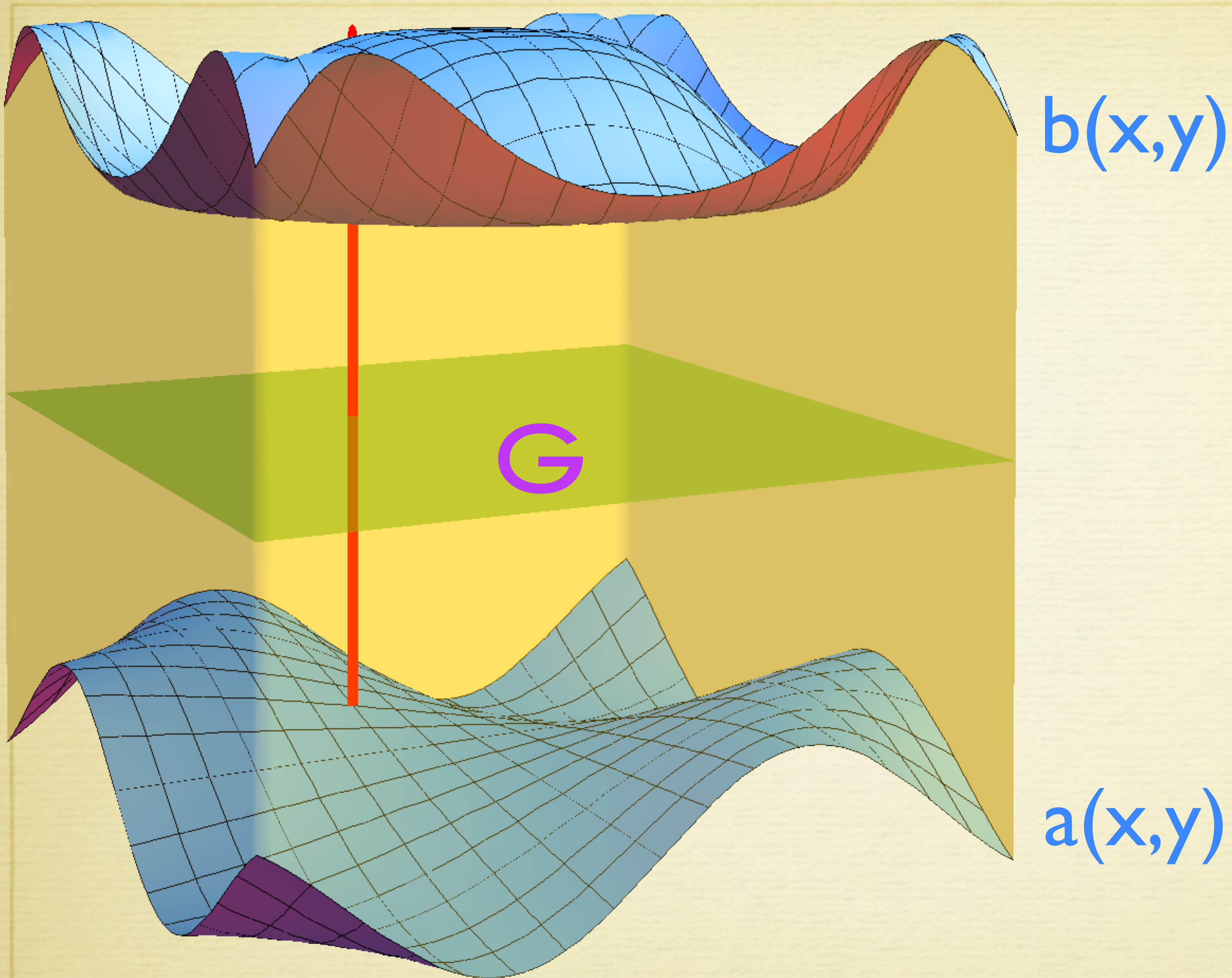
Math 21a Fall 11
Review II

Gauss

Stokes

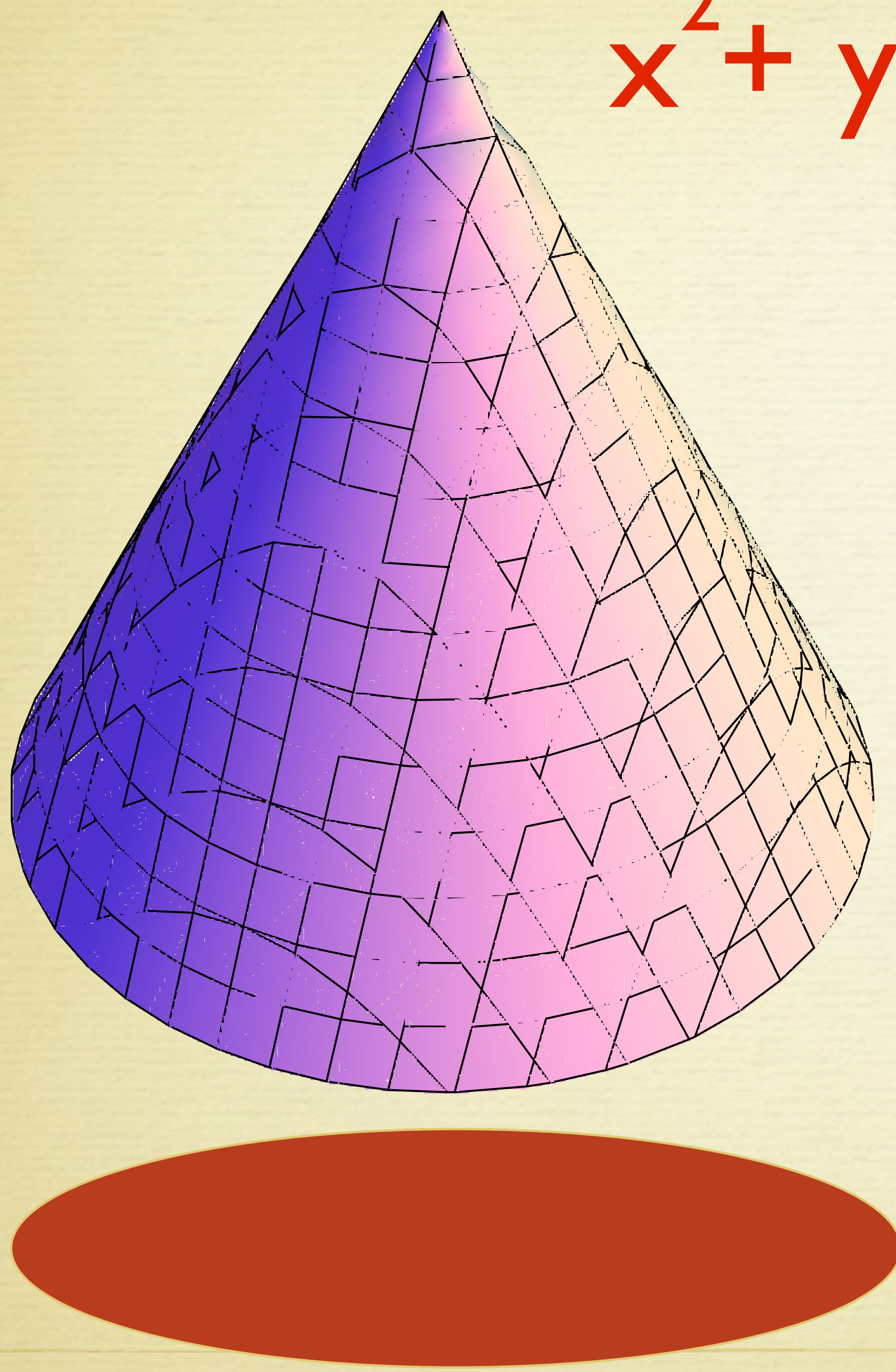
December 8, 2011

Oliver Knill



$$\iint_G \int_a^b f(x,y,z)\; dz\; dx\; dy$$

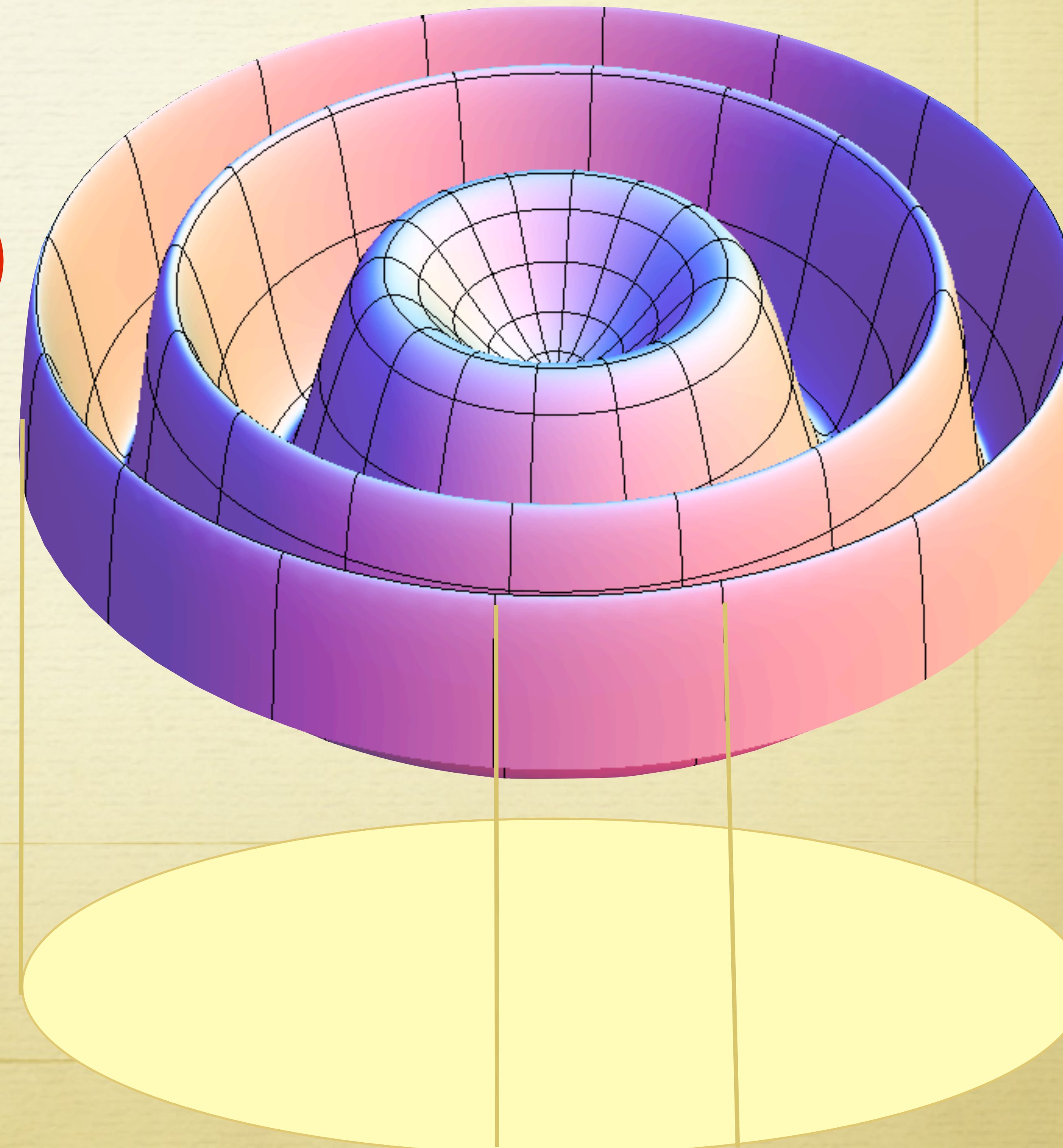
$$x^2 + y^2 = (z-1)^2 \qquad 1 > z > 0$$



