11/6/2012 SECOND HOURLY PRACTICE VII **Math 21a, Fall 2012**

Name:

MWF 9 Oliver Knill
MWF 10 Hansheng Diao
MWF 10 Joe Rabinoff
MWF 11 John Hall
MWF 11 Meredith Hegg
MWF 12 Charmaine Sia
TTH 10 Bence Béky
TTH 10 Gijs Heuts
TTH 11:30 Francesco Cavazzani
TTH 11:30 Andrew Cotton-Clay

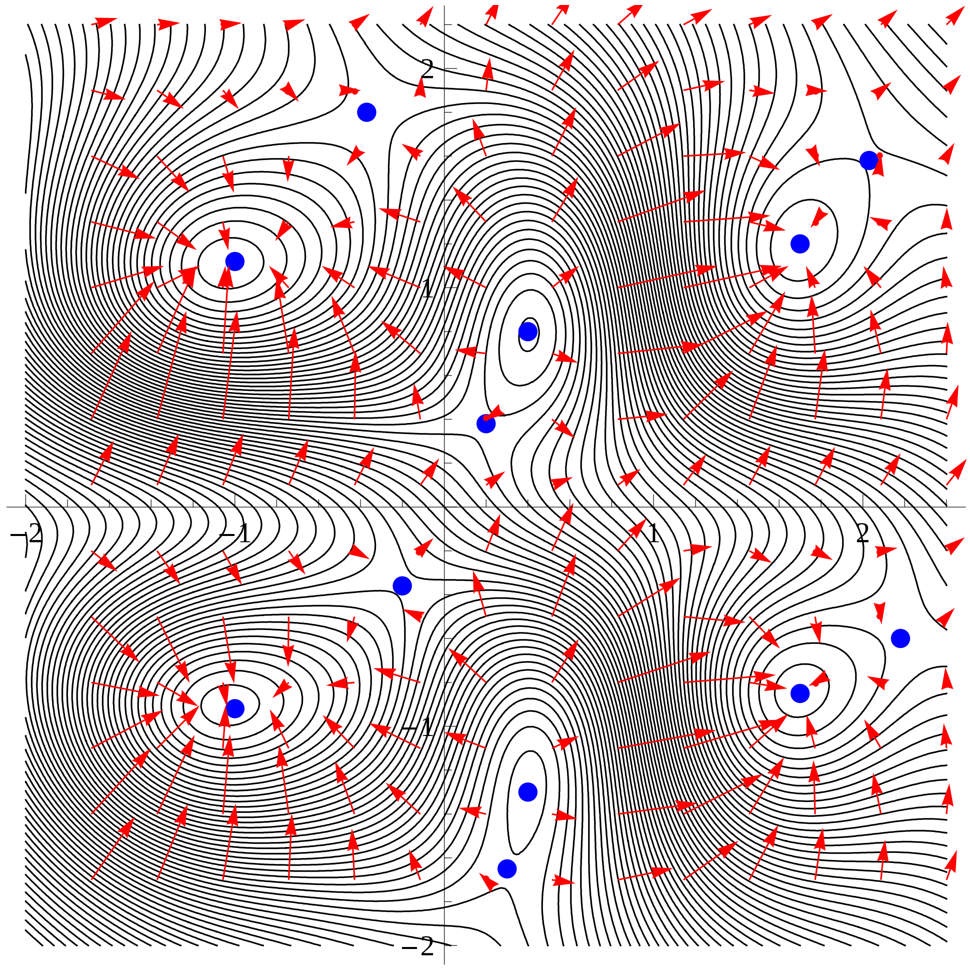
- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, and 5, we need to see **details** of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1)	T	F	Given a unit vector v, define $g(x) = D_v f(x)$. If at a critical point, for all vectors v we have $D_v g(x) > 0$, then f is a local maximum.
2)	T	F	Assume f satisfies the PDE $f_x = f_y$. If $g = f_x$, then $g_x = g_y$.
3)	T	F	The equation $\phi = \pi/4$ in spherical coordinates ($\rho \geq 0, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$ as usual) and the surface $x^2 + y^2 = z^2$ (with no further restrictions on x, y, z) are the same surface.
4)	T	F	Even with $f_x(a,b) = 0$ and $f_y(a,b) = 0$, it is possible that some directional derivative $D_{\vec{v}}(f)$ of $f(x,y)$ at (a,b) is non-zero.
5)	T	F	There exists a pair of different points on a sphere, for which the tangent planes are parallel.
6)	T	F	If $\vec{u}$ is a unit vector tangent at (x,y,z) to the level surface of $f(x,y,z)$ then $D_u f(x,y,z) = 0$.
7)	T	F	Assume we have a smooth function $f(x,y)$ for which the lines $x = 0, y = 0$ and $x = y$ are level curves $f(x,y) = 0$. Then $(0,0)$ is a critical point with $D < 0$.
8)	T	F	The gradient of $f(x,y)$ is perpendicular to the graph of f.
9)	T	F	The level curves of a linearization $L(x,y)$ of a function $f(x,y) = \sin(x+y)$ at $(0,0)$ consist of lines.
10)	T	F	If $x^4 y + \sin(y) = 0$ then $y' = 4x^3 y/(x^4 + \cos(y))$.
11)	T	F	The linearization $L(x,y)$ at a critical point (x_0, y_0) of a function $f(x,y)$ is a constant function.
12)	T	F	The surface $x^2 + y^2 - z^2 = 1$ has a parametrization of the form $\langle x(s,t), y(s,t), z(s,t) \rangle = \langle s, t, f(s,t) \rangle$ for some function $f(s,t)$ for which the parametrization covers the entire surface.
13)	T	F	The tangent plane to the graph of $f(x,y)$ at a point $(x_0, y_0, f(x_0, y_0))$ is a level surface of the linearization $L(x,y,z)$ of $z - f(x,y)$.
14)	T	F	The critical points of $F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$ are solutions to the Lagrange equations when extremizing the function $f(x,y)$ under the constraint $g(x,y) = 0$.
15)	T	F	The curve defined by $z = 1, \theta = \frac{\pi}{4}$ in cylindrical coordinates is a circle.
16)	T	F	If $(0,0)$ is a critical point of $f(x,y)$ and the discriminant D is zero but $f_{xx}(0,0) > 0$ then $(0,0)$ can not be a local maximum.
17)	T	F	If $f(x,y,z) = x^2 + y^2 + z^2$, then $\nabla f = 2x + 2y + 2z$.
18)	T	F	A function $f(x,y)$ in the plane always has a local minimum or a local maximum.
19)	T	F	For any smooth function $f(x,y)$, the inequality $\|\nabla f\| \geq |f_x + f_y|$ is true.
20)	T	F	If a function $f(x,y)$ satisfies $|\nabla f(x,y)| = 1$ everywhere in the plane, then f is constant.

