10/2/2012 FIRST HOURLY **Math 21a, Fall 2012**

Name:

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MWF 10 Hansheng Diao
MWF 10 Joe Rabinoff
MWF 11 John Hall
MWF 11 Meredith Hegg
MWF 12 Charmaine Sia
TTH 10 Bence Béky
TTH 10 Gijs Heuts
TTH 11:30 Francesco Cavazzani
TTH 11:30 Andrew Cotton-Clay

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False (TF) questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) | T | **F** | For any two vectors, the length of the sum is the sum of the lengths.

Solution:
Take $\vec{w} = -\vec{v}$, then the length of the sum is zero but the sum of the lengths is twice the length of $\vec{v}$.

2) | T | **F** | There are unit vectors $\vec{v}$ and $\vec{w}$ in space for which $|\vec{v} \times \vec{w}| = 2$.

Solution:
Use the formula $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\theta)$ to see that the length of the cross product is maximally 1.

3) | T | **F** | The vector $\langle 4, 5, 0 \rangle$ is perpendicular to the plane $-5x + 4y + z = 2$.

Solution:
We know that the normal vector to $ax + by + cz = d$ is $\langle a, b, c \rangle$.

4) | T | **F** | The distance between the cylinders $x^2 + z^2 = 1$ and $x^2 + (z-3)^2 = 1$ is 3.

Solution:
The two cylinders are parallel.

5) | **T** | F | The vector projection of $\langle 2, 3, 1 \rangle$ onto $\langle 1, 1, 1 \rangle$ is parallel to $\langle 1, 1, 1 \rangle$.

Solution:
Yes by definition of the projection.

6) | **T** | F | The equation $\rho \sin(\theta) \sin(\phi) = 2$ in spherical coordinates defines a plane.

Solution:
In spherical coordinates, we have $y = 2$.

7) | T | F | (T highlighted)

There is a planar curve for which the arc length is 2π and the curvature is constant 1.

Solution:
The unit circle

8) | T | F | (F highlighted)

If we know the intersection of a graph $z = f(x, y)$ with the coordinate planes $x = 0, y = 0$ and $z = 0$, the function f is determined uniquely.

Solution:
$f(x, y) = xy$ and $f(x, y) = x^2y^2$ have the same traces but the functions are different.

9) | T | F | (F highlighted)

The surface given in cylindrical coordinates as $r = z^2$ is a paraboloid.

Solution:
The surface $z = r^2$ is a paraboloid. This surface is a "pointy shape".

10) | T | F | (F highlighted)

If the curvature of a space curve is constant 2 everywhere along the curve then the curve is a circle.

Solution:
A spiral is a counter example.

11) | T | F | (T highlighted)

If $\vec{u}, \vec{v}$, and $\vec{w}$ are unit vectors then the volume of the parallelepiped spanned by $\vec{u}, \vec{v}$, and $\vec{w}$ is largest when the parallelepiped is a cube.

Solution:
Look at the formulas for the dot and cross product. The volume is $|u||v||w|\cos(\alpha)\sin(\beta)$ which is maximal if $\alpha = 0, \beta = \pi/2$.

12) | T | F | (T highlighted)

If a point is moving along a straight line parametrized by $\vec{r}(t)$ then the velocity $\vec{r}\,'(t)$ vector and acceleration vector $\vec{r}\,''(t)$ must be parallel.

Solution:
The curvature is zero. Look at the curvature formula

13) | T | **F** |

The parametrization $\vec{r}(u,v) = \langle v\cos(u), v\sin(u), v\rangle$ with $0 \le u < 2\pi$ and $v \in \mathbf{R}$ is a cylinder.

Solution:
It is cone

14) | T | **F** |

If two lines in space are not parallel, then they must intersect.

Solution:
They can be skew.

15) | **T** | F |

If two planes do not intersect, then their normal vectors are parallel.

Solution:
Yes, otherwise, the cross product between the two normal vectors and a common point defines the intersection line.

16) | **T** | F |

$(\vec{i} \times \vec{j})$ and $(\vec{i} \times (\vec{i} \times (\vec{i} \times \vec{j})))$ are parallel.

