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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) T F The angle between the vectors $\langle 1,2,3\rangle$ and $\langle 2,-1,-1\rangle$ is $\frac{\pi}{6}$.

2) T F $\langle 4,6,8\rangle$ is a normal vector for the plane $-2x-3y-4z=5$.

3) T F The plane tangent to the graph of $f(x,y)=x^2+y^2$ at $(3,4,25)$ is $6x+8y=50$.

4) T F The directional derivative of a function f in the direction of ∇f can never be negative.

5) T F The surface parameterized by $\langle \sin u, \cos v, u^2+v^2\rangle$, $0 \leq u,v \leq 1$ has the same surface area as the surface parameterized by $\langle \sin u^2, \cos v^2, u^4+v^4\rangle$, $0 \leq u,v \leq 1$.

6) T F By the chain rule, $\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))\ dt = f(\vec{r}(b)) - f(\vec{r}(a))$.

7) T F The function $u(x,y)=x^2+y^2$ is a solution of the wave equation $u_{xx}=u_{yy}$.

8) T F Let C be the unit circle parametrized counter-clockwise. If $\vec{F}(x,y)$ is a vector field and $\int_C \vec{F}\cdot d\vec{r}=0$, then $\vec{F}$ is a gradient vector field.

9) T F The vector field $\vec{F}(x,y,z)=\langle y^2-z^2, z^2-x^2, x^2-y^2\rangle$ is conservative.

10) T F The vector field $\vec{F}(x,y,z)=\langle x,y,z\rangle$ is the curl of a vector field.

11) T F Let $\vec{F}(x,y)=\langle x^2,y^2\rangle$ and $\vec{G}(x,y)=\langle x^2-y, y^2+x\rangle$. If C is the unit circle, traveled counter-clockwise, then $\int_C \vec{F}\cdot d\vec{r}=\int_C \vec{G}\cdot d\vec{r}$.

12) T F If $\vec{F}(x,y,z)=\langle 1,1,1\rangle$, and C is a curve, then $\int_C \vec{F}\cdot d\vec{r}$ is the arc length of C.

13) T F The function $f(x,y,z)=e^z$ is the divergence of a vector field $\vec{F}(x,y,z)$.

14) T F Given a function $f(x,y)=x^3+xy^2$ in the plane, then the flow lines of grad(f) are perpendicular to the level curves of f.

15) T F Given a function $f(x,y)$ without critical points, and a curve $\vec{r}(t)$ which is always perpendicular to grad(f), then $\vec{r}(t)$ is a piece of level curve of f.

16) T F The curl of a conservative vector field in space may not be conservative.

17) T F If a vector field $\vec{F}$ in space is incompressible (div$(\vec{F})=0$) and irrotational (curl$(\vec{F})=\vec{0}$), it is necessarily constant.

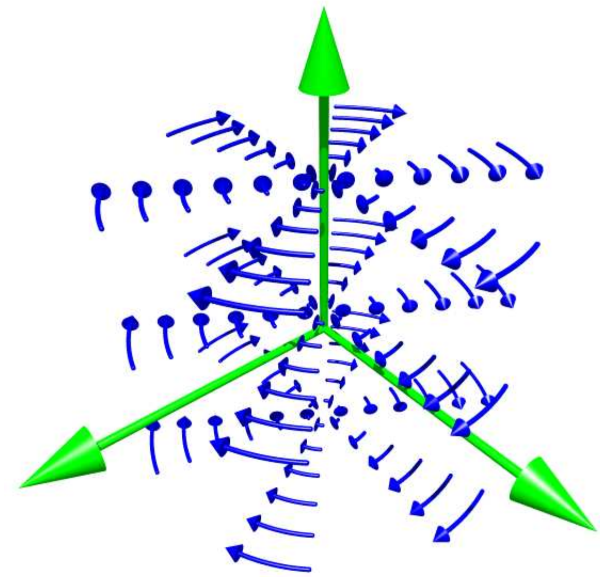
18) T F If a particle moves along a circle, its acceleration vector always points to the center of the circle.

19) T F $\int_0^1\int_0^{1-x}\int_0^{1-x-y} e^{-x^2-y^2-z^2}\,dz\,dy\,dx = \int_0^1\int_0^{1-z}\int_0^{1-z-y} e^{-x^2-y^2-z^2}\,dx\,dy\,dz$.