$$\iint_{\bullet} \int_{0} f \quad dz\, dx\, dy$$

FIND THE VOLUME OF THE SOLID ABOVE THE DISC OF RADIUS 5 AND BELOW THE GRAPH OF

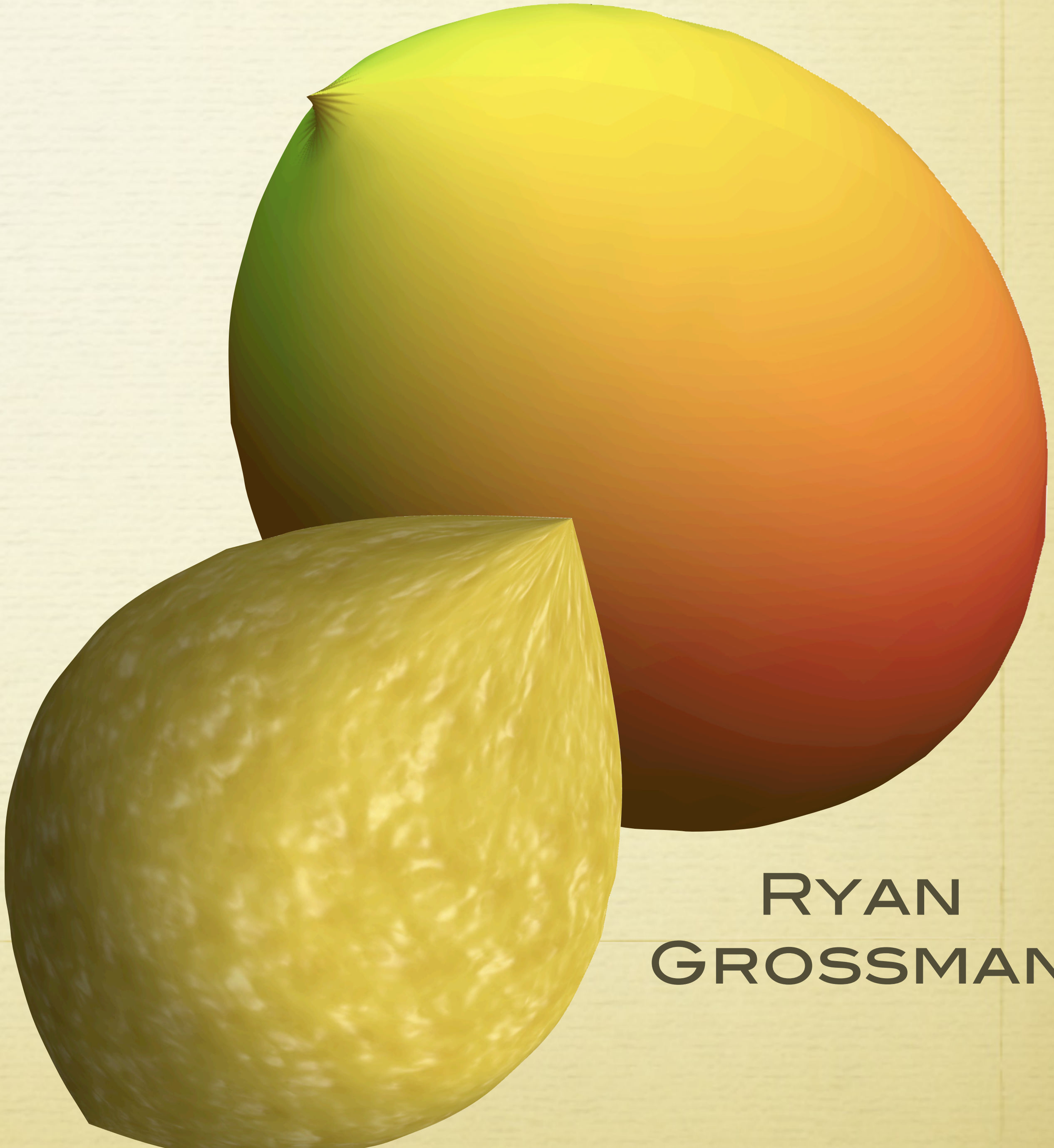
$$f(x,y) = (6+\sin(x^2+y^2))$$



Problem

$$\int\int_R (6 + \sin(x^2 + y^2))\; dx dy$$

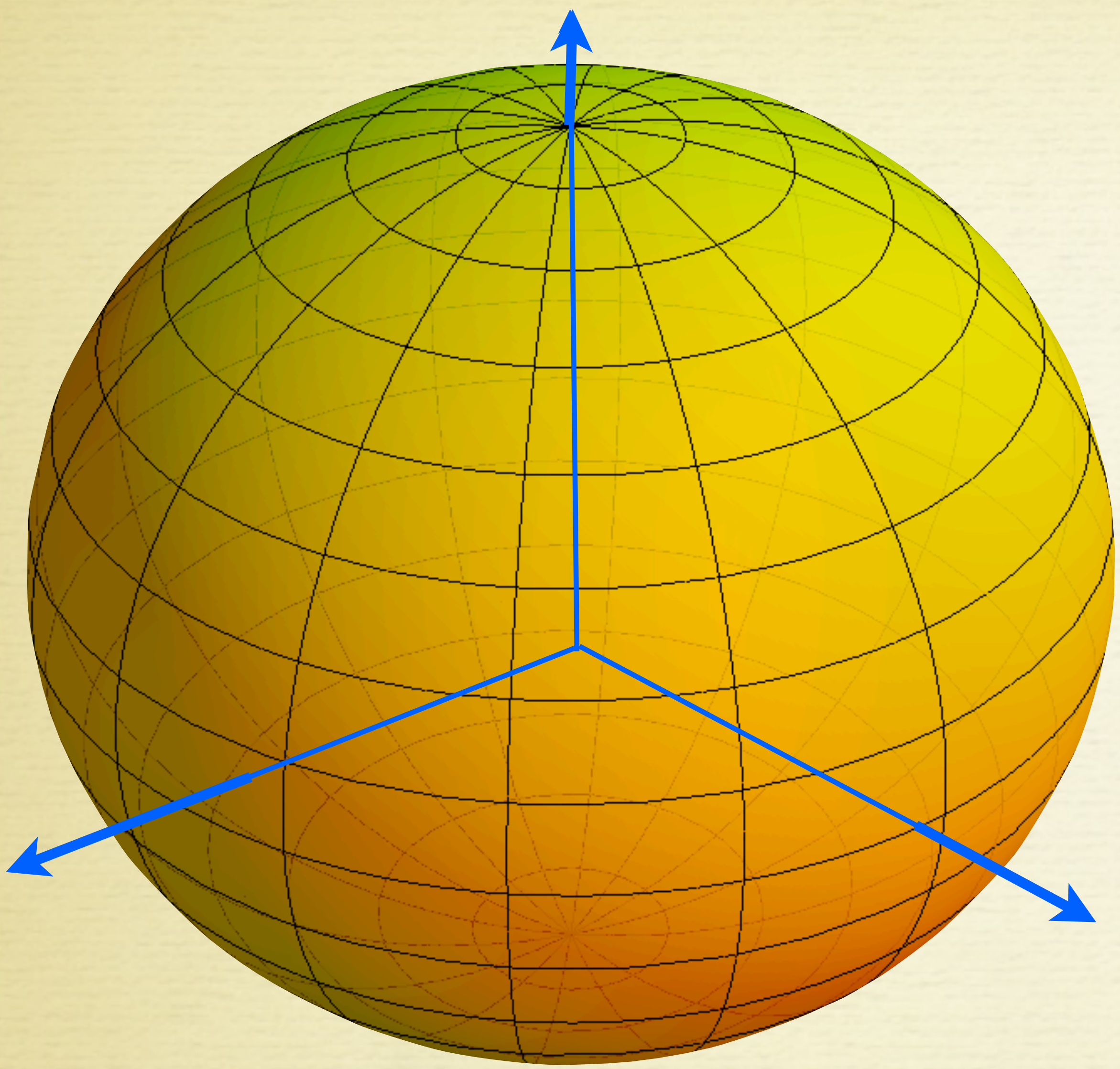
This smells like a situation for an other coordinate system



Ryan Grossman

David Zhang

Spherical Coordinates



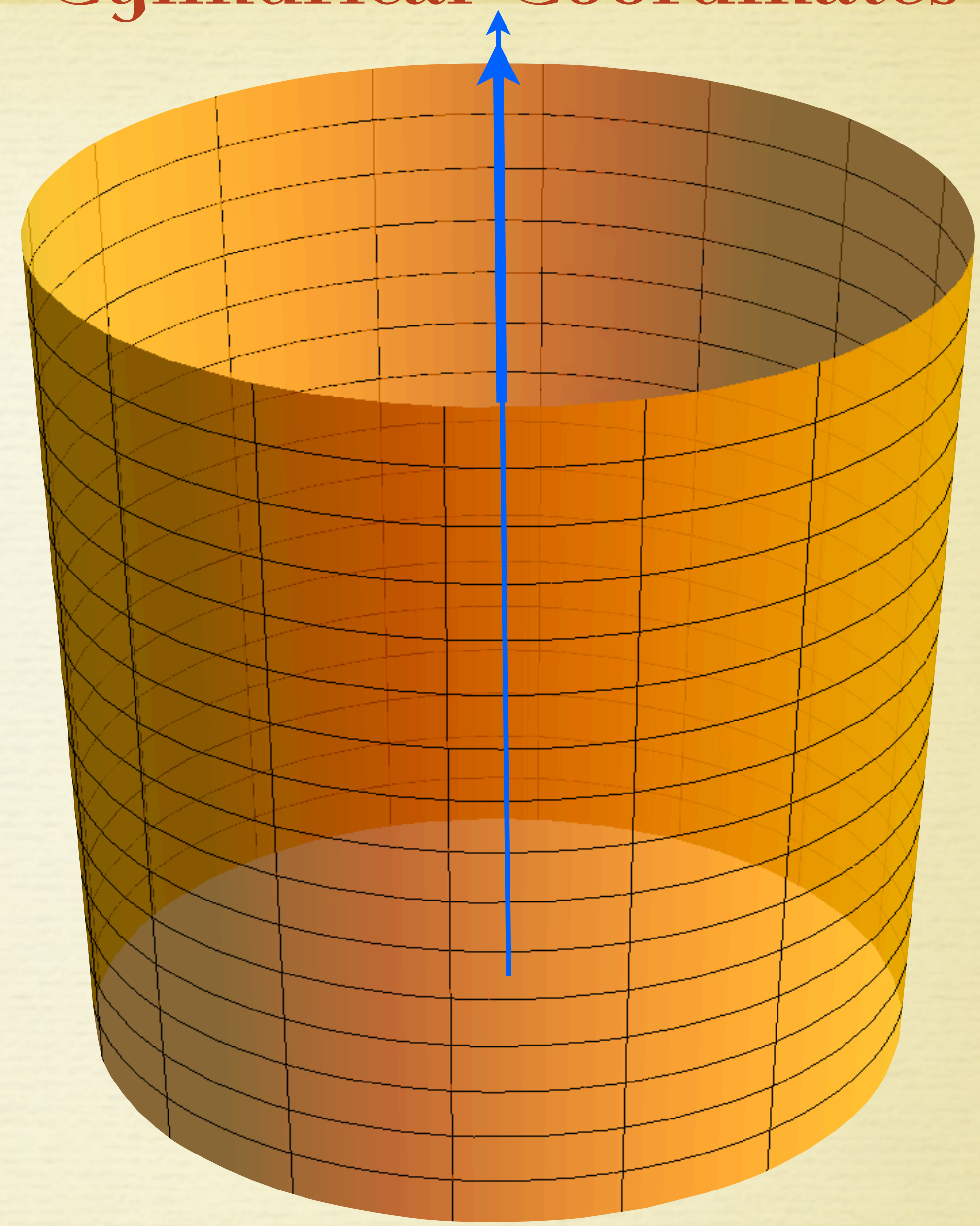
$$x = \rho \sin(\phi) \cos(\theta)$$

$$y = \rho \sin(\phi) \sin(\theta)$$

$$z = \rho \cos(\phi)$$

$$\rho^2 \sin(\phi)$$

Cylindrical Coordinates



$$x = r \cos(\theta)$$

$$y = r \sin(\theta)$$

$$z = z$$

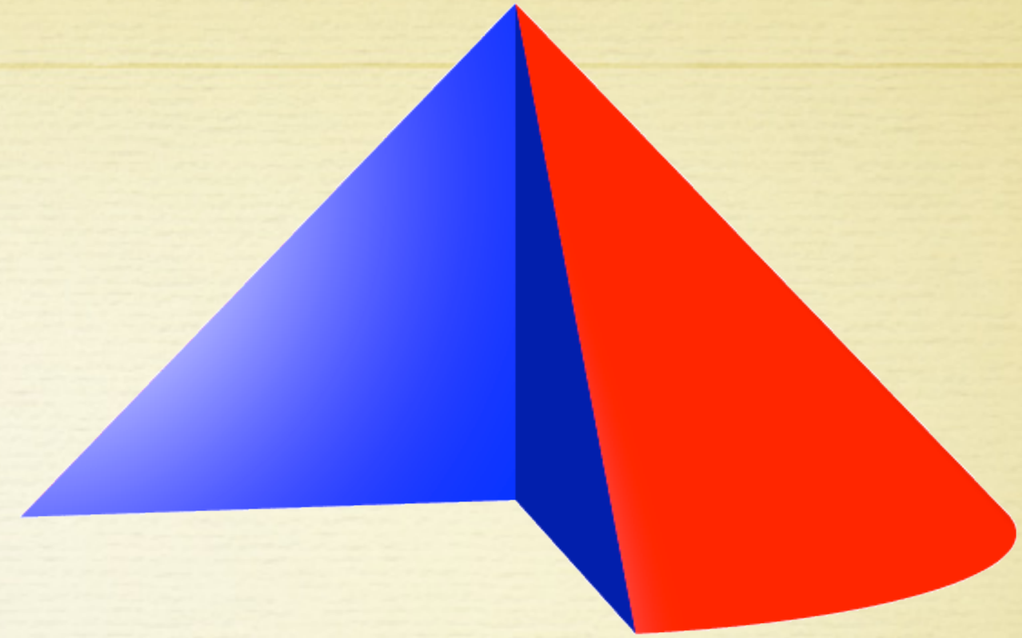
$$r$$

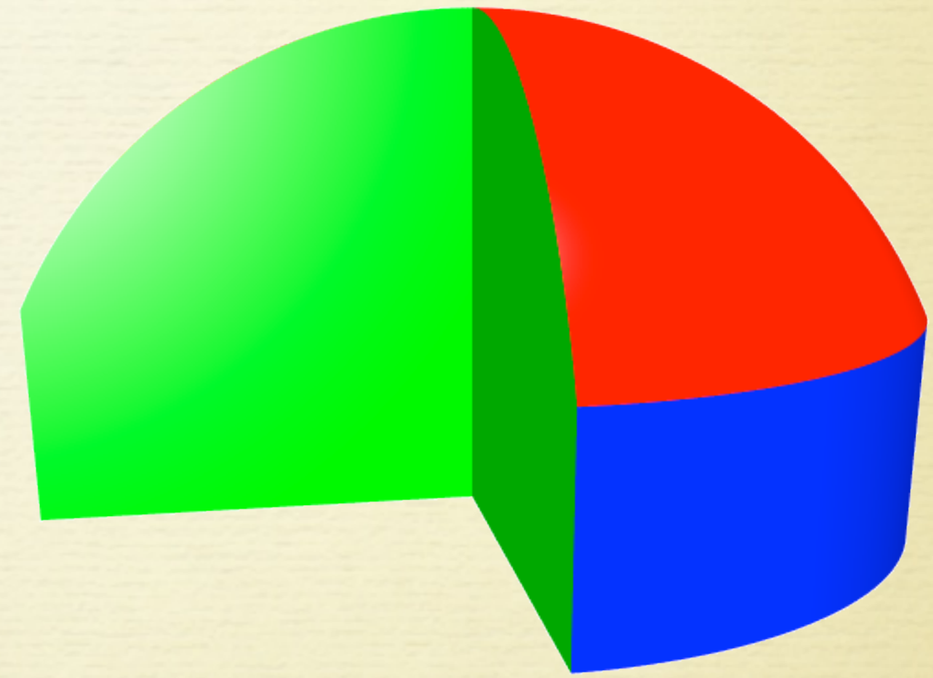
$$\int_0^{3\pi/2} \int_0^1 \int_0^{\sqrt{2-r^2}}$$

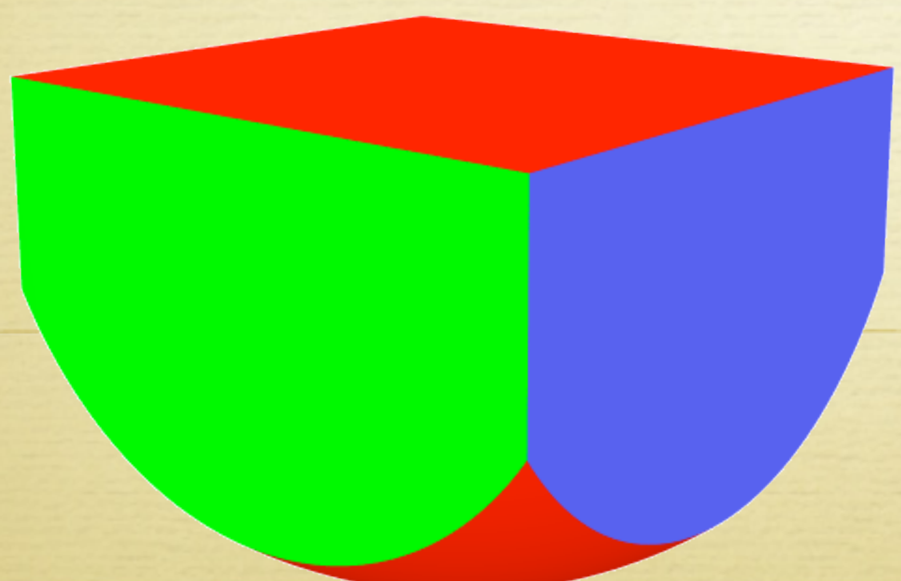
$$\int_0^{3\pi/2} \int_0^1 \int_0^{1-r}$$

