LAGRANGE

GAUSS

STOKES

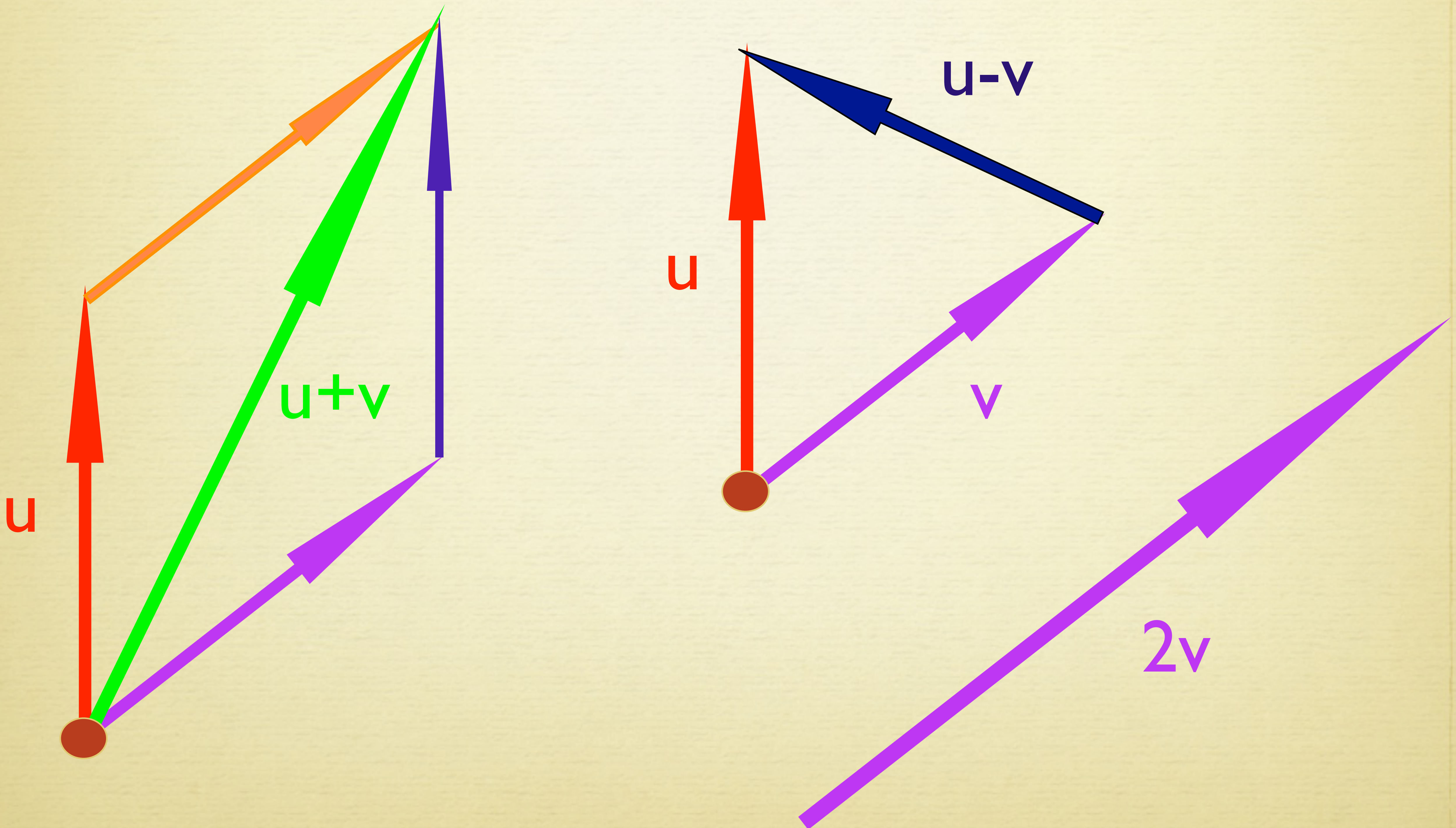
AMPERE

Math 21a Fall 2011

OLIVER KNILL

DECEMBER 7, 2011

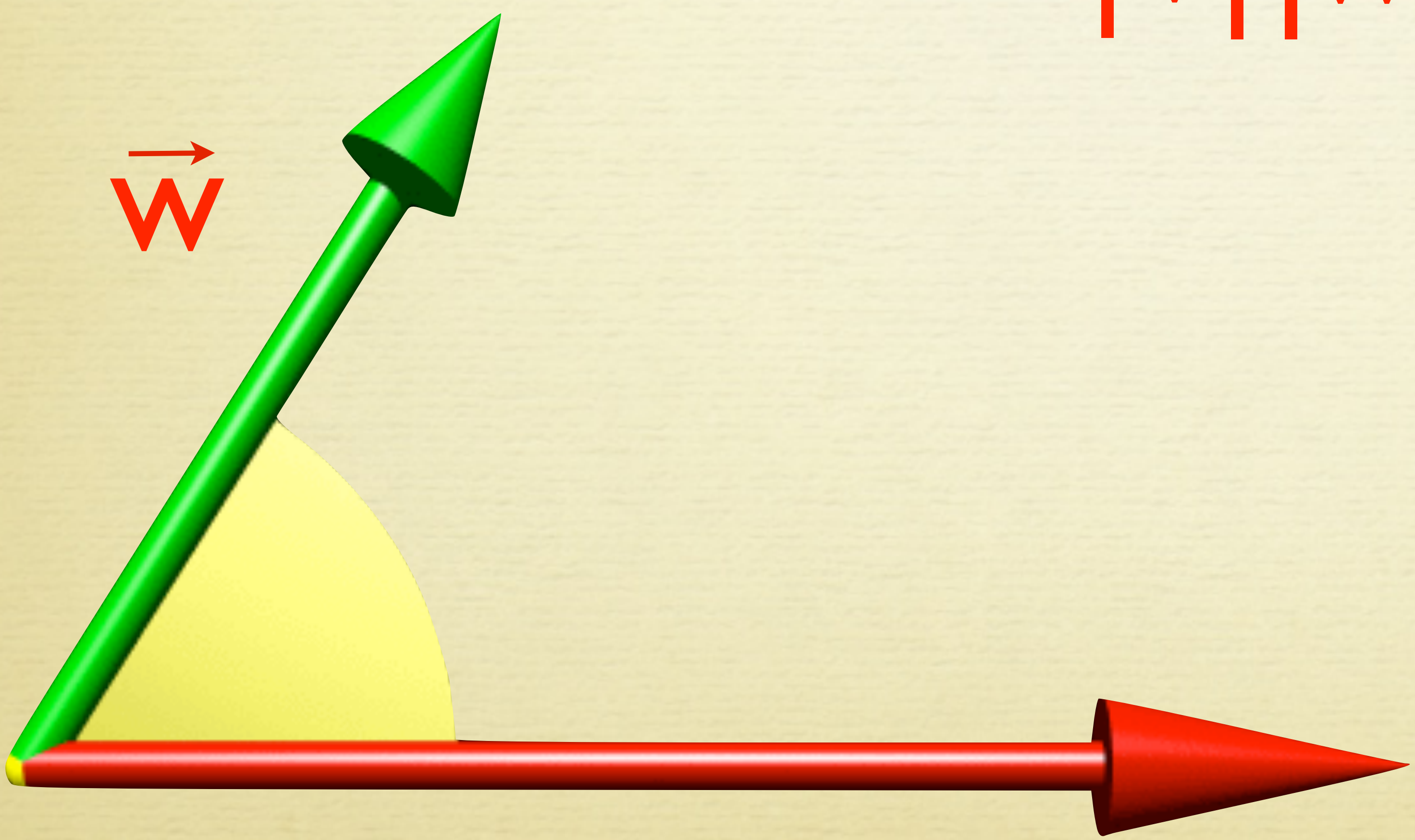
Vector Operations



Dot Product

$$\vec{v} \cdot \vec{w} = \vec{v}_1 \vec{w}_1 + \vec{v}_2 \vec{w}_2 + \vec{v}_3 \vec{w}_3$$