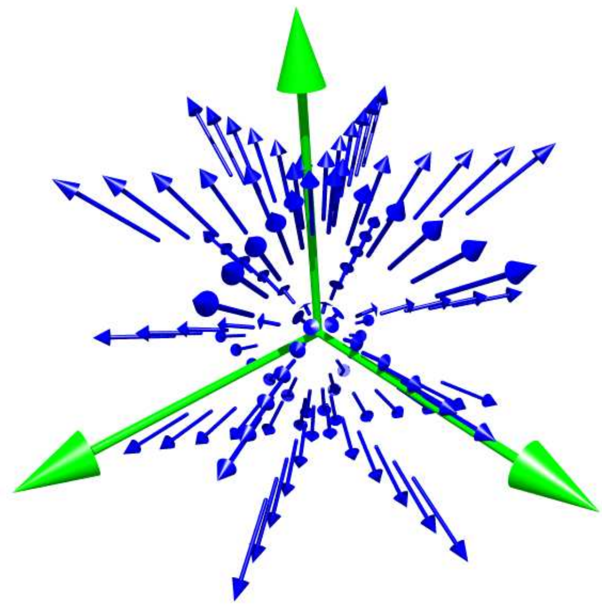
20) T F $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} 1\,dz\,dy\,dx = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^2\sin(\phi)\ d\rho\,d\phi\,d\theta$.

Problem 2) (10 points)

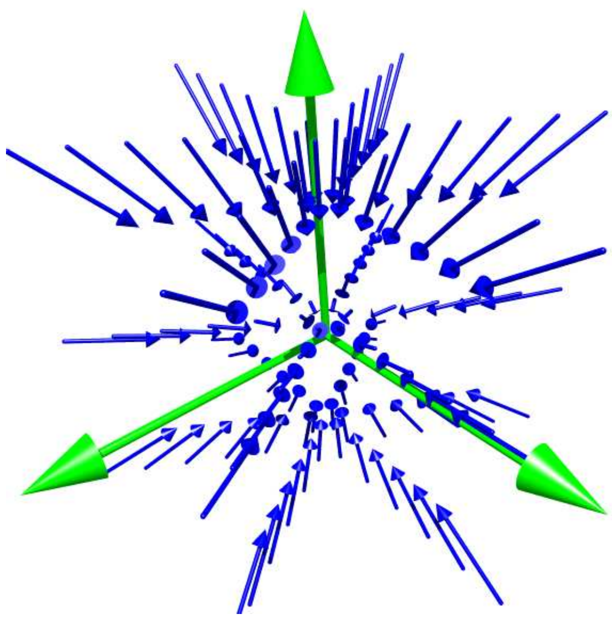
No justifications are needed in this problem. Match the following vector fields in space with the corresponding formulas:



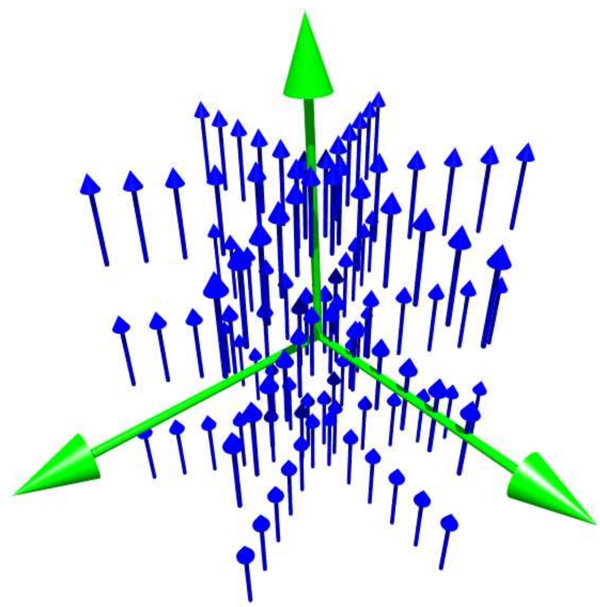
I



II



III

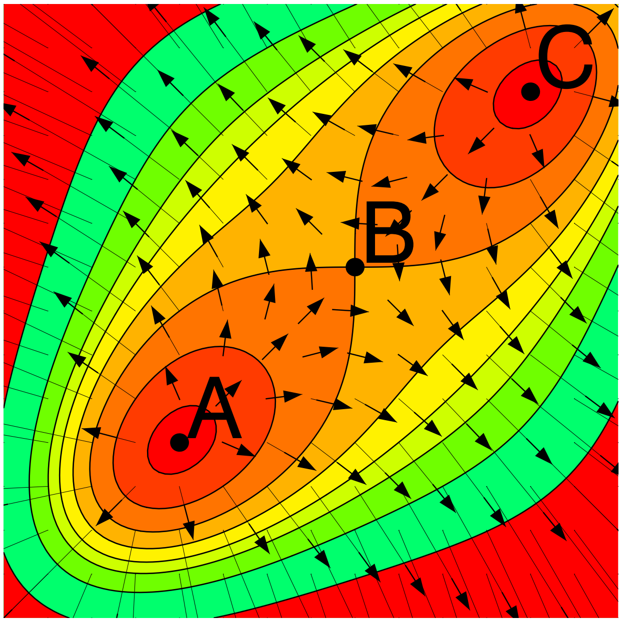


IV

Enter I,II,III,IV here	Vector Field
	$\vec{F}(x,y,z) = \langle y, -x, 0\rangle$
	$\vec{F}(x,y,z) = \langle x, y, z\rangle$
	$\vec{F}(x,y,z) = \langle 0, 0, 1\rangle$
	$\vec{F}(x,y,z) = \langle -x, -2y, -z\rangle$

Problem 3) (10 points)

No justifications are required in this problem. The first picture shows a gradient vector field $\vec{F}(x,y) = \nabla f(x,y)$.



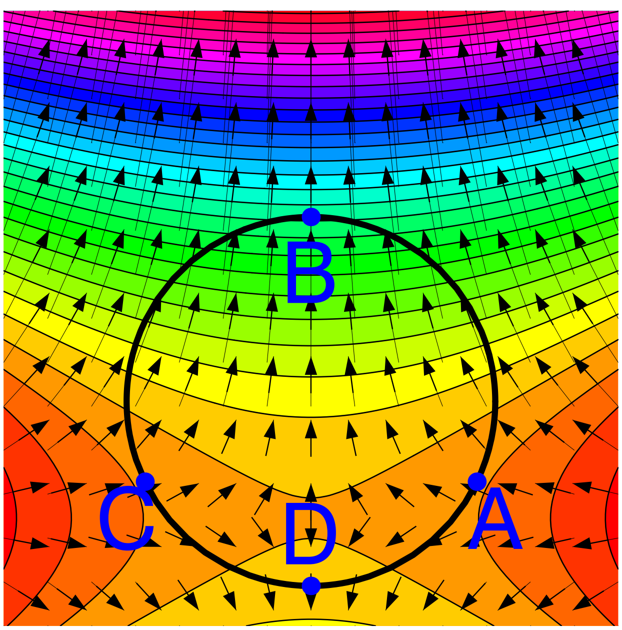
The critical points of $f(x,y)$ are called A, B and C. What can you say about the nature of these three critical points? Which one is a local max, which a local min, which a saddle.

point	local max	local min	saddle
A			
B			
C			

$\int_{P\to Q} \vec{F} \cdot d\vec{r}$ denotes the line integral of $\vec{F}$ along a straight line path from P to Q.

statement	True	False
$\int_{A\to B} \vec{F}\, d\vec{r} \geq 0$		
$\int_{A\to C} \vec{F}\, d\vec{r} \geq \int_{A\to B} \vec{F}\, d\vec{r}$		

The second picture again shows an other gradient vector field $\vec{F} = \nabla f(x,y)$ of a different function $f(x,y)$.



We want to identify the maximum of $f(x,y)$ subject to the constraint $g(x,y) = x^2 + y^2 = 1$. The solutions of the Lagrange equations in this case are labeled A, B, C, D. At which point on the circle if f maximal?

point	maximum
A	
B	
C	
D	

$\int_\gamma \vec{F} \cdot d\vec{r}$ denotes the line integral of $\vec{F}$ along the circle $\gamma : x^2 + y^2 = 1$, oriented counter clockwise.

