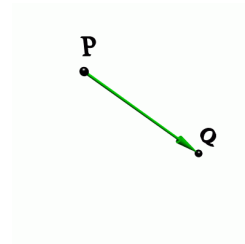


## Distances overview

DISTANCE POINT-POINT (3D). If  $P$  and  $Q$  are two points, then

$$d(P, Q) = |\vec{PQ}|$$

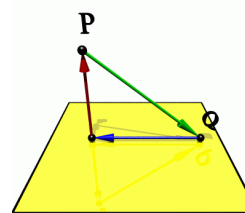
is the distance between  $P$  and  $Q$ . We use the notation  $|\vec{v}|$  instead of  $\|\vec{v}\|$  in this handout.



DISTANCE POINT-PLANE (3D). If  $P$  is a point in space and  $\Sigma : \vec{n} \cdot \vec{x} = d$  is a plane containing a point  $Q$ , then

$$d(P, \Sigma) = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

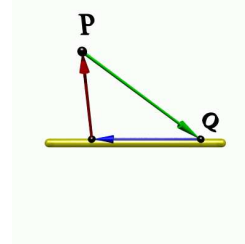
is the distance between  $P$  and the plane. Proof: use the angle formula in the denominator.



DISTANCE POINT-LINE (3D). If  $P$  is a point in space and  $L$  is the line  $\vec{r}(t) = Q + t\vec{u}$ , then

$$d(P, L) = \frac{|(\vec{PQ}) \times \vec{u}|}{|\vec{u}|}$$

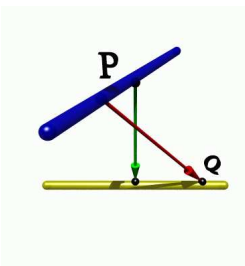
is the distance between  $P$  and the line  $L$ . Proof: the area divided by base length is height of parallelogram.



DISTANCE LINE-LINE (3D).  $L$  is the line  $\vec{r}(t) = Q + t\vec{u}$  and  $M$  is the line  $\vec{s}(t) = P + t\vec{v}$ , then

$$d(L, M) = \frac{|(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|}$$

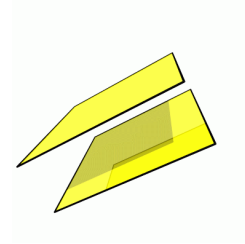
is the distance between the two lines  $L$  and  $M$ . Proof: the distance is the length of the vector projection of  $\vec{PQ}$  onto  $\vec{u} \times \vec{v}$  which is normal to both lines.



DISTANCE PLANE-PLANE (3D). If  $\vec{n} \cdot \vec{x} = d$  and  $\vec{n} \cdot \vec{x} = e$  are two parallel planes, then their distance is

$$\frac{|e - d|}{|\vec{n}|}$$

Non-parallel planes have distance 0. Proof: use the distance formula between point and plane.

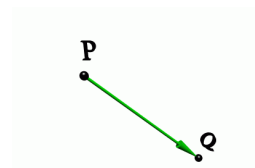


## EXAMPLES

DISTANCE POINT-POINT (3D).  $P = (-5, 2, 4)$  and  $Q = (-2, 2, 0)$  are two points, then

$$d(P, Q) = |\vec{PQ}| = \sqrt{(-5 + 2)^2 + (2 - 2)^2 + (0 - 4)^2} = 5.$$

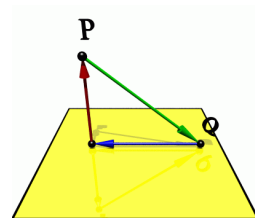
Question to the reader: what is the distance between the point  $(-5, 2, 4)$  and the sphere  $(x + 2)^2 + (y - 2)^2 + z^2 = 1$ ?



DISTANCE POINT-PLANE (3D).  $P = (7, 1, 4)$  is a point and  $\Sigma : 2x + 4y + 5z = 9$  is a plane which contains the point  $Q = (0, 1, 1)$ . Then

$$d(P, \Sigma) = \frac{|\langle -7, 0, -3 \rangle \cdot \langle 2, 4, 5 \rangle|}{|\langle 2, 4, 5 \rangle|} = \frac{29}{\sqrt{45}}$$

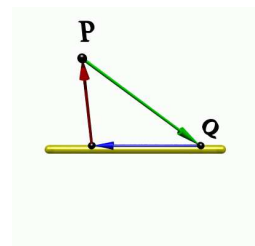
is the distance between  $P$  and  $\Sigma$ . Question to the reader: without the absolute value, the result is negative. What does this tell about the point  $P$ ?



DISTANCE POINT-LINE (3D).  $P = (2, 3, 1)$  is a point in space and  $L$  is the line  $\vec{r}(t) = (1, 1, 2) + t(5, 0, 1)$ . Then

$$d(P, L) = \frac{|\langle -1, -2, 1 \rangle \times \langle 5, 0, 1 \rangle|}{|\langle 5, 0, 1 \rangle|} = \frac{|\langle -2, 6, 10 \rangle|}{\sqrt{26}} = \frac{\sqrt{140}}{\sqrt{26}}$$

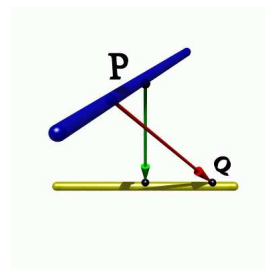
is the distance between  $P$  and  $L$ . Question to the reader: what is the equation of the plane which contains the point  $P$  and the line  $L$ ?



DISTANCE LINE-LINE (3D).  $L$  is the line  $\vec{r}(t) = (2, 1, 4) + t(-1, 1, 0)$  and  $M$  is the line  $\vec{s}(t) = (-1, 0, 2) + t(5, 1, 2)$ . The cross product of  $\langle -1, 1, 0 \rangle$  and  $\langle 5, 1, 2 \rangle$  is  $\langle 2, 2, -6 \rangle$ . The distance between these two lines is

$$d(L, M) = \frac{|(3, 1, 2) \cdot \langle 2, 2, -6 \rangle|}{|\langle 2, 2, -6 \rangle|} = \frac{4}{\sqrt{44}}.$$

Question to the reader: also here, without the absolute value, the formula can give a negative result. What happens with this sign, when  $P$  and  $Q$  are interchanged?



DISTANCE PLANE-PLANE (3D).  $5x + 4y + 3z = 8$  and  $5x + 4y + 3z = 1$  are two parallel planes. Their distance is

$$\frac{|8 - 1|}{|\langle 5, 4, 3 \rangle|} = \frac{7}{\sqrt{50}}.$$

Question for the reader: what is the distance between the planes  $x + 3y - 2z = 2$  and  $5x + 15y - 10z = 30$ ?