Solution:
The latter is a negative scalar multiple of the former.

17) | **T** | F |

The surface parametrized by $\vec{r}(u,v) = \langle \sin(u)\sin(v), \sin(u)\cos(v), \cos(u)\rangle$ with $0 \le v < 2\pi, 0 \le u \le \pi$ is a sphere.

Solution:
It looks as if something is false here, but it is the standard sphere. Just that u, v are switched.

18) | **T** | F |

The unit tangent vector $\vec{T}$ to a curve at a given point is independent of the parametrization up to a factor of -1.

Solution:
The curve at the point defines two unit tangent vectors. This is the only ambiguity.

19) | T | F | $z^2 = r^2(\cos^2(\theta) - \sin^2(\theta)) + 1$ is a one-sheeted hyperboloid.

Solution:
Convert into cartesian coordinates to get $z^2 = x^2 - y^2 + 1$.

20) | T | F | If $\vec{a} \cdot \vec{b} > 0$ and $\vec{b} \cdot \vec{c} > 0$, then $\vec{a} \cdot \vec{c} > 0$.

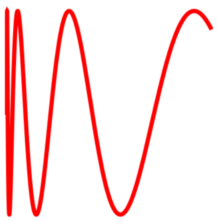
Solution:
Two acute angles can sum up to an obtuse angle.

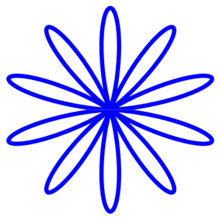
[ ] Total

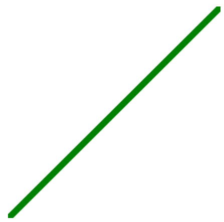
Problem 2) (10 points)

No explanations needed. I,II,III,O appear all once in each box.

a) (2 points) Match curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match.

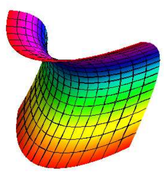

I

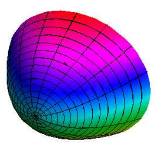

II

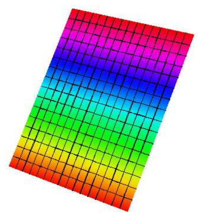

III

Parametrization $\vec{r}(t) =$	O, I,II,III
$\sin^2(5t)\langle\cos(t),\sin(t)\rangle$	
$\langle t^3, \sin(7t)\rangle$	
$\langle t^5, 1+t^5\rangle$	
$\langle\sin(t),\cos(t)\rangle$	

b) (2 points) Match the parametrization. Enter O, where no match.

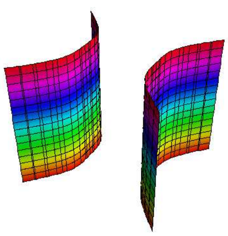

I

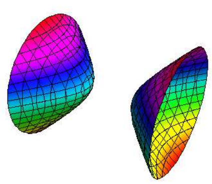

II

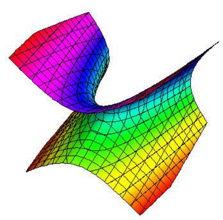

III

$\vec{r}(s,t)$	O,I,II,III
$\langle 1-t, 1+s, 2+s\rangle$	
$\langle s, t^2-s^2, t\rangle$	
$\langle t\cos(s), t\sin(s), s\rangle$	
$\langle s\cos(t), s^2, s\sin(t)\rangle$	

c) (2 points) The pictures show contour surfaces. Enter O, where no match.

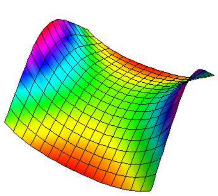

I

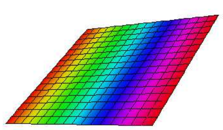

II

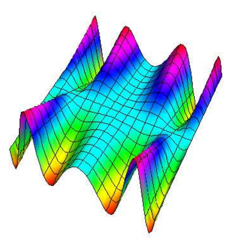

III

$g(x,y,z) =$	O, I,II,III
$x^2-y^2+z^2=-1$	
$x^2-y^2=1$	
$x^4+z=1$	
$x^2+y-z^2=1$	

d) (2 points) Match the graphs $z=f(x,y)$ with the functions. Enter O, where no match.