$\int_\gamma \vec{F} \cdot d\vec{r}$	> 0	< 0	$= 0$
Check if true:			

Problem 4) (10 points)

Given a tetrahedron with vertices $A = (-1, -1, -1)$, $B = (1, 0, 0)$, $C = (0, 1, 0)$ and $D = (0, 0, 1)$, find the distance between the edges AC and BD.

Problem 5) (10 points)

The vector field

$$\vec{F}(x, y) = \langle P, Q \rangle = \langle -\frac{x^5}{5} - 2y^2 x, -4x^2 \frac{y^3}{3} \rangle$$

has the divergence $f(x, y) = \text{div}(\vec{F})(x, y) = P_x(x, y) + Q_y(x, y)$. Classify all the critical points of this function f.

Problem 6) (10 points)

a) (4 points) Find the linearization $L(x, y)$ of the function $P(x, y) = x^2 - y$ and the linearization $K(x, y)$ of $Q(x, y) = \sin(\pi y) + x$ at the point $(0, 1)$.

b) (4 points) Estimate the **vector**

$$\vec{F}(0.01, 0.9999)$$

for the vector field

$$\vec{F}(x, y) = \langle x^2 - y, \sin(\pi y) + x \rangle = \langle M, N \rangle$$

using linear approximation.

c) (2 points) The vector field $\vec{G}(x, y) = \langle L(x, y), K(x, y) \rangle$ is called the **linearization** of the vector field $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$. Is the curl of $\vec{F}$ equal to the curl of $\vec{G}$ at $(0, 1)$?

Problem 7) (10 points)

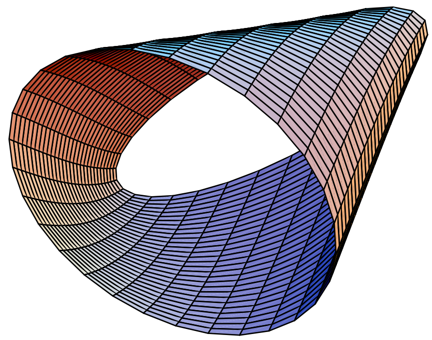
The parametrized surface

$$\vec{r}(u, v) = \langle \cos(u), v, \sin(u + v) \rangle$$

for $0 \leq u \leq 2\pi, 0 \leq v \leq \pi/3$ has a boundary which consists of two distinct curves C_1, C_2.

a) (5 points) Write down the integral for the surface area of S. You do not have to evaluate the integral in this problem a).

b) (5 points) Parameterize both boundaries and compute the arc length of one of the two boundary curves. The orientation of the parameterizations is not important. Of course, you have to find the value of the integral in this part b). It is possible to find it for one of the two boundary curves.



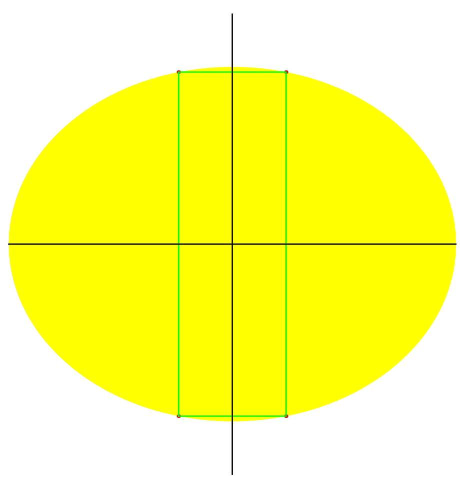
Problem 8) (10 points)

Which rectangle inscribed into the planar region

$$x^4 + y^8 = 17$$

has maximal area?

Note. As the picture indicates, the rectangle is symmetric both with respect to the x-axes and the y-axes and has all its sides parallel to the x axes or y axes.



