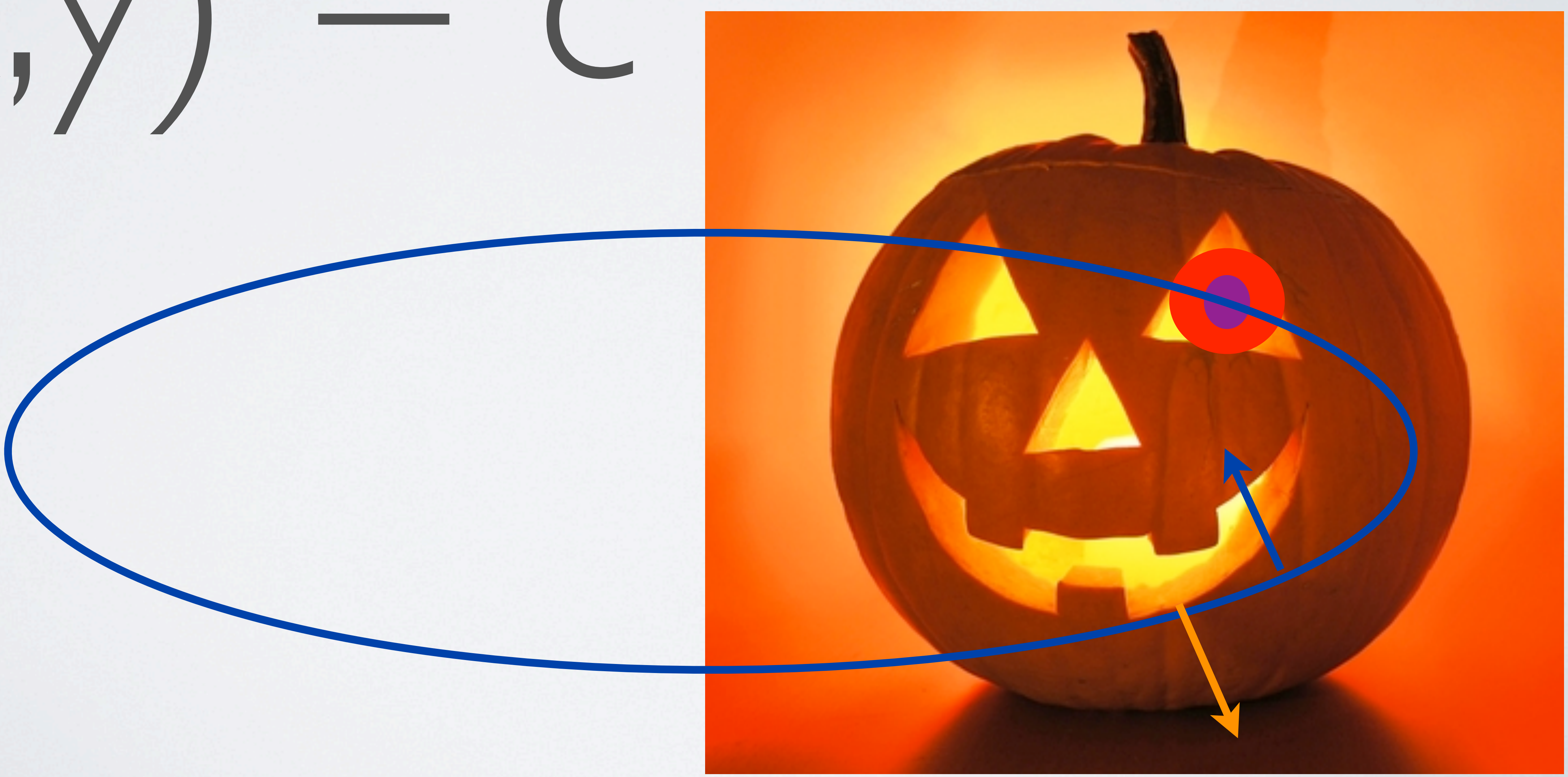
Shining

Drakula

LAGRANGE EQUATIONS

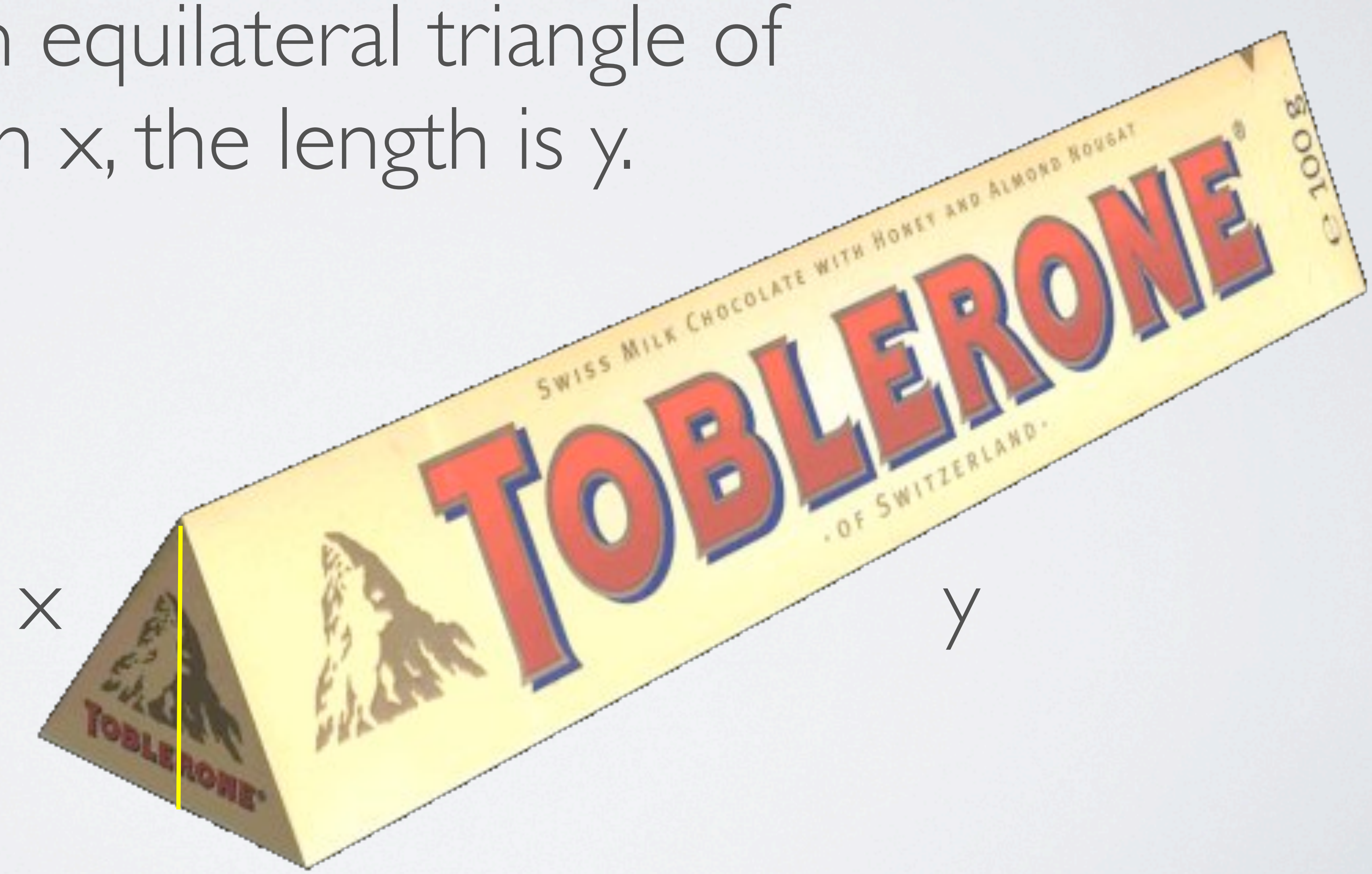
$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

$$g(x,y) = c$$



TOBLERONE PROBLEM

Find the toblerone chocolate shape which has maximal volume if the surface area is constant 1.
The base is an equilateral triangle of side length x, the length is y.