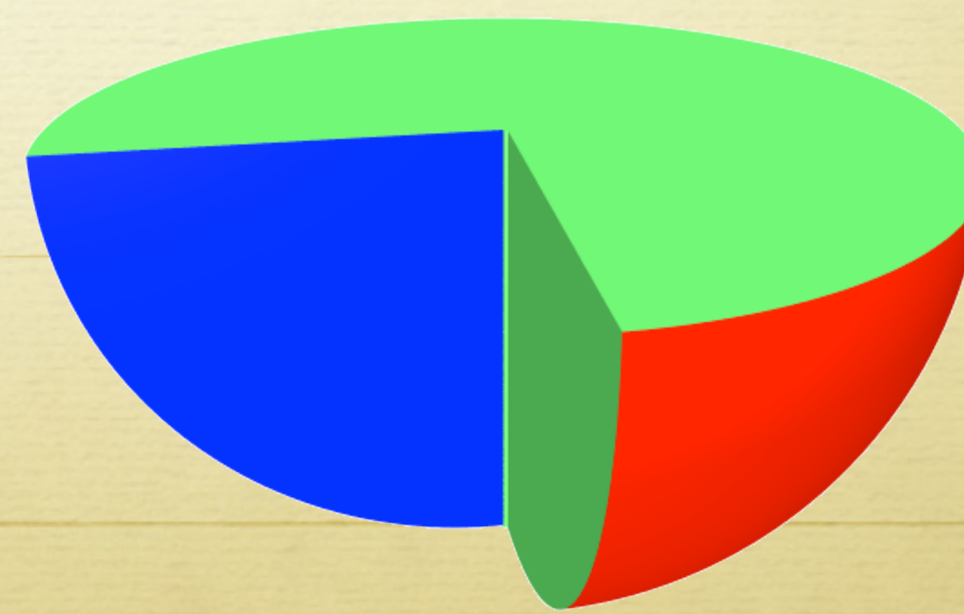
$$\int_0^{3\pi/2} \int_0^1 \int_{-\sqrt{1-r^2}}^{0}$$

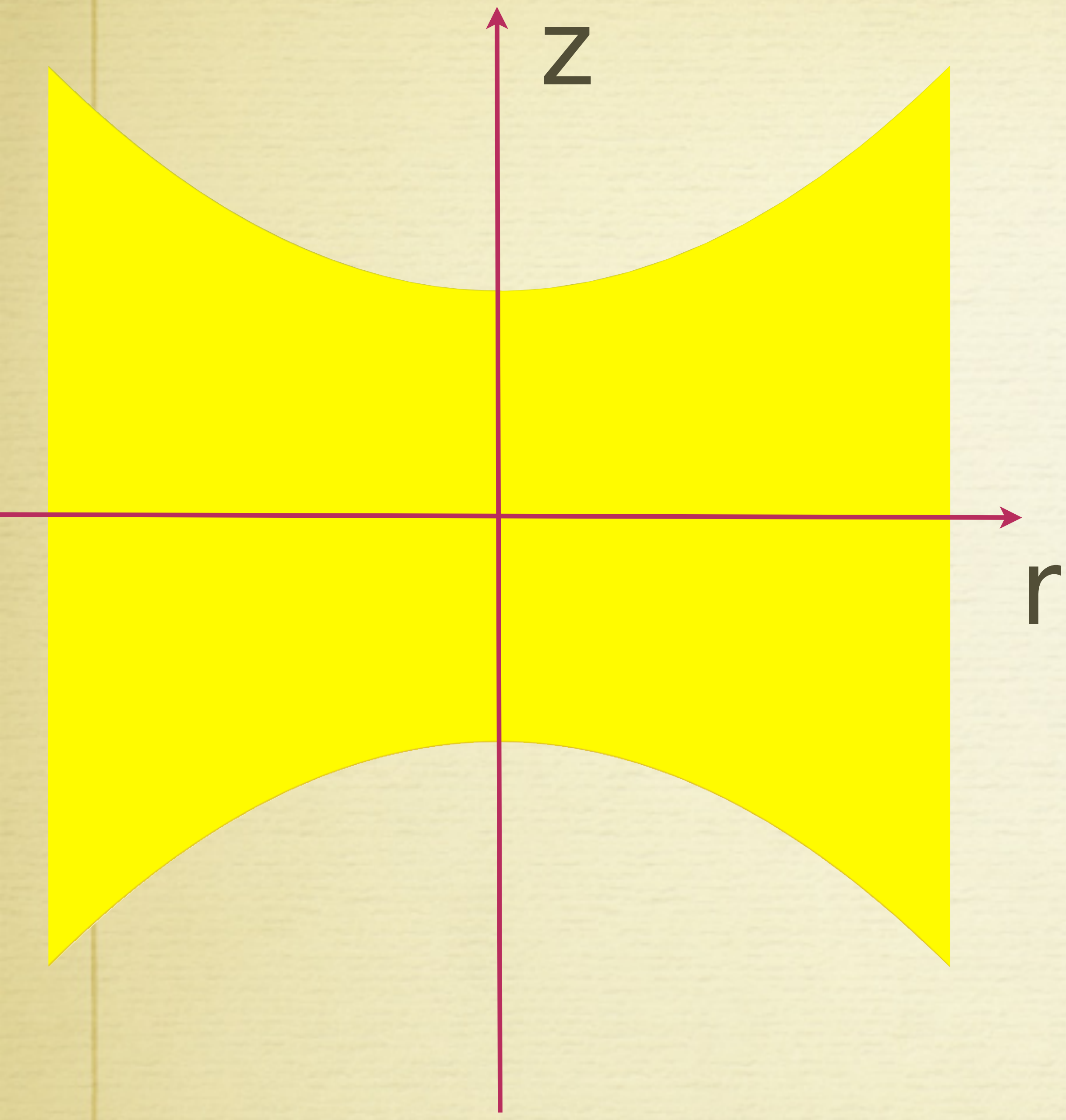
$$\int_{-1}^{1} \int_{-1}^{1} \int_{-\sqrt{2-x^2-y^2}}^{0}$$











Integrate

$f(x,y,z) = 14+x$

over the solid

$$x^2 + y^2 < 1$$

$$z \leq 1 + x^2 + y^2$$

$$z \geq -1 - x^2 - y^2$$

Problem

Vector Fields

$$F = \langle P, Q \rangle$$

$$F = \langle P, Q, R \rangle$$

$$Q_x - P_y$$

Curl in 2D

$$P_x + Q_y$$

Div in 2D

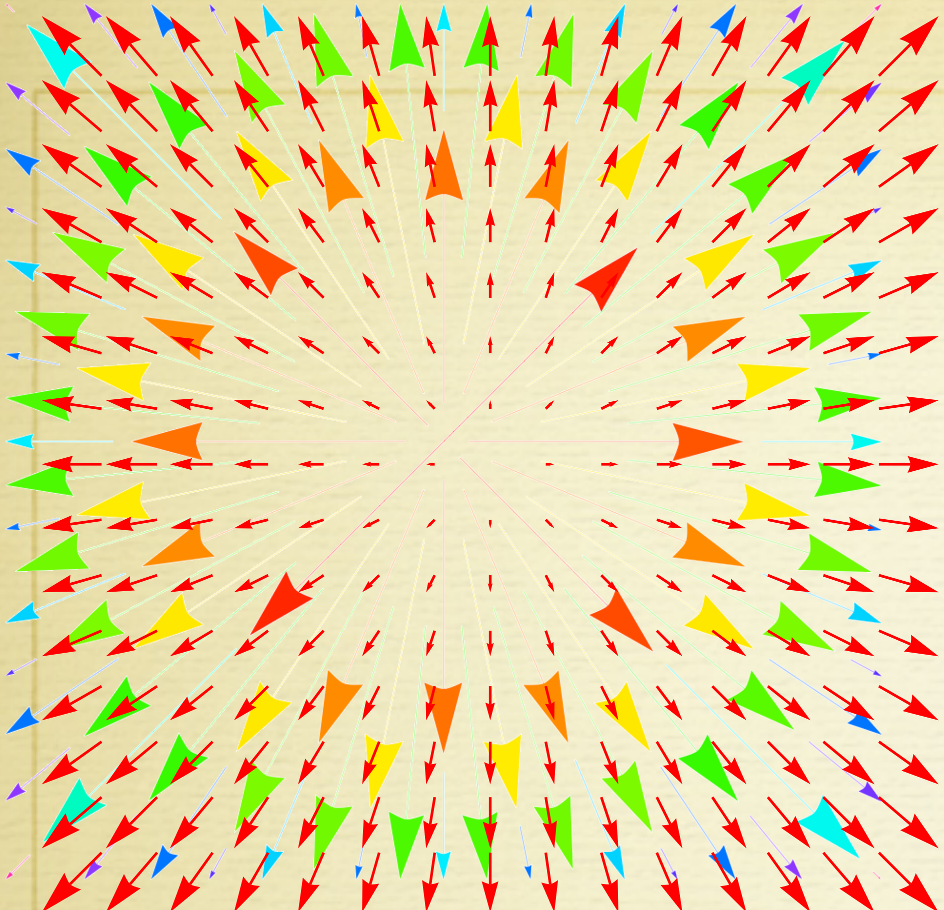
$$\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