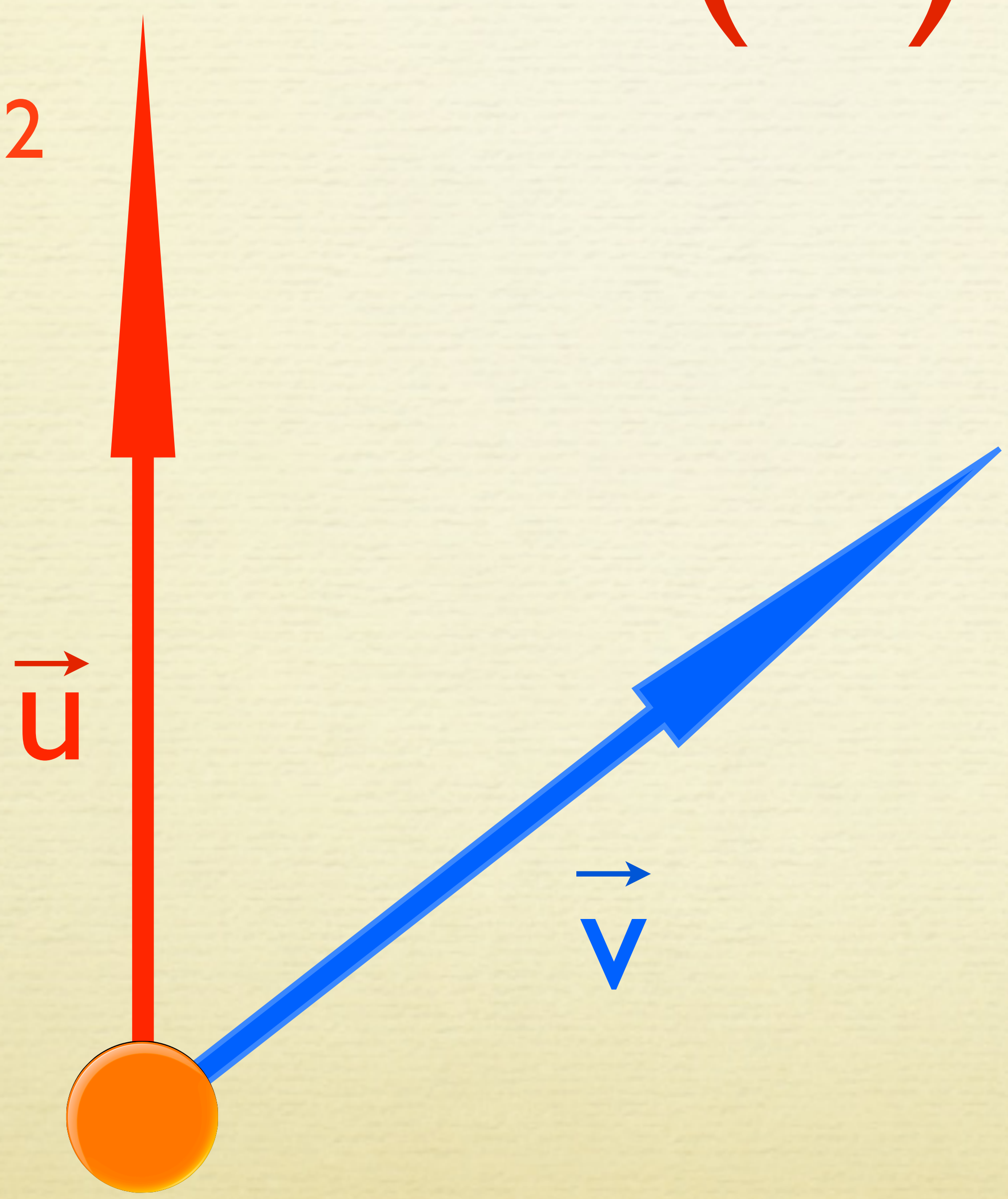
$$= |\vec{v}|\,|\vec{w}| \cos(\alpha)$$

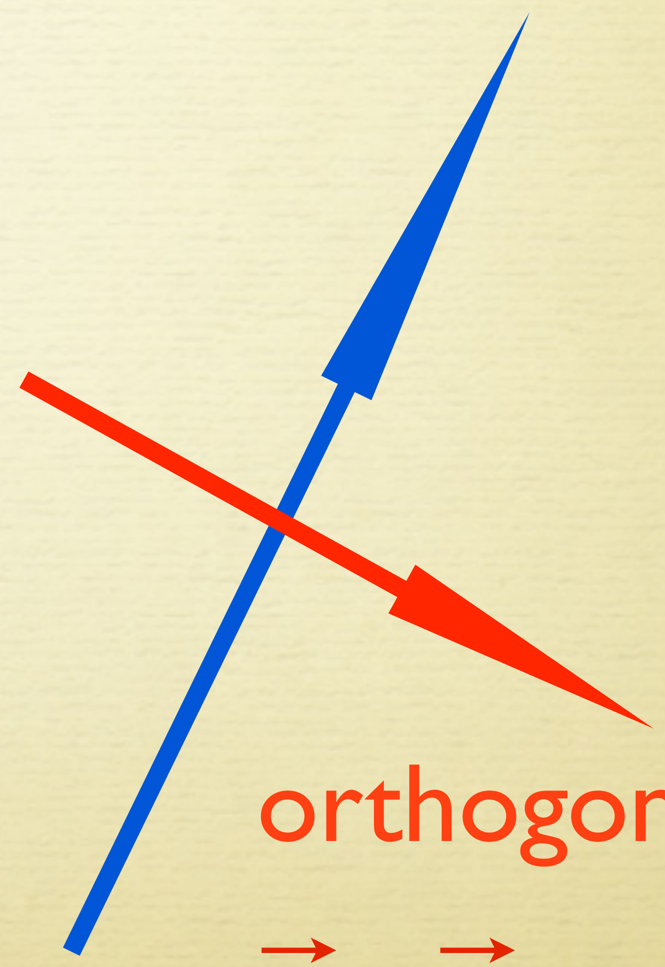


Angle formula

$$\cos(\alpha) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|}$$

$$\vec{u} \cdot \vec{u} = |\vec{u}|^2$$



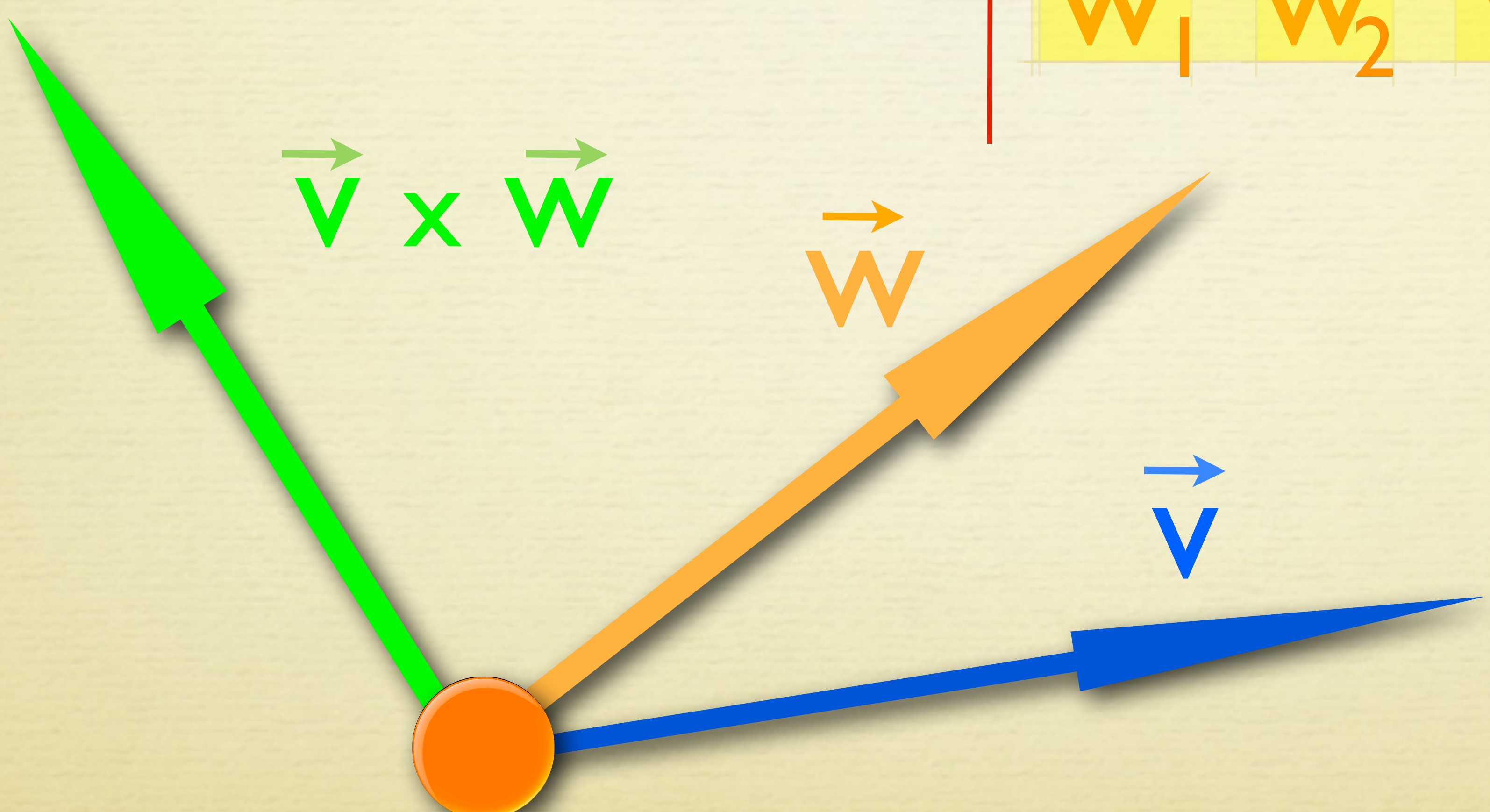


orthogonal

$\vec{u} . \vec{v} = 0$

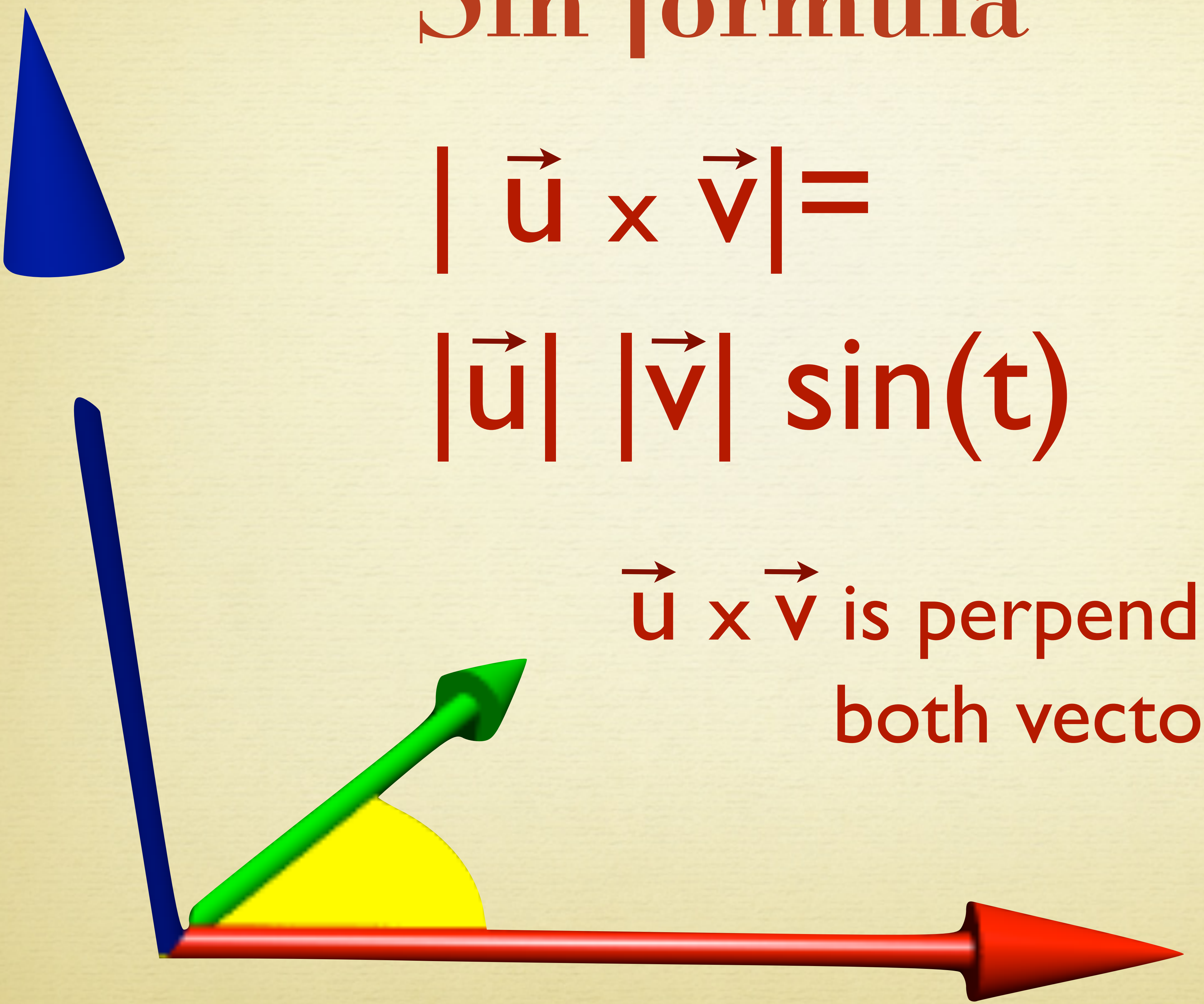
Cross Product

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$



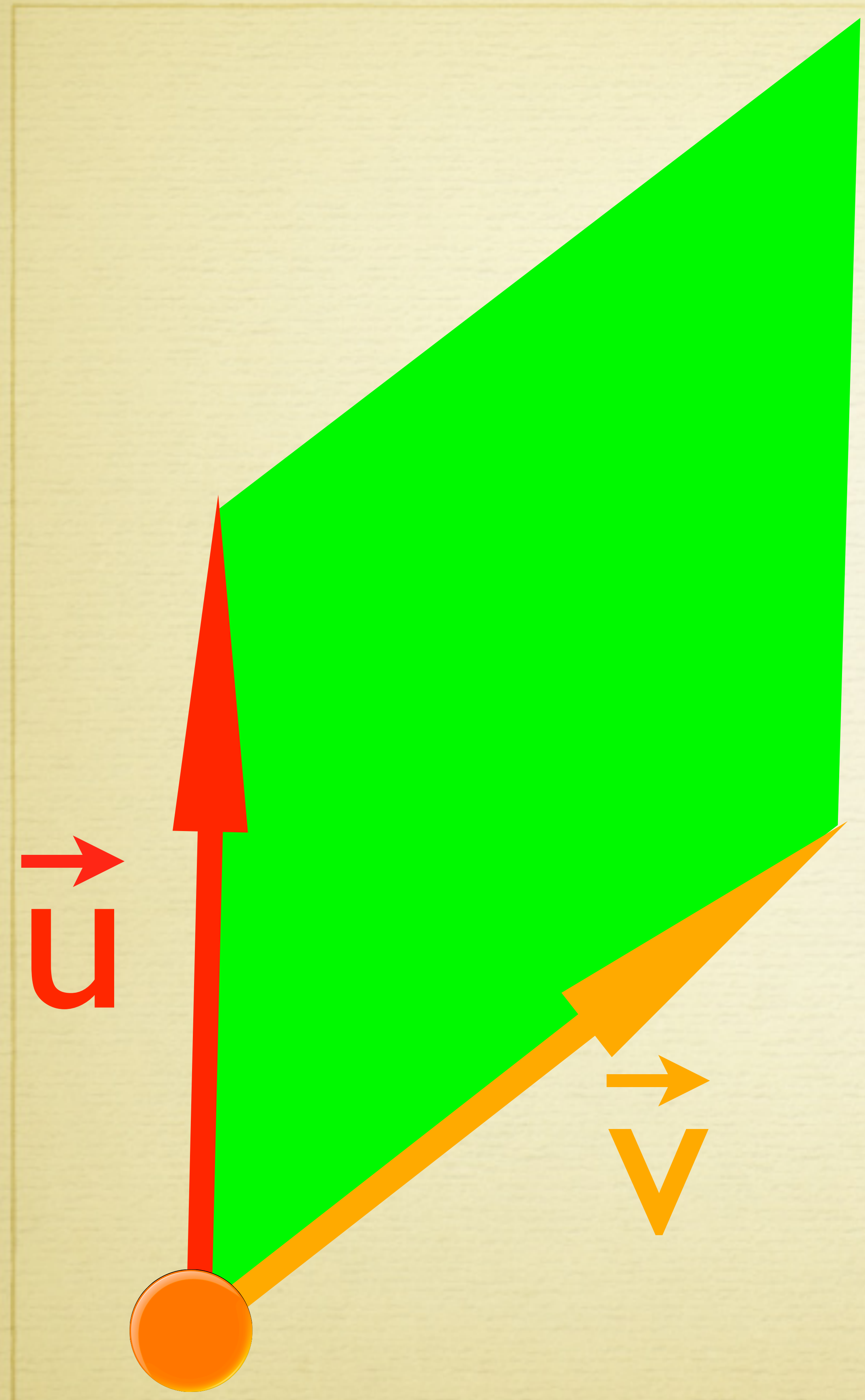
Sin formula

$$|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}|\,\sin(t)$$



$\vec{u} \times \vec{v}$ is perpendicular to both vectors.

Area



$$|\vec{u} \times \vec{v}| =$$

$$|\vec{u}|\ |\vec{v}|\ \sin(t)$$

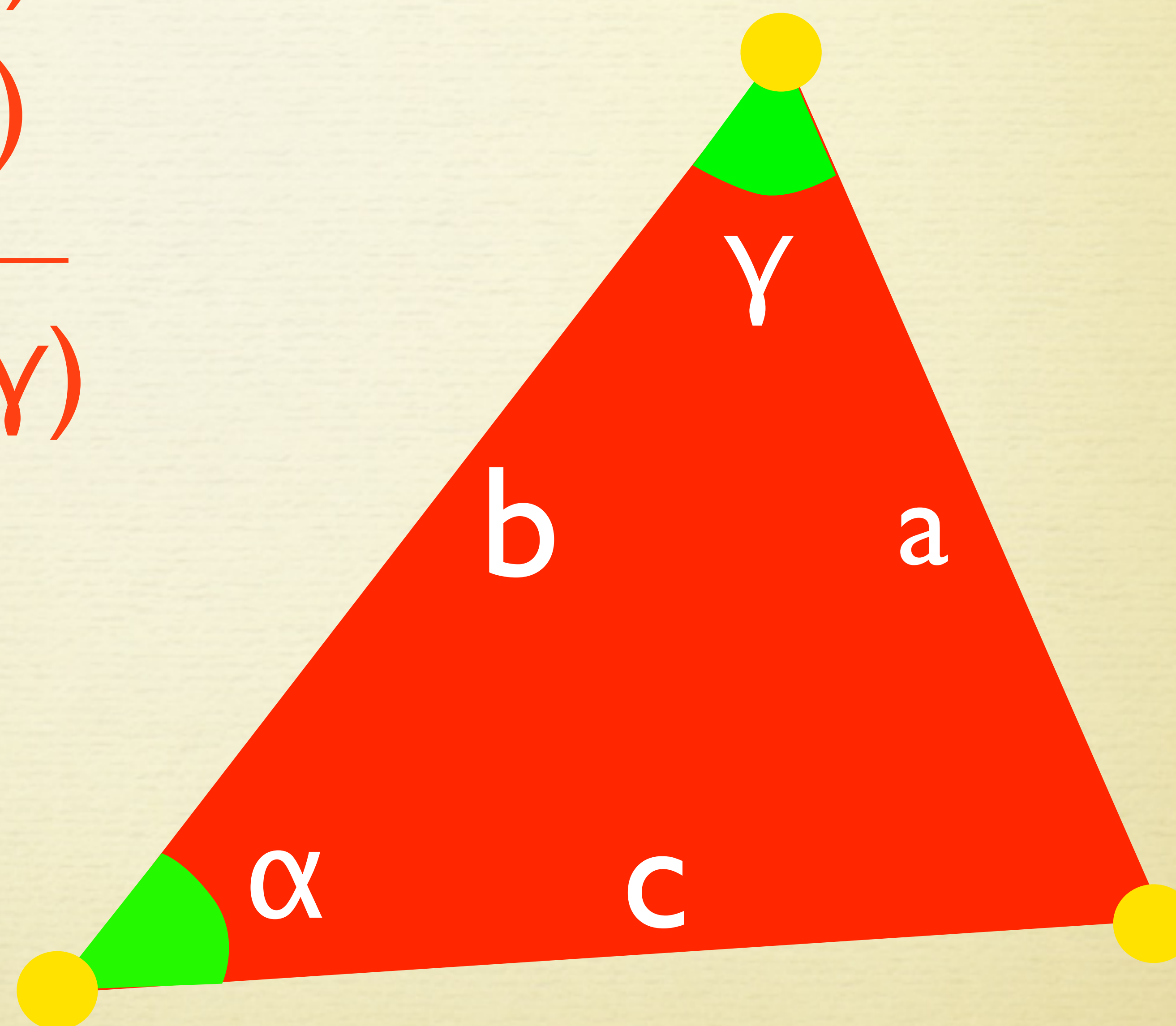
Sin formula

$$|b \times c| = |b|\,|c| \sin(\alpha)$$

$$|b \times a| = |b|\,|a| \sin(\gamma)$$

$$1 = |c| \sin(\alpha) / |a| \sin(\gamma)$$

$$\frac{|c|}{\sin(\gamma)} = \frac{|a|}{\sin(\alpha)}$$



Formula	to compute
$\vec{u} \cdot \vec{v} = \lvert\vec{u}\rvert\ \lvert\vec{v}\rvert \cos(\alpha)$	ANGLE
$\lvert\vec{u} \times \vec{v}\rvert = \lvert\vec{u}\rvert\ \lvert\vec{v}\rvert \sin(\alpha)$	AREA
$\text{proj}_{\vec{w}}(\vec{v}) = \dfrac{\vec{v}\cdot\vec{w}}{\lvert\vec{w}\rvert^2}\,\vec{w}$	Length
$\lvert\vec{u} \cdot \vec{v} \times \vec{w}\rvert$	Volume

