10/2/2012 FIRST HOURLY PRACTICE III **Math 21a, Fall 2012**

Name:

MWF 9 Oliver Knill
MWF 10 Hansheng Diao
MWF 10 Joe Rabinoff
MWF 11 John Hall
MWF 11 Meredith Hegg
MWF 12 Charmaine Sia
TTH 10 Bence Beky
TTH 10 Gijs Heuts
TTH 11:30 Francesco Cavazzani
TTH 11:30 Andrew Cotton-Clay

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-2 and 8, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False (TF) questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) T F The surface $x^2 + y^2 + z^2 + 2z = 0$ is a sphere.

Solution:
Complete the square: add 1 on both sides, to get $x^2 + y^2 + (z+1)^2 = 1$ which is a sphere.

2) T F The length of the vector $\langle 1, 2, 2 \rangle$ is an integer.

Solution:
Indeed, $1 + 4 + 4 = 9$ is a perfect square.

3) T F The vector $\langle 3, 4 \rangle$ appears as a velocity vector of the curve $\vec{r}(t) = \langle \cos(5t), \sin(5t) \rangle$. Namely, there is a t such that $\vec{r}\,'(t) = \langle 3, 4 \rangle$.

Solution:
The velocity vector of that curve has length 5 and takes any possible direction, so also the direction of the vector $\langle 3, 4 \rangle$.

4) T F If $\vec{T}$ is the unit tangent vector, $\vec{N}$ is the unit normal vector, and $\vec{B}$ is the binormal vector, then $\vec{B} \times \vec{N} = \vec{T}$.

Solution:
The sign is wrong.

5) T F The curvature of a larger circle $r = 2$ is greater than the curvature of a smaller circle $r = 1/2$.

Solution:
False. The curvature of a circle of radius r is 1/r.

6) T F The surface $x^2 - y^2 - z^2 - 1 = 0$ is a one sheeted hyperboloid.

Solution:
It is a two sheeted one. Make a completion of a square

7) T **F** The function $f(x, y) = y^2 - x^2$ has a graph that is an elliptic paraboloid.

Solution:
It is a hyperbolic paraboloid.

8) **T** F Let $\vec{r}(t)$ be a parametrization of a curve. If $\vec{r}(t)$ is always parallel to the tangent vector $\vec{r}\,'(t)$, then the curve is part of a line through the origin.

Solution:
$\vec{T}\,'$ is zero at all times, so since $\vec{r}\,'(t)$ is parallel to the position vector $\vec{r}(t)$, the curve must lie on a line through the origin.

9) T **F** If $\text{proj}_{\vec{k}}(\vec{u})$ is perpendicular to $\vec{u}$, then $\vec{u}$ is the zero vector.

Solution:
Take $\vec{u} = \vec{i}$.

10) **T** F If $\text{proj}_{\vec{k}}(\vec{u})$ is perpendicular to $\vec{u}$, then $\text{proj}_{\vec{k}}(\vec{u})$ is the zero vector.

Solution:
Assume $\mathbf{proj}_{\vec{k}}(\vec{u})$ is a nonzero vector in the direction of $\vec{k}$. This means that $\vec{k}$ is perpendicular to $\vec{u}$, so $\text{proj}_{\vec{k}}(\vec{u}) = \vec{0}$ by the definition of the projection.

11) T **F** If $\vec{u} \times \vec{v} = \vec{0}$ then $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$.

Solution:
The two vectors can be parallel.

12) T F

There are two vectors $\vec{a}$ and $\vec{b}$ such that the scalar projection of $\vec{a}$ onto $\vec{b}$ is 100 times the magnitude of $\vec{b}$.

Solution:
Take $\vec{b} = 100\vec{a}$.

13) T F

The curve $\vec{r}(t) = \langle \cos(t), e^t + 10, t^2 \rangle, 2 \leq t \leq 6$ and the curve $\vec{r}(t) = \langle \cos(2t), e^{2t}, 4t^2 \rangle, 1 \leq t \leq 3$ have the same length.

Solution:
Make a change of variables.

14) T F

The equation $\rho \sin(\phi) - 2\sin(\theta) = 0$ in spherical coordinates defines a two sheeted hyperboloid.

