10/2/2012 FIRST HOURLY PRACTICE II **Math 21a, Fall 2012**

Name:

MWF 9 Oliver Knill
MWF 10 Hansheng Diao
MWF 10 Joe Rabinoff
MWF 11 John Hall
MWF 11 Meredith Hegg
MWF 12 Charmaine Sia
TTH 10 Bence Beky
TTH 10 Gijs Heuts
TTH 11:30 Francesco Cavazzani
TTH 11:30 Andrew Cotton-Clay

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-2 and 8, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False (TF) questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) **T** F The length of the vector $\langle 4,2,4\rangle$ is an integer.

Solution:
It is the square root of $4^2+2^2+4^2$ which is 6.

2) T **F** Any three distinct points A,B,C in space determine a unique plane which passes through these points.

Solution:
The three points can be on a line.

3) **T** F For any two non-intersecting lines L,K, there are two parallel planes Σ,Δ whose distance $d(\Sigma,\Delta)$ is equal to the distance $d(L,K)$ such that L is in Σ and K is in Δ.

Solution:
The two planes are spanned by the vectors in the lines.

4) **T** F If $z-f(x,y)=g(x,y,z)$ then the graph of $f(x,y)$ is a level surface $g(x,y,z)=c$ of $g(x,y,z)$.

Solution:
It is the level surface to $c=0$.

5) T **F** The graph of the function $f(x,y)=x^2+y$ is called an elliptic paraboloid.

Solution:
It is a parabolic cylinder

6) **T** F The equation $\rho\sin(\phi)\sin(\theta)=1$ in spherical coordinates defines a plane.

Solution:
Use spherical coordinates. We have $y = 1$.

7) T **F** The vector $\langle 1, 2, 3\rangle$ is parallel to the plane $2x + 4y + 6z = 4$.

Solution:
The vector $\langle 2, 4, 6\rangle$ is the normal vector.

8) T **F** The cross product between $\langle 2, 3, 1\rangle$ and $\langle 1, 1, 1\rangle$ is 6.

Solution:
It is the dot product which is 6. The cross product is a vector.

9) **T** F The curve $\vec{r}(t) = \langle \cos(t), t^2, \sin(t)\rangle, 1 \leq t \leq 9$ and the curve $\vec{r}(t) = \langle \cos(t^2), t^4, \sin(t^2)\rangle, 1 \leq t \leq 3$ have the same length.

Solution:
This is a change of parametrization

10) T **F** The point $(1, -1, 1)$ has the spherical coordinates of the form $(\rho, \theta, \phi) = (\sqrt{3}, \pi/4, \pi/4)$.

Solution:
Apply the transformation formulas. We have $\theta = -\pi/4$.

11) **T** F The distance between two parallel lines in space is the distance of any point on one line to the other line.

Solution:
Note that this is only true for parallel lines.

12) **T** F For two nonzero vectors $\vec{v}$ and $\vec{w}$, the identity $\text{Proj}_{\vec{w}}(\vec{v} \times \vec{w}) = \vec{0}$ holds.

Solution:
the vector $(\vec{v} \times \vec{w})$ is projected onto a vector perpendicular to it.

13) **T** F The vector projection of $\langle 2, 3, 4\rangle$ onto $\langle 1, 0, 0\rangle$ is $\langle 2, 0, 0\rangle$.

Solution:
Apply the formula. Because the vector on which we project has length 1, the result is the dot product times this vector.

14) T **F** The triple scalar product $\vec{u} \cdot (\vec{v} \times \vec{w})$ between three vectors $\vec{u}, \vec{v}, \vec{w}$ is zero if and only if two or more of the 3 vectors are parallel.

Solution:
They can be nonparallel but in the same plane.

15) **T** F There are two vectors $\vec{v}$ and $\vec{w}$ so that the dot product $\vec{v} \cdot \vec{w}$ is equal to the length of the cross product $|\vec{v} \times \vec{w}|$.

Solution:
Take two vectors which make an angle of 45 degrees. Then $\sin(\theta) = \cos(\theta)$.