MOZART KUGELN PROBLEM

Maximize Taste value — Cost

$$f(x,y,z)=3x^3+y^3+z^3; \quad g(x,y,z)=x+3y+27z=253$$

MOZART KUGELN PROBLEM

Maximize Taste value

Cost

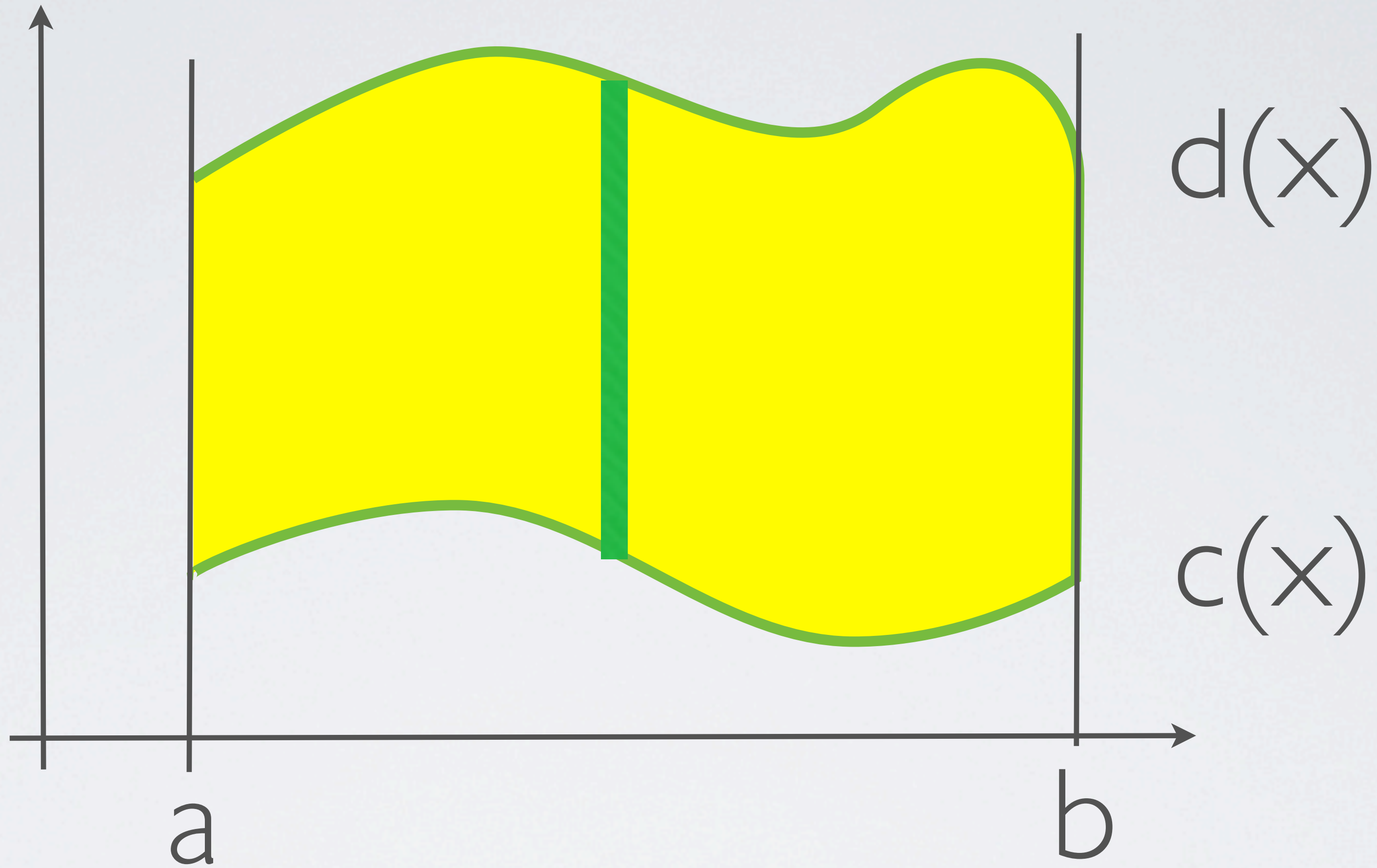
$$f(x,y,z)=3x^3+y^3+z^3; \quad g(x,y,z)=x+3\,y+27\,z=253$$