Solution:
The equation means $r = 2\sin(\theta)$ or $x^2 + y^2 = 2y$.

15) T F

If triple scalar product of three vectors $\vec{u}, \vec{v}, \vec{w}$ is larger than $|\vec{u} \times \vec{v}|$ then $|\vec{w}| > 1$.

Solution:
Think of the triple scalar product as the volume. If that is larger than the base area, the height has to be bigger than than 1.

16) T F

The distance between the x-axis and the line $x = y = 1$ is $\sqrt{2}$.

Solution:
The distance is 1. The distance between $x = y = 1$ and the z-axis would be $\sqrt{2}$.

17) T F

The vector $\langle -1, 2, 3 \rangle$ is perpendicular to the plane $x - 2y - 3z = 9$.

Solution:
It is. Because $\langle 1, -2, -3 \rangle$ is perpendicular to the plane.

18) T F The curve $\vec{r}(t) = t^3\langle 1,2,3\rangle$ is a line.

Solution:
Even so t appears not in a linear way, the curve is still a line.

19) T F The point $(1,1,-\sqrt{3})$ is in spherical coordinates given by $(\rho,\theta,\phi) = (\sqrt{5},\pi/4,2\pi/3)$.

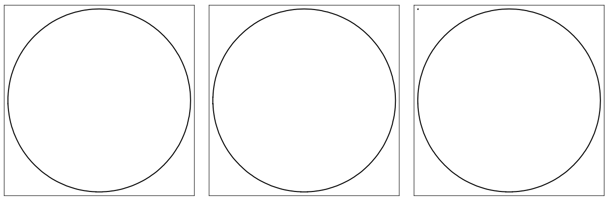
Solution:
Also the z coordinate is wrong.

20) T F If the cross product satisfies $(\vec{v} \times \vec{w}) \times \vec{v} = \vec{0}$ then $\vec{v}$ and $\vec{w}$ are orthogonal.

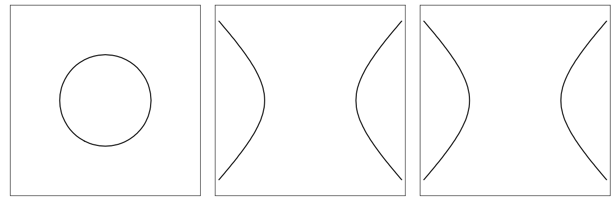
Solution:
One can have $\vec{v} = \vec{w}$ in which case the two vectors are not orthogonal and still the product in question is zero.

Problem 2a) (6 points)

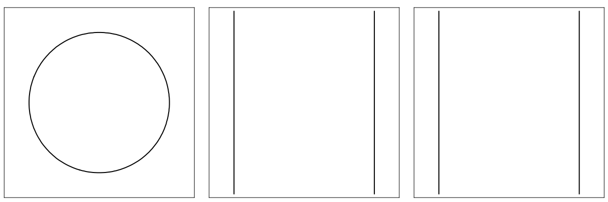
xy-trace yz-trace xz-trace xy-trace yz-trace xz-trace