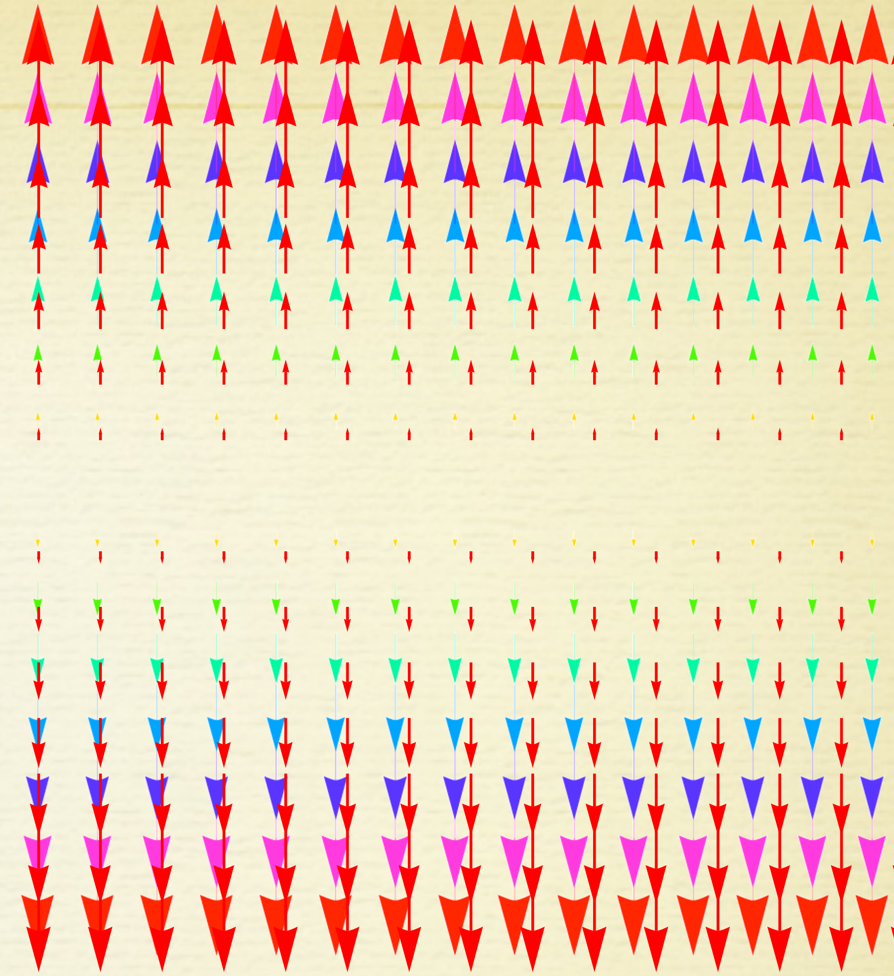
Curl in 3D

$$P_x + Q_y + R_z$$

Div in 3D

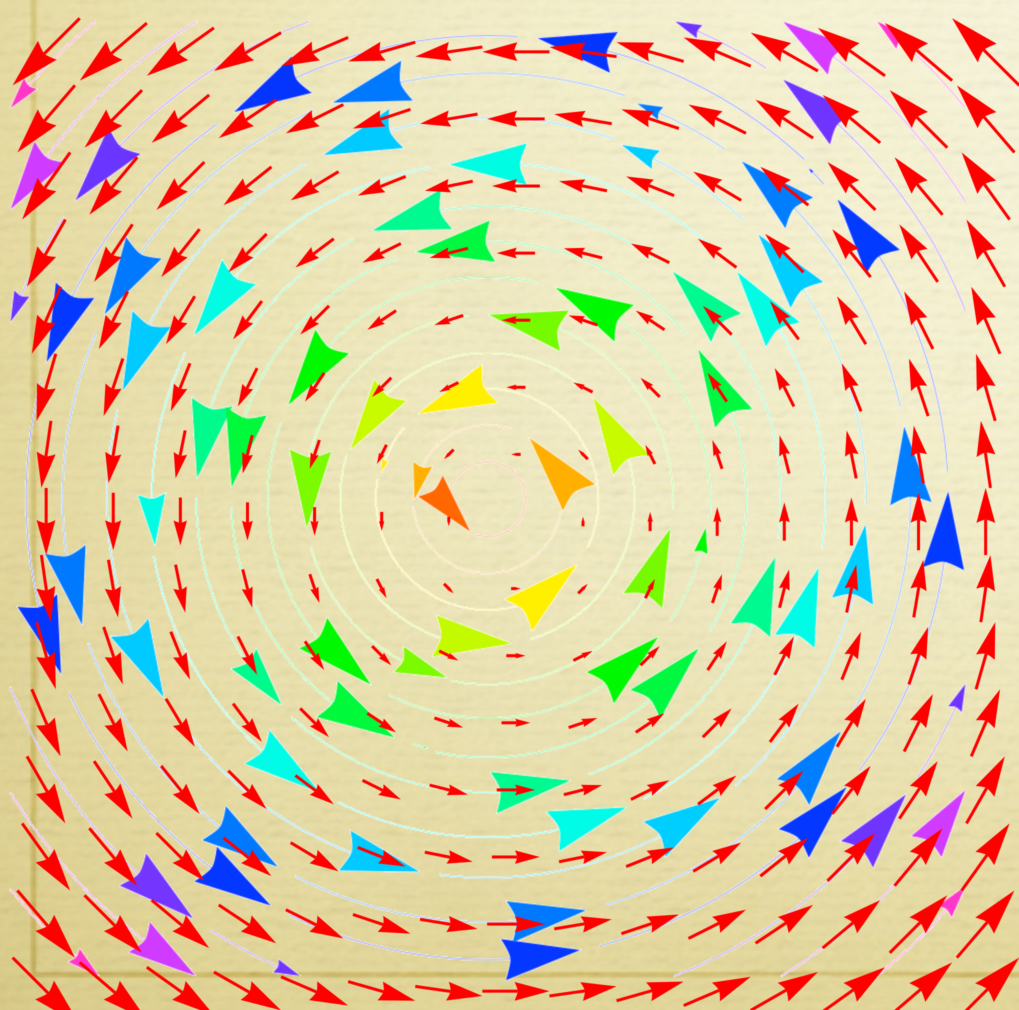
Div and Curl

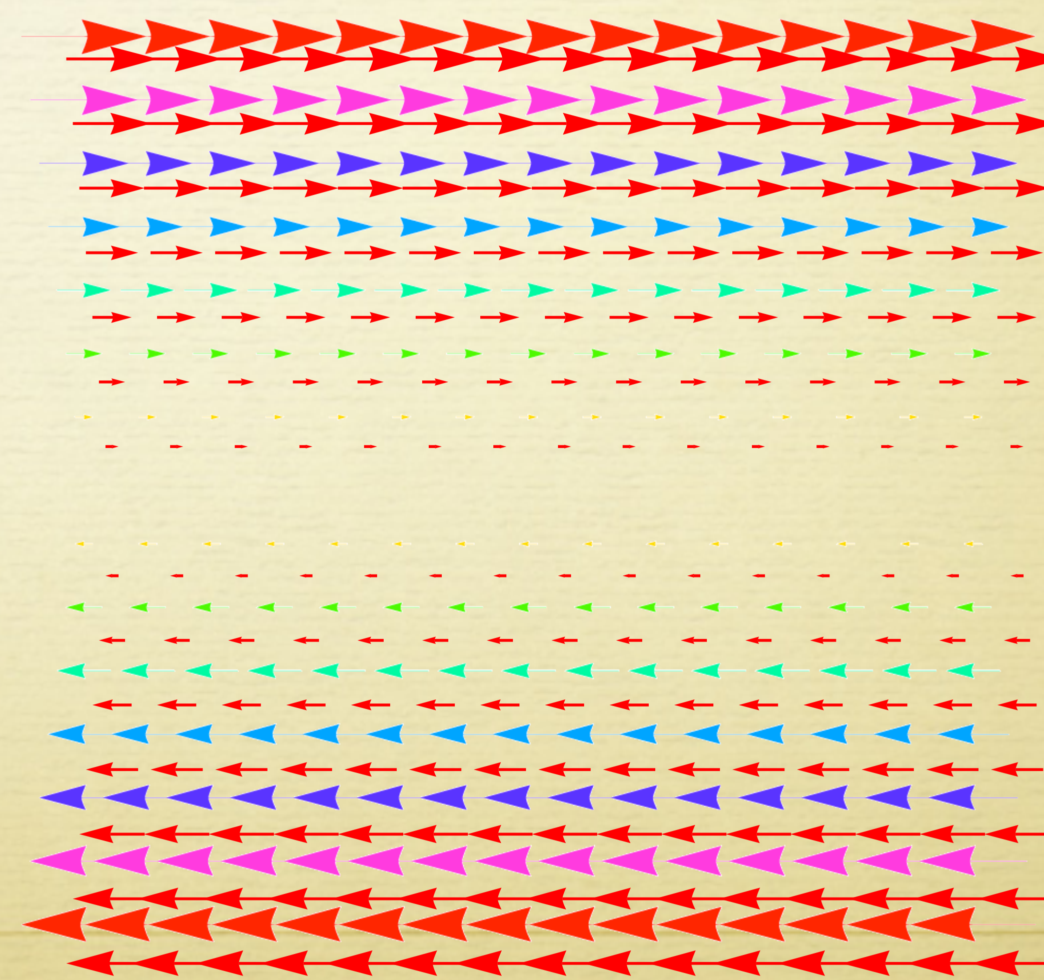




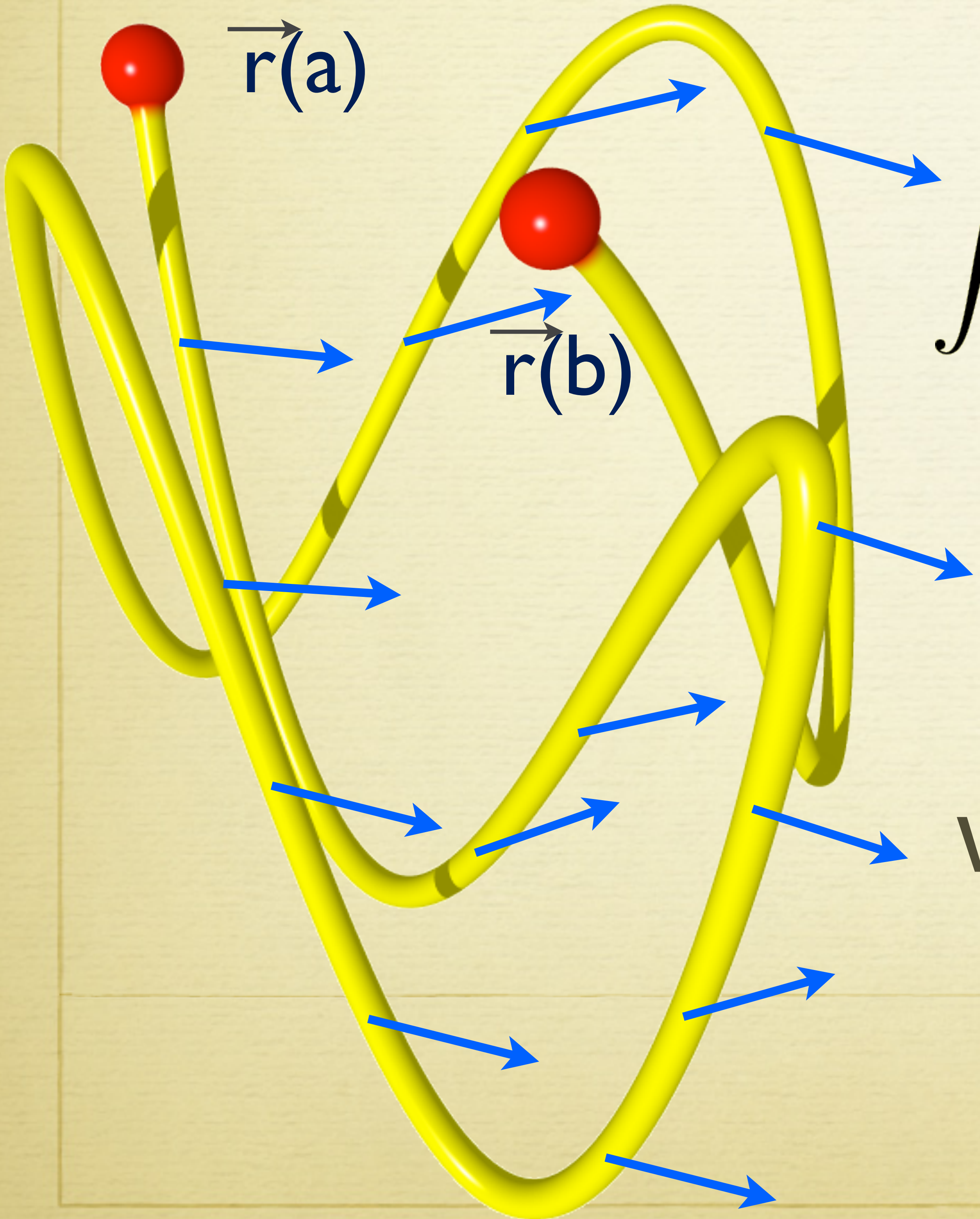
Curl?

Div?





Line Integrals



$$\int_C \vec{F} \cdot \vec{ds} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

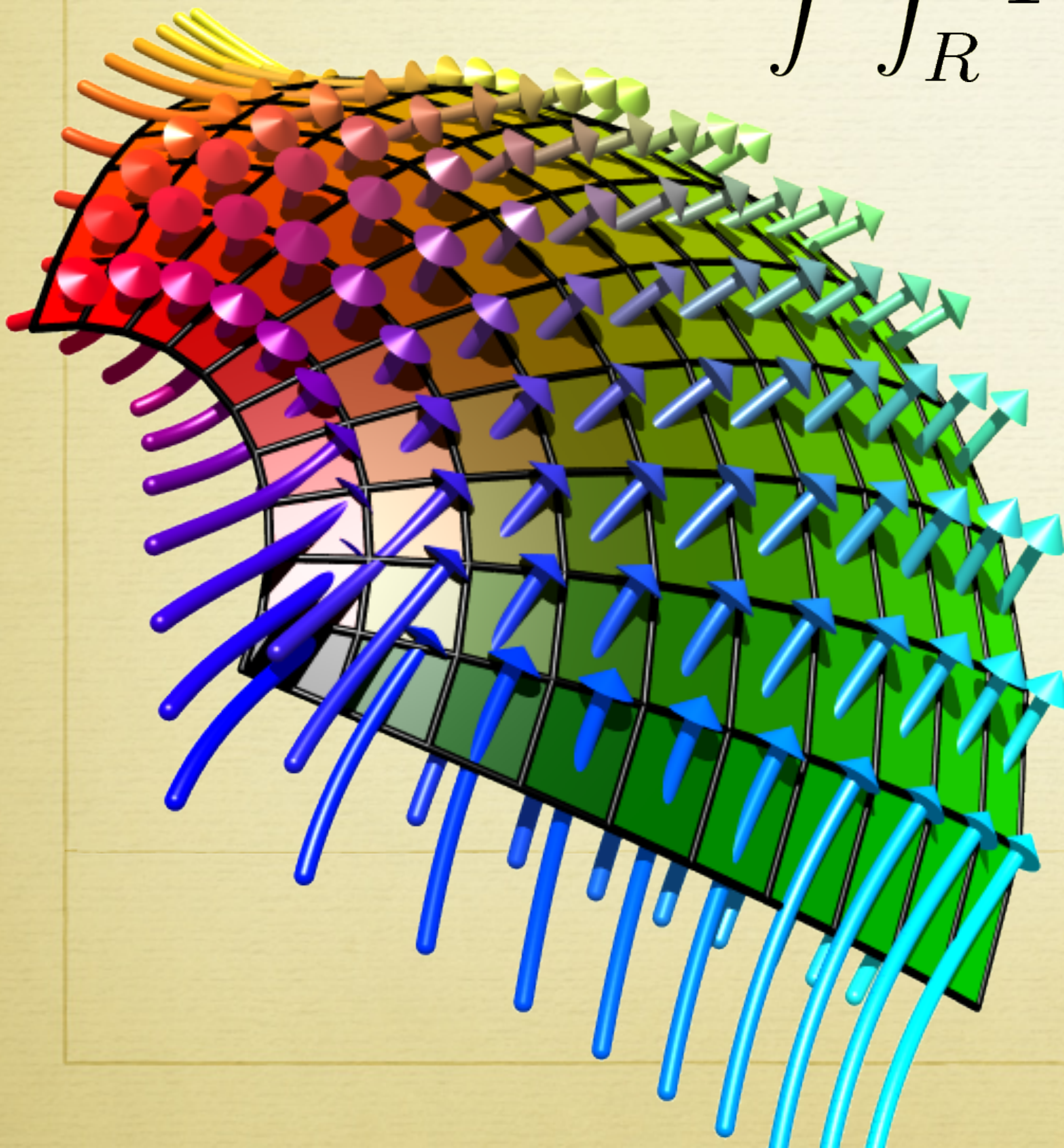
POWER

MEASURES THE WORK OF THE VECTOR FIELD F DONE ON THE BODY ALONG THE PATH C

Flux Integrals

$$\int\int_S \vec{F} \cdot d\vec{S} =$$

$$\int\int_R \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \, du dv$$



Measures The Amount of field passing through S in Unit time.

$$\int_C \vec{F} \cdot d\vec{s} =$$

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt$$

$$\int\int_S \vec{F} \cdot d\vec{S} =$$

$$\int\int_R \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \; dudv$$

These two integrals is all we need.

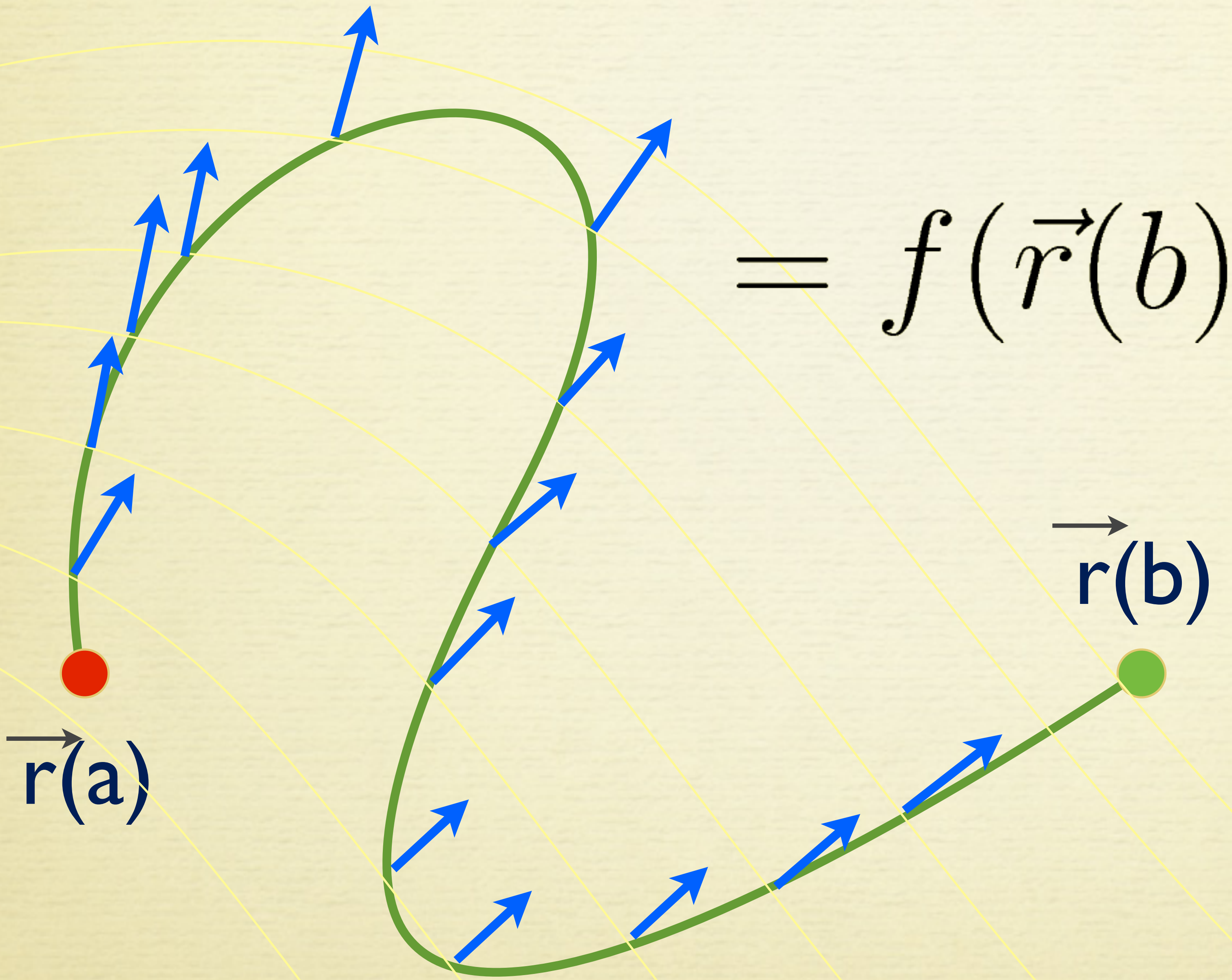
FT Lineintegrals

- The theorem
- Closed Loop
- Conservative
- Examples

Theorem 1

$$\int_a^b \nabla f(\vec{r}(t))\cdot\vec{r}\,'(t)\;dt$$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$