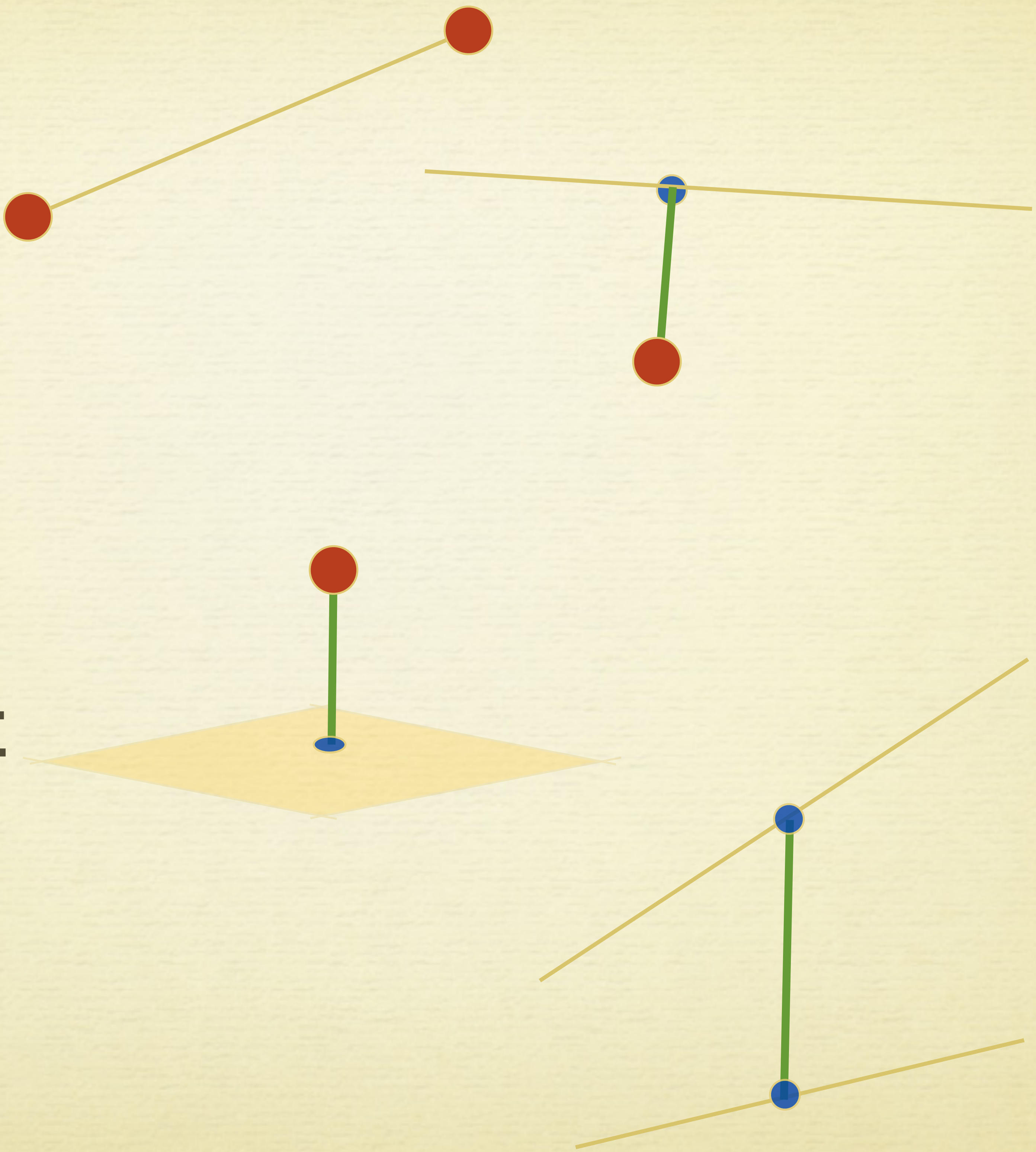
4 multiplications to remember

Point Point

Point Line

Point Plane

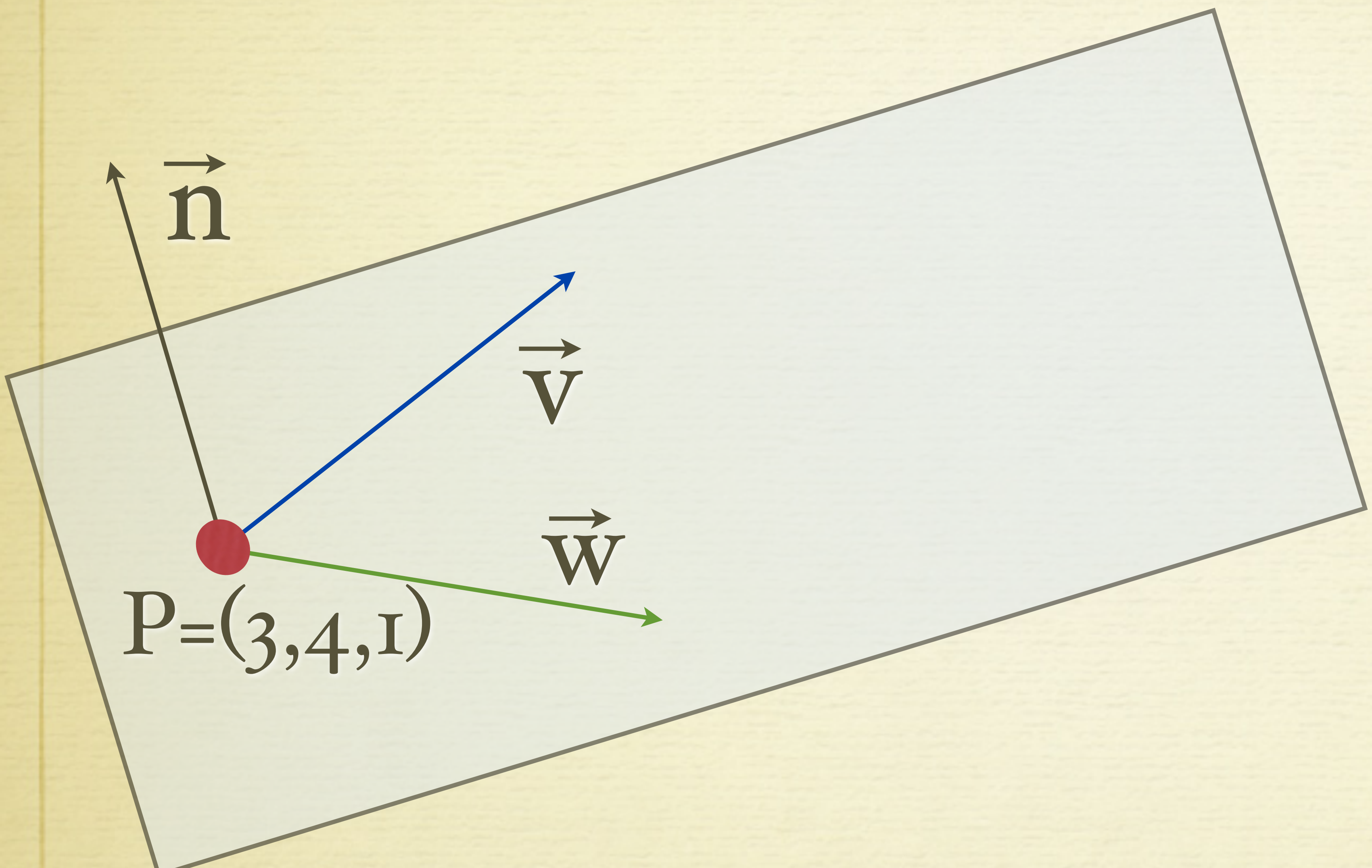
Line Line



4 distance formulas to remember

Distances Review

First Lines and Planes



$\overrightarrow{OP} = <3,4,1>$

$\vec{v} = <1,1,-3>$

$\vec{w} = <1,-2,1>$

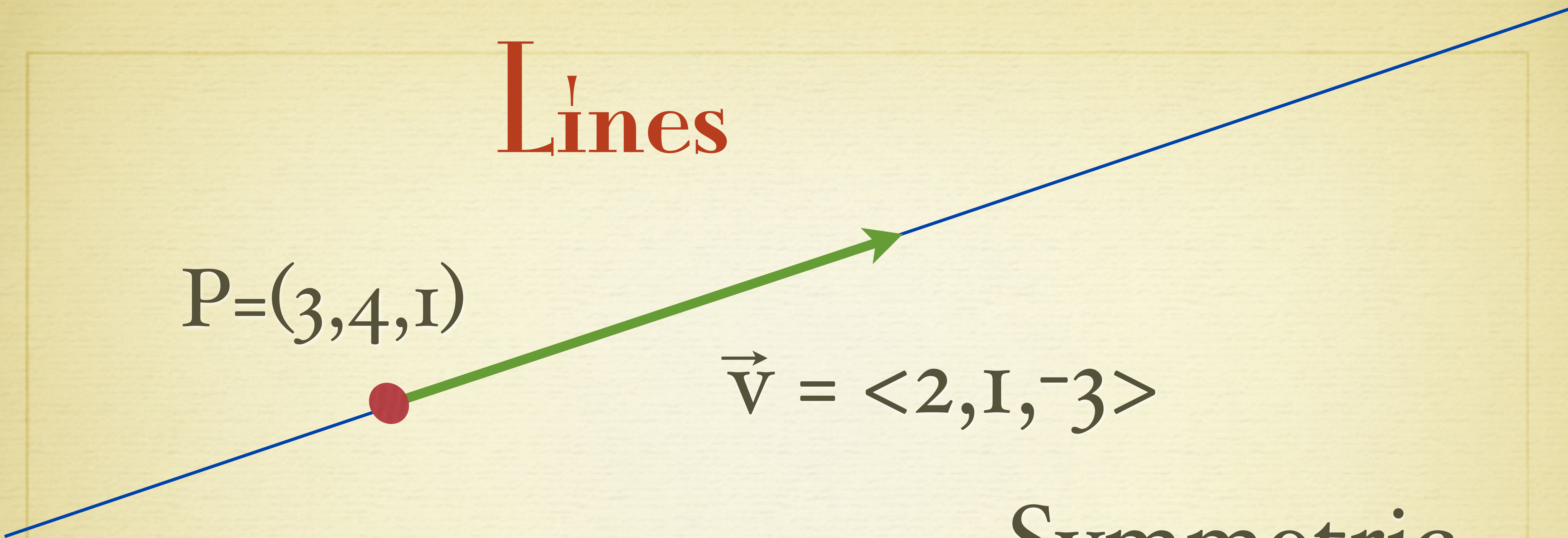
Parametrization

$\vec{n} = <7,-4,-3>$

$\vec{r}(t,s) = < 3+t+s,4+t-2s,1-3t+s>$

$7x-4y-3z=2$

Lines



Parametrization

$$\vec{r}(t) = \langle 3+2t, 4+t, 1-3t \rangle$$

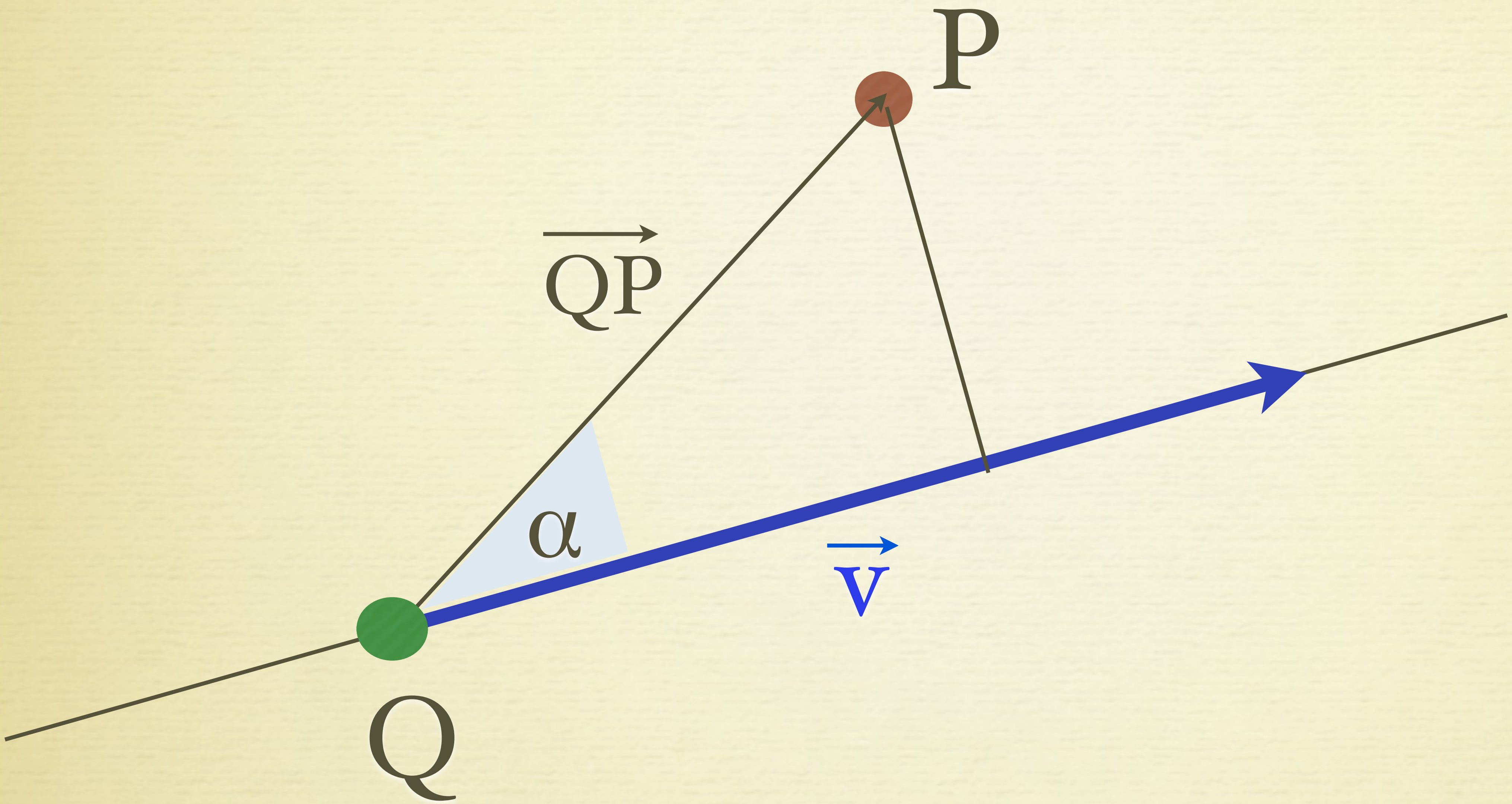
$$= \langle x, y, z \rangle$$

Symmetric equations

$$\frac{x-3}{2} = \frac{y-4}{1} = \frac{z-1}{-3}$$

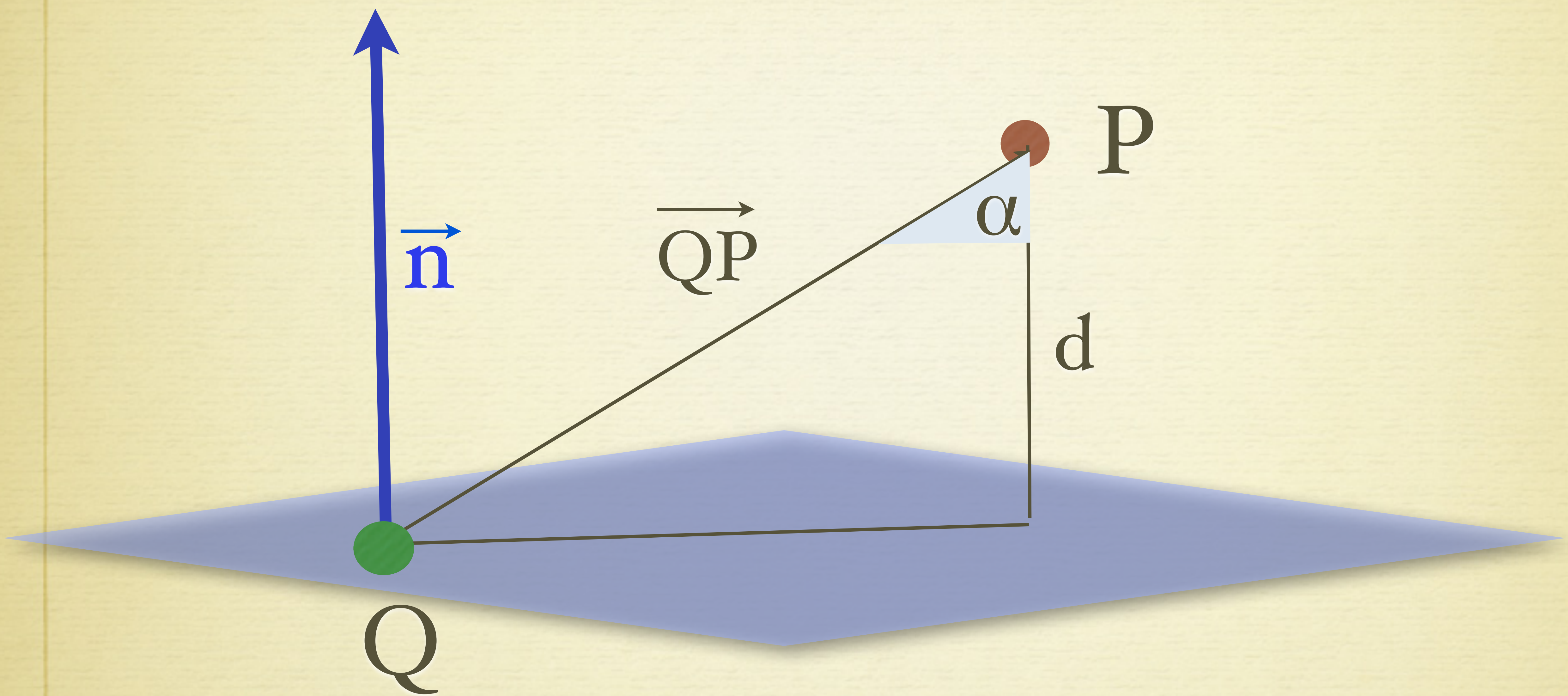
(solve for t)

Point/Line Distance



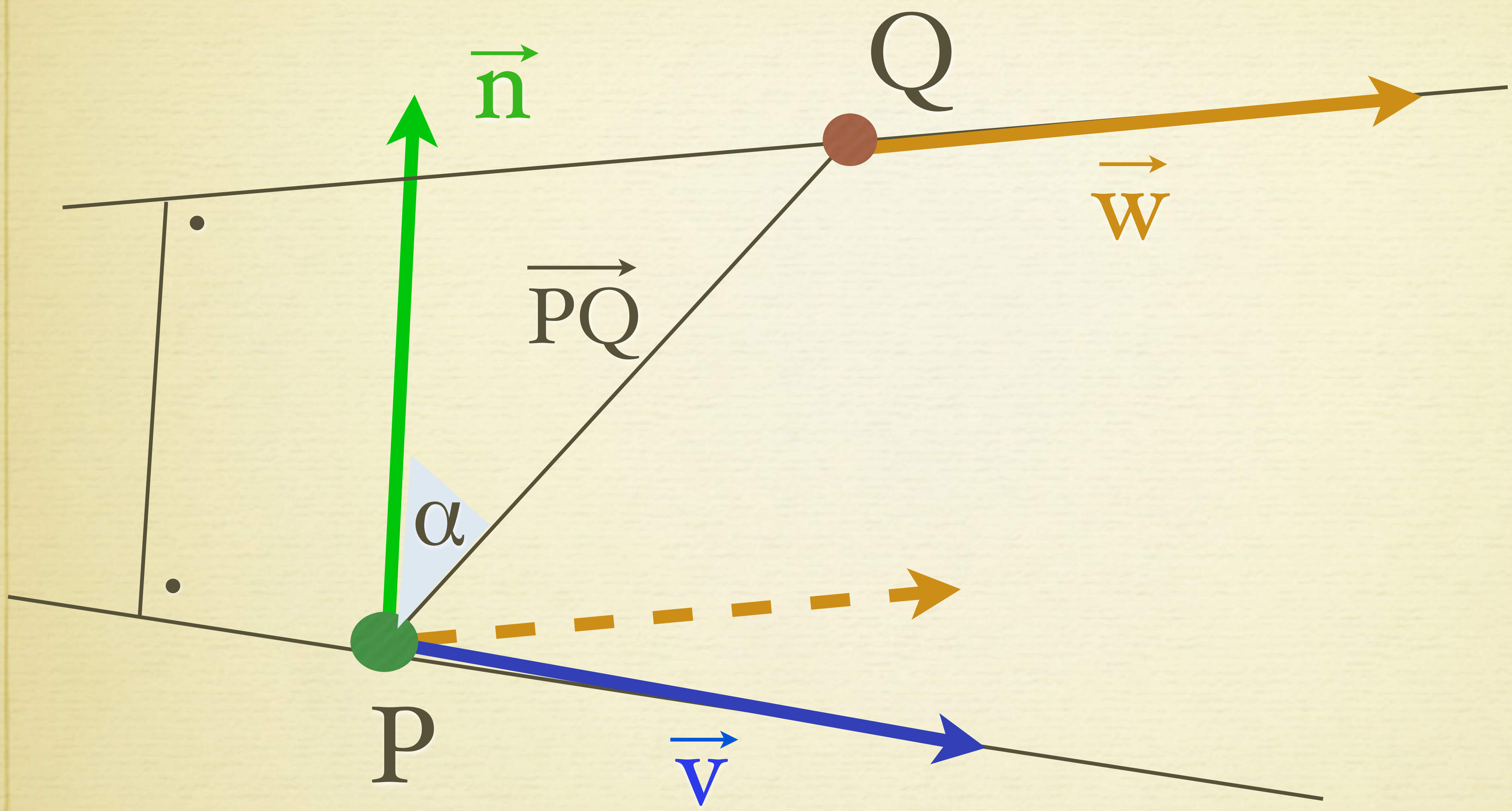
$$\frac{|\overrightarrow{QP}|\sin(\alpha)\ |\vec{v}|}{|\vec{v}|} = \frac{|\overrightarrow{QP} \times \vec{v}\,|}{|\vec{v}|}$$

Point/Plane Distance



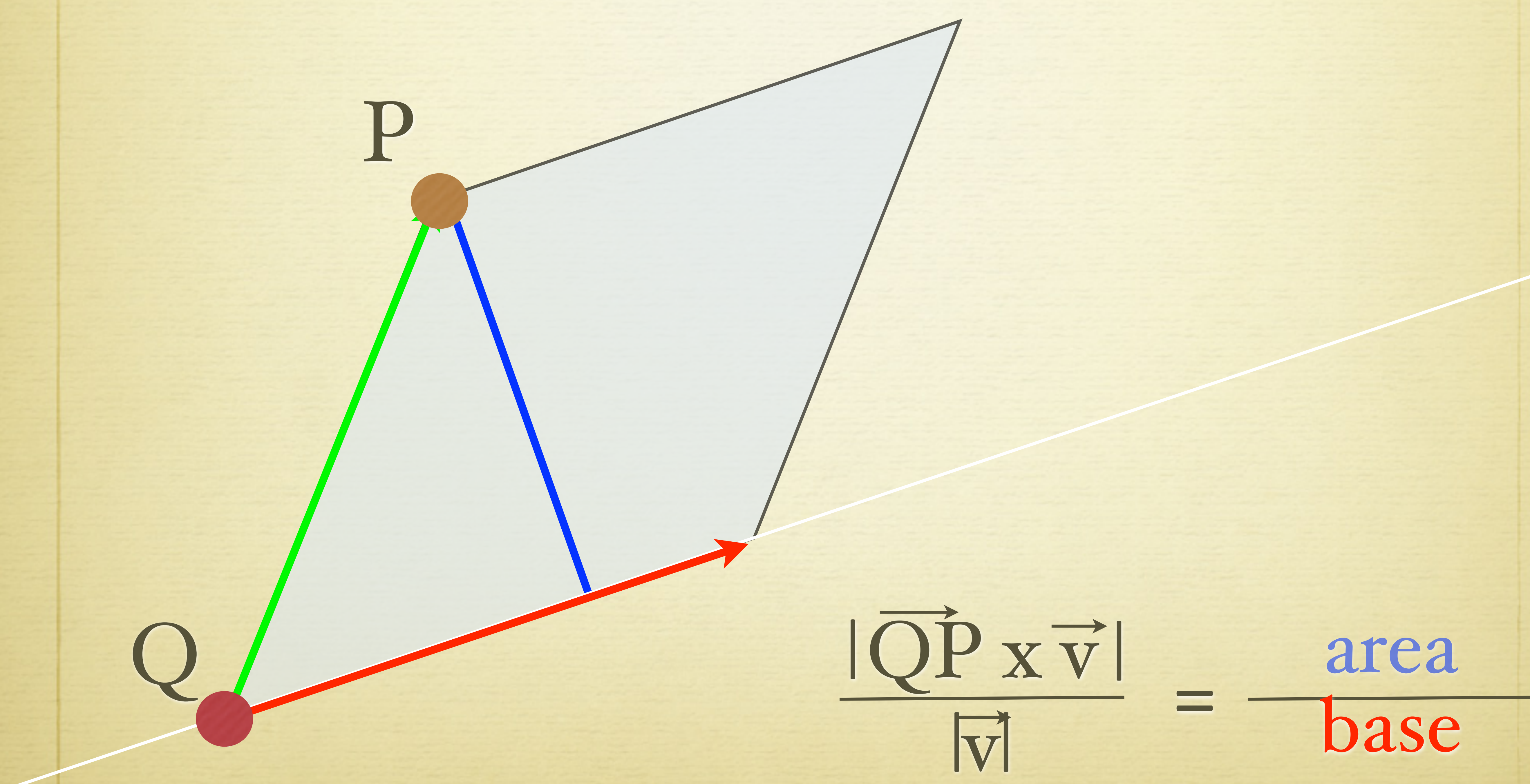
$$d = \frac{|\overrightarrow{PQ}| \cos(\alpha)\, |\vec{n}|}{|\vec{n}|} = \frac{|\overrightarrow{PQ} \cdot \vec{n}\,|}{|\vec{n}|}$$

Line/Line Distance



$$\frac{|\overrightarrow{PQ}|\cos(\alpha)\,|\vec{n}|}{|\vec{n}|} = \frac{|\overrightarrow{QP}\cdot \vec{v}\times\vec{w}|}{|\vec{v}\times\vec{w}|}$$

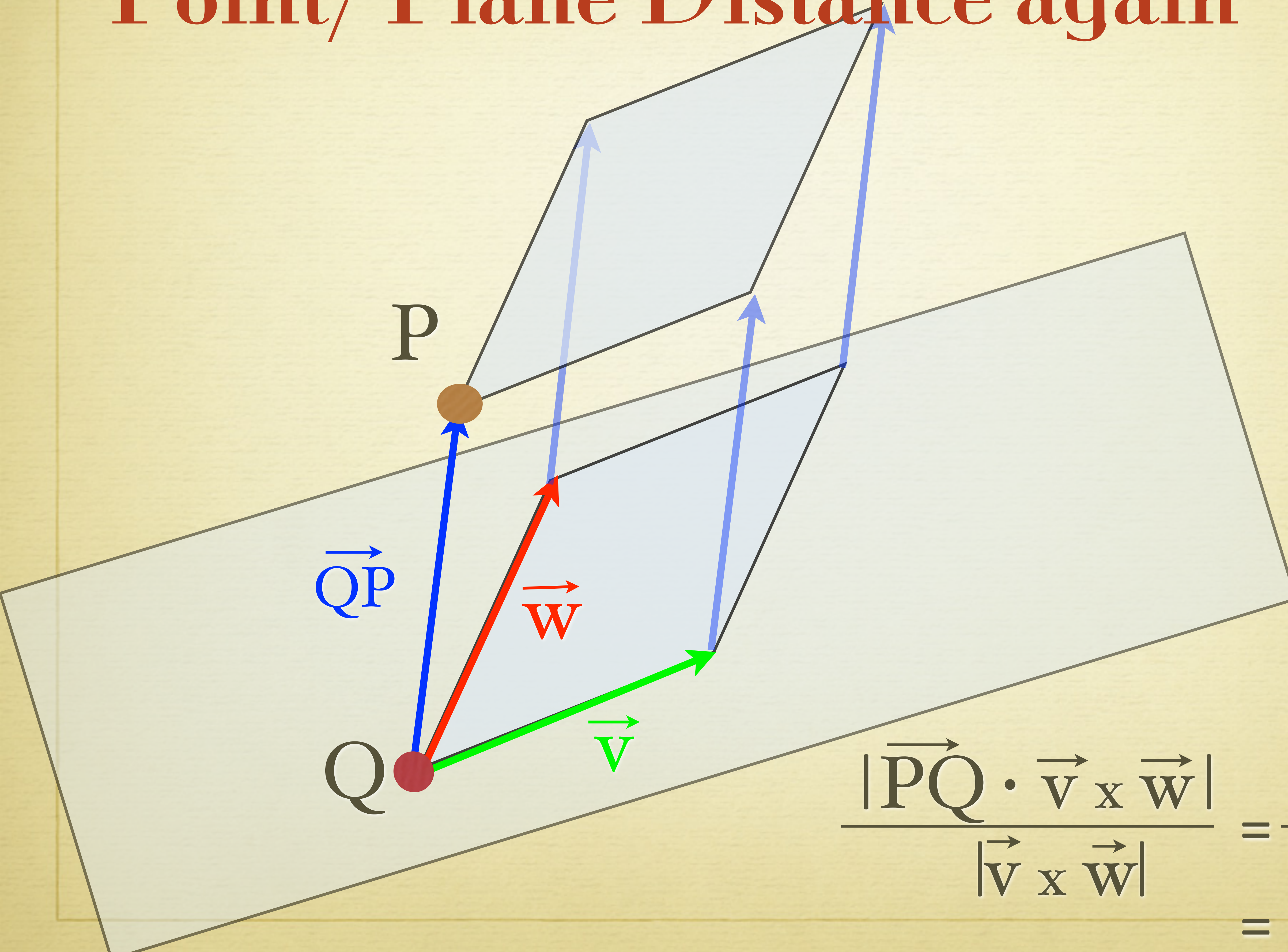
Point/Line Distance again



$$\frac{|\overrightarrow{QP} \times \vec{v}|}{|\vec{v}|} = \frac{\text{area}}{\text{base}}$$

$$= \text{height}$$

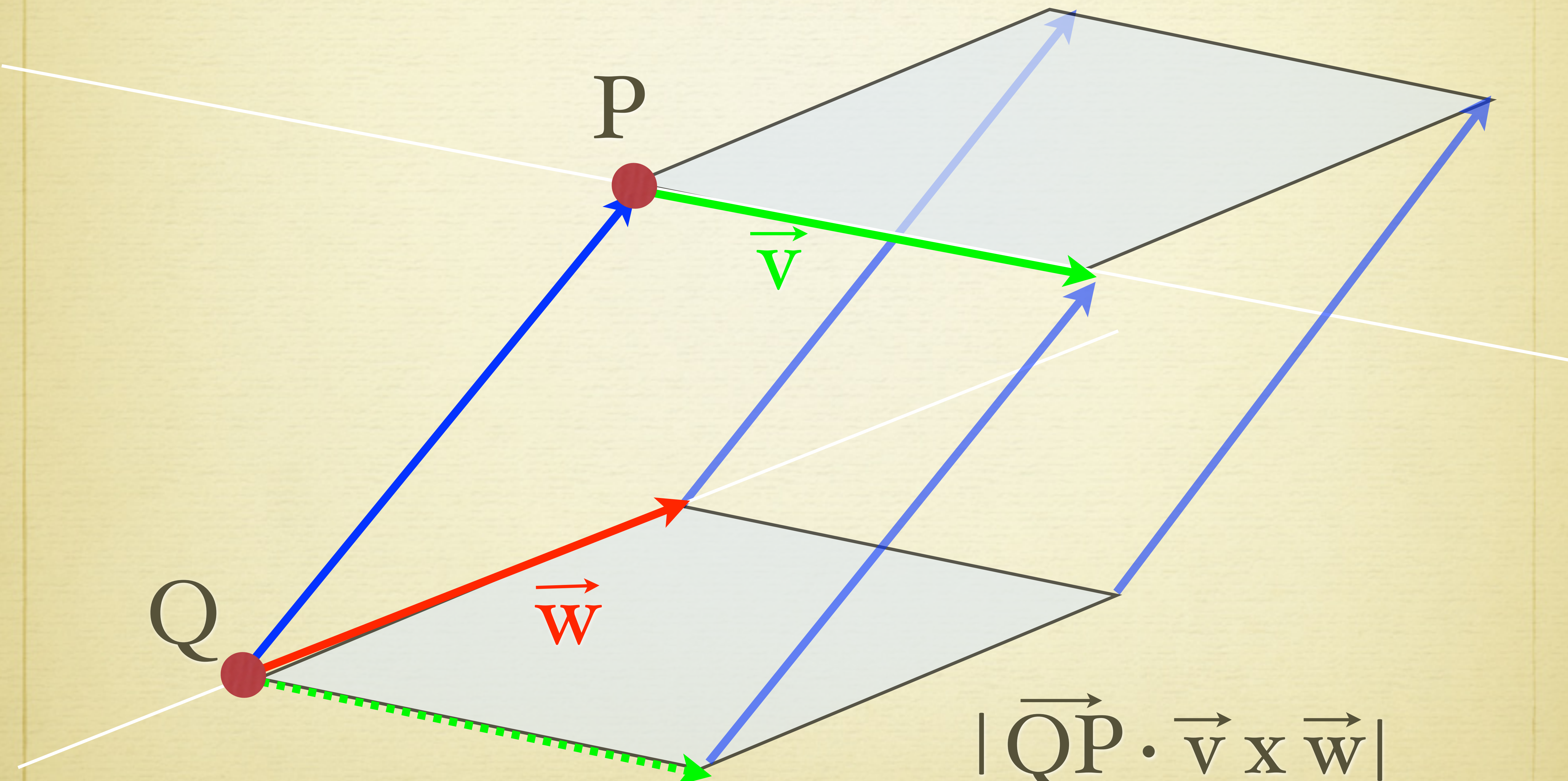
Point/Plane Distance again



$$\frac{|\overrightarrow{PQ} \cdot \vec{v} \times \vec{w}|}{|\vec{v} \times \vec{w}|} = \frac{\text{volume}}{\text{area}}$$

$$= \text{height}$$

Line/Line Distance again



$$\frac{|\overrightarrow{QP} \cdot \vec{v} \times \vec{w}|}{|\vec{v} \times \vec{w}|}$$