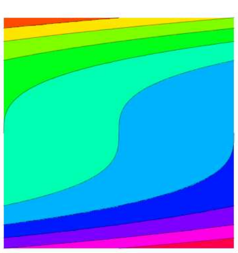

I

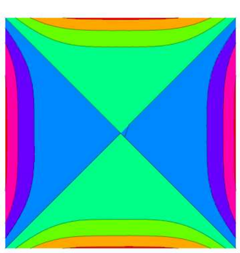

II


III

function $f(x,y) =$	O,I,II,III
$2x$	
$e^{-2x^2-2y^2}$	
$e^{x^2-y^2}$	
$y\sin(x^2)$	

e) (2 points) Match the family of level curves with $f(x,y)$. Enter O, where no match.


I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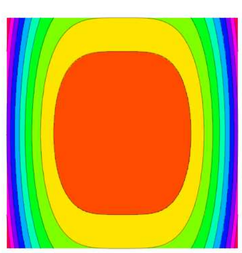

III

Function $f(x,y) =$	O,I,II,III
x^4+y^2	
x^4-y^4	
$x-y$	
$x-y^3$	

Solution:
II,I,III,O
III,I,O,II
II,I,0,III
II,0,I,III
III,II,0,I

Problem 3) (10 points)

No explanations needed. In 3a), in each row check only one box.

a) (4 points) The intersection of a plane with a cone $S : x^2 + y^2 - z^2 = 0$ is called a **conic section**. What curve do we get?

Intersect S with	hyperbola	parabola	circle	line
$z = 1$ gives a				
$z = x$ gives a				
$z = x + 1$ gives a				
$x = 1$ gives a				

b) (3 points) By intersecting the upper half sphere $x^2 + y^2 + z^2 = 5$, $z > 0$ with the hyperboloid $x^2 + y^2 - z^2 = -3$ we get a curve. Which one? Check exactly one box.

$\vec{r}(t) = \langle \cos(t), \sin(t), 2 \rangle$	
$\vec{r}(t) = \langle 0, 0, t \rangle$	
$\vec{r}(t) = \langle \cos(t), \sin(t), 2t \rangle$	

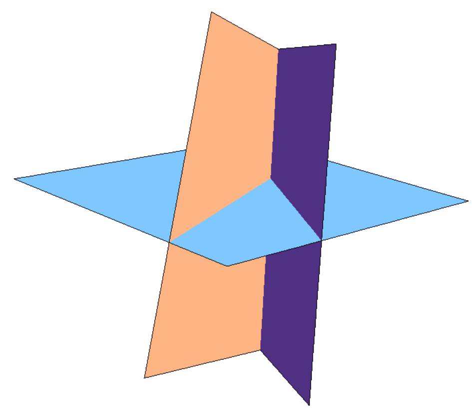
c) (3 points) Which of the following surface parametrizations gives a one sheeted hyperboloid? Check exactly one box.

$\vec{r}(t, s) = \langle s, t, s^2 - t^2 \rangle$	
$\vec{r}(t, s) = \langle \sqrt{1 + s^2} \cos(t), \sqrt{1 + s^2} \sin(t), s \rangle$	
$\vec{r}(t, s) = \langle \sqrt{1 - s^2} \cos(t), \sqrt{1 - s^2} \sin(t), s \rangle$	

Solution:
a) Circle, line, parabola, hyperbola. This constructions are important and give the etymology for the name "conic sections". We cut a cone with a plane and can get all the basic quadratic curves.
b) first box
c) second box.

Problem 4) (10 points)

We are given two planes $x+y+z=1$ and $x-y-z=2$. Find a third plane which contains the point $(1,0,0)$ and which is perpendicular to both.