Fundamental Theorem of Line integrals

The nano car is exposed to a force field

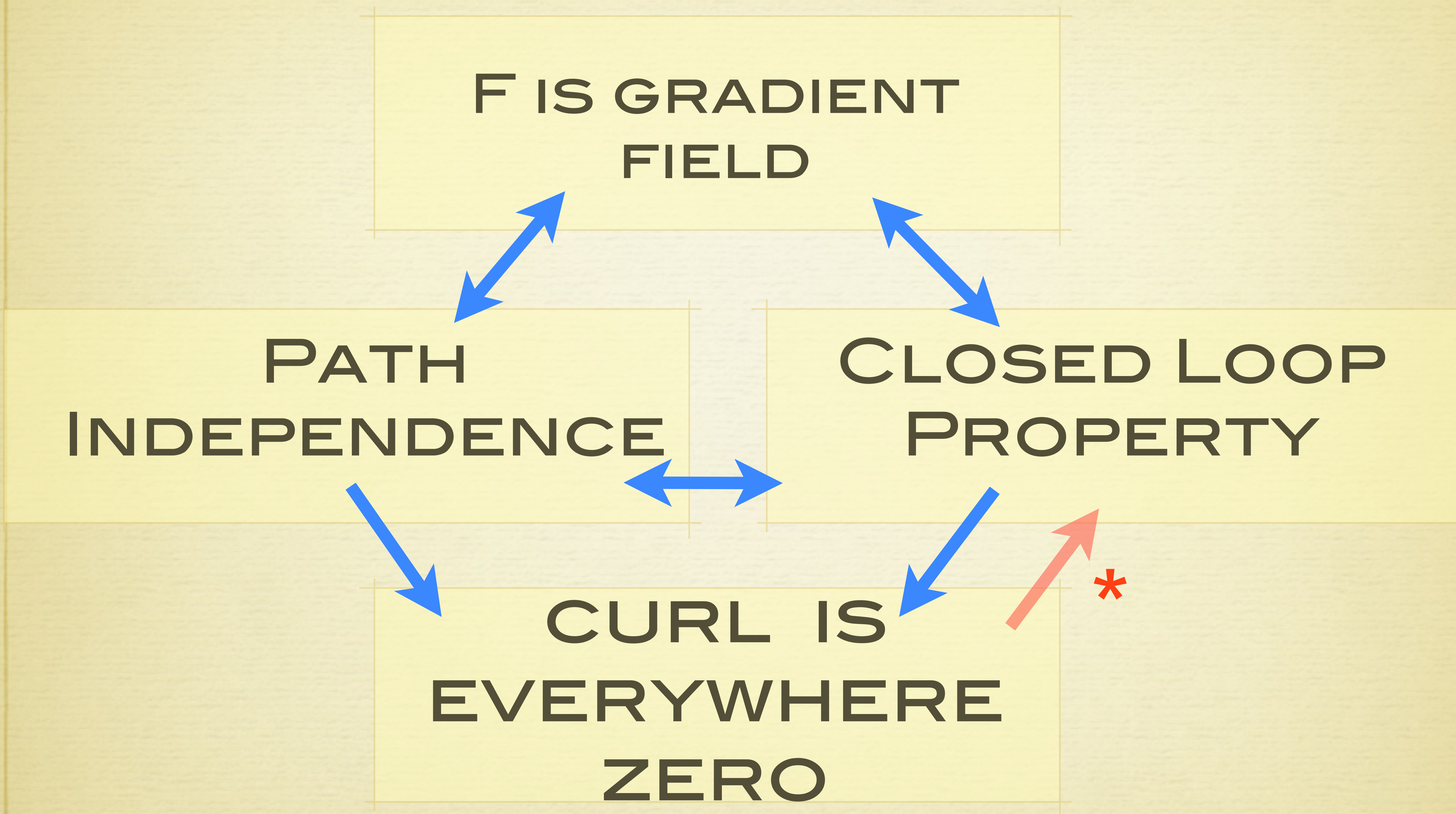
$$\vec{F}(x,y,z) = \langle yz\cos(xyz), xz\cos(xyz), xy\cos(xyz)+x \rangle$$

from the surface and pushed along a path

$$\vec{r}(t) = \langle t, \cos(t), \sin(t) \rangle$$

where t goes from 0 to π. What work is done on the car?

Problem



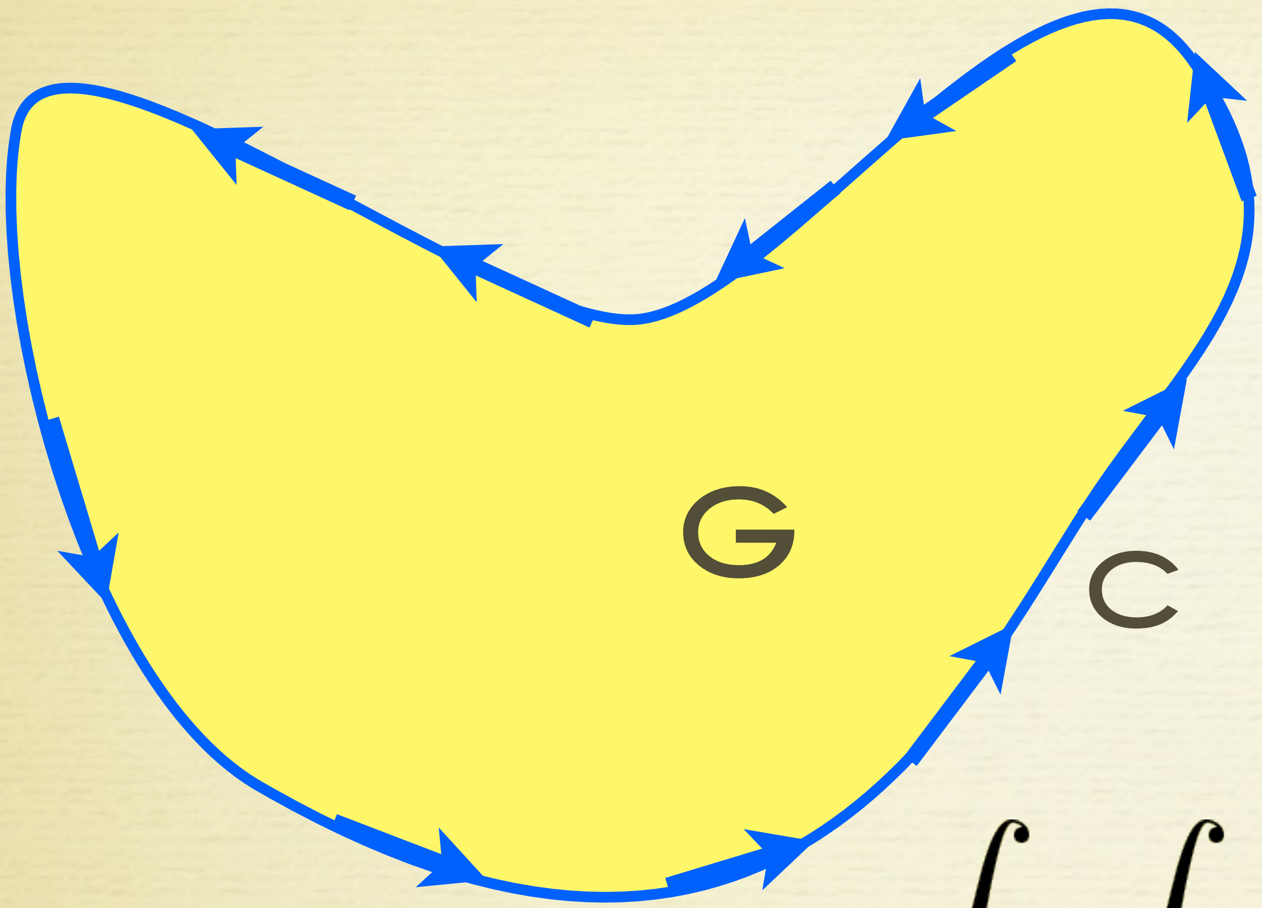
Properties of Gradient fields

* IF DEFINED IN SIMPLY CONNECTED REGION

Greens Theorem

- The Theorem
- Consequences
- Examples

Theorem 2

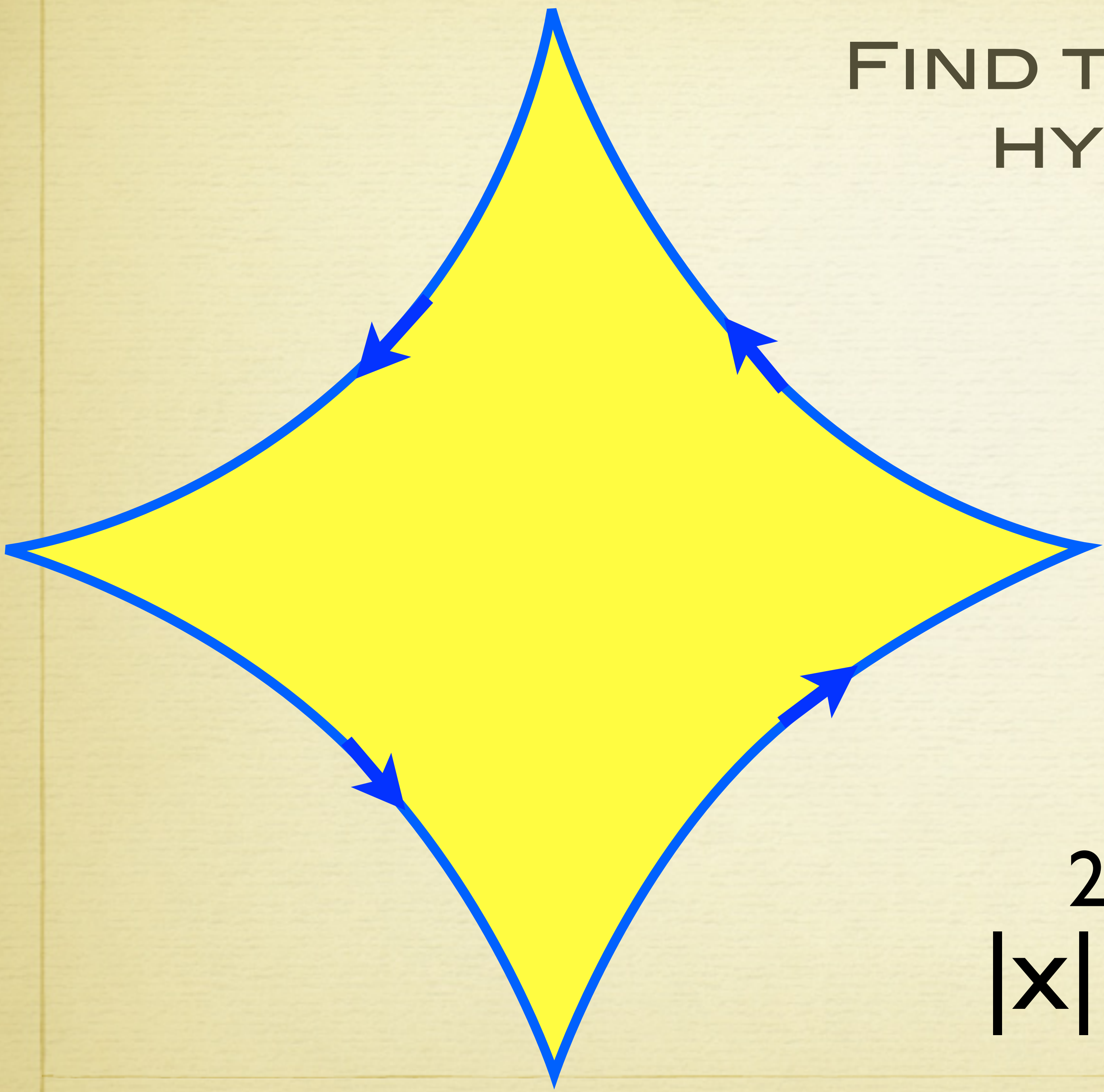


$$\int\int_G \operatorname{curl}(\vec{F})\ dA$$

$$= \int \vec{F}\ \vec{ds}$$

Greens Theorem

Find the area of the hypercylcoid



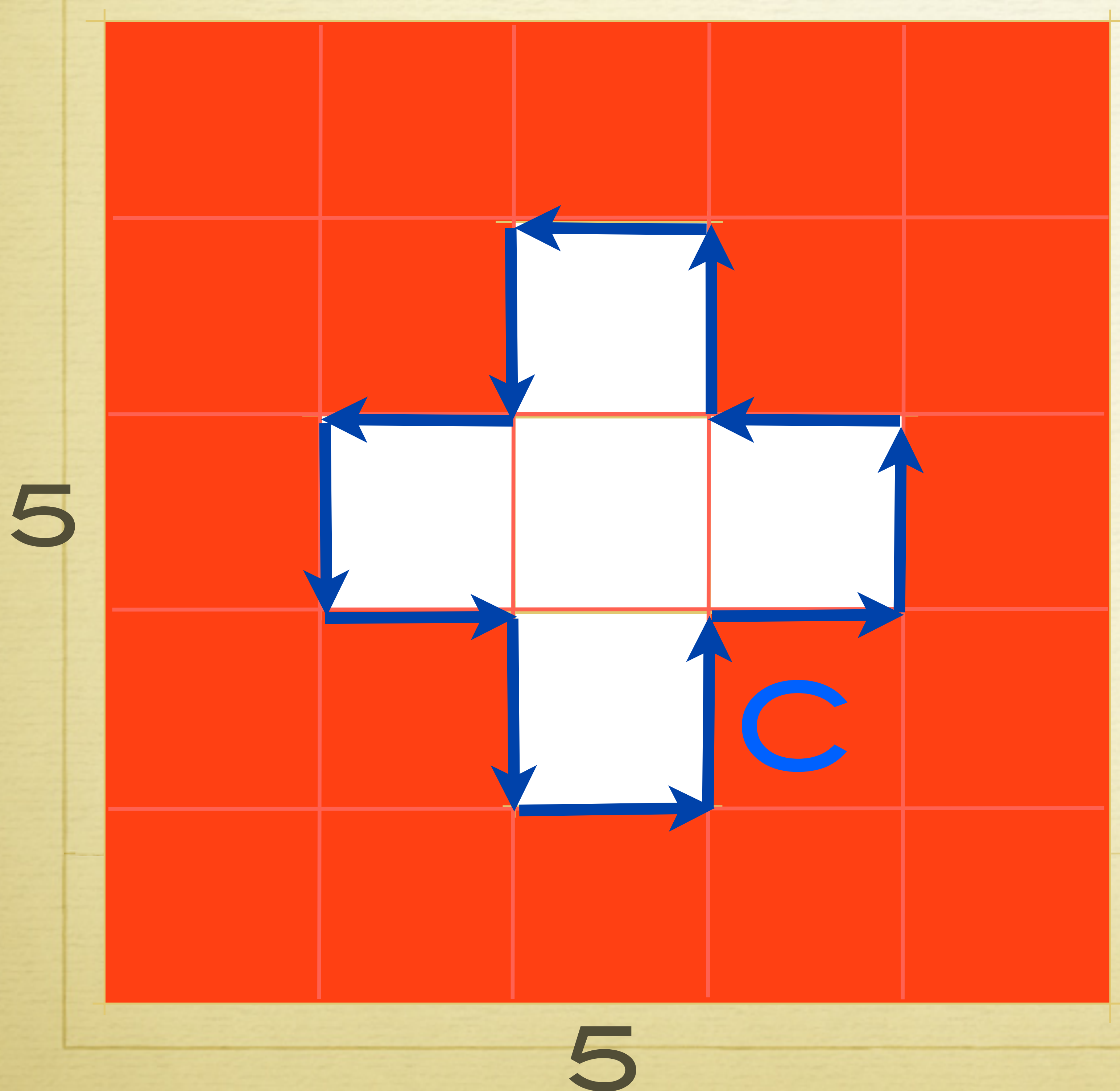
$$|x|^{2/3} + |y|^{2/3} < 1$$

Problem

Find the line integral of the vector field

$$\vec{F}(x,y) = \langle y^2\cos(x) + 3y, x + 2y \sin(x) \rangle$$

along the curve C.