FUBINIS  THEOREM

$$\int_a^b \int_c^d f(x,y)\ dy\ dx$$

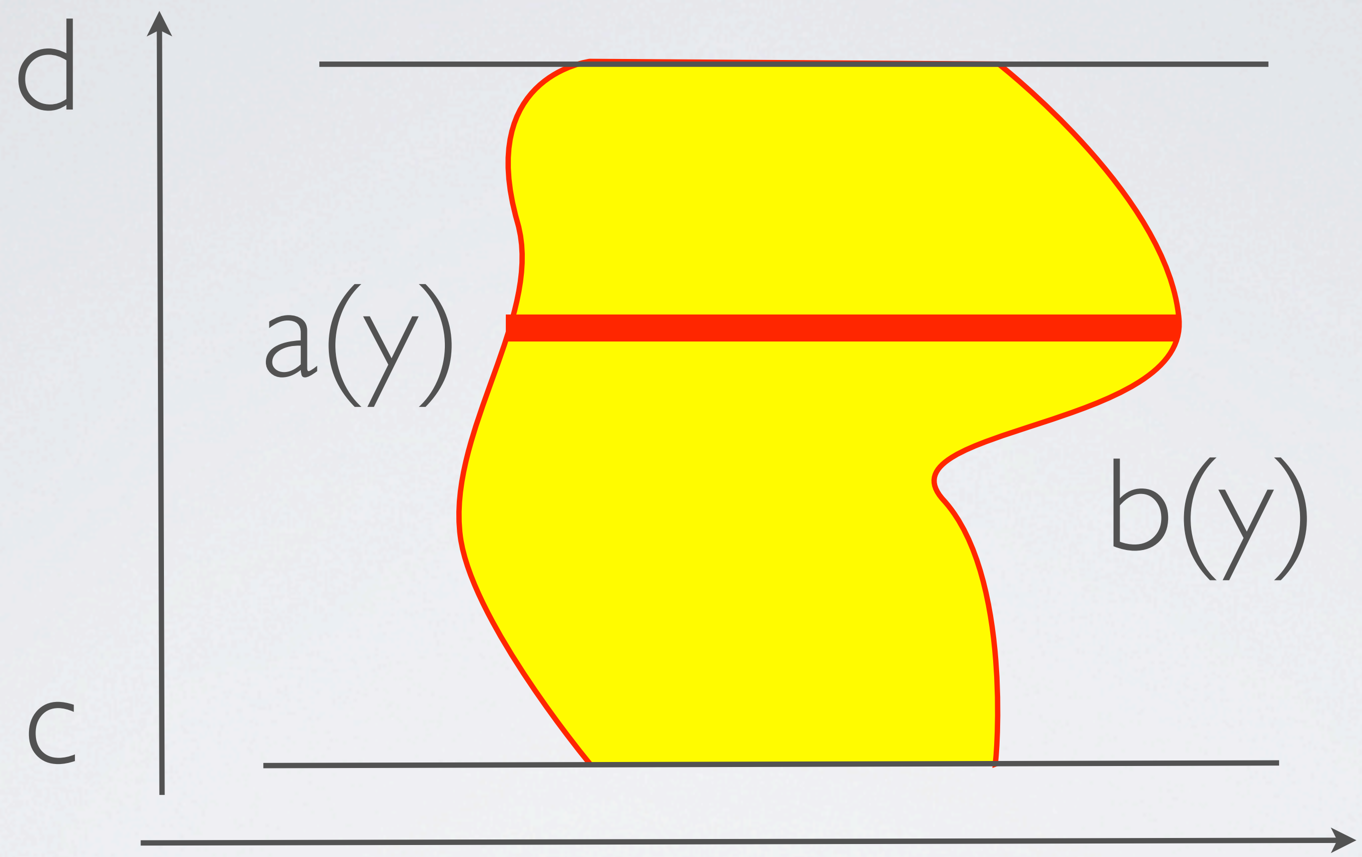
$$\int_c^d \int_a^b f(x,y)\ dx\ dy$$

TYPE I INTEGRALS



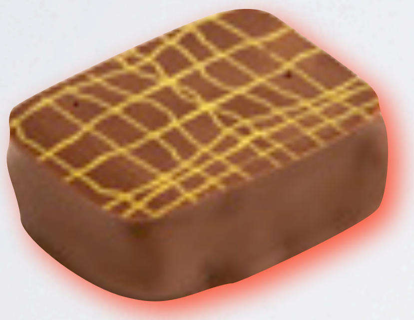
$$\int_a^b \int_{c(x)}^{d(x)} f(x,y)\, dy\, dx$$

TYPE II INTEGRALS

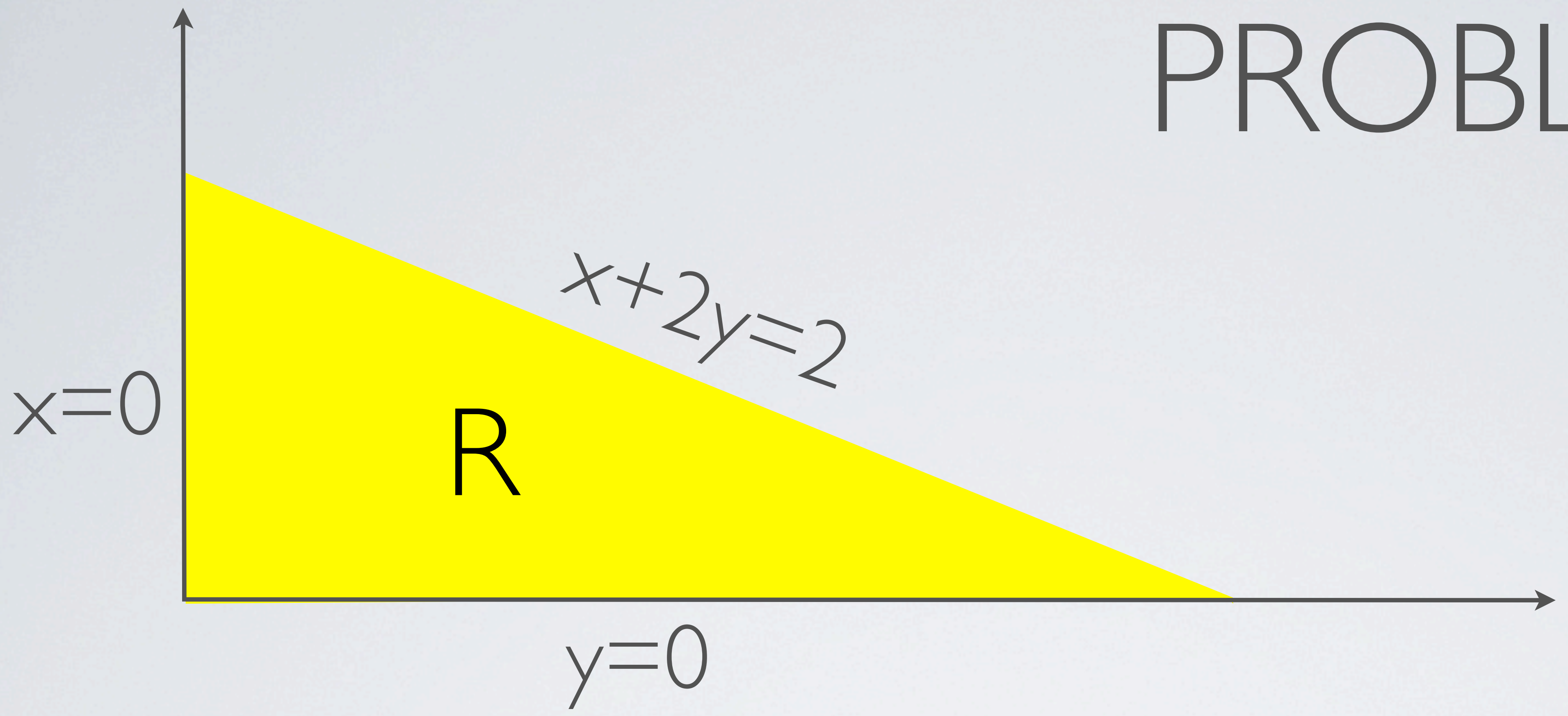


$$\int_{c}^{d} \int_{a(y)}^{b(y)} f(x,y)\, dx\, dy$$

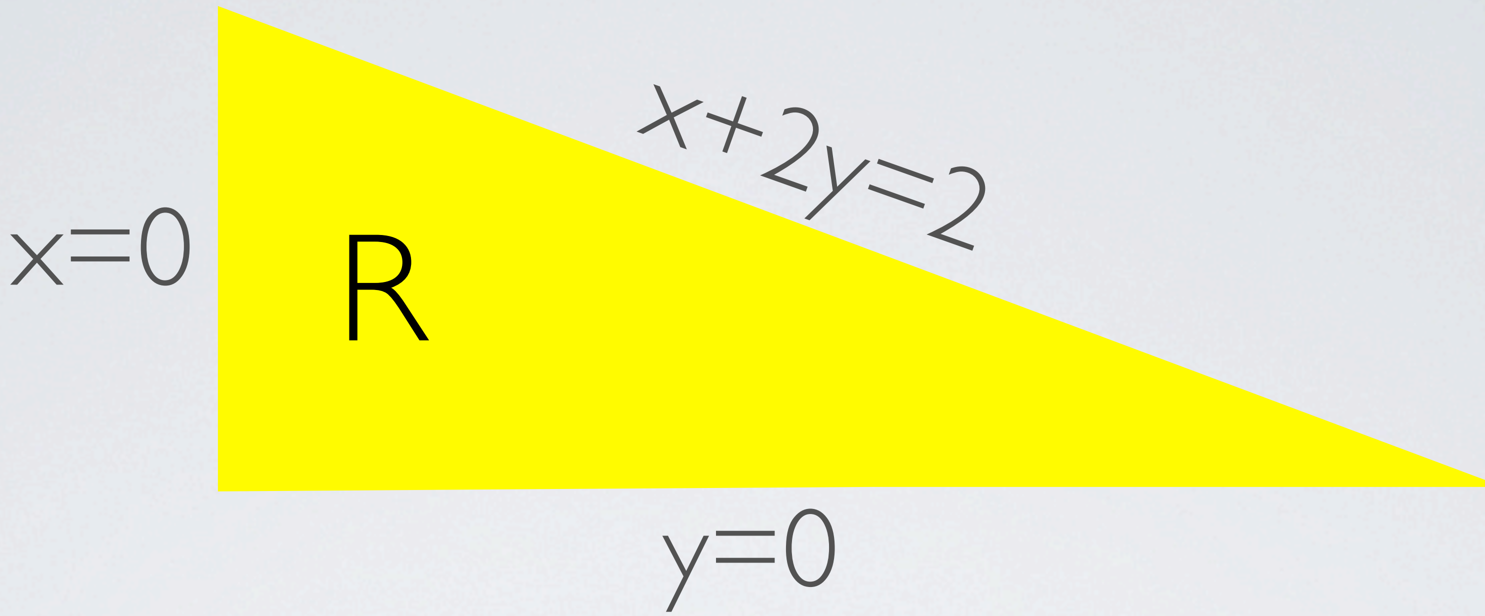
MIND THE ORDER OF INTEGRATION!