16) **T** F The distance between two spheres of radius 2 whose centers have distance 8 is 4.

Solution:
The connection between the centers is also the connection between the nearest points on the sphere.

17) **T** F If two vectors $\vec{v}$ and $\vec{w}$ are both parallel and perpendicular, then at least one of the vectors must be the zero vector.

Solution:
If $\vec{v} = \lambda\vec{w}$, then $\vec{v} \cdot \vec{w} = |\vec{v}|\vec{w}| = 0$ implies that one must be empty.

18) T **F** The curvature $\kappa(\vec{r}(t))$ is always smaller than or equal to the length $|\vec{r}\,''(t)|$ of the acceleration vector $\vec{r}\,''(t)$.

Solution:
If you drive along a circle very slowly the acceleration is small but the curvature is the same.

19) **T** F The curve $\vec{r}(t) = \langle \cos(t^2)\sin(t^2), \sin(t^2)\sin(t^2), \cos(t^2) \rangle$ is located on a sphere.

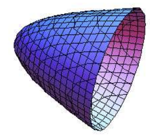
Solution:
Check $x^2 + y^2 + z^2 = 1$.

20) **T** F The surface $x^2 + y^2 + z^2 = 2z$ is a sphere.

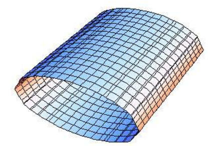
Solution:
Complete the square to see that it is indeed a sphere centered at $(0, 0, 1)$ with radius 1.

Problem 2) (10 points)

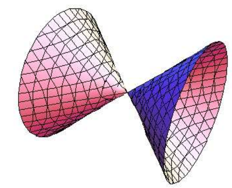
a) (6 points) Match the surfaces the equations $g(x, y, z) = 0$.



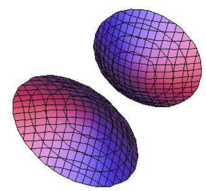
I



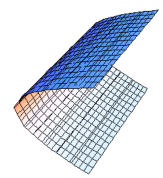
II



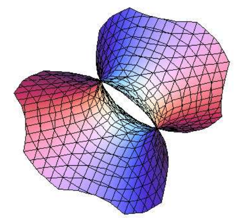
III



IV



V



VI

Function $g(x, y, z) = 0$	Enter I,II,III,IV,V,VI
$y^2 + z^2 - x$	
$y^2/4 + z^2/4 - 1$	
$x^2 - y^2 - z^2 - 1$	

Function $g(x, y, z) =$	Enter I,II,III,IV,V,VI
$x^2 - y^2 - z^2 + 1$	
$x - z^2$	
$y^2 - z^2 + x^2$	

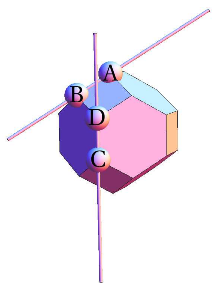
b) (4 points) Match the surfaces given in cylindrical and spherical coordinates with the surfaces given in Cartesian coordinates:

	surface
A	$r = 1$
B	$\sin(\theta) = 0$
C	$\cos(2\phi) = 0$
D	$\rho = 1$

Enter A-D	surface
	$x^2 + y^2 = 1$
	$x^2 + y^2 = z^2$
	$x^2 + y^2 + z^2 = 1$
	$y = 0$

Solution:
a) left: I II IV right: VI,V,III
b) ACDB

Problem 3) (10 points)



A **truncated octahedron** has an edge connecting the vertices $A = (-1, 3, 0), B = (-1, 1, -1)$ and an edge connecting the vertices $C = (-3, -1, 0), D = (-3, 1, 0)$.

a) (5 points) Find the distance of C to the line through A, B.

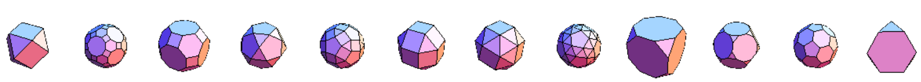
b) (5 points) Find the distance between the line L through A, B and the line K through C, D.