$$= \frac{\text{Volume}}{\text{Area}}$$

Variants

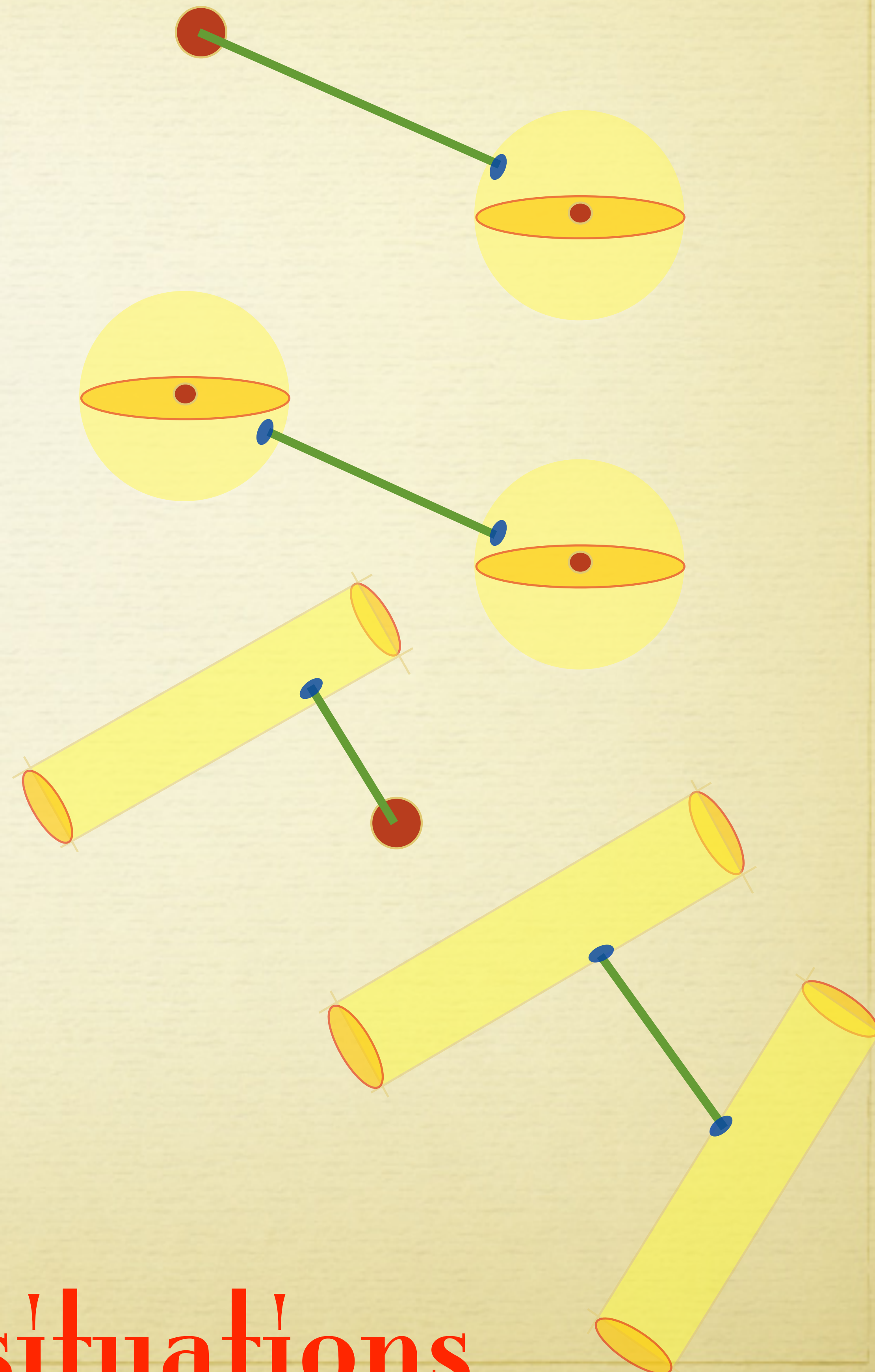
Point Sphere

Sphere Sphere

Point Cylinder

Cylinder Cylinder

etc.



4 additional situations

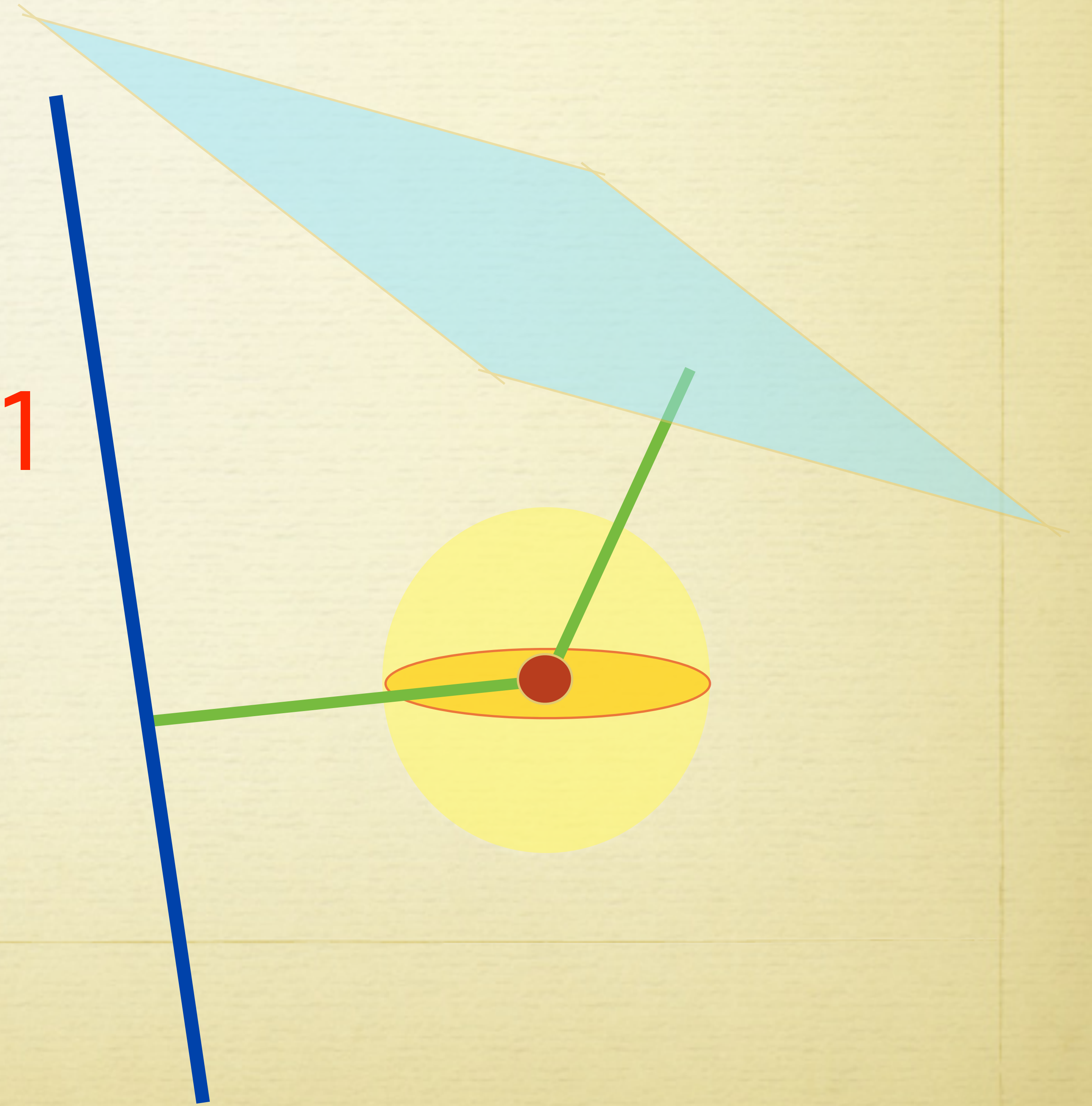
Problem

FIND ALL THE POSSIBLE DISTANCES

$$(x-1)^2 + (y-2)^2 + (z+3)^2 = 1$$

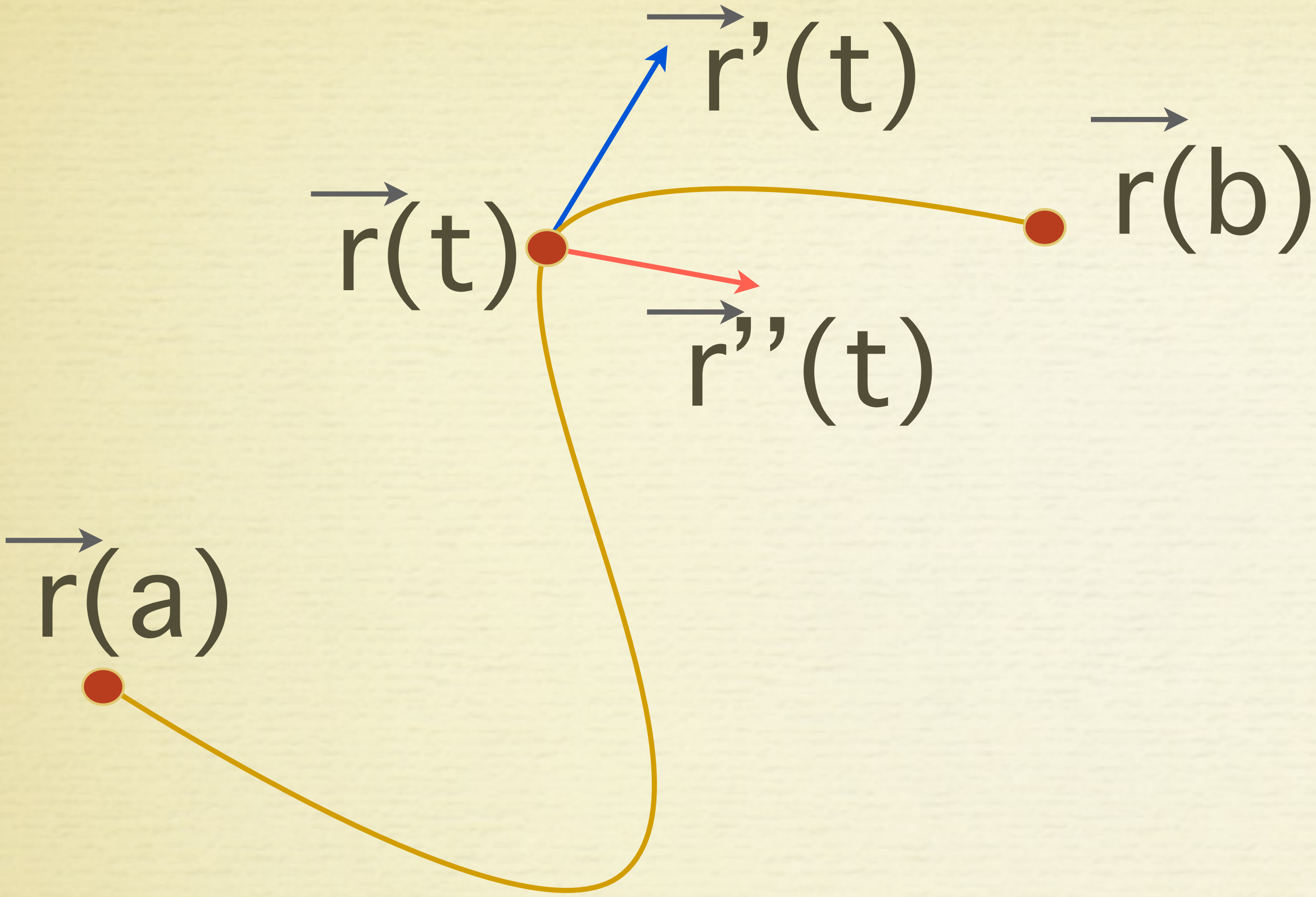
$$x-y+z=3$$

$$x-1=y-3=z-5$$



4 Concepts on Curves

Velocity and Acceleration



Arc length

$$L = \int_a^b |\vec{r}\,'(t)|\, dt$$

TNB

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{r}\,'(t)|}$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

curvature

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

$$\vec{r}(t) = \vec{r}(0) + t\vec{r}\,'(0) + \vec{R}(t)$$

$$\vec{v}(t) = \int_0^t \vec{r}\,''(s)\,ds, \vec{R}(t) = \int_0^t \vec{v}(s)\,ds$$

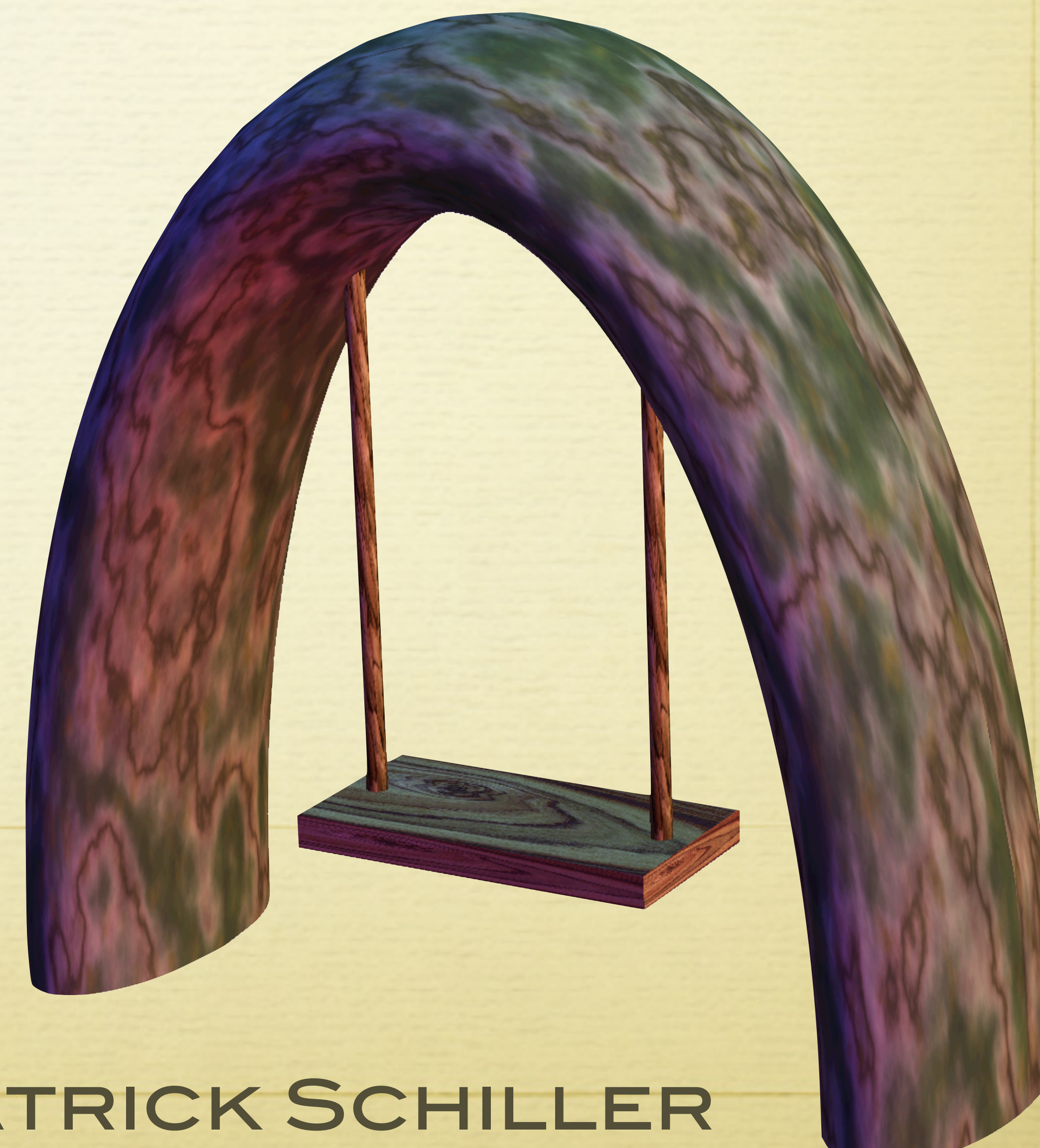
Integration

Your initial velocity is $<1,0,0>$ at $t=0$.

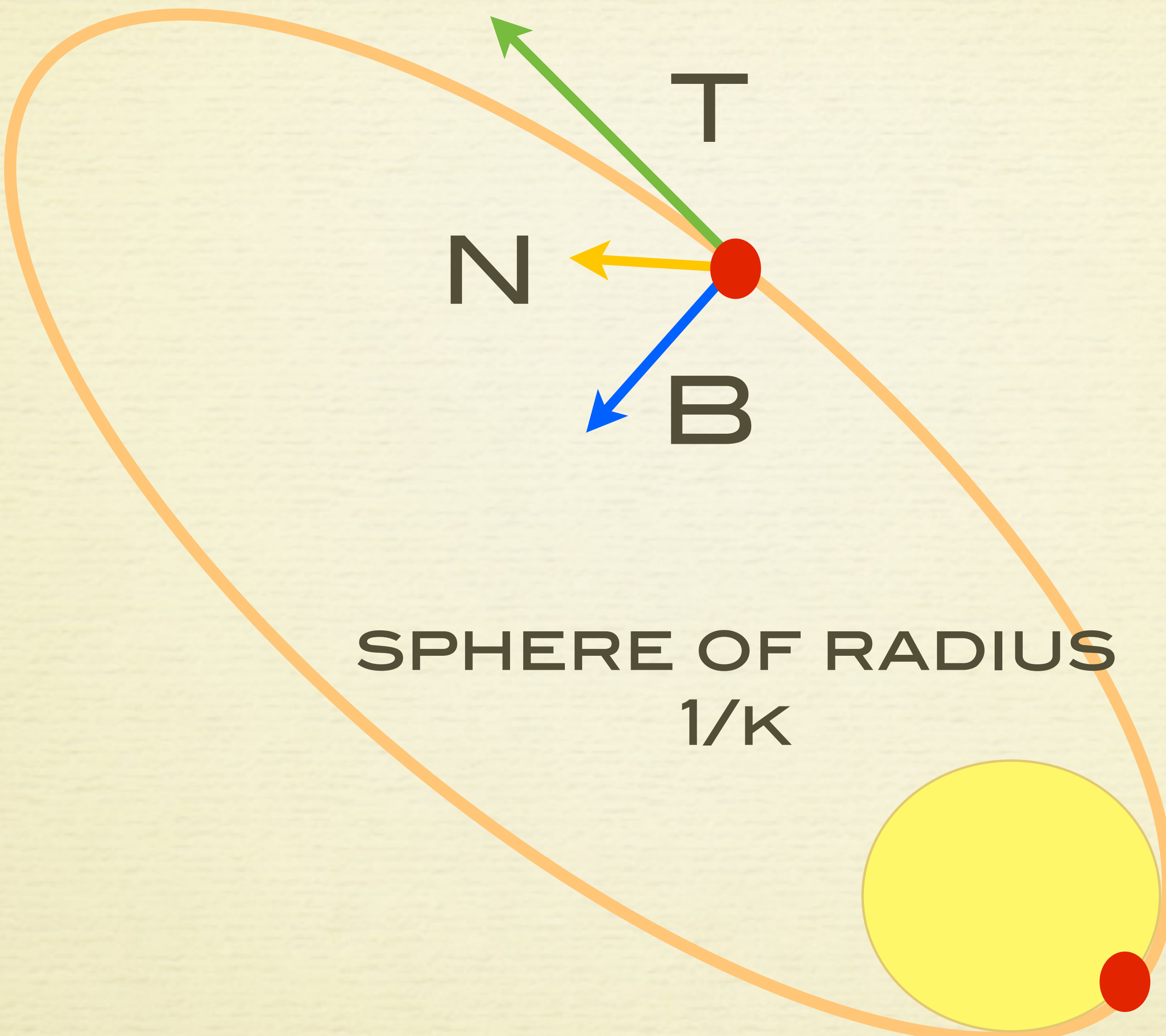
You start at $(2,3,5)$ at $t=0$.

Where are you at time $t=1$?