PROBLEM:

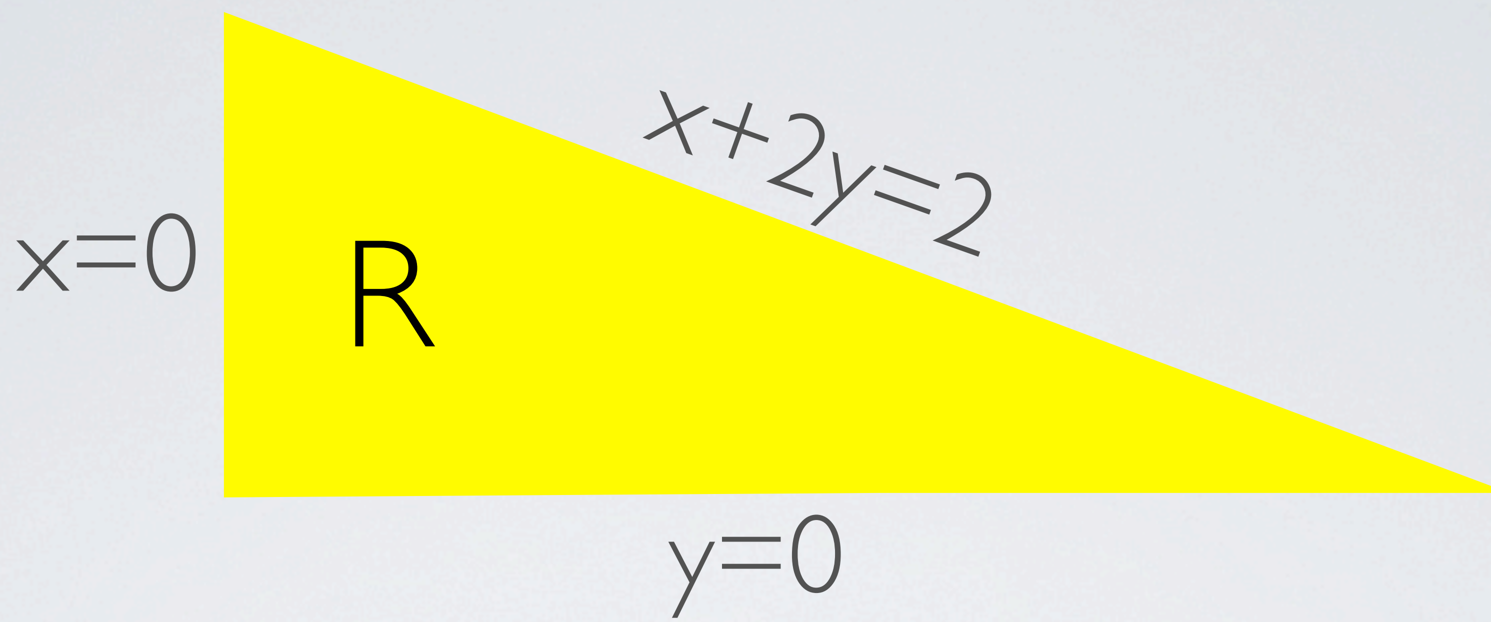


Find $\iint_R e^y/(y-1)\,dA$



As a type I region

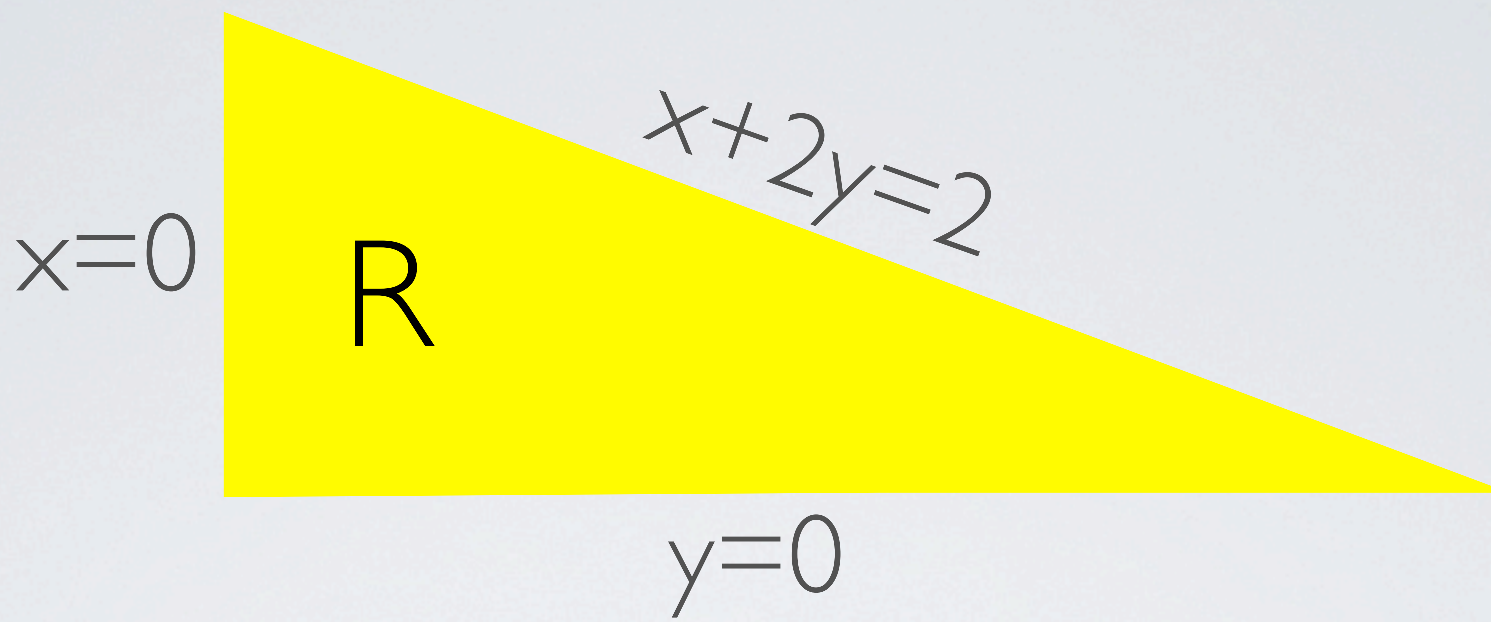
As a type II region



As a type I region

$$\int_0^2 \int_0^{x/2-1} e^y/(y+1)\, dy\, dx$$

As a type II region



As a type I region

$$\int_0^2 \int_0^{x/2-1} e^y/(y+1)\ dy\ dx$$

As a type II region

$$\int_0^1 \int_0^{2+2y} e^y/(y+1)\ dx\ dy$$

PROBLEM ON SETTING UP INTEGRALS:

Let R be the region bounded by

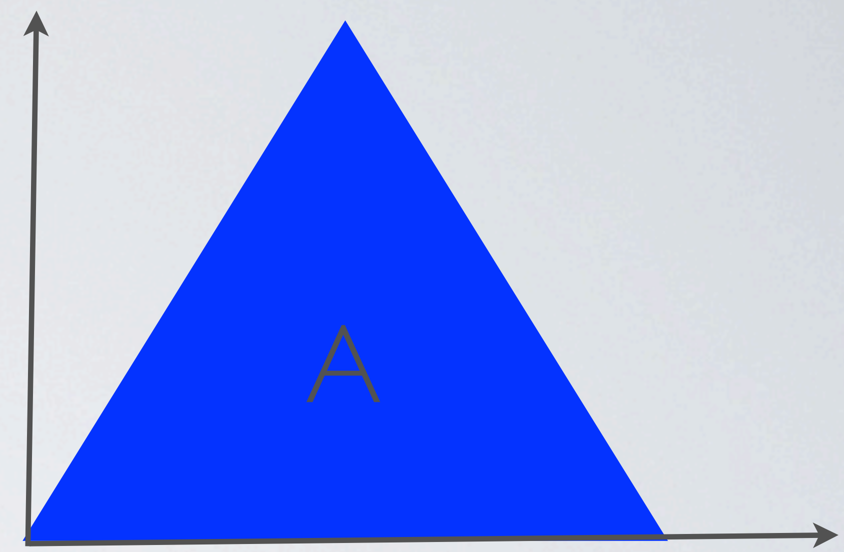
$$y = x^2 - 1$$

and the unit circle $x^2 + y^2 = 1$. Find

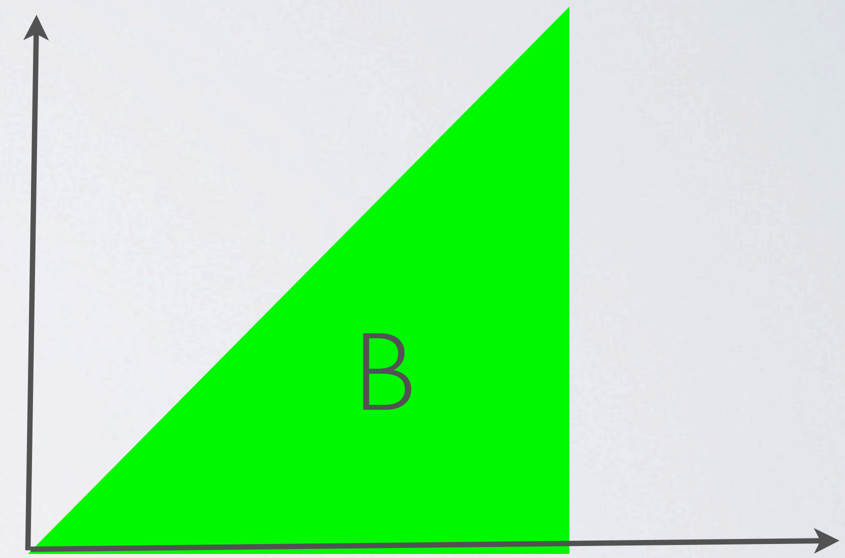
$$\iint_R f(x,y)\, dx\, dy$$

MATCHING REGIONS

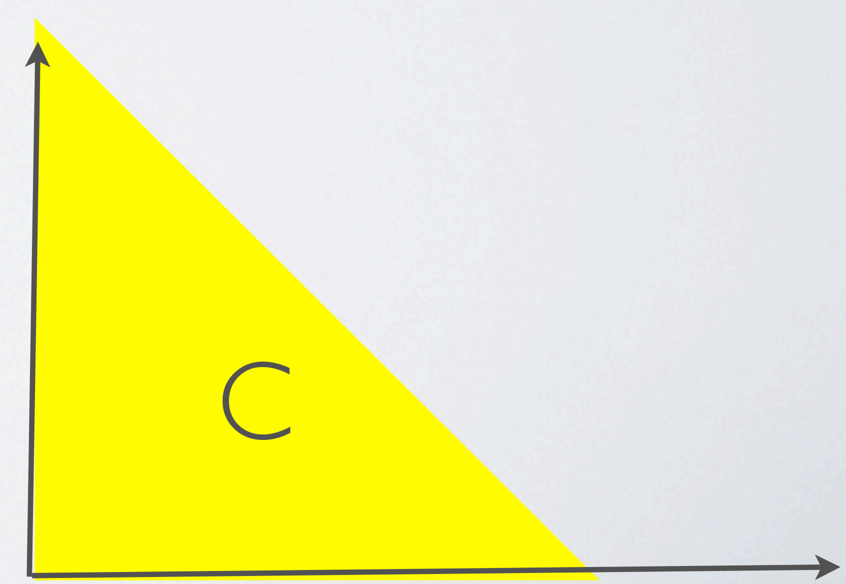
$$\int_0^1 \int_0^{1-x} f \; dy\, dx$$



$$\int_0^1 \int_y^1 f \; dx\, dy$$



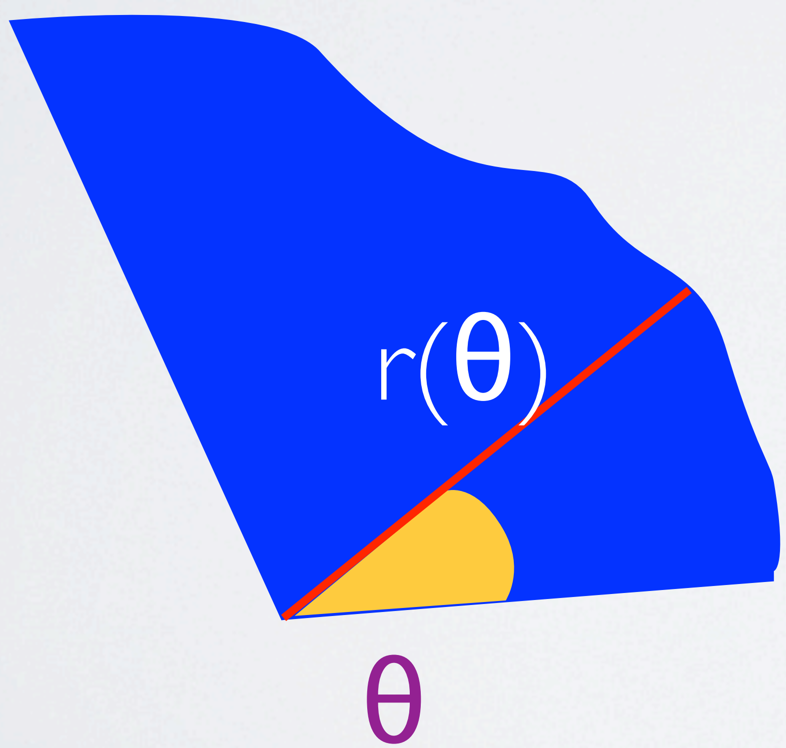
$$\int_0^1 \int_{1-y}^{y-1} f \; dx\, dy$$