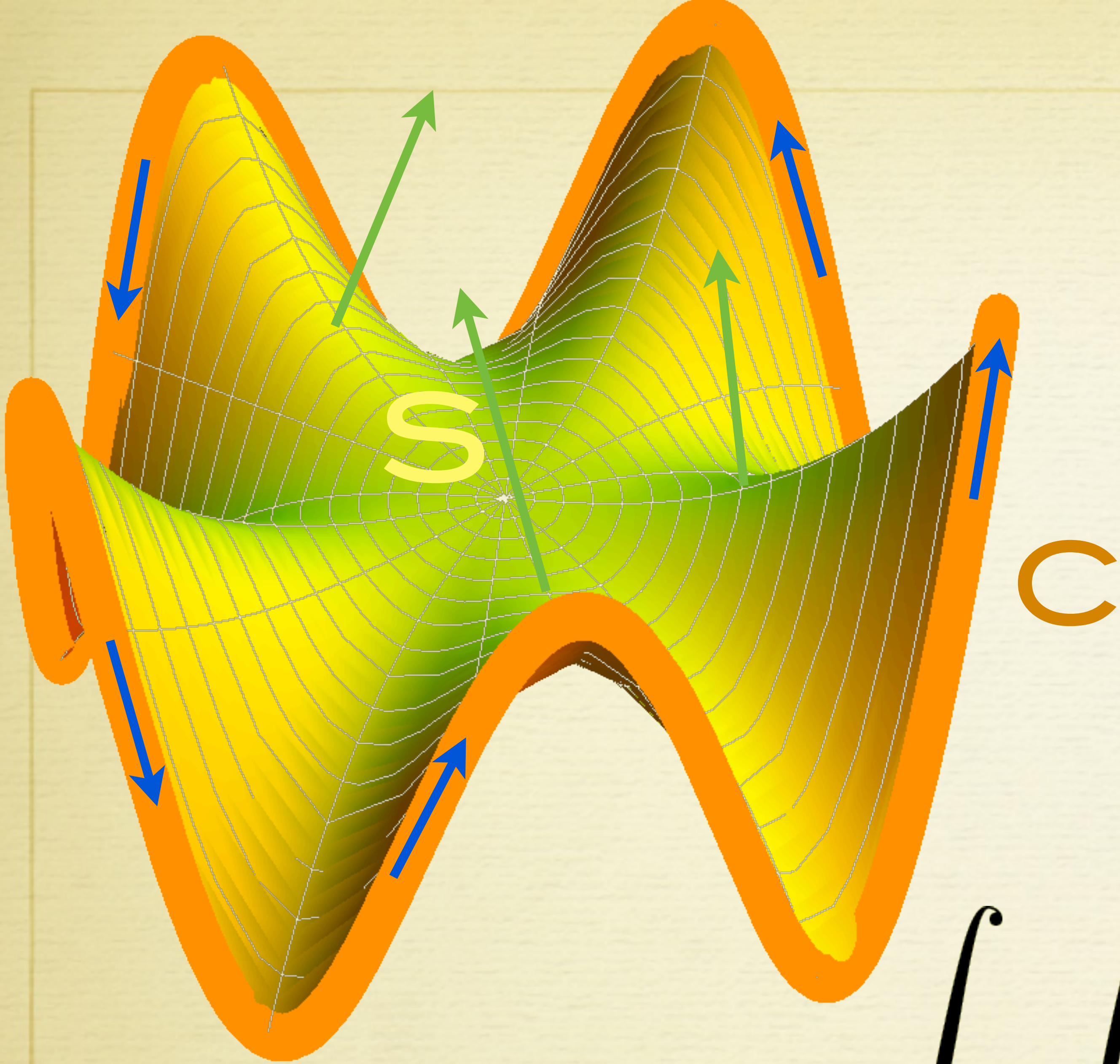


Stokes Theorem

- The Theorem
- Consequences
- Examples

George Gabriel Stokes

Theorem 3



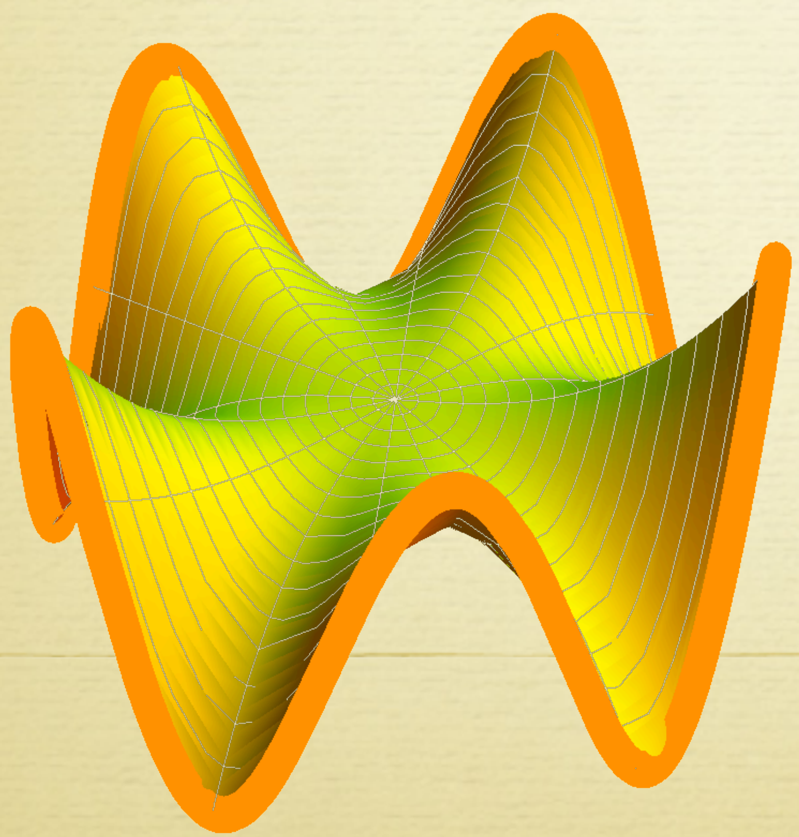
$$\int\int_S \operatorname{curl}(\vec{F}) \cdot d\vec{S}$$

$$= \int_C \vec{F} \cdot d\vec{s}$$

Stokes Theorem

Consequence

If Curl(F) = 0 everywhere, then the field has the closed loop property.
The vector field is a Gradient Field.

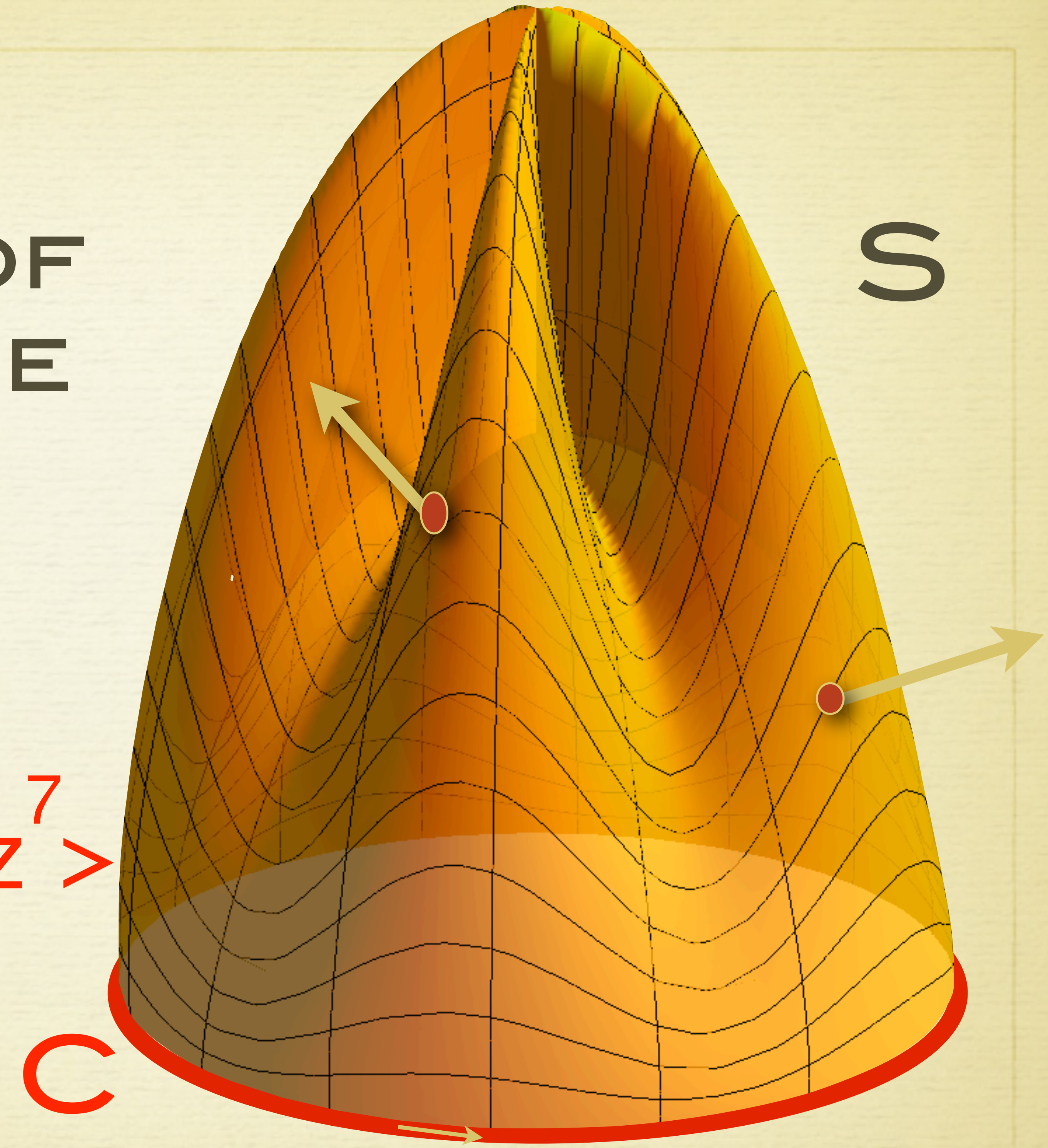


$$\int\int_S \mathrm{curl}(\vec{F}) \cdot d\vec{S}$$

$$= \int_C \vec{F} \cdot d\vec{s}$$

Problem 8

Find the flux of the curl of the vector field through the surface S



$$\vec{F}(x,y,z) = \langle\, x+zy^9, zy, z^7 \,\rangle$$

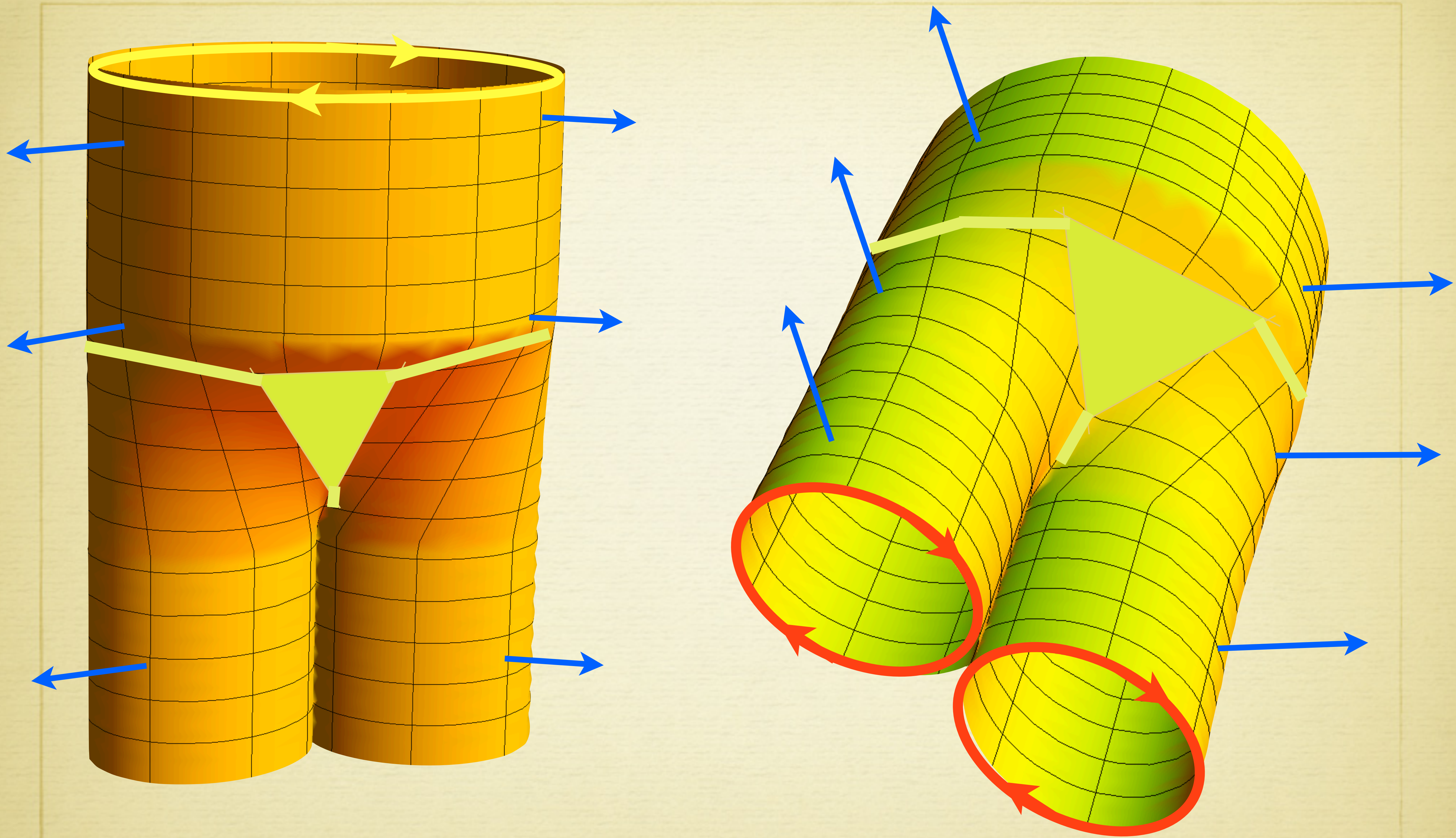
s=π/2 gives C

$0 \le t < 2\pi$

$0 \le s \le \pi/2$

$$\vec{r}(s,t) =$$

$$\langle \cos(t)\sin(s), \sin(t)\sin(s), \cos(3s) + (2+\cos(5t))\cos(s) \rangle$$