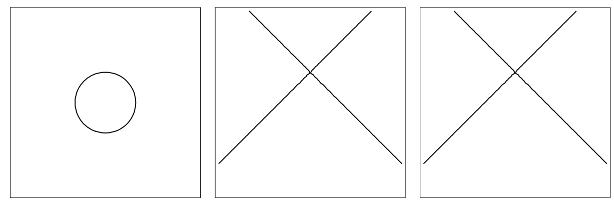
A



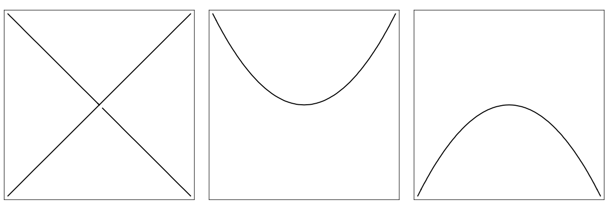
B



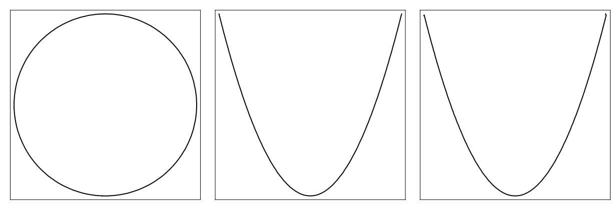
C



D



E

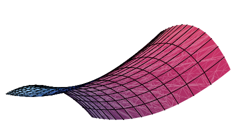


F

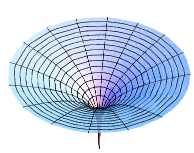
The figures above show the xy-trace, (the intersection of the surface with the xy-plane), the yz-trace (the intersection of the surface with the yz-plane), and the xz-trace (the intersection of the surface with the xz-plane). Match the following equations with the traces. No justifications required.

Enter A,B,C,D,E,F here	Equation
	$x^2 + y^2 - (z - 1/3)^2 = 0$
	$x^2 + y^2 + z^2 - 1 = 0$
	$x^2 - y^2 - z = 0$
	$x^2 + y^2 - 1 = 0$
	$x^2 + y^2 - z^2 - 1 = 0$
	$x^2 + y^2 - z = 1$

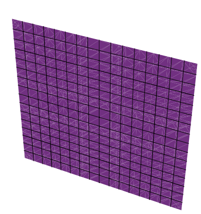
Problem 2b) (4 points)



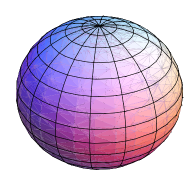
I



II



III



IV

Match the parametric surfaces with their parameterization. No justifications are needed.

Enter I,II,III,IV here	Parameterization
	$\vec{r}(u,v) = \langle u^2, v^2, u^4 - v^4 \rangle$
	$\vec{r}(u,v) = \langle \cos(u)\sin(v), 1 + \sin(u)\sin(v), \cos(v) \rangle$
	$\vec{r}(u,v) = \langle v\cos(u), v\sin(u), v^{1/4} \rangle$
	$\vec{r}(u,v) = \langle u, 3, v \rangle$

Solution:
a) DAECBF.
b) I,IV,II,III

Problem 3) (10 points)

Find the distance of the point $P = (3, 4, 5)$ to the line

$$\frac{x-1}{4} = \frac{y-2}{5} = \frac{z-3}{6} .$$

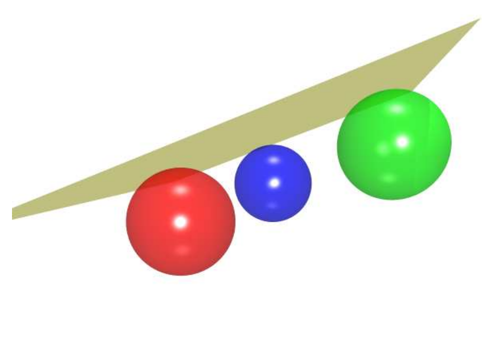
Solution:
First parametrize the line as $\vec{r}(t) = \langle 1, 2, 3\rangle + t\langle 4, 56\rangle = \langle 1, 2, 3\rangle + t\vec{v}$. We also need the vector $\vec{QP} = \langle 2, 2, 2\rangle$ connecting a point $Q = (1, 2, 3)$ on the line with the point $P = (3, 4, 5)$. Use the distance formula

$$d = |\langle 4, 5, 6\rangle \times \langle 2, 2, 2\rangle|/|\langle 4, 5, 6\rangle| = |\langle -2, 4, -2\rangle|/|\langle 4, 5, 6\rangle|$$

$\boxed{= \sqrt{24/77}}$.

Problem 4) (10 points)

Given three spheres of radius 1 centered at $A = (1, 2, 0), B = (4, 5, 0), C = (1, 3, 2)$. Find a plane $ax + by + cz = d$ which touches each of three spheres from the same side.



Solution:
The normal vector to the plane is $\vec{n} = \langle 3, 3, 0 \rangle \times \langle 0, 1, 2 \rangle = \langle 6, -6, 3 \rangle$. The plane touching the three spheres has the equation $6x - 6y + 3z = d$, where d is a constant still to be determined. To find this constant, we have to find a point P on the plane. We do that by going from the point A by 1 unit in the direction of the normal vector. The point $P = A + \vec{n}/|\vec{n}| = (1, 2, 0) + \langle 6/9, -6/9, 3/9 \rangle = (5, 4, 1)/3$ is on the plane. Plug in this point into the equation gives $d = 3$. The equation of the plane is $\boxed{6x - 6y + 3z - 3 = 0}$.