POLAR INTEGRATION

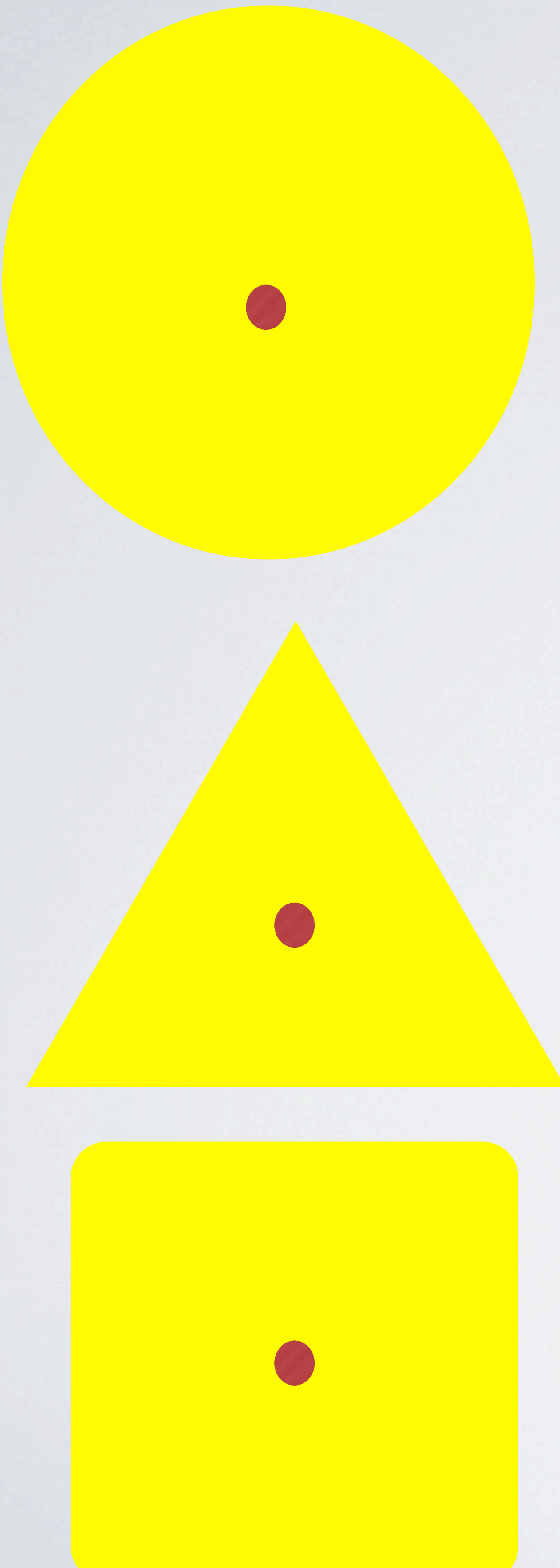
$$\iint f(x,y)\, dx\, dy = \iint f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta$$