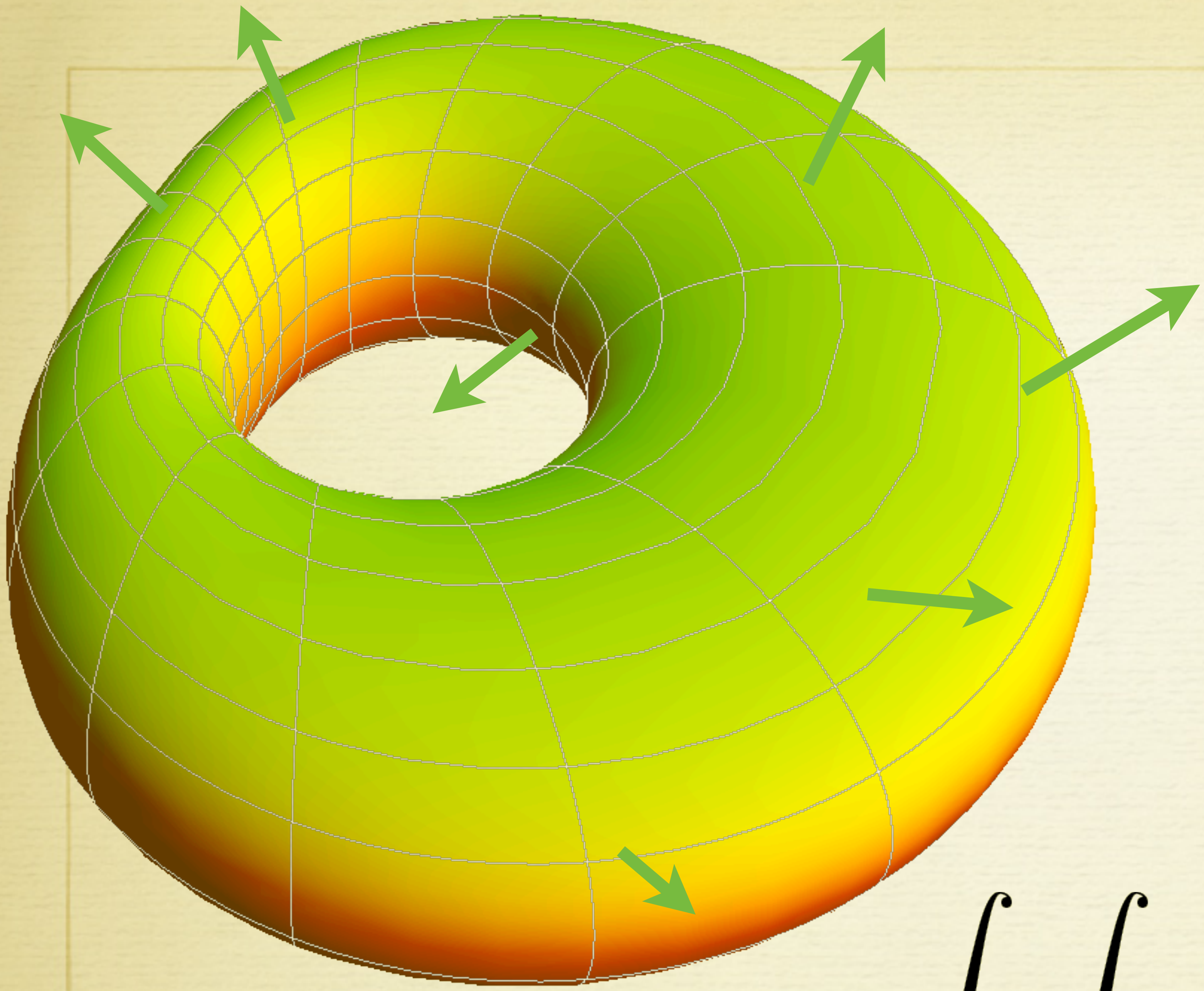
Orientation

Divergence Theorem

- The Theorem
- Consequences
- Examples

Carl Friedrich Gauss

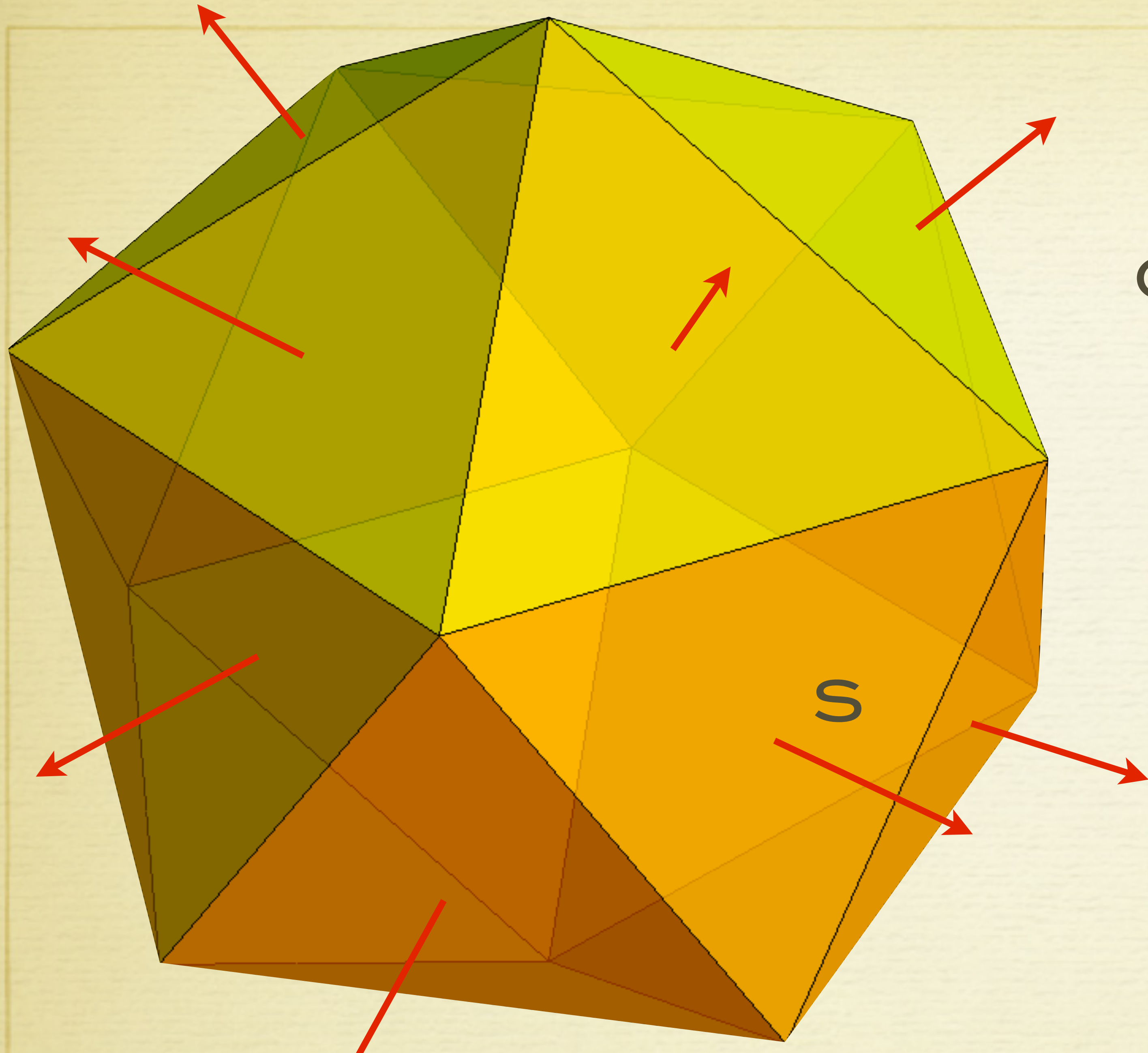
Theorem 4



$$\int\int\int_E \operatorname{div}(\vec{F})\, dV$$

Divergence Theorem

$$= \int\int_S \vec{F} \cdot d\vec{S}$$



S: Side length 1
Centered at origin

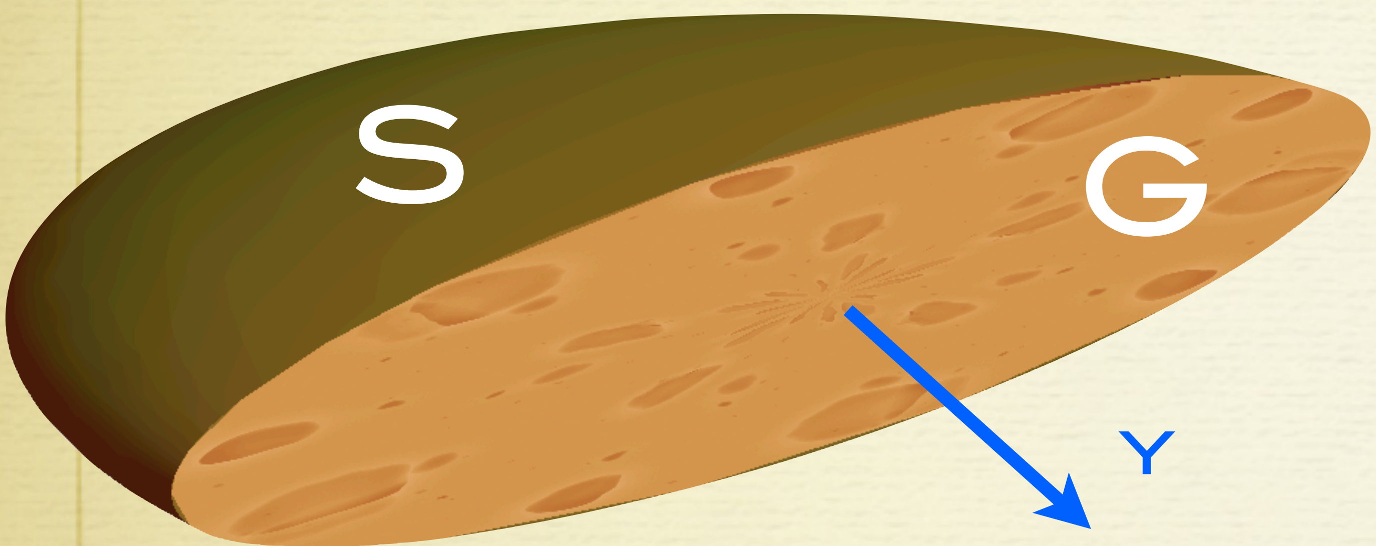
Find the flux of the vector field

$$\vec{F}(x,y,z) = < \sin(y\,z), \cos(x\,z), \tan(x\,y) >$$

Problem 9 Through the Icosahedron

$$\frac{x^2}{25} + \frac{y^2}{25} + z^2 \le 1$$

$$y \ge 0$$

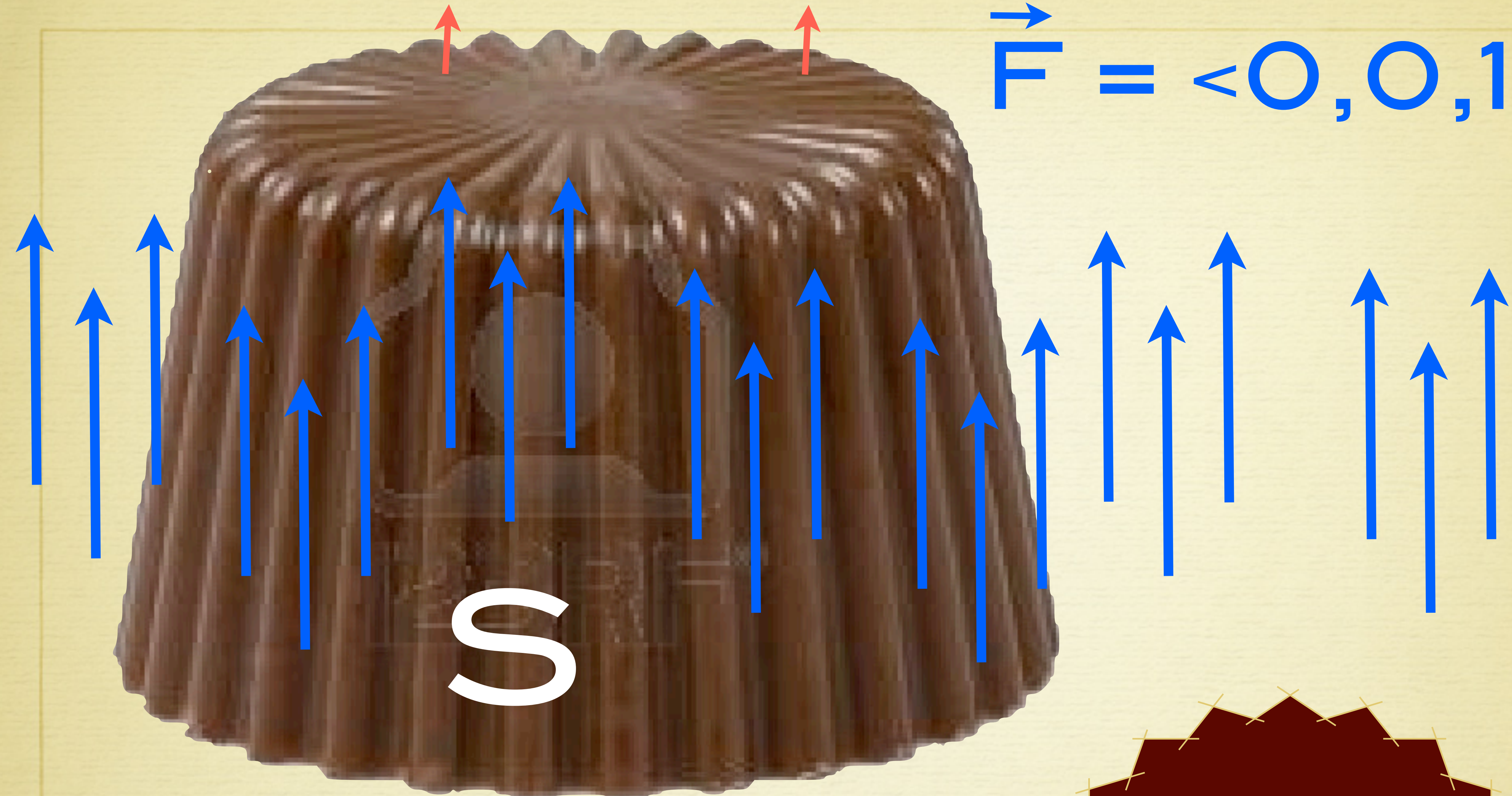


Find the flux of the vector field F through the cheese both S and G oriented outwards