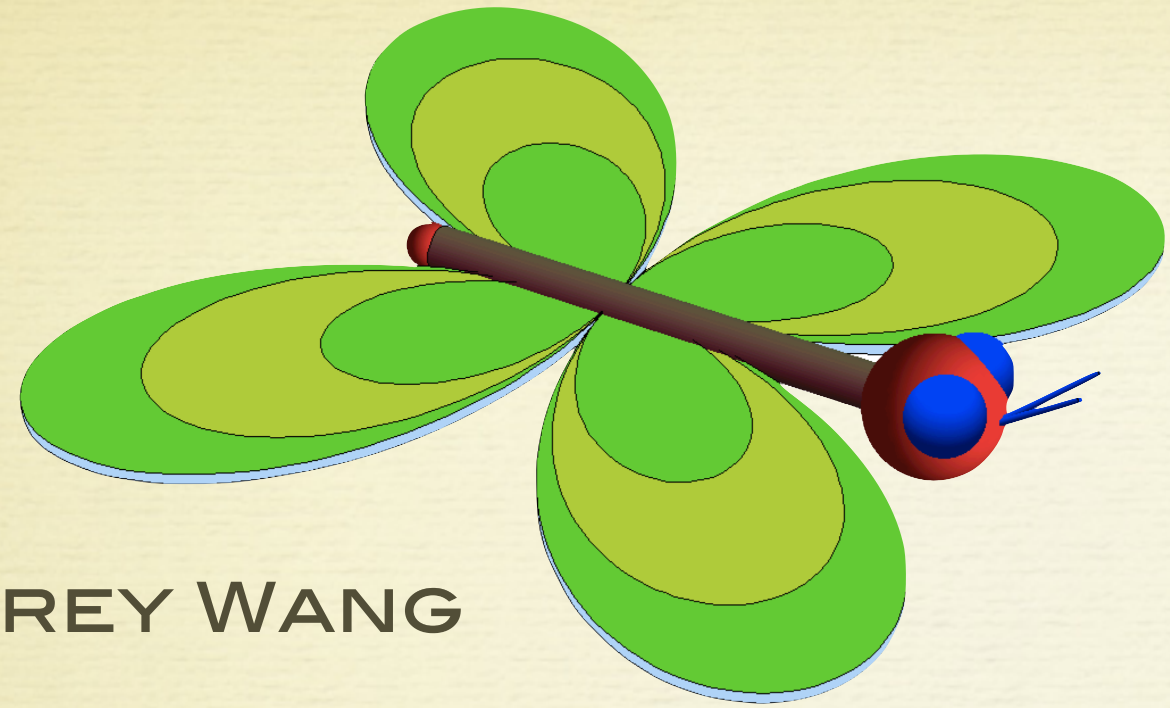
$$\vec{r}''(t) = <0, \sin(t), 0>$$



Patrick Schiller



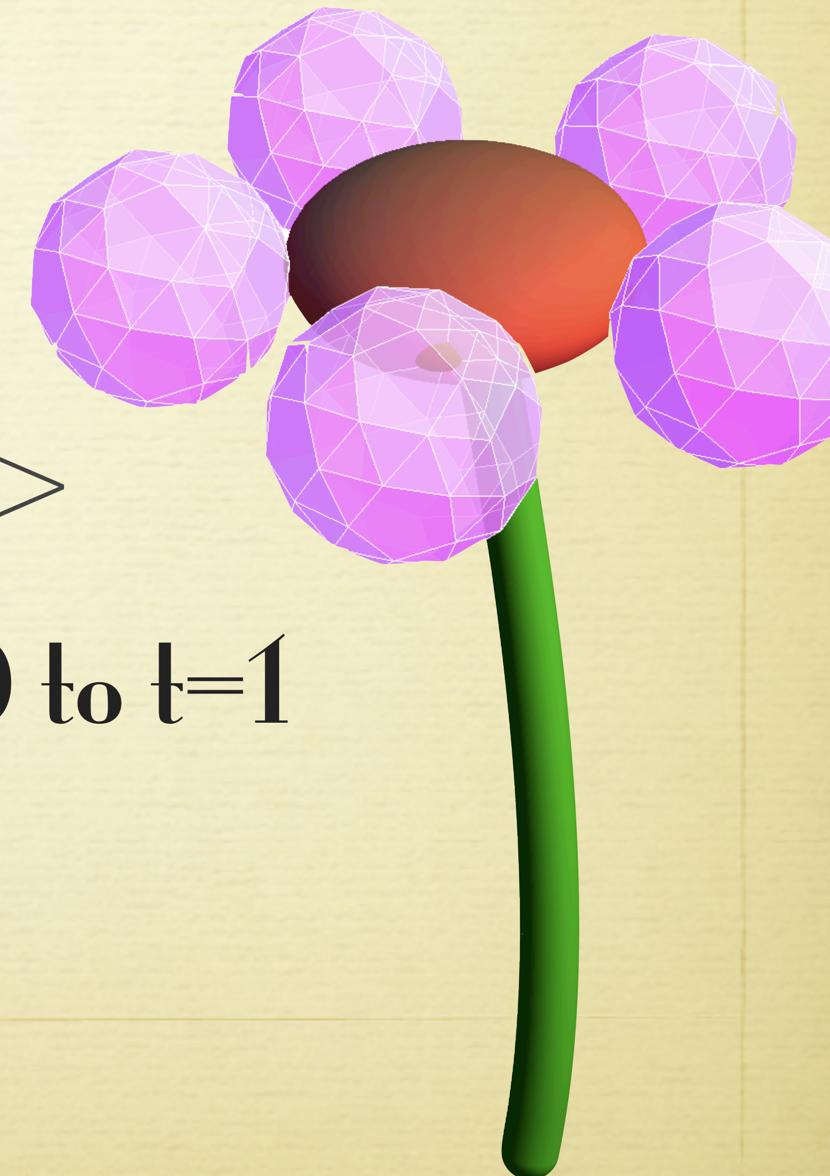
TBN and Curvature



JEFFREY WANG

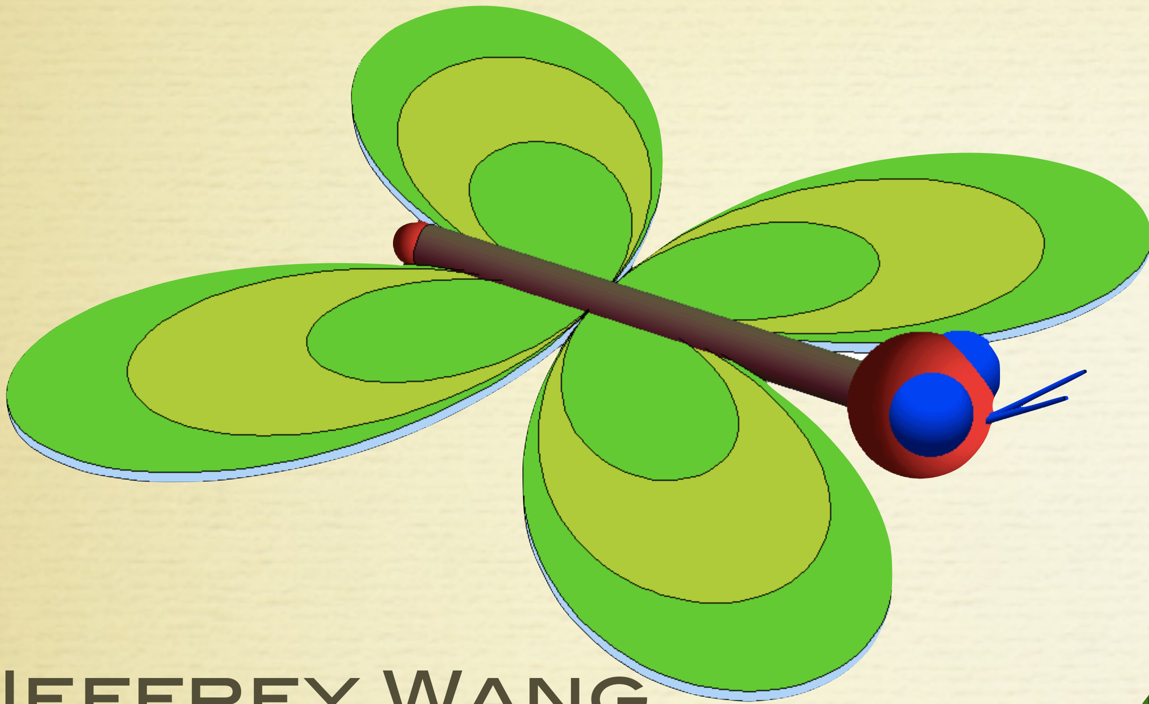
$$\vec{r}(t) = < e^{t}, \sqrt{2}t, e^{-t} >$$