Problem 2) (10 points)



a) The picture above shows a contour map of a function $f(x,y)$ of two variables. This function has 12 critical points and all of them are marked. Each of them is either a local max, a local min or a saddle point. The picture shows also some gradient vectors. Count the number of critical points in the following table. No justifications are necessary.

The function $f(x,y)$ has		local maxima
The function $f(x,y)$ has		local minima
The function $f(x,y)$ has		saddle points

b) (4 points) Match the following partial differential equations with the names. No justifications are needed.

Enter A,B,C,D here	PDE
	$u_{xx} + u_{yy} = 0$
	$u_{xx} - u_{yy} = 0$

Enter A,B,C,D here	PDE
	$u_x - u_{yy} = 0$
	$u_x - u_y = 0$

A) Wave equation	B) Heat equation	C) Transport equation	D) Laplace equation

Problem 3) (10 points)

Find the cos of the angle between the sphere

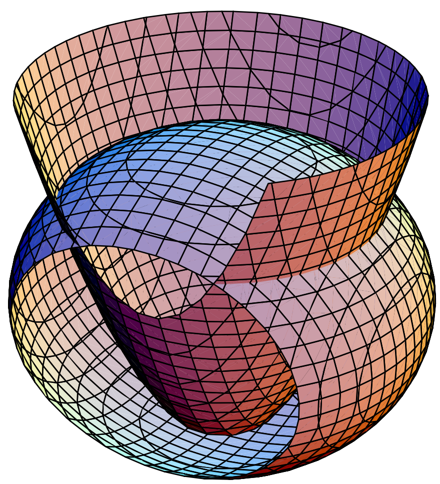
$$x^2 + y^2 + z^2 - 9 = 0$$

and the paraboloid

$$z - x^2 - y^2 + 3 = 0$$

at the point $(2, -1, 2)$.

Note: The angle between two general surfaces at a point P is defined as the angle between the tangent planes at the point P.



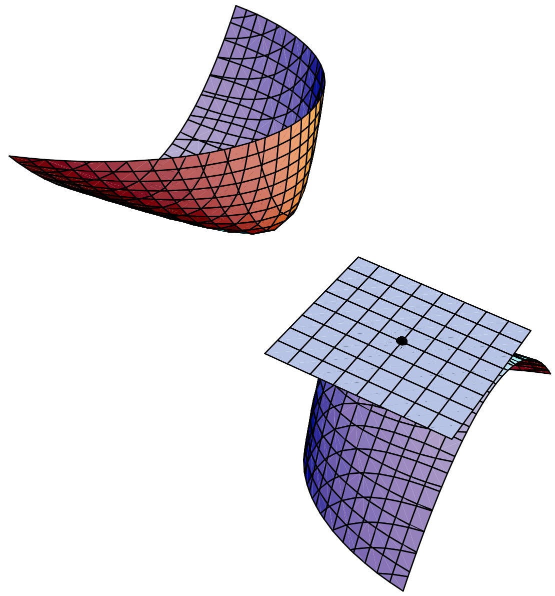
Problem 4) (10 points)