Solution:
a) These are typical distance formula problems: the first is a distance between a point and a line

$$d = \frac{|\vec{AC} \times \vec{AB}|}{|\vec{AB}|} = 6/\sqrt{5} \; .$$

b) The second problem asks for the distance between a line and a point:

$$d = \frac{|\vec{AC} \cdot (\vec{AB} \times \vec{CD})|}{|\vec{AB} \times \vec{CD}|} | = |-4|/2 = 2 \; .$$



Here are the remaining 12 **Archimedean solids**. These are polyhedra bound by different types of regular polygons but for which each vertex of the polyhedron looks the same. There are 13 such semiregular polyhedra. Archimedes studied them first in 287BC. Kepler was the first to describe the complete set of 13 in his work **"Harmonices Mundi"** of 1619.

Problem 4) (10 points)

a) (3 points) Give a parametrization $\vec{r}(\theta, z) = \langle x(\theta, z), y(\theta, z), z(\theta, z) \rangle$ of the surface which is in cylindrical coordinates given by

$$r = z^4 \; .$$

b) (2 points) Find a parametrization $\vec{r}(u, v)$ of the graph $z = \sin(xy)$.

c) (2 points) Find a parametrization $\vec{r}(u, v)$ of the yz-plane $x = 0$.

d) (3 points) Give a parametrization $\vec{r}(\phi, \theta)$ of the surface which is in spherical coordinates given by

$$\rho = 2 + \cos(13\phi) \; .$$

Solution:
a) This is a typical surface of revolution

$$\vec{r}(\theta, z) = \langle z^4 \cos(\theta), z^4 \sin(\theta), z \rangle \; .$$

b) This is a typical graph of a function of two variables:

$$\vec{r}(u, v) = \langle u, v, \sin(uv) \rangle$$

c) This is a typical plane

$$\vec{r}(u, v) = \langle 0, u, v \rangle$$

d) This is a typical modification of a sphere. It is called a bumpy sphere:

$$\vec{\rho}(\phi, \theta) = \langle (2 + \cos(13\phi) \sin(\phi)) \cos(\theta), (2 + \cos(13\phi)) \sin(\phi) \sin(\theta), (2 + \cos(13\phi)) \cos(\phi) \rangle$$

Problem 5) (10 points)

a) (7 points) Find the arc length of the curve

$$\vec{r}(t) = \langle \cos(t^2/2), \sin(t^2/2), (1/3)(1 - t^2)^{3/2} \rangle$$

from $0 \leq t \leq 1$.

b) (3 points) Decide whether the function

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

is continuous.

Solution:
a) We have

$$\vec{r}\,'(t) = \langle -t\sin(t^2/2), t\cos(t^2/2), t(1-t^2)^{1/2}$$

so that the speed is

$$|\vec{r}(t)| = t\sqrt{2-t^2}$$

and

$$\int_0^1 |\vec{r}(t)|\; dt = \int_0^1 t\sqrt{2-t^2}\; dt = -(1/3)(2-t^2)^{3/2}|_0^1 = (2\sqrt{2}-1)/3$$

b) Yes, the function is continuous. It is obviously continuous everywhere except the origin. To investigate the origin, us polar coordinates. We have

$$f(r,\theta) = r\cos(\theta)\sin^2(\theta)\ .$$

Since $\cos(\theta)\sin^2(\theta)$ stays bounded as $r \to 0$, the value of f approaches 0 as we approach the origin.

Problem 6) (10 points)

Wall-e explores a planet, where a strong solar wind produces a time-dependent magnetic field and where the combined force of gravity and magnetic lift produces a time dependent vertical acceleration

$$\vec{r}\,''(t) = \langle 0, 0, 10\cos(t)\rangle\ .$$

a) (6 points) Wall-e knows that he is at time $t = 0$ at $\vec{r}(0) = \langle 1, 2, 3\rangle$ with velocity $\langle 0, 1, 2\rangle$. Where is he at time $t = \pi$?

b) (4 points) What speed does he have at time $t = \pi$?