Problem 5) (10 points)

Find the arc length of the curve

$$\vec{r}(t) = \langle t^3/3, t^4/2, 2t^5/5 \rangle$$

from $0 \leq t \leq 1$.

Solution:
We have $\vec{r}\,'(t) = \langle t^2, 2t^3, 2t^4 \rangle$ and $|\vec{r}\,'(t)| = \sqrt{t^4 + 4t^6 + 4t^8} = t^2 + 2t^4$. The arc length is

$$\int_0^1 t^2 + 2t^4 \, dt = 1/3 + 2/5 = 11/15 \, .$$

The arc length is $\boxed{11/15}$.

Problem 6) (10 points)

An apple at position $(0, 0, 20)$ rests 20 meters above Newton's head, the tip of whose nose is at $(1, 0, 0)$. The apple falls with constant acceleration $\vec{r}\,''(t) = \langle a, 0, -10 \rangle$ (where $\langle 0, 0, -10 \rangle$ is caused by gravity and $\langle a, 0, 0 \rangle$ by the wind) precisely onto the nose of Newton. Find the wind force $\langle a, 0, 0 \rangle$ which achieves this. Give a parametrization for the path along which the apple falls.

Solution:
From
$$\vec{r}\,''(t) = \langle a, 0, -10 \rangle$$
we get by integration
$$\vec{r}\,'(t) = \langle at, 0, -10t \rangle + \langle 0, 0, 0 \rangle$$
and
$$\vec{r}\,(t) = \langle at^2/2, 0, -5t^2 \rangle + \langle 0, 0, 20 \rangle \ = \langle at^2/2, 0, -5t^2 + 20 \rangle \ .$$
Now, in order that we reach the nose $(1, 0, 0)$, we have to get the time t such the apple is at the ground $5t^2 = 20$ gives $t = 2$. In order that $at^2/2 = 1$ we have $a = 1/2$. The wind force is $\langle 1/2, 0, 0 \rangle$. The path is
$$\vec{r}\,(t) = \langle t^2/4, 0, 20 - 5t^2 \rangle \ .$$

Problem 7) (10 points)

a) (5 points) A red maple leaf falls to the ground $z = 0$. It falls along the curve $\vec{r}(t) = \langle 3\sqrt{3}\cos(t), 3\sqrt{3}\sin(t), 5-t-4t^2 \rangle$. At which angle does it hit the xy-plane?

b) (5 points) Find the tangent line to the curve at the impact point.

Solution:
a) The leaf hits the ground at the point $\vec{r}(1) = \langle 3\sqrt{3}\cos(1), 3\sqrt{3}\sin(1), 0 \rangle$ at time $t = 1$. We compute the velocity vector at that time
$$\vec{r}\,'(1) = \langle -3\sqrt{3}\sin(1), 3\sqrt{3}\cos(1), -9 \rangle \ .$$
In order to compute the impact angle, we compute the angle with the normal vector $\langle 0, 0, 1 \rangle$ which is $\cos(\alpha) = -9/(6\sqrt{3}) = -\sqrt{3}/2$ so that $\alpha = 5\pi/6$. The angle between the plane and the velocity vector is $5\pi/6 - \pi/2 = \pi/3$. The result is $\boxed{\pi/3}$.
b) the tangent line has the parametrization $\vec{r}(t) = \vec{r}(1) + t\vec{r}\,'(1)$ which is
$$\vec{r}(t) = \langle 3\sqrt{3}\cos(1), 3\sqrt{3}\sin(1), 0 \rangle + t\langle -3\sqrt{3}\sin(1), 3\sqrt{3}\cos(1), -9 \rangle \ .$$

$\boxed{\text{Problem 8) (10 points)}}$

a) (5 points) The surface

$$\vec{r}(t,s) = \langle 1+t+s, 1-t-2s, 1+t-s \rangle$$

with $0 \leq t \leq 1, 0 \leq s \leq 1$ is a parallelogram in space. Find the area of this parallelogram.

b) (5 points) Another surface is given in spherical coordinates by $\rho = 2\sin(\phi)\cos(\theta)$. Write down the equation of this surface in rectangular coordinates as well as in cylindrical coordinates.