Find the arc length from t=0 to t=1



Problem

ALEXANDRA ROJEK

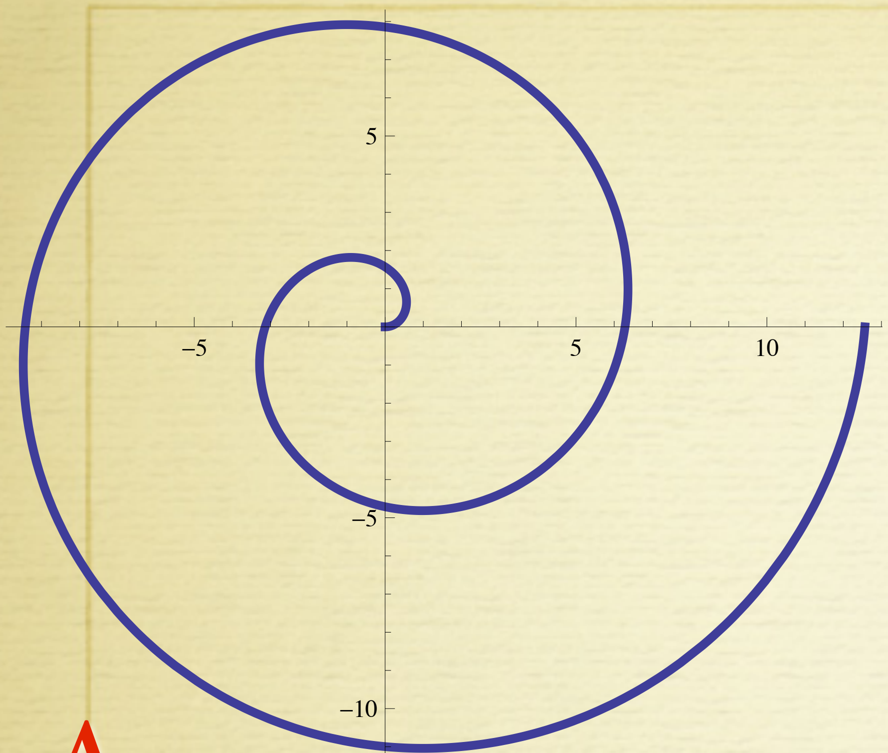


Jeffrey Wang

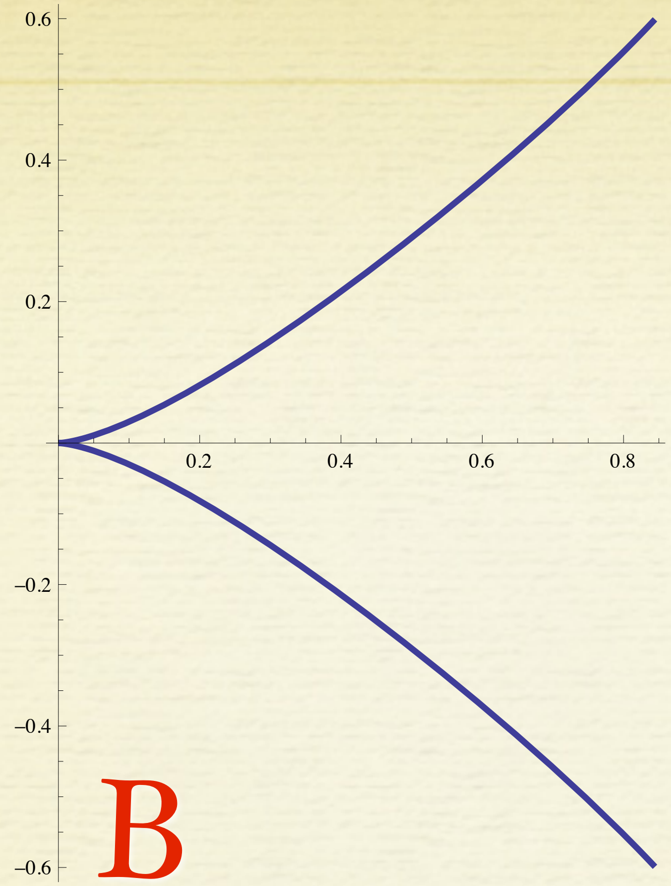
Charlie Wolock

No Happy End

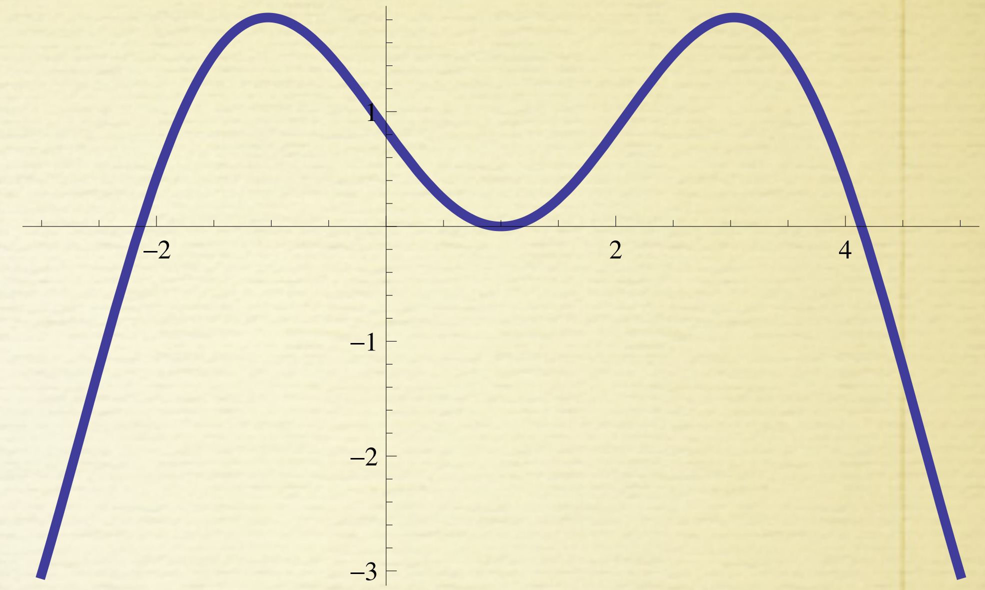
Matching Curves



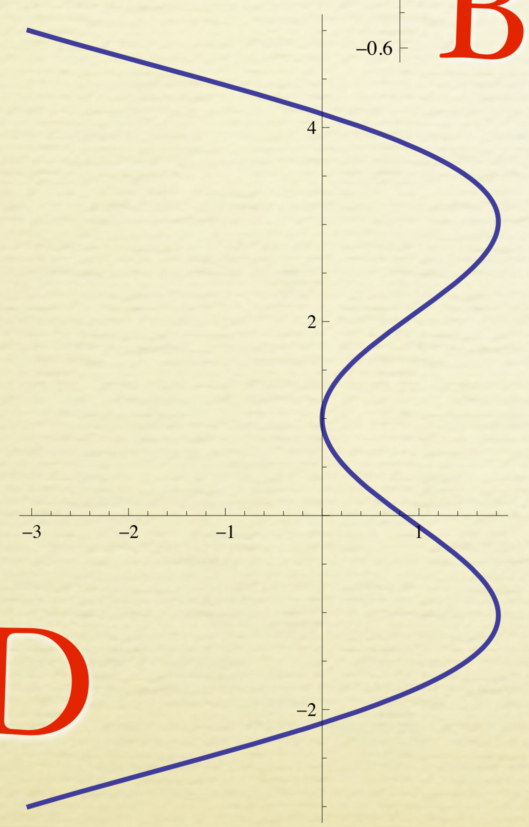
A



B



C



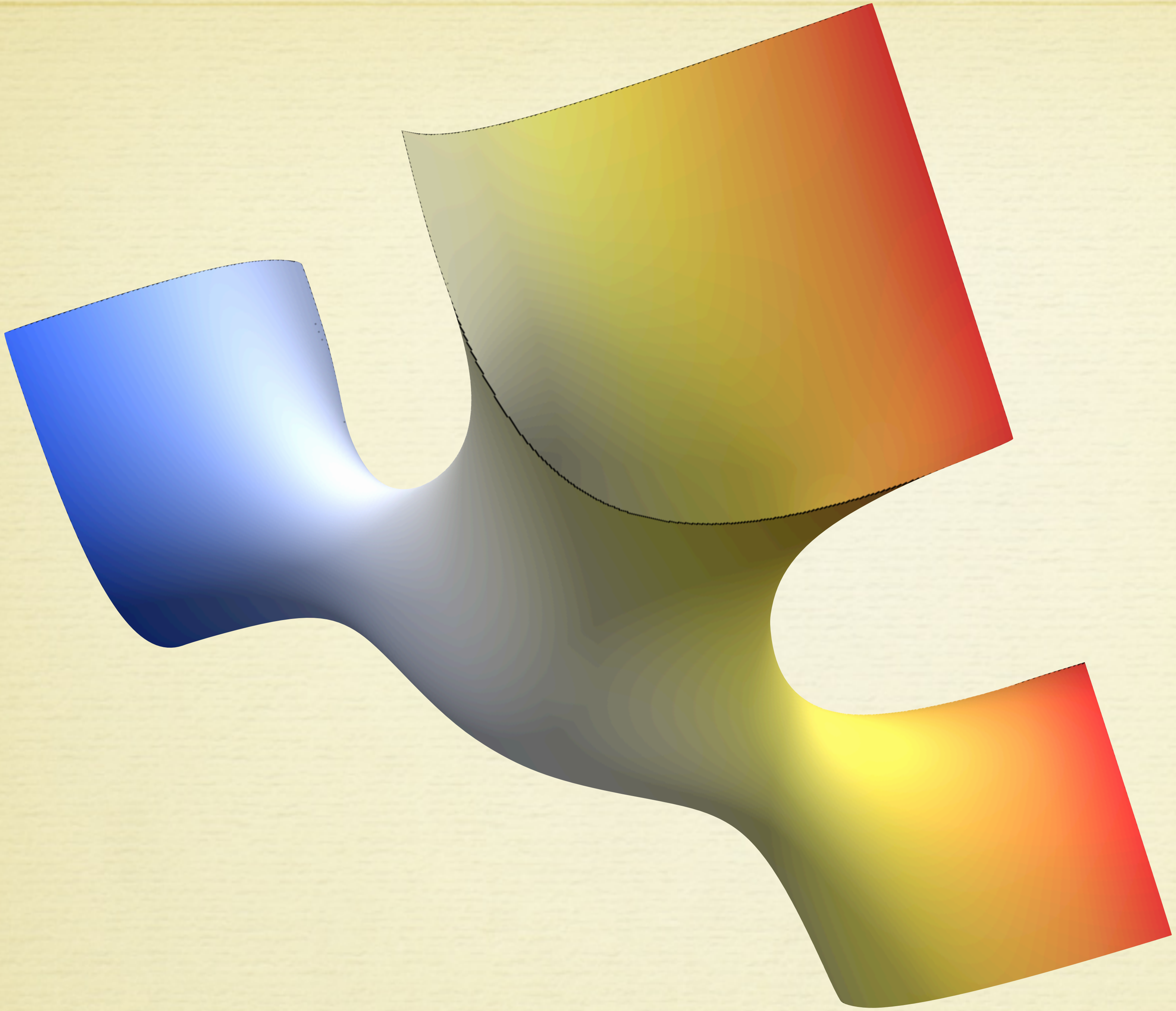
D

$r(t) = < t+1, t\sin(t) >$

$r(t) = < t\cos(t), t\sin(t) >$

$r(t) = < \sin(t^2), \sin(t^3) >$

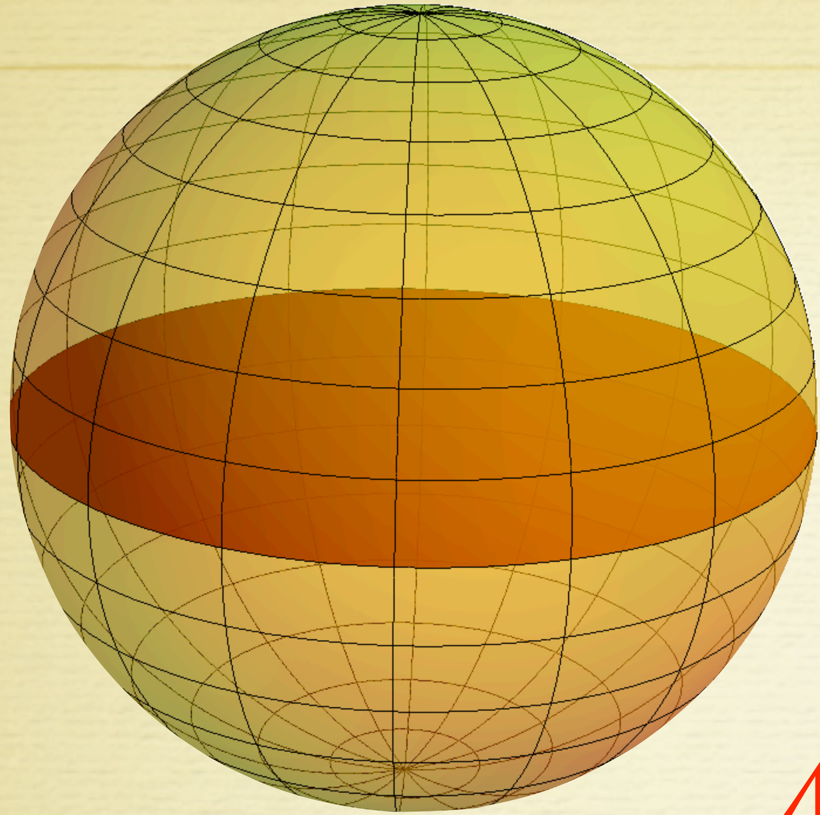
$r(t) = < t\sin(t), t+1 >$



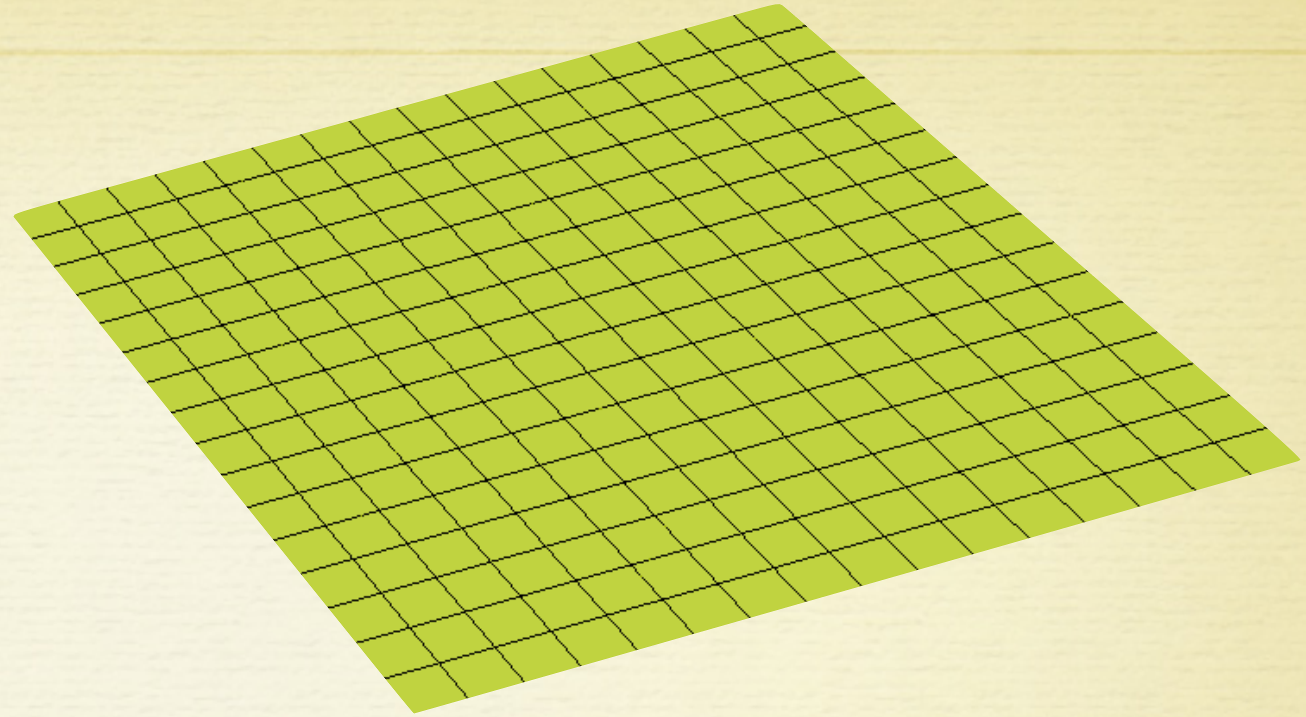
$$x^2+y^2+z^2+xyz+yz+xy+xz+y=0$$

Implicit Surface

4 cases

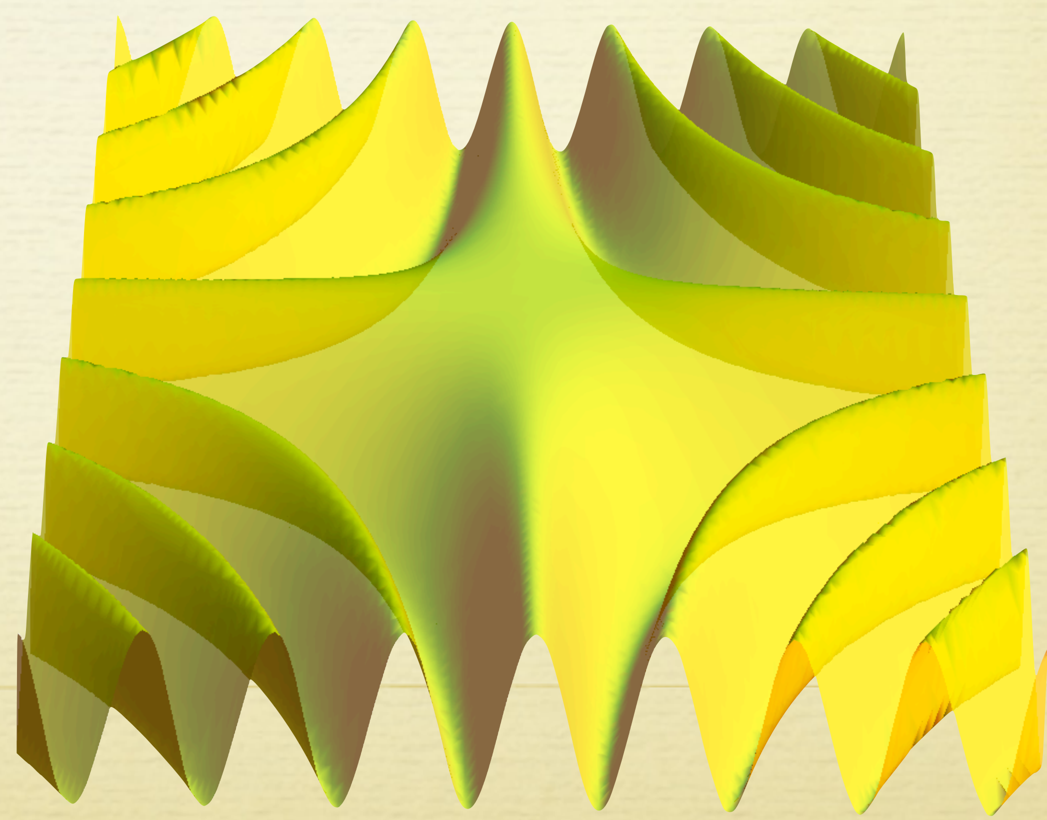


Spheres

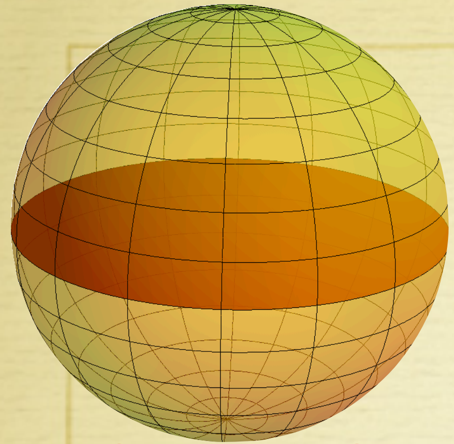


Planes

Surface of revolution

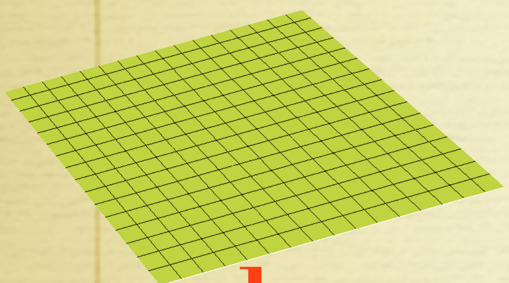


Graphs



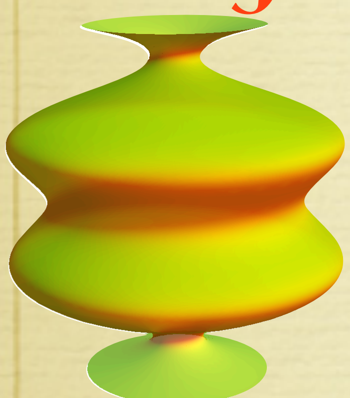
$$\vec{r}(t,s) = < \sin(s)\cos(t), \sin(s)\sin(t), \cos(s) >$$

$$x^2 + y^2 + z^2 = 1$$



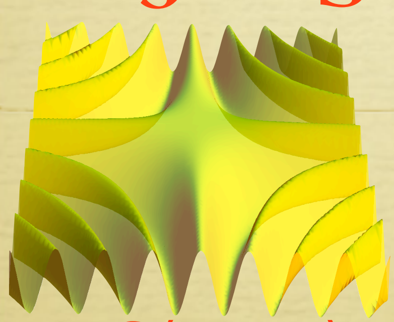
$$\vec{r}(t,s) = < x + v_1 s + w_1 t, y + v_2 s + w_2 t, z + v_3 s + w_3 t >$$

$$ax + by + cz = d$$



$$\vec{r}(t,s) = < r(z)\cos(t), r(z)\sin(t), z >$$

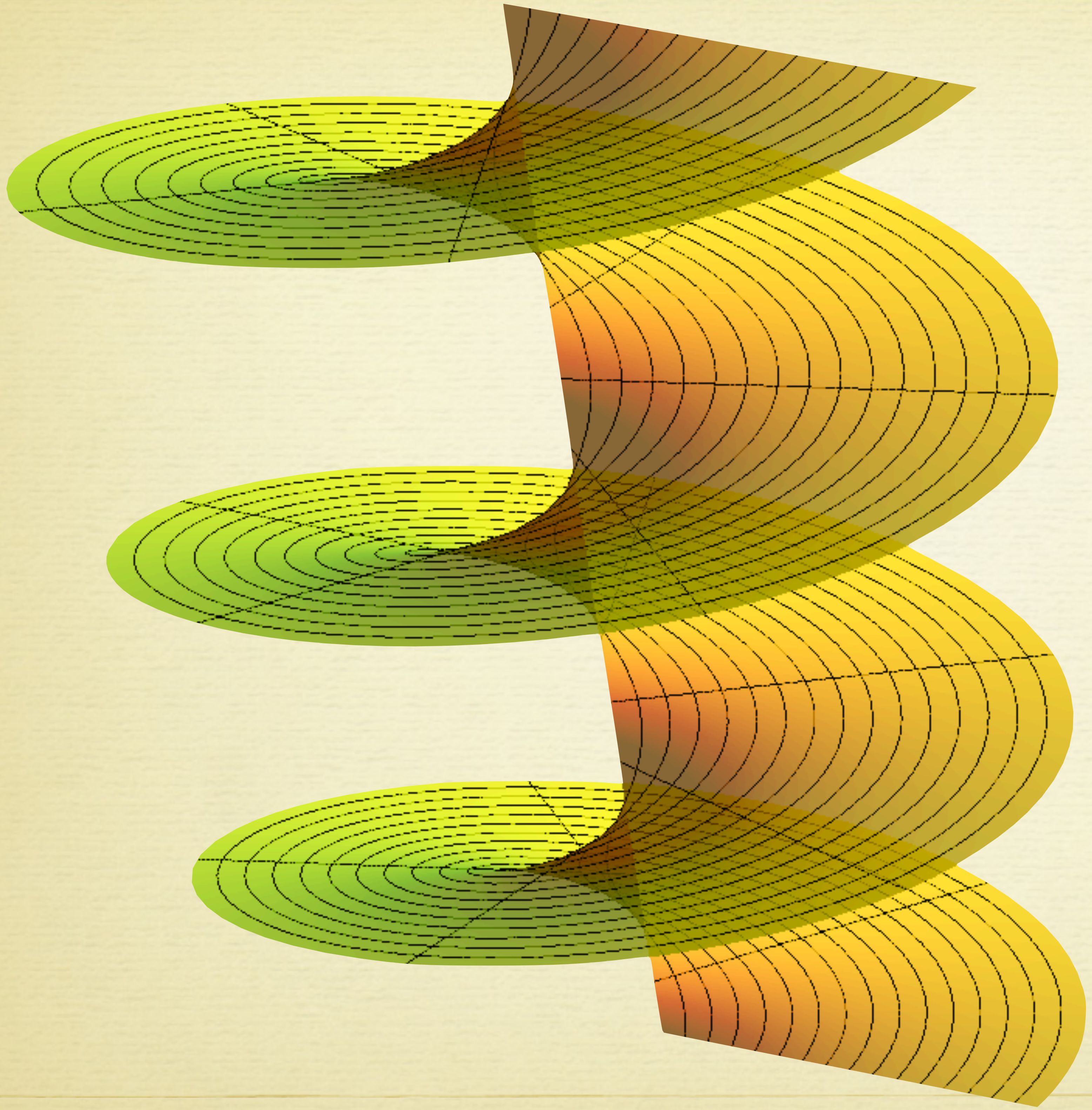
$$x^2 + y^2 = g(z)$$



$$\vec{r}(t,s) = < s, t, f(s,t) >$$

4 Surface types

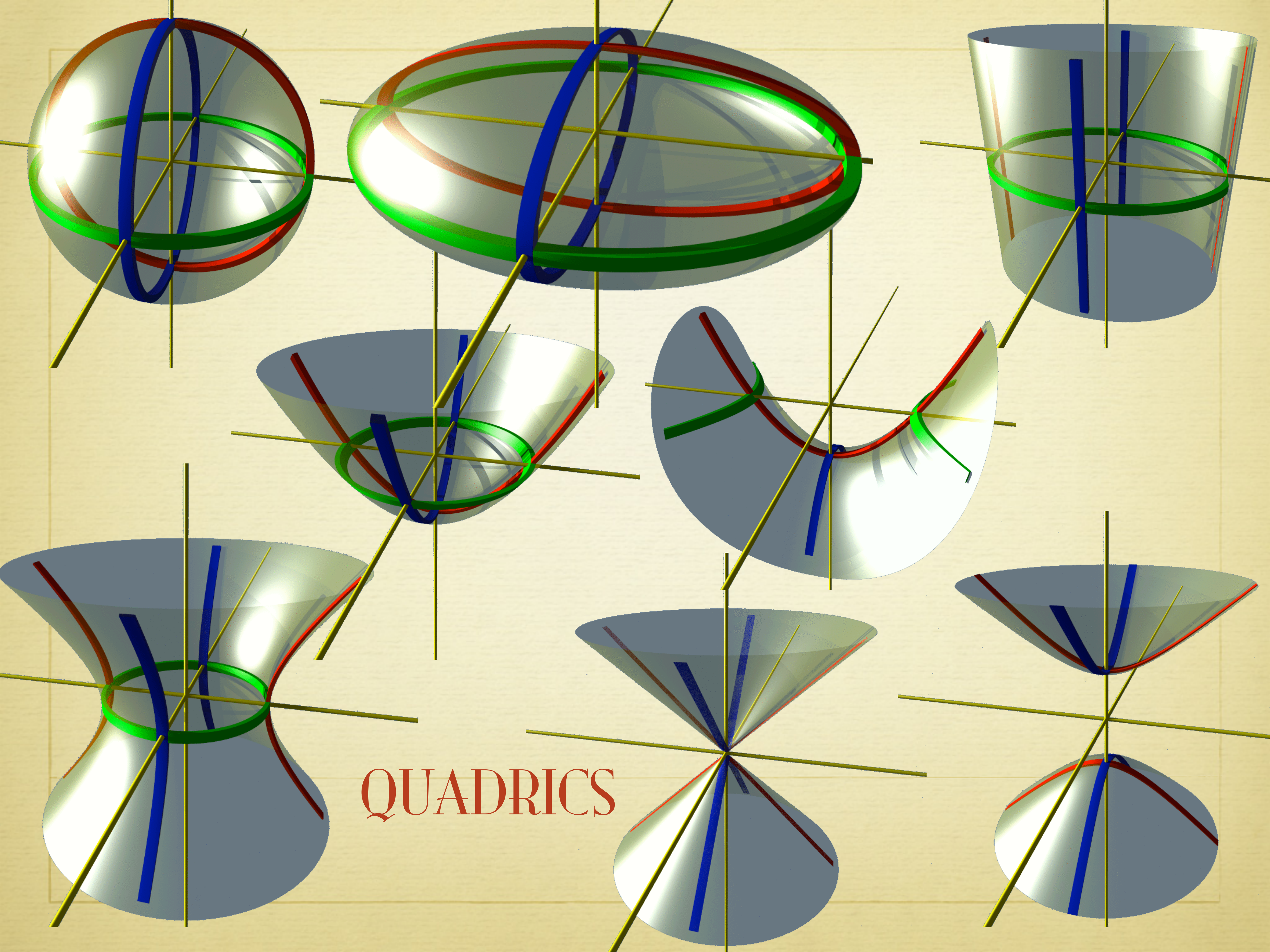
$$z = f(x,y)$$

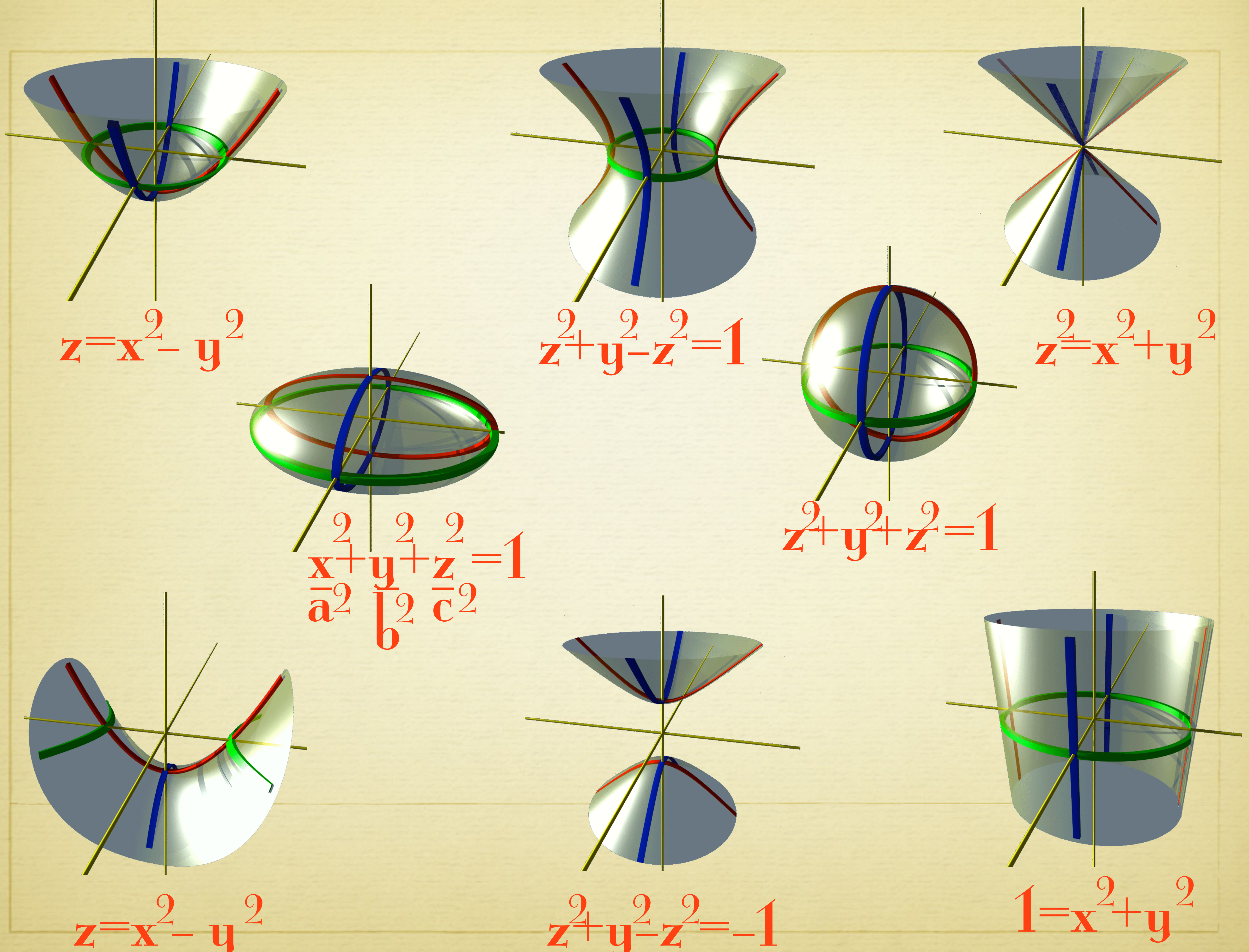


Grid Curves are Lines or Helix Curves

$$\vec{r}(t,s) = \langle s\cos(t), s\sin(t), t \rangle$$

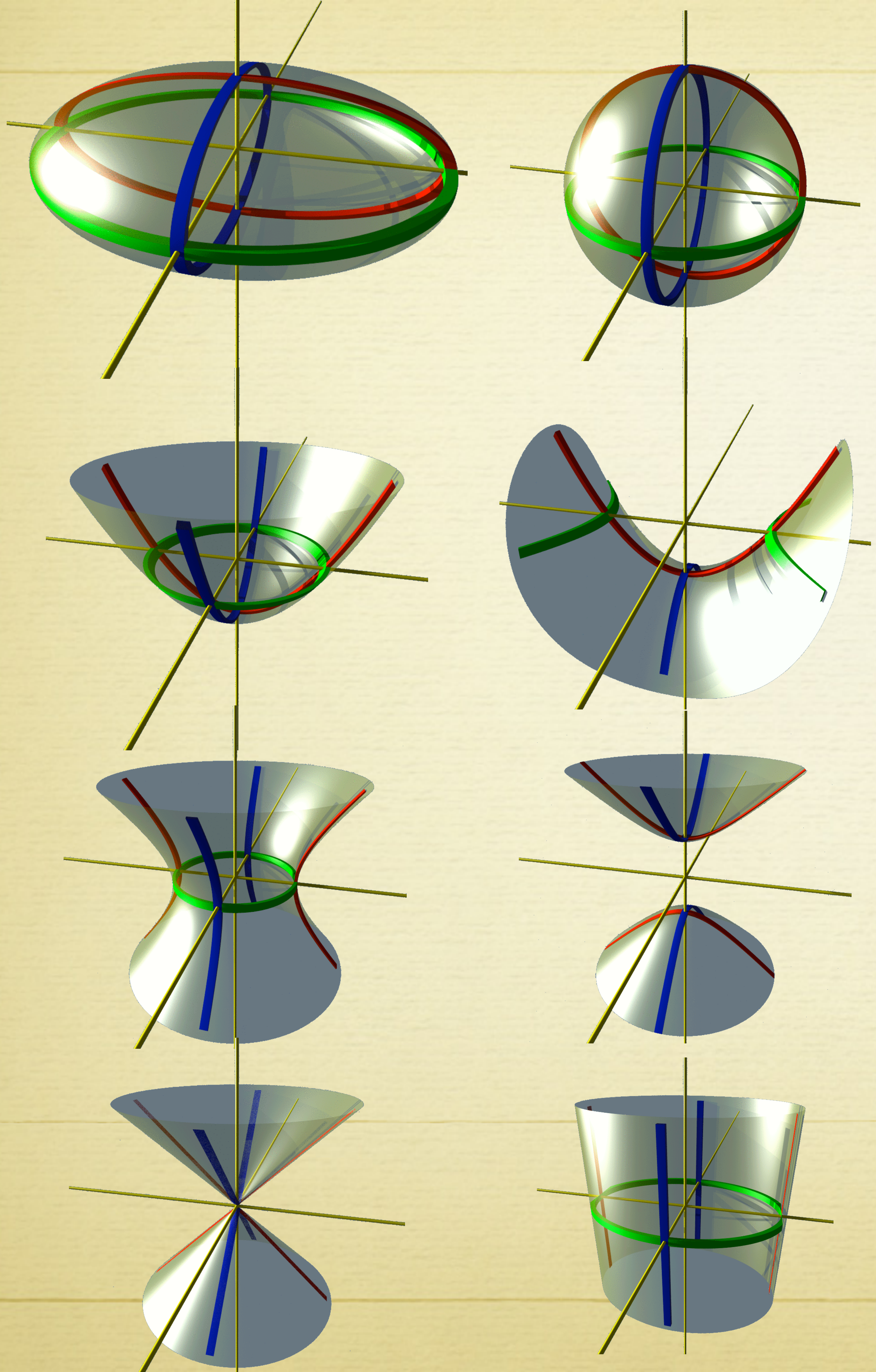
QUADRICS





$z=x^2-y^2$
$z^2+y^2-z^2=1$
$z^2=x^2+y^2$
$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$
$z^2+y^2+z^2=1$
$z=x^2-y^2$
$z^2+y^2-z^2=-1$
$1=x^2+y^2$