Solution:
a) Integrate up the equation

$$\vec{r}\,''(t) = \langle 0, 0, 10\cos(t)\rangle$$

to get

$$\vec{r}\,'(t) = \langle 0, 0, 10\sin(t)\rangle + \langle 0, 1, 2\rangle$$

The constant part to the right was obtained by comparing with $\vec{r}\,'(0)$.
Now integrate again:

$$\vec{r}\ (t) = \langle 0, 0, -10\cos(t)\rangle + \langle 0, t, 2t\rangle + \langle 1, 2, 3\rangle\ ,$$

where the constant part to the right was obtained by comparing with $\vec{r}(0)$. At time $t = \pi$, he is at

$$\vec{r}(\pi) = \langle 0, 0, 10\rangle + \langle 0, \pi, 2\pi\rangle + \langle 1, 2, 13\rangle = \langle 1, 2+\pi, 2\pi+23\rangle\ .$$

b) We obtain the speed, the length of the velocity vector $\vec{r}\,'(t)$ at $t = \pi$ as follows:

$$\vec{r}\,'(\pi) = |\langle 0, 1, 2\rangle| = \sqrt{5}\ .$$

Problem 7) (10 points)

Potter plays Quidditch. At time $t = 0$ he is at $P = (1, 3, 5)$. At time $t = 1$ he is at $Q = (0, 1, 3)$. Harry is spell-bound and can not change direction, nor change speed and crashes into a tilted side wall of the stadium crushing his knee (*).

a) (3 points) If Potter flies on a straight line through PQ, find a parametrization for that line.

b) (4 points) Where and when does he hit the tilted side wall $x + y + z = 1$ of the stadium?

c) (3 points) What is the angle between Harry's velocity vector and the upwards pointing normal vector of the side wall?

(*) Don't worry, Madam Pomfrey will fix it.

Solution:
a) $\vec{r}(t) = \langle 1, 3, 5\rangle + t\langle -1, -2, -2\rangle = \langle 1 - t, 3 - 2t, 5 - 2t\rangle$.
b) We look for the time t, where $x(t)+y(t)+z(t) = 1$. This is $(1-t)+(3-2t)+(5-2t) = 9 - 5t = 1$. which gives $t = 8/5$. Now $\vec{r}(8/5) = \langle -3, -1, 9\rangle/5$.
c) The velocity vector is $\langle -1, -2, -2\rangle$. The upwards pointing normal vector to the side wall is the vector $\langle 1, 1, 1\rangle$. The dot product is -5 and the cos of the angle is $-5/(3\sqrt{3})$. The angle is $\arccos(-5/(3\sqrt{3}))$.

Problem 8) (10 points)

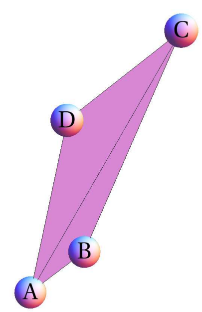
No justifications are needed in this problem. All vectors $\vec{v}, \vec{w}, \vec{r}\,', \vec{r}\,'', \vec{T}, \vec{T}\,'$ can be assumed to be nonzero. The vector $\vec{N} = \vec{T}'/|\vec{T}'|$ is the normal vector and $\vec{B}$ is the binormal vector. Recall that two vectors are perpendicular, if and only if their dot product is zero and that two vectors are parallel if and only if their cross product is the zero vector.

first vector	second vector	always parallel	always perpendicular	neither
$\vec{T}'(0)$	$\vec{r}''(0)$			
$\vec{w}$	$(\vec{v} \times \vec{w}) \times \vec{v}$			
$\vec{v}$	$\vec{v}/\|\vec{v}\|$			
$\vec{v} = \langle a, b, c\rangle$	normal to $ax + by + cz = 4$			
$\vec{T}$	$\vec{r}\,'$			
$\vec{T}$	$\vec{T}\,'$			
$\text{Proj}_{\vec{v}}(\vec{w})$	$\vec{v}$			
$\text{Proj}_{\vec{v}}(\vec{w})$	$\vec{w}$			
$(\vec{v} + \vec{w}) \times \vec{w}$	$\vec{v} \times \vec{w}$			
$\vec{B}$	$\vec{N}$			