Solution:
A normal vector to the third plane can be obtained by taking the cross product of the two normal vectors of the first two planes.

$$\langle 1,1,1\rangle \times \langle 1,-1,-1\rangle = \langle 0,2,-2\rangle \ .$$

The third plane has the equation $2y-2z=d$. In order that the point $(1,0,0)$ is there, the constant $d=0$. The equation is $2y-2z=0$ or $\boxed{y=z}$
A completely different solution was also valid: we can directly write down a parametrization of the plane

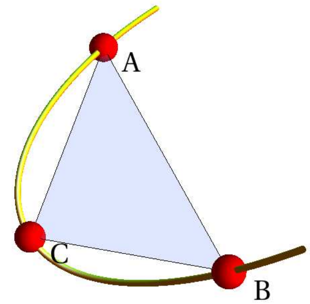
$$\vec{r}(s,t)\langle 1,0,0\rangle + s\langle 1,1,1\rangle + t\langle 1,-1,-1\rangle \ .$$

Problem 5) (10 points)

We are given a curve $\vec{r}(t) = \langle 1+t, t^2, t^3 \rangle$.

a) (5 points) Find the area of the triangle with vertices $A = \vec{r}(-1), B = \vec{r}(1)$ and $C = \vec{r}(0)$.

b) (5 points) Find an equation $ax + by + cz = d$ for the plane through A, B, C.

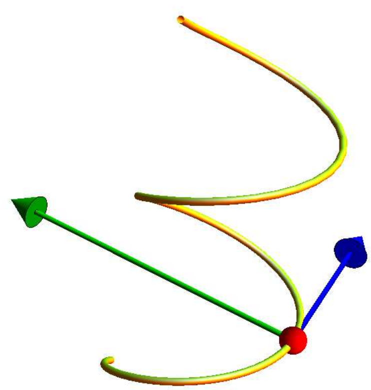


Solution:
a) $\vec{r}(0) = \langle 1, 0, 0 \rangle$ $\vec{r}(1) = \langle 2, 1, 1 \rangle$ $\vec{r}(-1) = \langle 0, 1, -1 \rangle$. We have $\vec{v} = \vec{r}(1) - \vec{r}(0)$ and $\vec{w} = \vec{r}(-1) - \vec{r}(0)$ and $\vec{n} = \vec{v} \times \vec{w} = \langle 1, 1, 1 \rangle \times \langle -1, 1, -1 \rangle = \langle -2, 0, 2 \rangle$. The area of the triangle is one half of the area of the parallelogram which is $\sqrt{4+4}/2 = 2\sqrt{2}/2 = \sqrt{2}$.
b) $-2x + 2z = d$. We get the constant by plugging in the point $(1, 0, 0)$. The solution is $-2x + 2z = -2$ or $\boxed{x - z = 1}$.

Problem 6) (10 points)

a) (3 points) Find the unit tangent vector $\vec{T}(t)$ of the curve $\vec{r}(t) = \langle t^2, \cos(t^2\pi), \sin(t^2\pi) \rangle$ at $t = 1$.
b) (3 points) What is the acceleration vector $\vec{r}\,''(t)$ at $t = 1$?
c) (4 points) Find the curvature at the time $t = 1$. You may us the formula

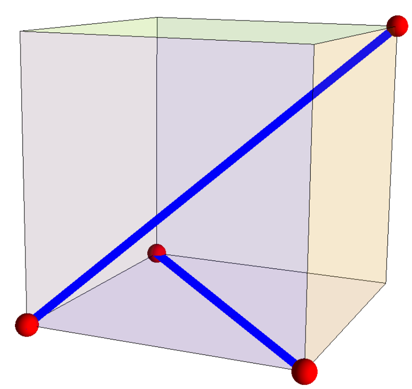
$$\kappa = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} .$$