4 types



Ellipsoids

Paraboloids

Hyperboloids

Specials

$$\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$$

Gradient

$$D_{\vec{v}} f(x,y,z) = \langle f_x, f_y, f_z \rangle \cdot \vec{v}$$

Directional Derivative

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(x,y,z) \cdot \vec{r}'(t)$$

Chain Rule

$\nabla f(x,y,z) = \langle a,b,c \rangle$ normal vector

$ax + by + cz = d$ tangent plane

$L(x,y,z) = f(x_o, y_o, z_o) = a(x-x_o) + b(y-y_o) + c(z-z_o)$ linearization

Estimate $f(x,y,z)$ by $L(x,y,z)$ near (x_o, y_o, z_o) estimation

4 concepts are related

Problem

a) Estimate $0.999 * 3.0004^2$

b) Find the tangent plane of

$f(x,y) = x y^2 - z = 0$ at $(1,3,9)$

ESTIMATE $0.999 * 3.0004^2$

$f(x,y) = x y^2$ $\nabla f(x,y) = \langle y^2, 2xy \rangle$

$\nabla f(1,3) = \langle 9,6 \rangle$ $f(1,3) = 9$

$L(0.999, 3.004) = 9 - 9 * 0.001 + 6 * 0.0004$

$= 8.9934$

ACTUAL: 8.99339775

$u_t(t,x) = u_x(t,x)$ transport

$u_t(t,x) = u_{xx}(t,x)$ heat

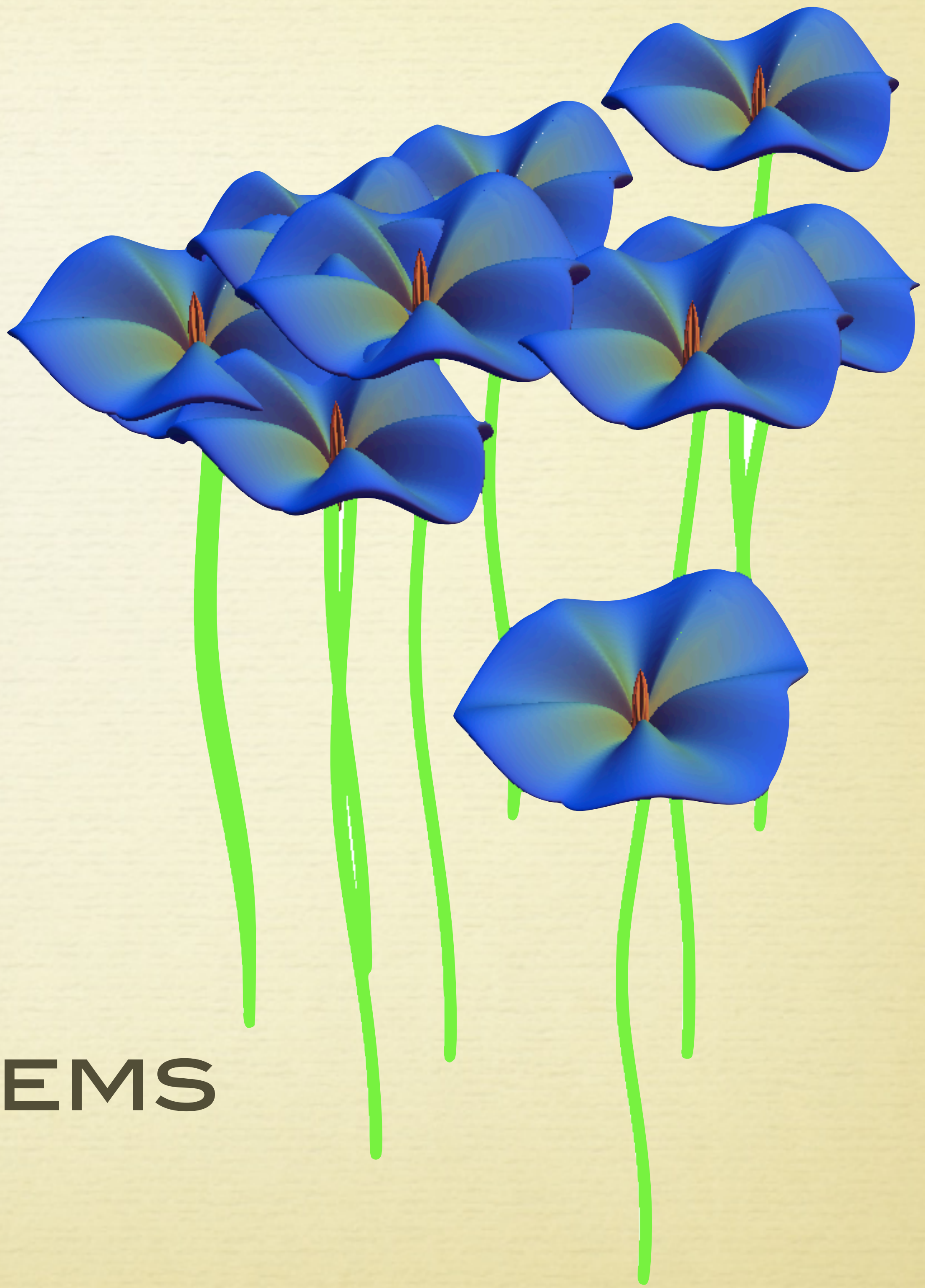
4 PDE's

$u_{tt}(t,x) = u_{xx}(t,x)$ wave

$u_{xx} + u_{xx} = 0$ Laplace

$u_{tx}(t,x) = u_{xt}(t,x)$ Clairot

Extrema



- Local Extrema
- Lagrange Problems
- Global Extrema

Jenna McGugan

$$\nabla f(x,y)=\langle 0,0\rangle$$

$$D=f_{xx} f_{yy} - f_{xy}^2$$

$D>0$	$f_{xx} < 0$	maximum
$D>0$	$f_{xx} > 0$	minimum
$D<0$		saddle point

Second Derivative Test

Problem:

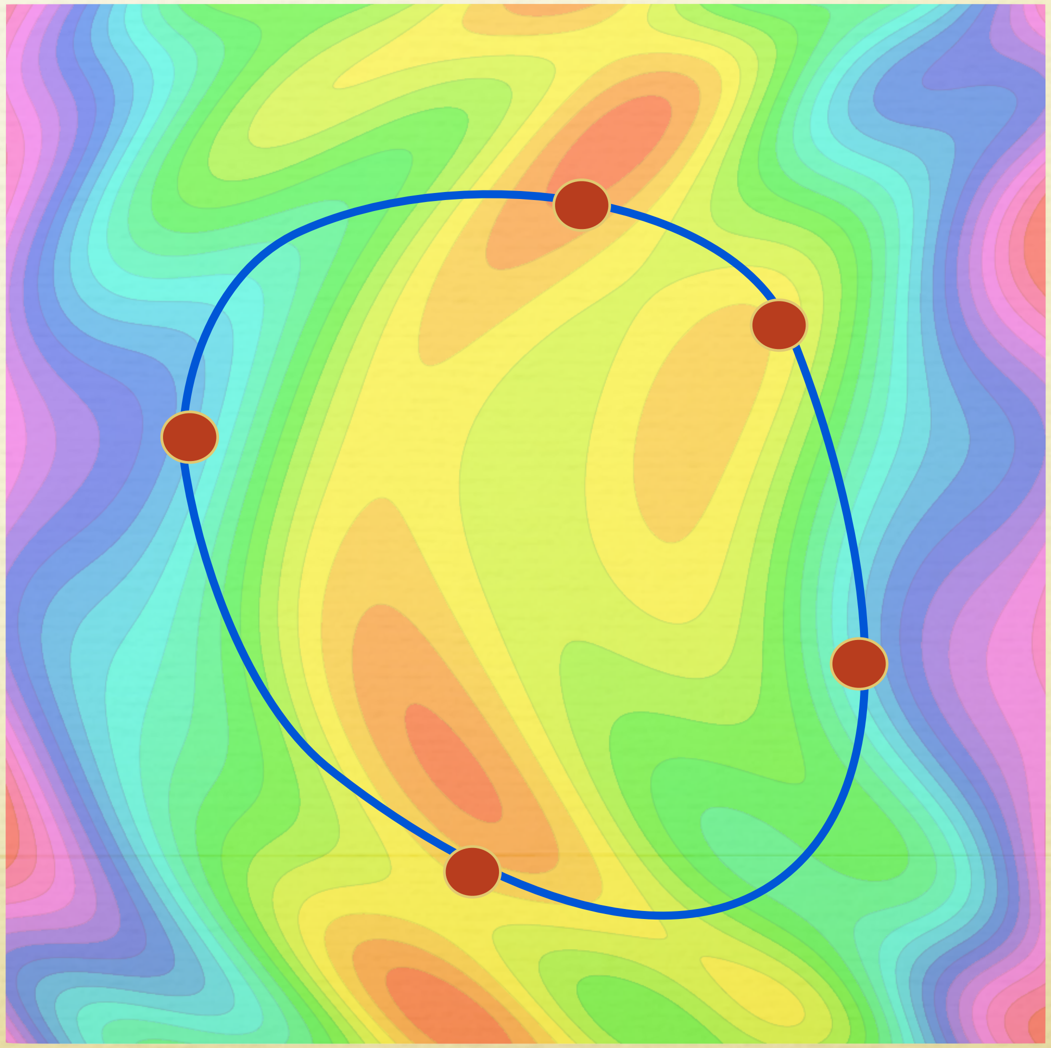
Find the extrema of the function

$$f(x,y) = x^2 y + 3y^3 - 6xy$$

and classify them.

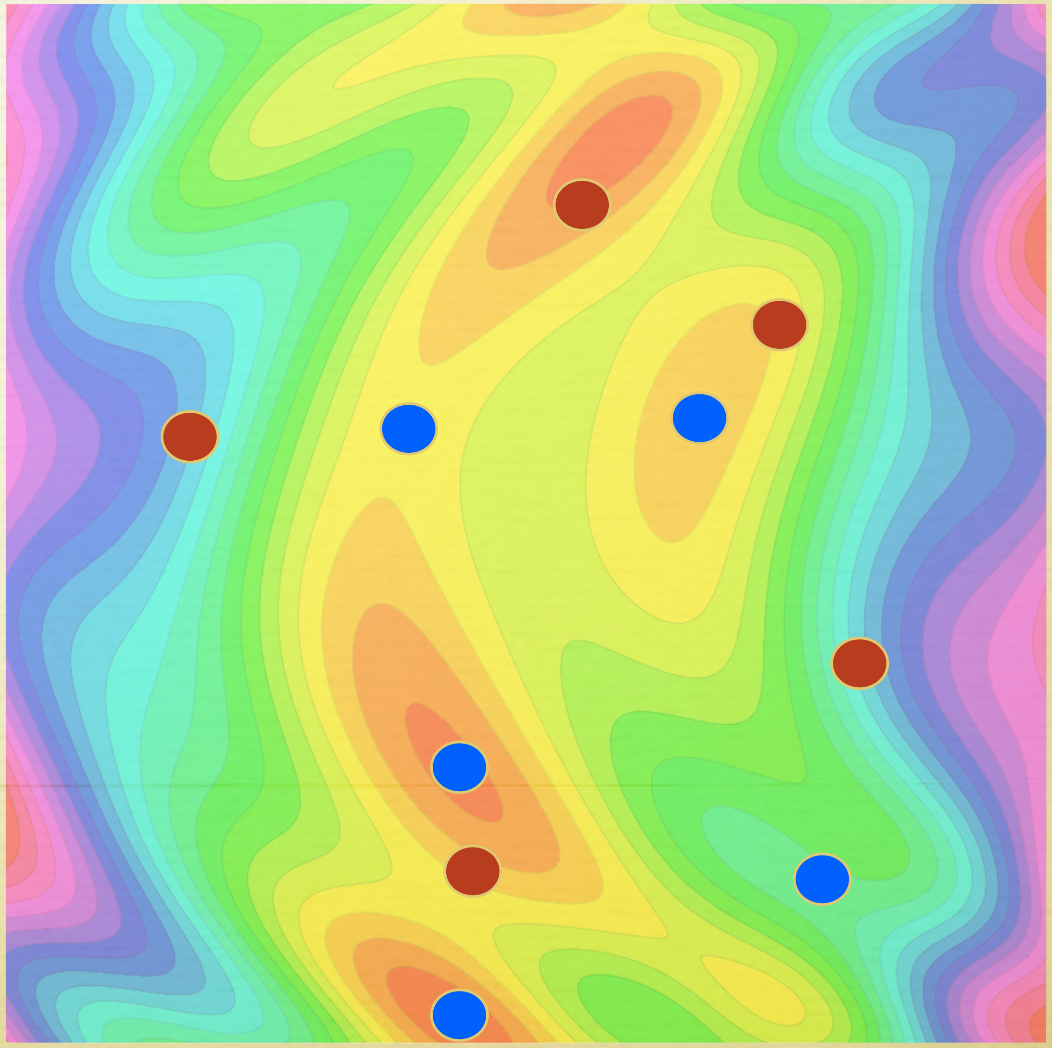
(0,0) saddle
(6,0) saddle
(3,1) minimum
(3,-1) maximum

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$
$$g(x,y) = c$$

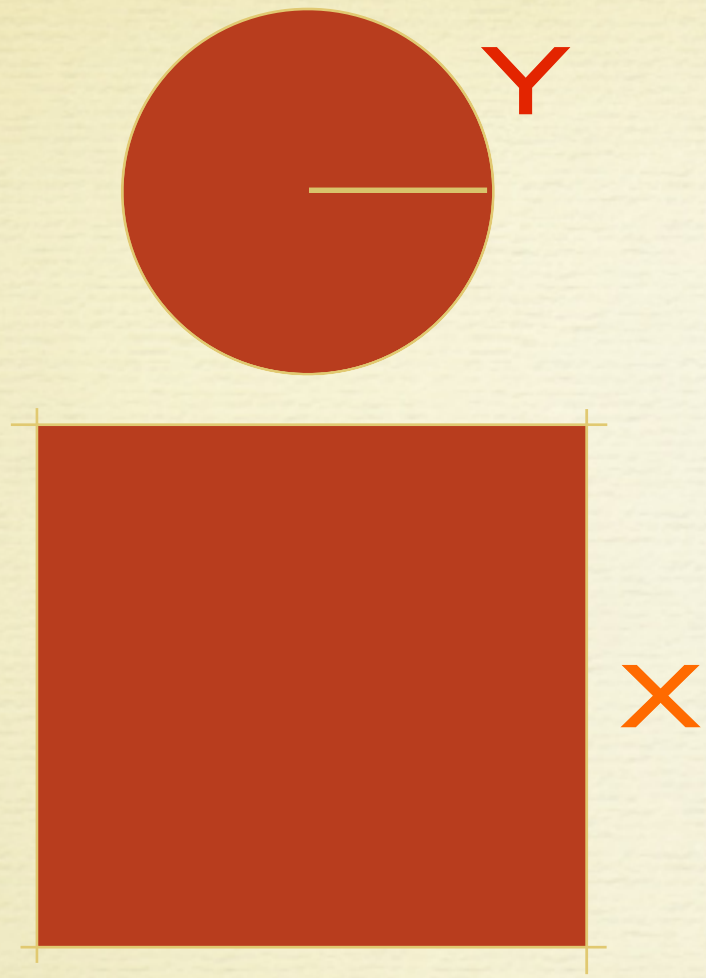


Lagrange equations

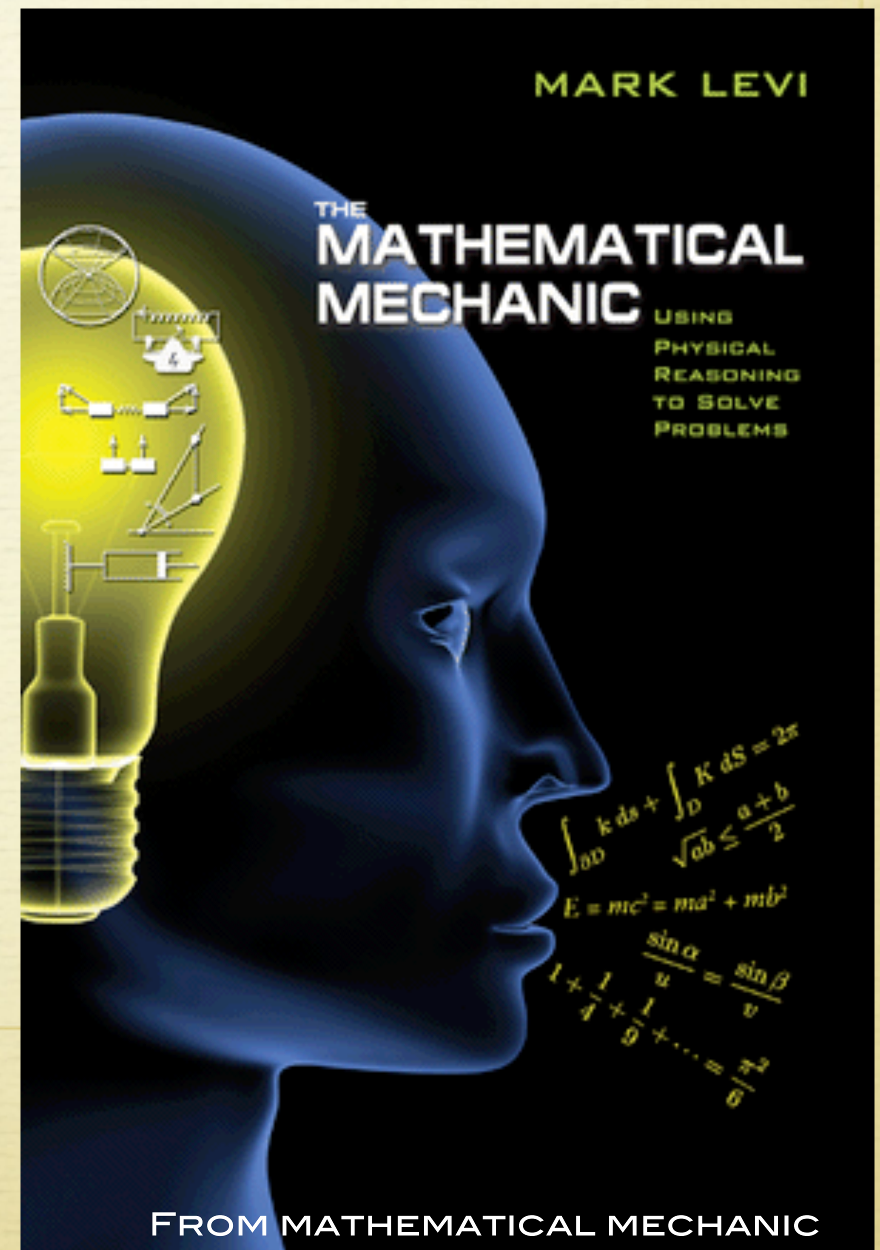
- Find local extrema inside
- Find extrema on boundary
- possibly at Infinity
- Compare to find largest