$$\vec{F}(x,y,z) = < x^3\text{-}\sin(y), y^3+1, -z(x^2+y^2)3+2z>$$

Problem

$\vec{F} = \langle 0,0,1 \rangle$



Area π

Find the Flux of F through S oriented upwards

1

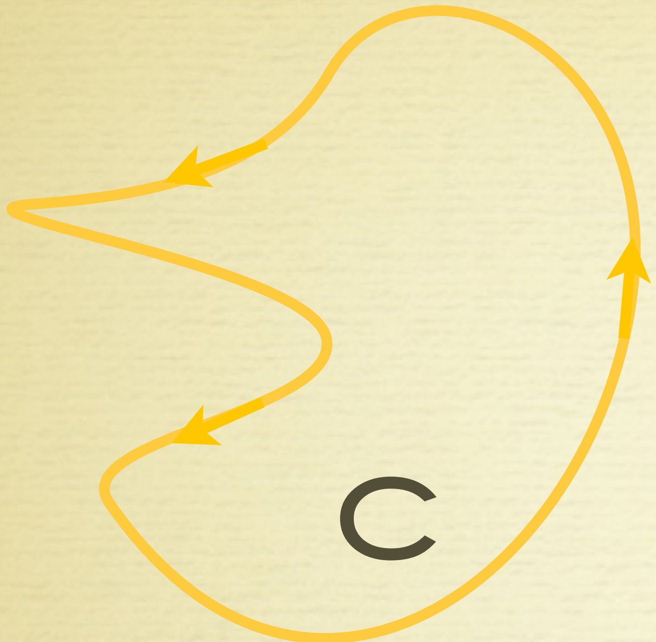
1 f’ 1

1 grad(f) 2 curl(F) 1

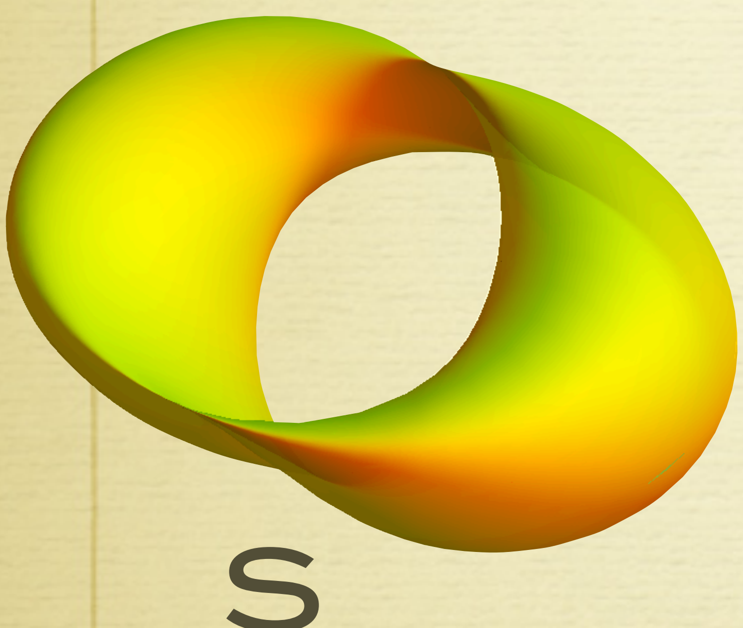
1 grad(f) 3 curl(F) 3 div(F) 1

1 4 6 4 1

Derivatives



$$\int_C \mathrm{grad}(f)\,(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = 0$$



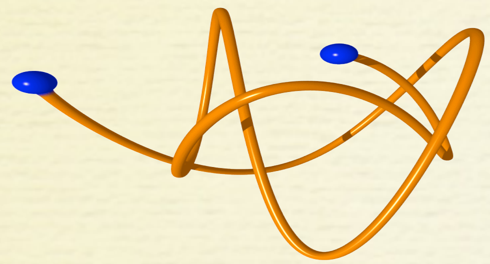
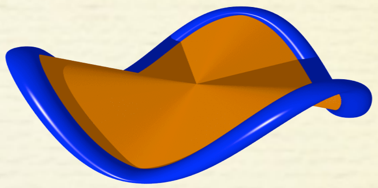
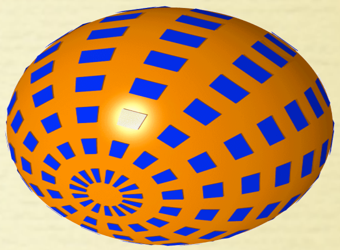
$$\iint_S \mathrm{curl}(\vec{F})(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \, du\,dv = 0$$

identities

$$\mathrm{curl\ grad}\ (f) = 0$$

$$\mathrm{div\ curl}\ (\vec{F}) = 0$$

Overview Theorems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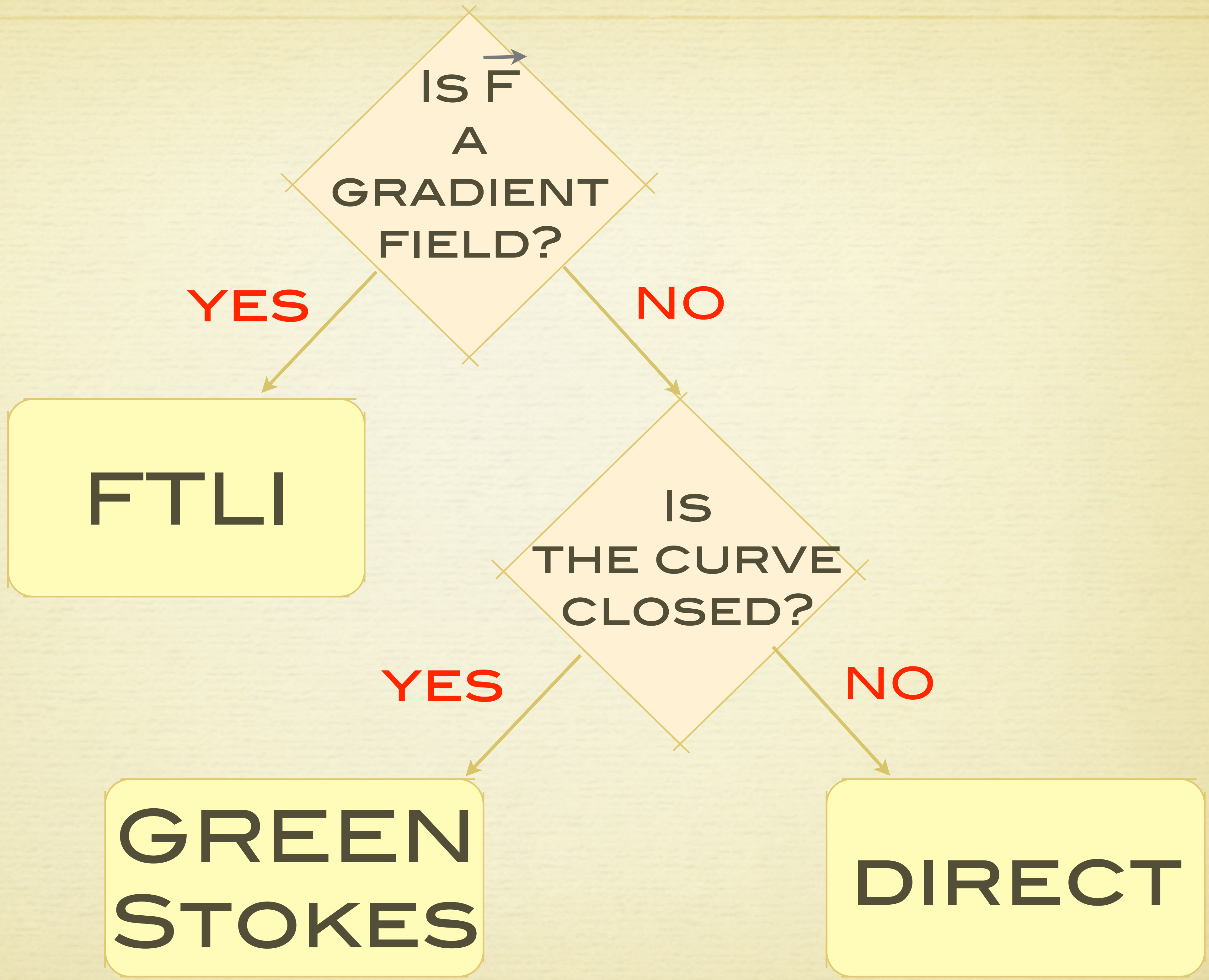
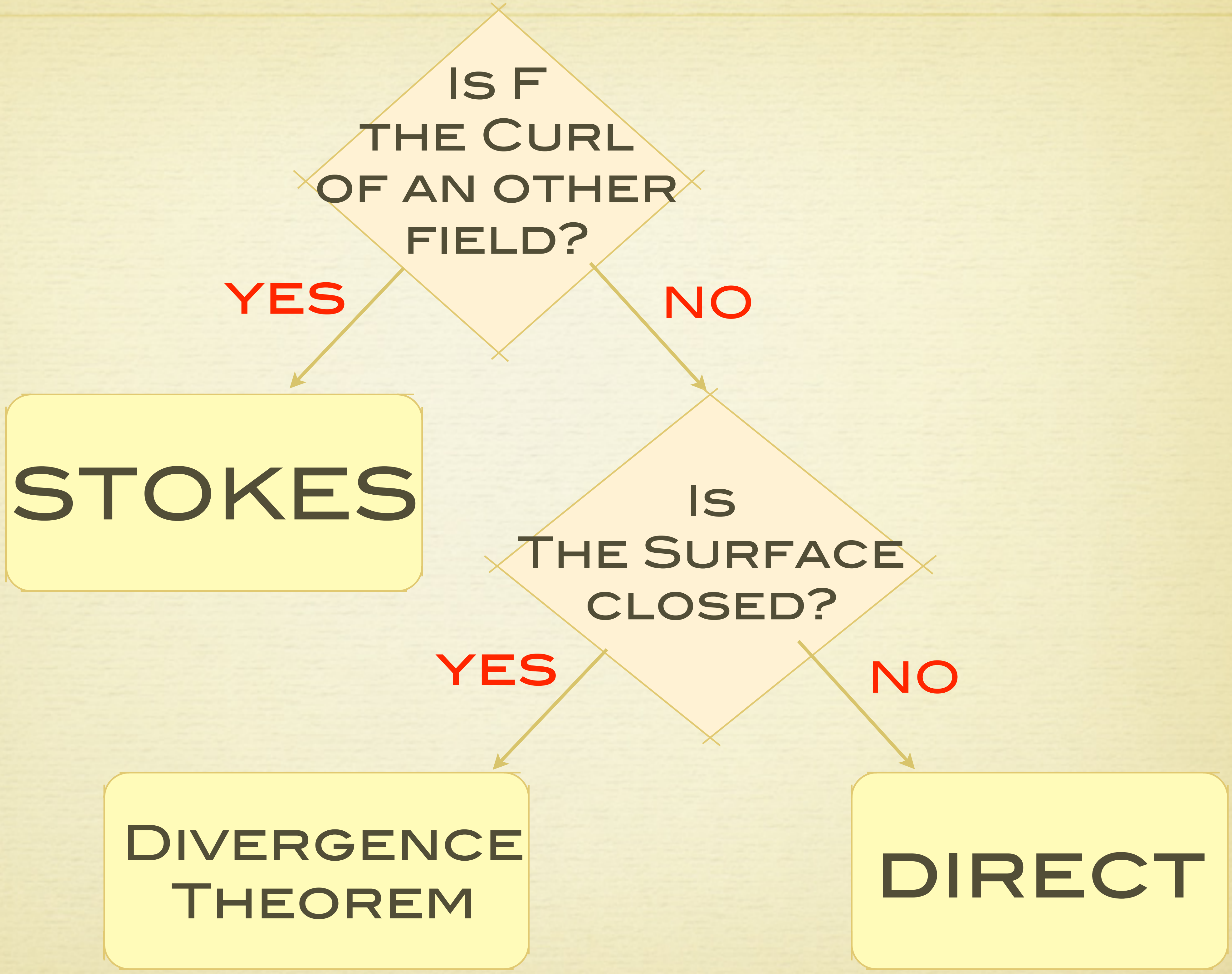


Decision Tree for Line integrals

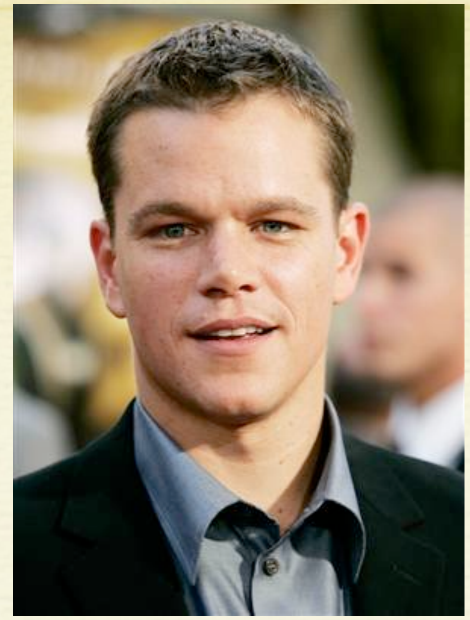


Decision Tree for Flux integrals

CARL FRIEDRICH GAUSS

1777-1855

GEORGE GREEN



1793-1841

GEORGE STOKES

1819-1903

JEAN MARY AMPERE

1775-1846

MIKHAEL OSTROGRADSKY

1801-1862

AUGUSTINE CAUCHY

1789-1857

1750 1800 1850 1900

GAUSS THEOREM

GREEN THEOREM

STOKES THEOREM