Solution:
a) $\vec{r}\ '(t) = \langle 2t, -2t\pi\sin(t^2\pi), 2t\pi\cos(t^2\pi)\rangle$ is $\vec{r}\ '(1) = \langle 2, 0, 2\pi\rangle$. We have $\boxed{\vec{T}(1) = \langle 1, 0, \pi\rangle/\sqrt{1+\pi^2}}$.
b) We have $\vec{r}\,''(t) = \langle 2, -2\pi\sin(t^2\pi) - 4t^2\pi^2\cos(t^2\pi), 2\pi\cos(t^2\pi) - 4t^2\pi^2\sin(t^2\pi)\rangle$ and $\vec{r}\,''(1) = \boxed{\langle 2, 4\pi^2, -2\pi\rangle}$.
c) $|\langle 1, 0, \pi\rangle \times \langle 2, 4\pi^2, -2\pi\rangle|/(1+\pi^2)^{3/2}$ simplifies to $\boxed{\pi^2/(1+\pi^2)}$.
Bu the way, the figure to this problem as well as in the previous figure has been produced in Mathematica. The computer algebra system would by default not make such shiny curves. We computed the T,N,B frame using the formulas and use it to create a "tube" around the curve. The circles which make that tube use the N,B directions. This is an example of a situation, where the TNB frame is useful!

Problem 7) (10 points)

What's the closest that the long diagonal of the unit cube connecting the corners $(0,0,0)$ to $(1,1,1)$, comes to the diagonal of a face connecting the corners $(1,0,0)$ and $(0,1,0)$?



Solution:
This is a distance problem. $\vec{n} = \langle 1, 1, 1\rangle \times \langle 1, -1, 0\rangle = \langle 1, 1, -2\rangle$ and

$$d = |\langle 1, 0, 0\rangle \cdot \langle 1, 1, -2\rangle|/|\langle 1, 1, -2\rangle| = 1/\sqrt{6}\ .$$

The distance is $\boxed{1/\sqrt{6}}$.

Problem 8) (10 points)

In a parallel universe of ours, the inhabitants live under a "Newton's law" of gravity in which the "jerk" $\vec{r}\,'''(t)$ rather than the acceleration is constant. Suppose that $\vec{r}\,'''(t) = \langle 0, 0, -10 \rangle$ for all t.

a) (3 points) Find $\vec{r}\,''(t)$ if you know $\vec{r}\,''(0) = \langle 0, 0, 0 \rangle$.
b) (3 points) Now find $\vec{r}\,'(t)$ if we know also $\vec{r}\,'(0) = \langle 1, 0, 0 \rangle$.
c) (4 points) Finally find $\vec{r}(t)$ if we know additionally $\vec{r}\,(0) = \langle 0, 0, 10 \rangle$.

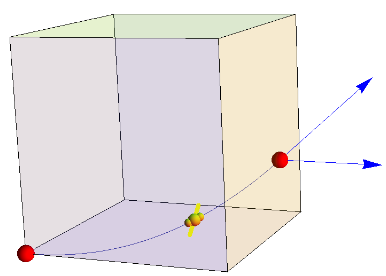
Solution:
Integrate

$$\vec{r}\,''(t) = \langle 0, 0, -10t \rangle$$
$$\vec{r}\,'(t) = \langle 0, 0, -5t^2 \rangle + \langle 1, 0, 0 \rangle$$
$$\vec{r}\,(t) = \langle 0, 0, -5t^3/3 \rangle + \langle t, 0, 0 \rangle + \langle 0, 0, 10 \rangle = \langle t, 0, 10 - 5t^3/3 \rangle \ .$$

Problem 9) (10 points)

a) (2 points) A fly is trapped inside a unit cubicle made of planar glass panes. It flies, starting at $t = 0$ at the origin $(0, 0, 0)$ along the curve

$$\vec{r}(t) = \langle t, \frac{t^2}{\sqrt{2}}, \frac{t^3}{3} \rangle \ .$$



At what time does it bump into the glass wall $x = 1$?
b) (4 points) Find the impact angle (= the angle between the normal vector of the plane and the velocity vector).
c) (4 points) How long is the path it has followed from $t = 0$ to the impact point?

Solution:
a) Compare the first coordinates to get $\boxed{t = 1}$.
b) The velocity vector is $\langle 1, 2t/\sqrt{2}, t^2 \rangle$. At time $t = 1$, it is $\boxed{\langle 1, \sqrt{2}, 1 \rangle}$. The angle satisfies $\cos(\theta) = 1/2$ which is $\boxed{\theta = \pi/3}$.
c) $\int_0^1 (1 + t^2)\ dt = 1 + 1/3 = \boxed{4/3}$.