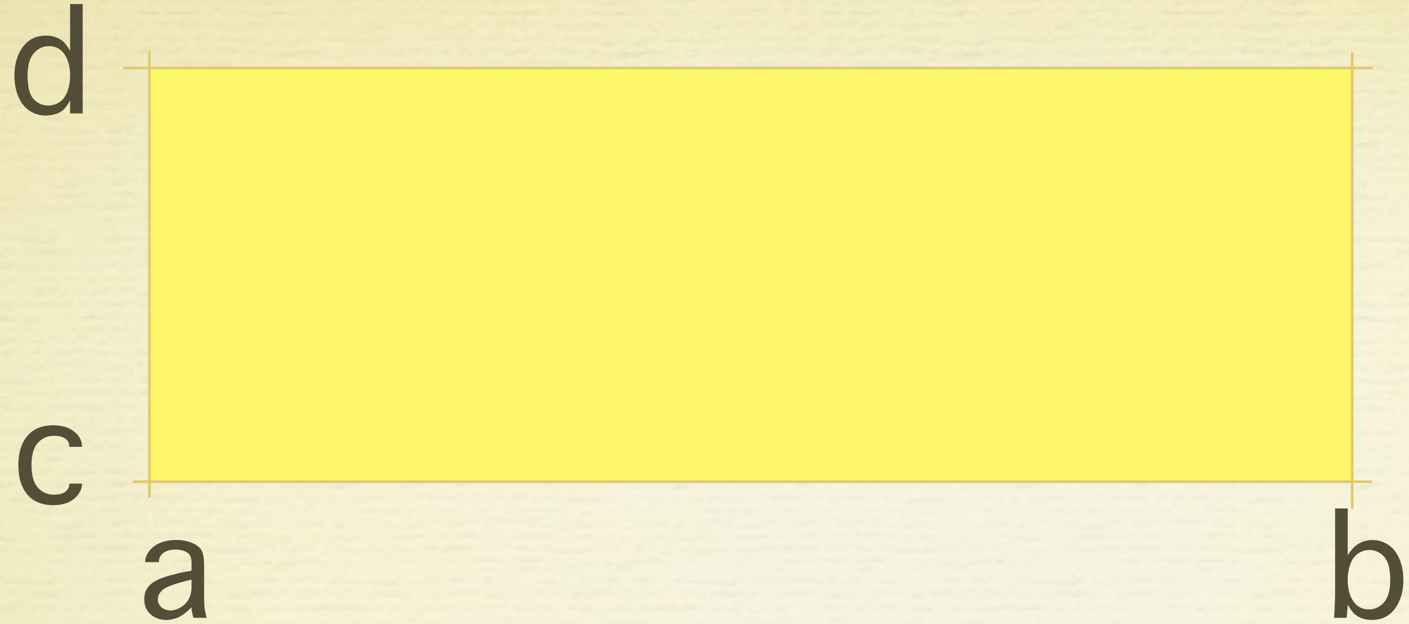
Global extrema



$$g(x,y) = 4x+2\pi y = 1$$
$$f(x,y) = x^2 + y^2\pi$$



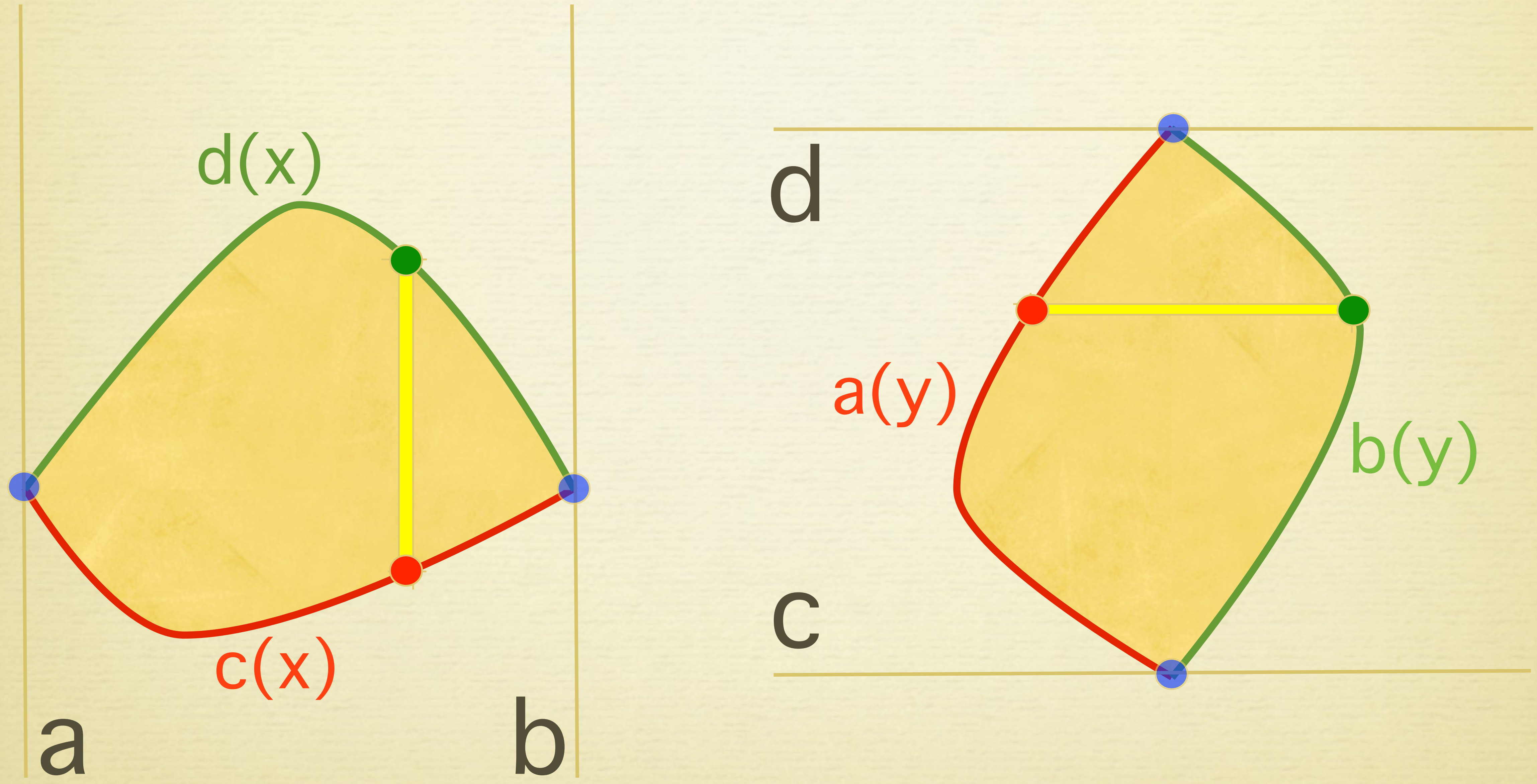
Problem: minimize the area under fixed circumference.



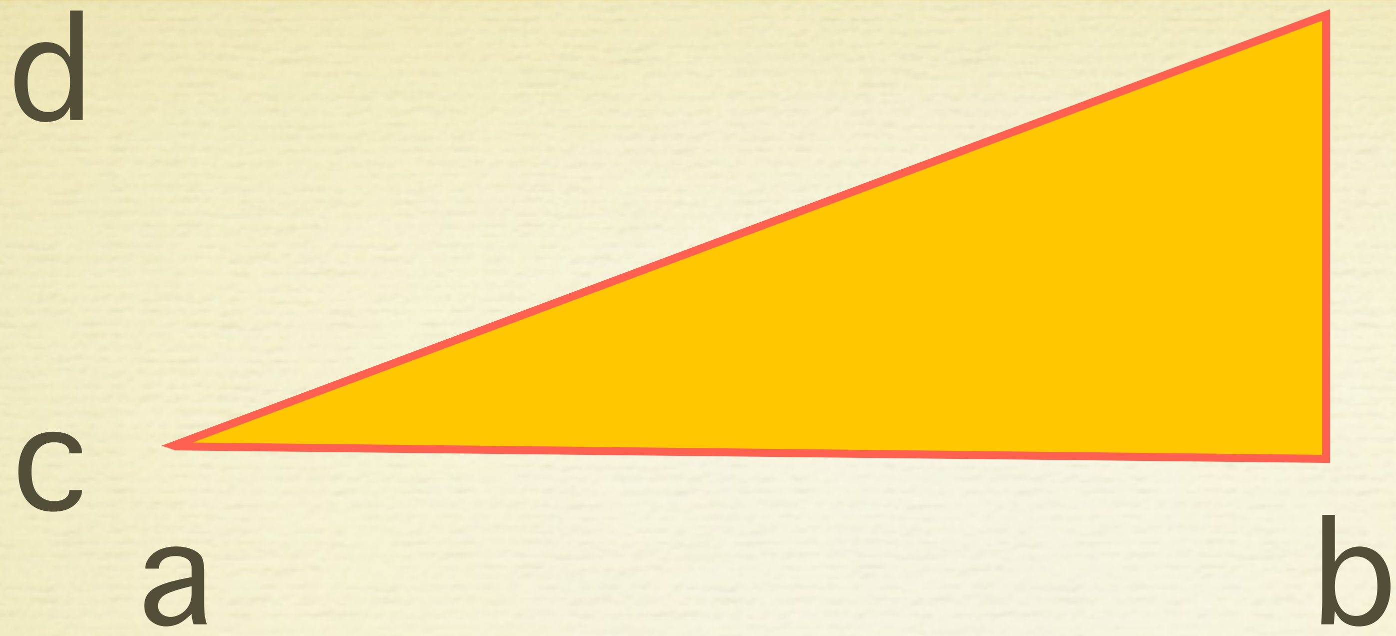
$$\int_a^b \int_c^d f(x,y)\,dydx = \int_c^d \int_a^b f(x,y)\,dxdy$$

Fubini Theorem

ONLY FOR RECTANGLES



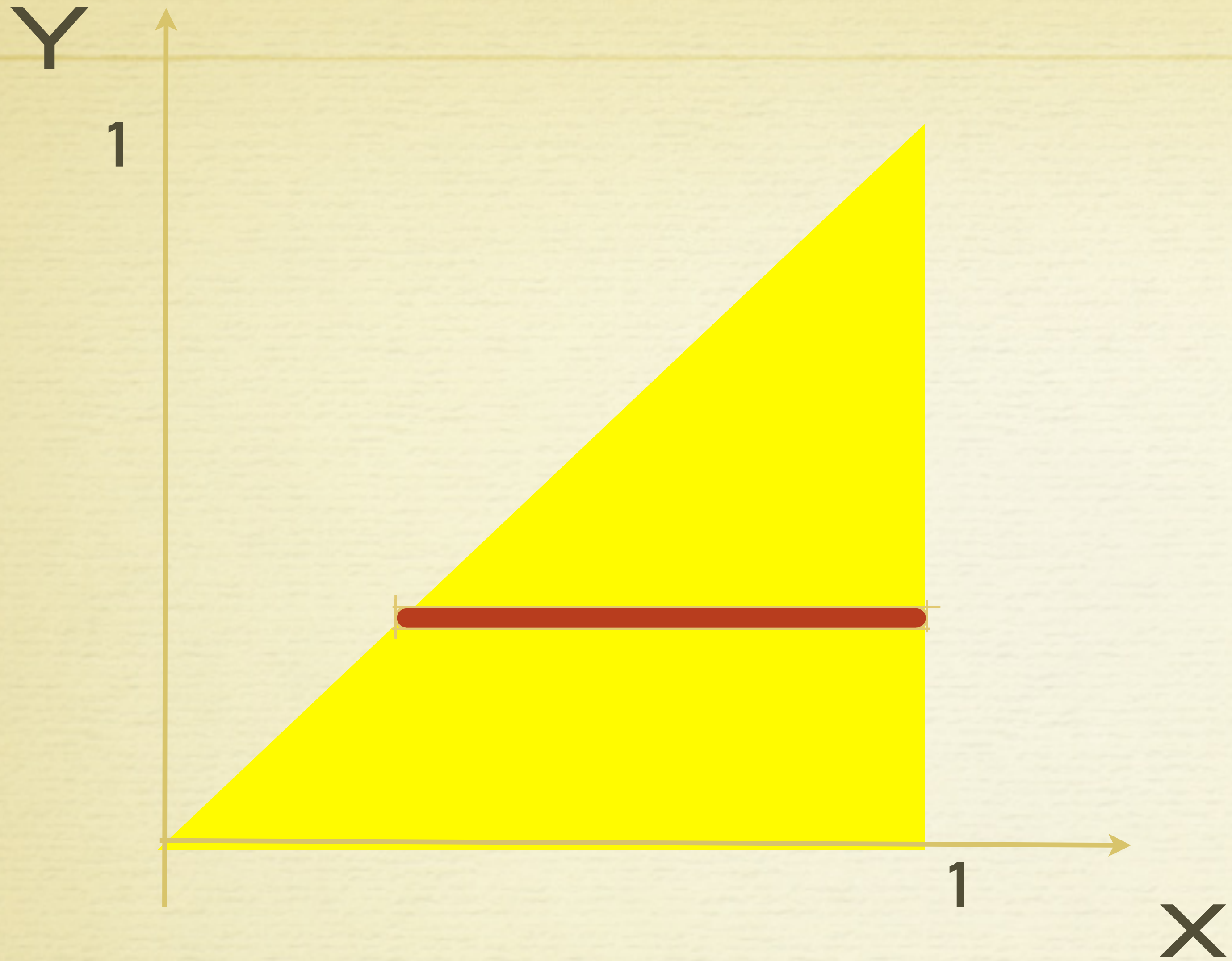
Type I and Type II



$$\int_a^b \int_{c(x)}^{d(x)} f(x,y)\, dy\, dx = \int_c^d \int_{a(y)}^{b(y)} f(x,y)\, dx\, dy$$

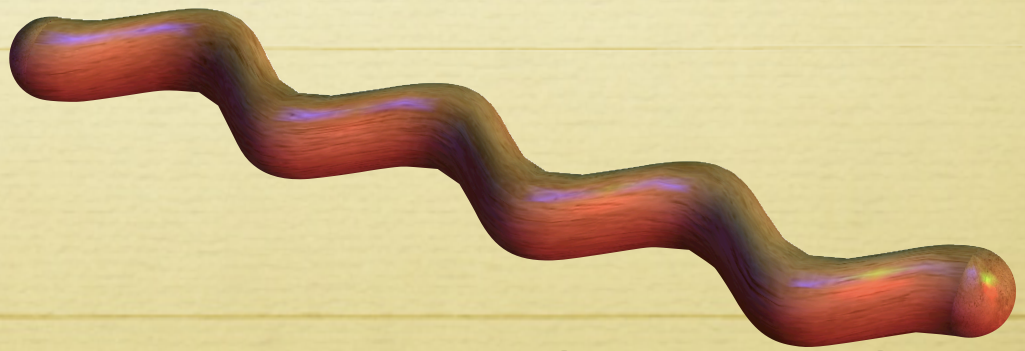
Switch Order

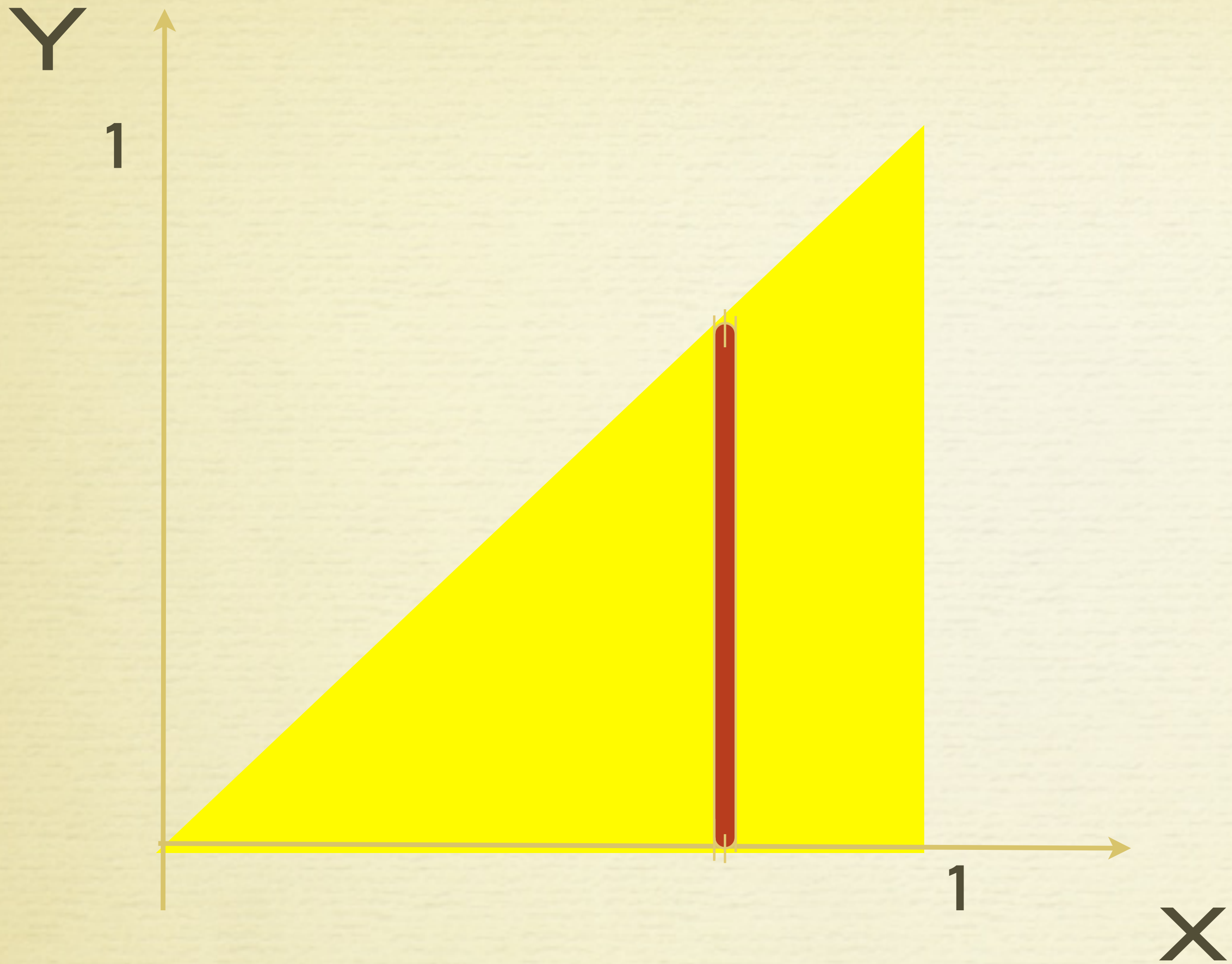
Make Picture!



Example

$$\int_0^1 \int_y^1 e^{-x^2}\, dx\, dy$$

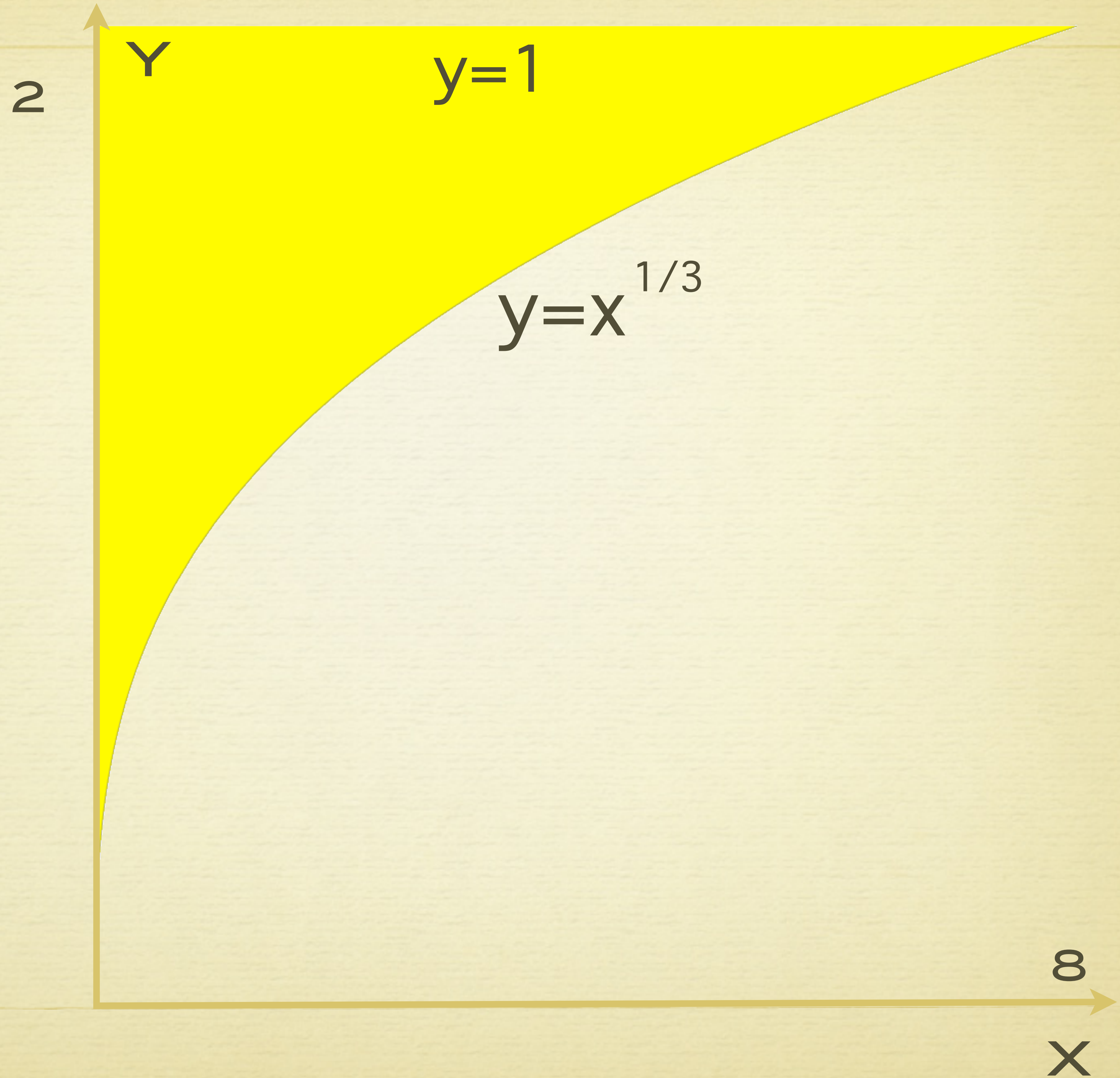




$$\int_0^1 \int_0^x e^{-x^2} \, dy \, dx$$

$$\int_0^8 \int_{x^{1/3}}^2 \cos(y^4 + 1)\ dydx$$

Problem



Make a picture

$$\int\int_R |\vec{r}_u \times \vec{r}_v| \, du dv$$

COMPARE:

$$\int_I |\vec{r}\,'(t)| \, dt$$

Surface area is an application of double integrals

Integration in Polar Coordinates

DANIEL
Backman's
Olive

Eli Leavit's
Clover

$$z = x^2 + y^2$$

$$x^2 + y^2 < 1$$

Problem: find the surface area

What are the top 21a Mistakes?

Taewen Shin

1) Nonlinear Linearization
2) Division by Zero in Lagrange
3) Not Trying an Integral Theorem
4) Lack of Figure in Integration
5) Mixup of DImension for Graphs
6) Forgetting the -1 in cross product
7) Use Pdx+Qdy for Line integrals
8) Forgetting integration Factor
9) Wrong Orientation
10) Final Result is not written

Top 10 Mistakes in 21a