Typical region



$$\int_a^b \int_0^{r(\theta)}$$

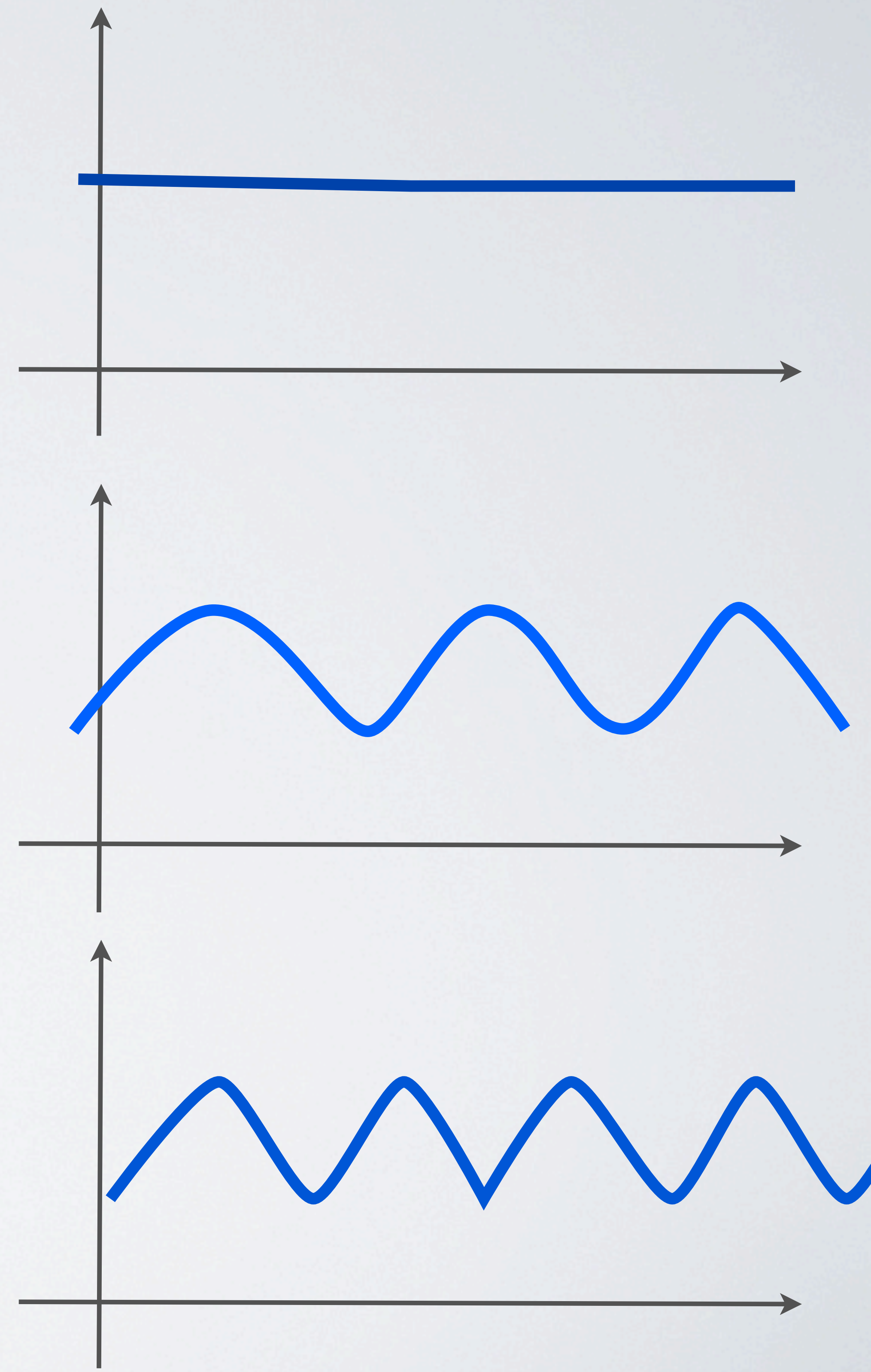
MATCHING REGIONS



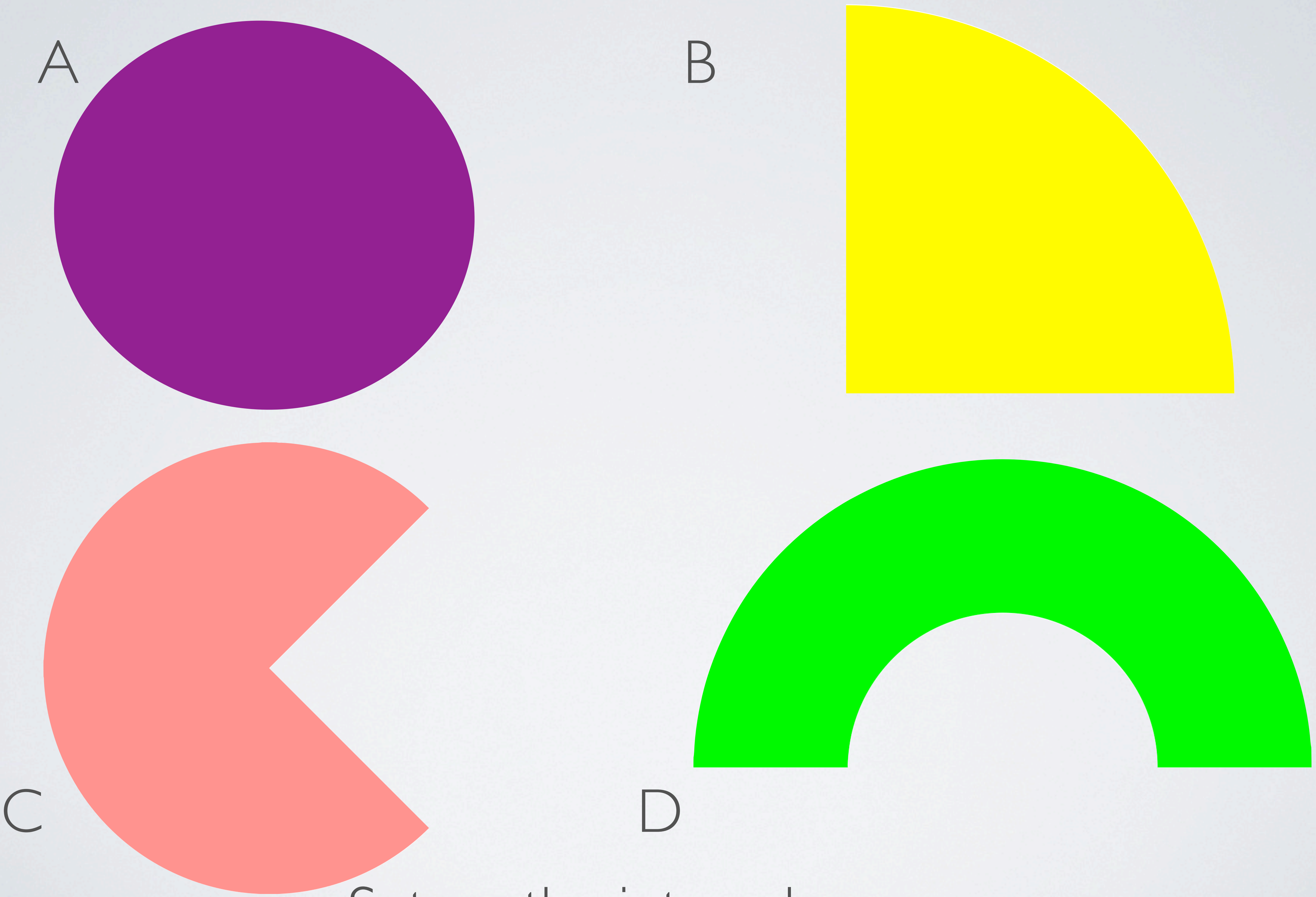
A $\int_0^{2\pi}\int_0^{f(\theta)}$

B $\int_0^{2\pi}\int_0^{g(\theta)}$

C $\int_0^{2\pi}\int_0^{h(\theta)}$



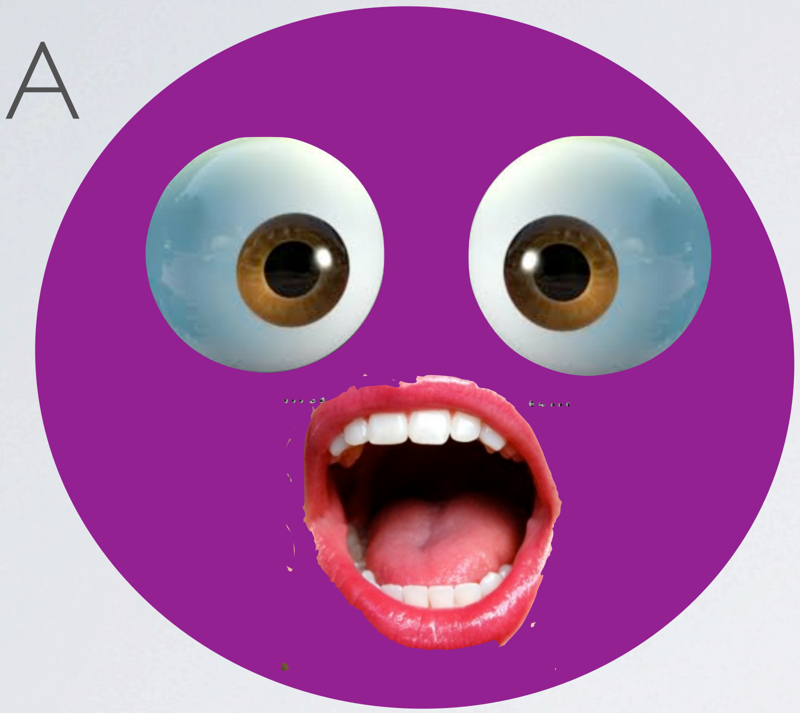
HALLOWEEN CANDIES



Set up the integrals

HALLOWEEN CANDIES

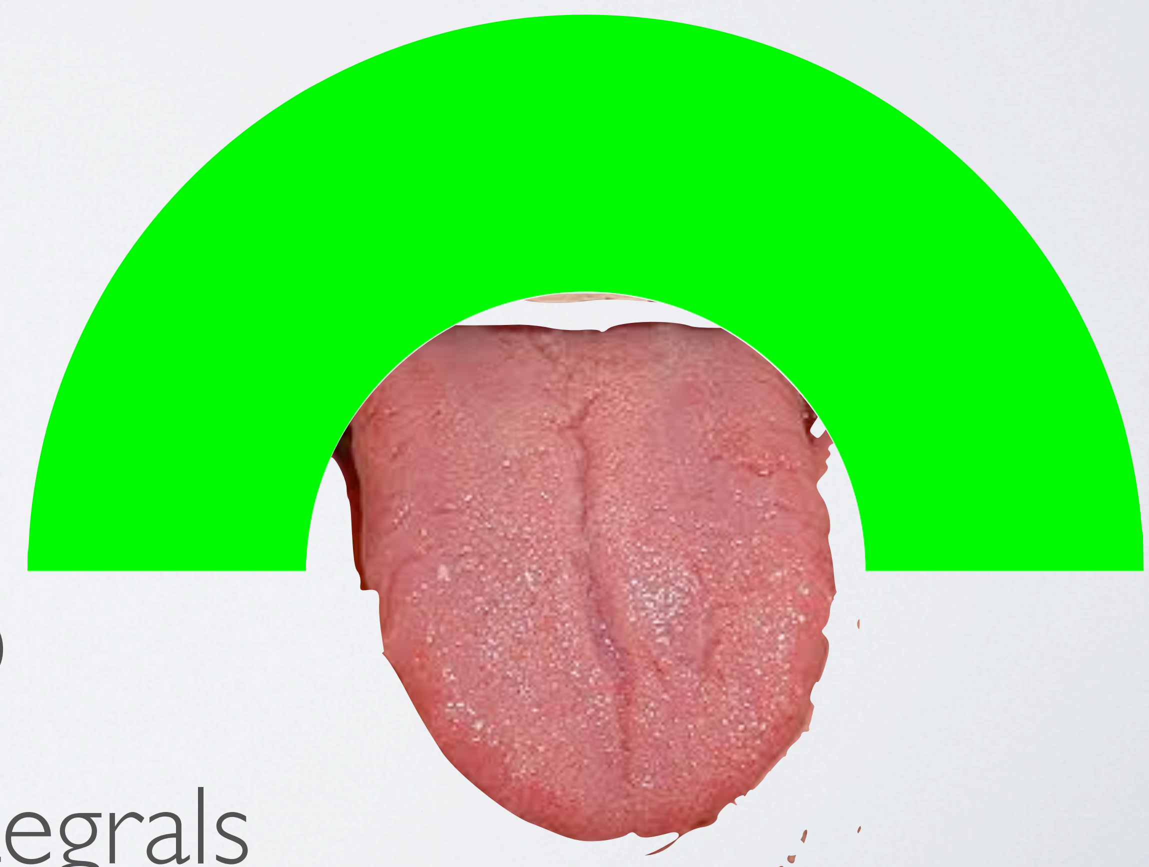
A



B

C

D



Set up the integrals

ADVISE

make a picture - in any case!

consider change order of integration

are Polar coordinates better?