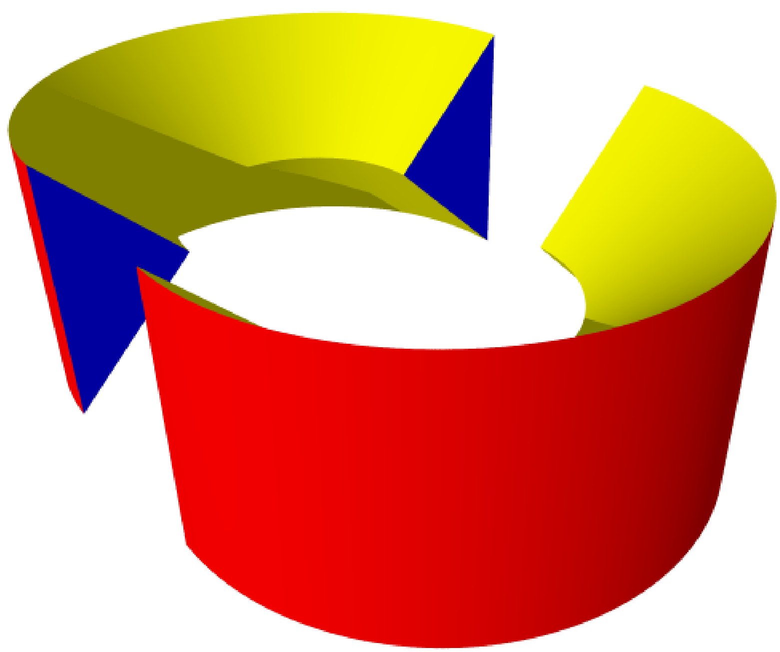
Problem 9) (10 points)

Evaluate the double integral

$$\int_0^2 \int_{x^2}^4 \frac{x}{e^{y^2}} \, dydx \, .$$

Problem 10) (10 points)

A solid is made by intersecting the solid cylinder $x^2 + y^2 \leq 4$ with $x^2 + y^2 \geq (z+1)^2$ and $x^2 + y^2 \geq (z-1)^2$. Find the volume of this body. The picture shows the body sliced along the xz-plane to show the crosssection. It actually is in one piece.



Problem 11) (10 points)

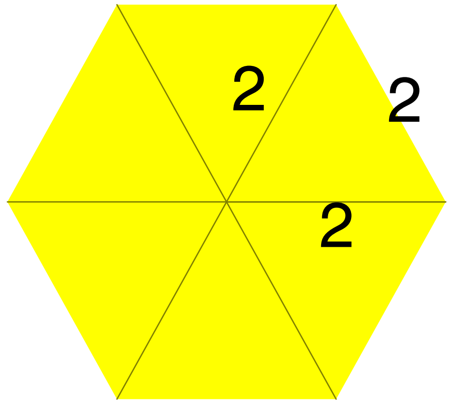
Compute the line integral of the vector field

$$\vec{F}(x, y) = \langle 0, x + e^{\sin(e^y)} \rangle$$

along the boundary of a regular hexagon with vertices

$$(2, 0), (1, \sqrt{3}), (-1, \sqrt{3}), (-2, 0), (-1, -\sqrt{3}), (1, -\sqrt{3}) \, .$$

Hint. As the picture indicates, the hexagon is the union of 6 identical equilateral triangles with side length 2.



Problem 12) (10 points)

Evaluate the line integral

$$\int_C \langle y^3, 3xy^2 + e^z, ye^z \rangle \cdot d\vec{r}$$

where C is the curve parameterized by $\vec{r}(t) = \langle e^t, t^5, \log(t^{10} - t^5 + 1) \rangle$, and where the parameter t satisfies $0 \le t \le 1$.

Problem 13) (10 points)

Evaluate the integral

$$\int\int_S \operatorname{curl}(\langle y + e^{\cos(z)}, e^{\sin(z)}, ze^{e^x} \rangle) \cdot d\vec{S} \ ,$$

where S is the part of the paraboloid $16 - x^2 - y^2$ above the xy- plane with orientation so that the normal vector points **downward**.

Hint: Relate this integral to an integral over the disc $x^2 + y^2 \le 16, z = 0$.

Problem 14) (10 points)

Find the flux integral

$$\int\int_S \vec{F} \cdot d\vec{S} \ ,$$

where $\vec{F}$ is the vector field

$$\vec{F}(x, y, z) = \langle y^2 - xz + e^y, -yz + z^{3x+6z^2}, x^4 + y^2 + z^2 \rangle \ ,$$

and where S is the surface given by

$$f(x,y,z) = (x^2+y^2)^2 - (x^2-y^2) - z^2 + \frac{1}{5} + \frac{(x^6+y^6+z^6)}{10} = 0$$

with orientation pointing outward.