Solution:

first vector	second vector	always parallel	always perpendicular	neither
$\vec{T}'(0)$	$\vec{r}''(0)$			*
$\vec{w}$	$(\vec{v} \times \vec{w}) \times \vec{v}$			*
$\vec{v}$	$\vec{v}/\|\vec{v}\|$	*		
$\vec{v} = \langle a, b, c\rangle$	normal to $ax + by + cz = 4$	*		
$\vec{T}$	$\vec{r}\,'$	*		
$\vec{T}$	$\vec{T}\,'$		*	
$\text{Proj}_{\vec{v}}(\vec{w})$	$\vec{v}$	*		
$\text{Proj}_{\vec{v}}(\vec{w})$	$\vec{w}$			*
$(\vec{v} + \vec{w}) \times \vec{w}$	$\vec{v} \times \vec{w}$	*		
$\vec{B}$	$\vec{N}$		*	

Problem 9) (10 points)



The four points $A = (0, 0, 5), B = (1, 1, 6), C = (2, 4, 11), D = (0, 2, 9)$ are in a plane.

a) (5 points) Find the equation $ax + by + cz = d$ for this plane.

b) (5 points) The quadrilateral $ABCD$ is the union of two triangles ABC and ACD. Find the area of the quadrilateral.

Solution:
a) To get the equation of the plane, we find the normal vector

$$\vec{n} = \vec{AB} \times \vec{AC} = \langle 2, -4, 2 \rangle \ .$$

The equation is $2x - 4y + 2z = d$, where d can be obtained by plugging in one point. It is 10. The plane is $x - 2y + z = 5$.
b) The first triangle area is half the length of the vector $\vec{AB} \times \vec{AC}$ which is $\sqrt{6}$. The second triangle area is half the length of the vector $\vec{AC} \times \vec{AD}$ which is half the length of $\langle 4, 8, 4 \rangle$ and which is $2\sqrt{6}$. The quadrilateral therefore has the area $3\sqrt{6} = \sqrt{54}$.

Problem 10) (10 points)

In this problem we find some parametrizations of surfaces which is of the form

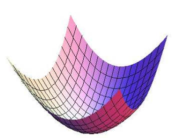
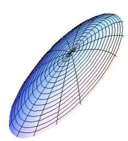
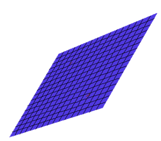
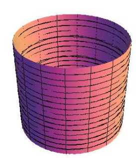
$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle \ .$$

a) (2 points) Parametrize the **paraboloid** $z = x^2 - y^2$.

b) (3 points) Parametrize (the entire!) **ellipsoid** $(x - 1)^2 + \frac{(y-2)^2}{4} + z^2 = 1$.

c) (2 points) Parametrize the **plane** $x + y + z = 3$.

d) (3 points) Parametrize the **cylinder** $x^2 + z^2 = 1$.

Solution:
a) $\vec{r}(u, v) = \langle u, v, u^2 - v^2 \rangle$. This is a graph.
b) $\vec{r}(u, v) = \langle 1 + \cos(u)\sin(v), 2 + 2\sin(u)\sin(v), \cos(v) \rangle$. In this problem, a graph parametrization like $\langle u, v, f(u, v) \rangle$ would not give the entire ellipsoid.
c) $\vec{r}(u, v) = \langle 3, 0, 0 \rangle + u\langle 3, -3, 0 \rangle + v\langle 3, 0, -3 \rangle$. There are of course many possibilities here. An other simple solution is $\langle u, v, 3 - u - v \rangle$.
d) $\vec{r}(u, v) = \langle \cos(u), v, \sin(u) \rangle$. In this problem, the order was often incorrect and the standard cylinder $\langle \cos(u), \sin(u), v \rangle$ along the z-axes taken.