Solution:
a) The parallelogram is spanned by $\langle 1,-1,1 \rangle$ and $\langle 1,-2,-1 \rangle$. The area is the length of the cross product $\langle 3,2,-1 \rangle$ which is $\boxed{\sqrt{14}}$.
b) To get the equation in rectangular coordinates, multiply both sides of the equation with ρ, this gives

$$x^2 + y^2 + z^2 = \rho^2 = 2\rho\sin(\phi)\cos(\theta) = 2x \ .$$

Complete the square to get $\boxed{(x-1)^2 + y^2 + z^2 = 1}$.
To get from rectangular to cylindrical coordinates, just replace x^2+y^2 with r^2 and x with $r\cos(\theta)$ and leave z as it is. In cylindrical coordinates the surface is given by the equation $\boxed{r^2 - 2r\cos(\theta) + z^2 = 0}$.

$\boxed{\text{Problem 9) (10 points)}}$

a) (5 points) Parametrize the curve obtained by intersecting the surface $z - x^2 + y^3 = 0$ with the cylindrical surface $x^2/4 + 9y^2 = 1$.

b) (5 points) Find the unit tangent vector $\vec{T}$ and the normal vector $\vec{N}(t) = \vec{T}\,'(t)/|\vec{T}\,'(t)|$ to the curve

$$\vec{r}(t) = \langle 3, t^2, t \rangle$$

at the point $(3,0,0)$. What is the binormal vector $\vec{B} = \vec{T} \times \vec{N}$?

Solution:
a) First parametrize the first two coordinates: $\vec{r}(t) = \langle 2\cos(t), 1/3\sin(t), ... \rangle$, then fill in the third coordinate $z = x^2 - y^3$ to get

$$\vec{r}(t) = \langle 2\cos(t), 1/3\sin(t), 4\cos^2(t) - \sin^3(t)/27 \rangle$$

b) $\vec{T}(t) = \langle 0, 2t, 1 \rangle / \sqrt{1+4t^2}$. At $t = 0$ we have $\boxed{\vec{T} = \langle 0, 0, 1 \rangle}$. Now $\vec{T}\ '(t) = \langle 0, 2, 0 \rangle / \sqrt{1+4t^2} - 4t \langle 0, 2t, 1 \rangle / (1+4t^2)^{3/2}$. Which is at $t = 0$ equal to $\langle 0, 2, 0 \rangle$. Normalized, we get $\boxed{\vec{N} = \langle 0, 1, 0 \rangle}$. The third vector is just the cross product of the first two $\boxed{\vec{B} = \langle -1, 0, 0 \rangle}$.

Problem 10) (10 points)

a) (4 points) Give a parametrization of the hyperboloid $x^2 + y^2 = z^2 + 1$.
b) (3 points) Give a parametrization of the plane $x + y = 1$.
c) (3 points) Give a parametrization of the ellipsoid $x^2 + y^2 + z^2/4 = 1$.

Solution:
a) Since the hyperboloid is in cylindrical coordinates given as $r^2 = z^2 + 1$, we have

$$\vec{r}(z, \theta) = \langle \sqrt{z^2+1}\cos(\theta), \sqrt{z^2+1}\sin(\theta), z \rangle \ .$$

b) Since the points $(1, 0, 0), (0, 1, 0), (1, 0, 1)$ are in the plane, we have two vectors $\vec{v} = \langle -1, 1, 0 \rangle, \vec{w} = \langle 0, 0, 1 \rangle$ parallel to the plane and can parametrize

$$\vec{r}(s, t) = \langle 1, 0, 0 \rangle + t\langle -1, 1, 0 \rangle + s\langle 0, 0, 1 \rangle \ .$$

c) The ellipsoid is a deformation of the sphere. We just have to make it higher:

$$\vec{r}(\theta, \phi) = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), 2\cos(\phi) \rangle \ .$$