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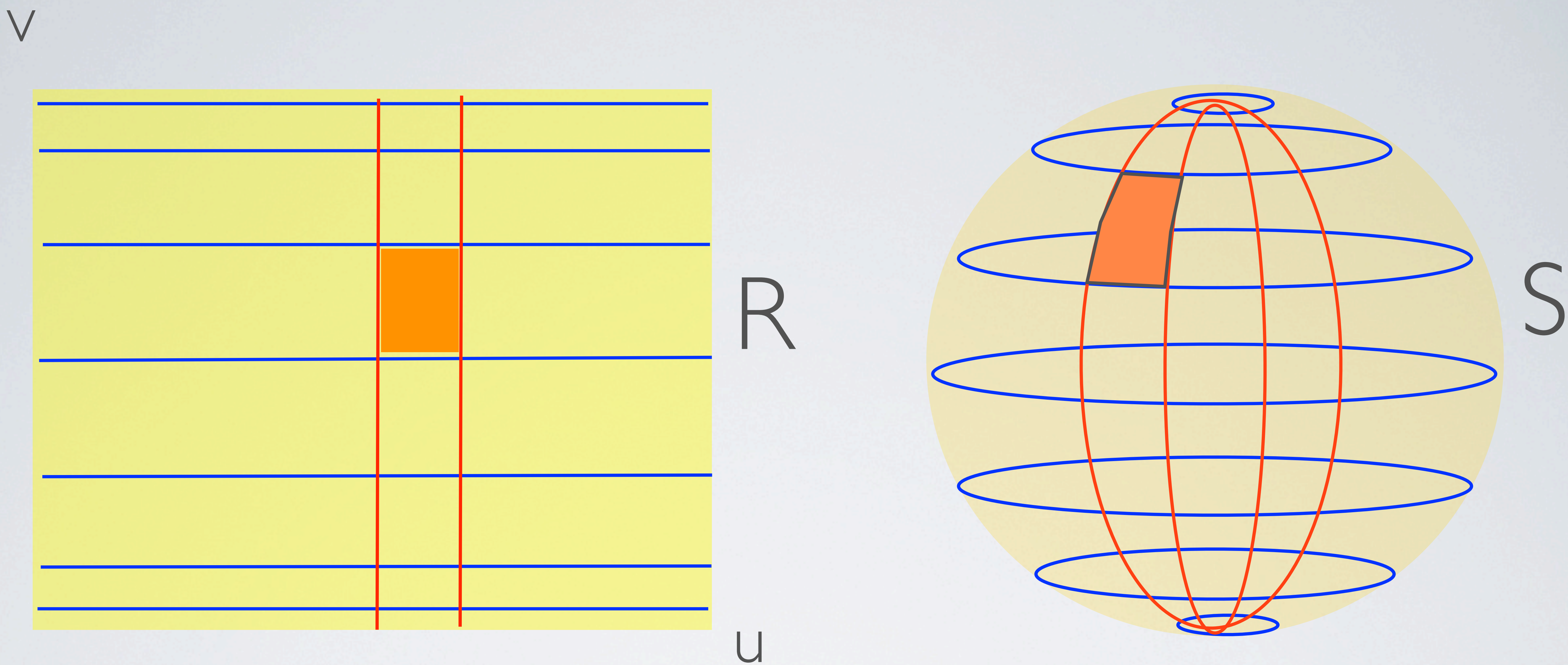
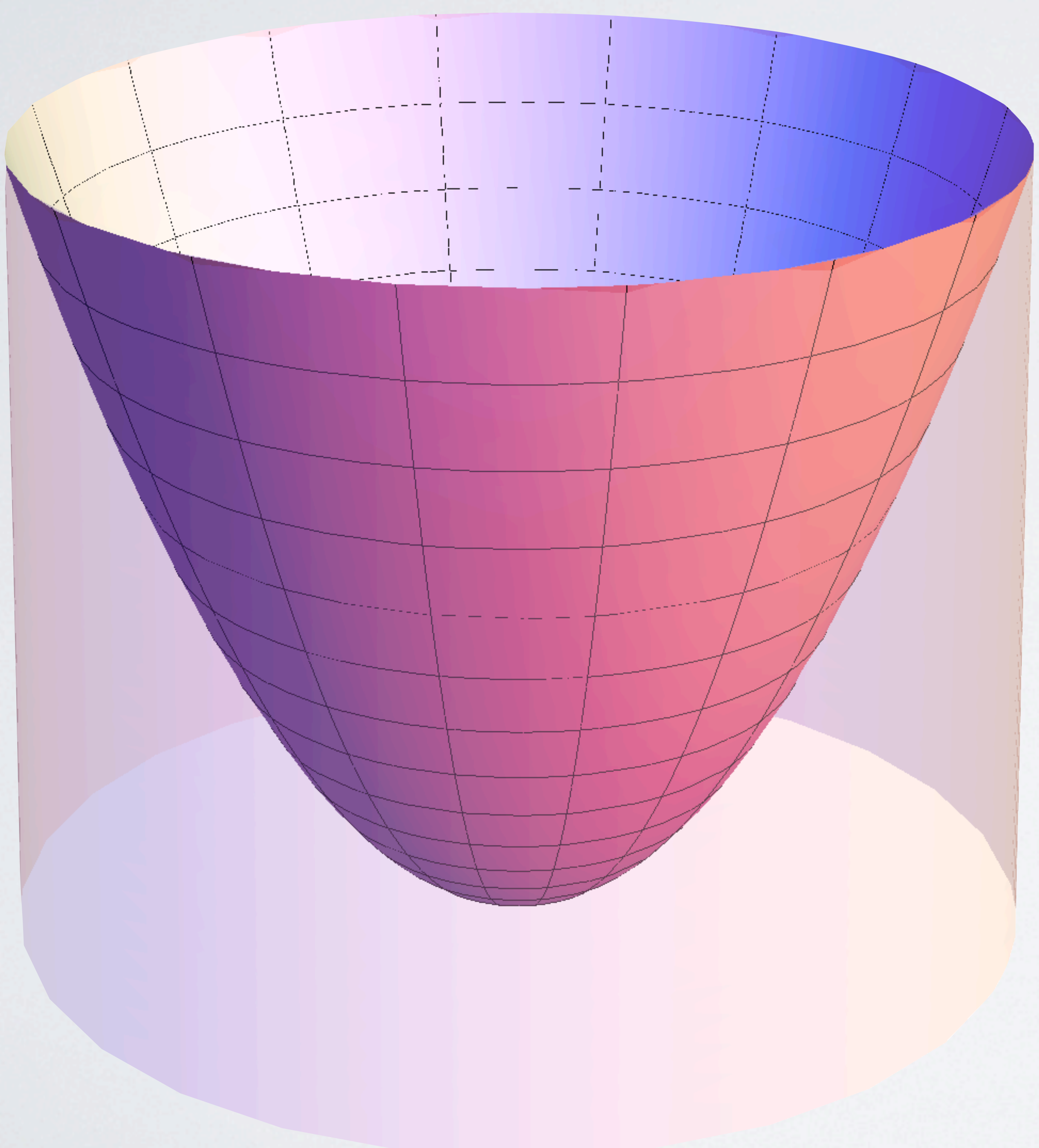


$$A = \iint_R |r_u \times r_v| \, du \, dv$$

FIND THE SURFACE AREA OF A PARABOLOID



$$x^2 + y^2 = z$$

$$x^2 + y^2 \leq 1$$