a) You know that

$$-2x + 5y + 10z = 2$$

is the equation of the tangent plane to the graph of $f(x, y)$ at the point $(-1, 2, -1)$. Find the gradient $\nabla f(-1, 2)$ at the point $(-1, 2)$ and Estimate $f(-0.998, 2.0001)$ using linear approximation.

b) Let $f(x, y, z) = x^2 + 2y^2 + 3xz + 2$. Find the equation of the tangent plane to the surface $f(x, y, z) = 0$ at the point $(2, 0, -1)$ and estimate $f(2.001, 0.01, -1.0001)$.



Problem 5) (10 points)

a) (4 points) Find all the critical points of the function $f(x, y) = xy$ in the interior of the elliptic domain

$$x^2 + \frac{1}{4}y^2 < 1 \; .$$

and decide for each point whether it is a maximum, a minimum or a saddle point.

b) (4 points) Find the extrema of f on the boundary

$$x^2 + \frac{1}{4}y^2 = 1 \; .$$

of the same domain.

c) (2 points) What is the global maximum and minimum of f on $x^2 + \frac{1}{4}y^2 \leq 1$.

Problem 6) (10 points)

a) Assume $f(x, y) = e^{2x-y-2} + y + \sin(x - 1)$ and $x(t) = \cos(5t), y(t) = \sin(5t)$. What is

$$\frac{d}{dt} f(x(t), y(t))$$

at time $t = 0$.

b) The relation

$$xyz + z^3 + xy + yz^2 = 4$$

defines z as a function of x and y near $(x, y, z) = (1, 1, 1)$. Find the gradient

$$\langle \frac{\partial z}{\partial x}(1,1), \frac{\partial z}{\partial y}(1,1) \rangle$$

of $z(x, y)$ at the point $(1, 1)$.

Problem 7) (10 points)

The temperature in a room is given by $T(x, y, z) = x^2 + 2y^2 - 3z + 1$.

a) Barry B. Benson is hovering at the point $(1, 0, 0)$ and feels cold. Which direction should he go to heat up most quickly? Make sure that your answer is a unit vector.

b) At some later time, Barry arrives at the point $(3, 2, 1)$ and decides that this is a nice temperature. Find a direction (a unit vector) in which he can go, to stay at the same temperature and the same altitude.

Problem 8) (10 points)

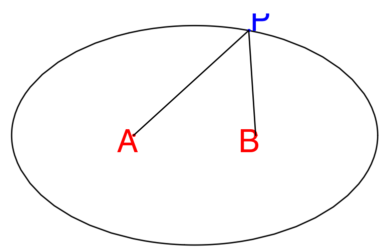
Let $g(x, y)$ denote the distance of a point $P = (x, y)$ to a point A and $h(x, y)$ the distance from P to a point B. The set of points (x, y) for which $f(x, y) = g(x, y) + h(x, y)$ is constant, forms an ellipse. In other words, the level curves of f are ellipses.

a) (4 points) Why is $\nabla g + \nabla h$ perpendicular to the ellipse?

b) (3 points) Show that if $\vec{r}(t)$ parametrizes the ellipse, then $(\nabla g + \nabla h) \cdot \vec{r}' = 0$ or $\nabla g \cdot r' = -\nabla h \cdot r'$.

c) (3 points) Conclude from this that the lines AP and BP make equal angles with the tangent to the ellipse at P. (Hint: check that $|\nabla f| = |\nabla g| = 1$).

You have now shown that light rays originating at focus A will be reflected from the ellipse to focus at the point B.

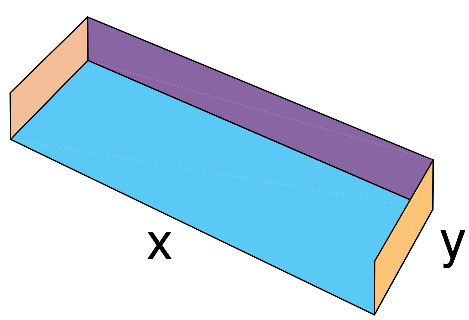


Problem 9) (10 points)

Minimize the material cost of an office tray

$$f(x, y) = xy + 2x + 2y$$

of length x, width y and height 1 under the constraint that the volume $g(x, y) = xy$ is constant and equal to 4.



Problem 10) (10 points)

A beach wind protection is manufactured as follows. There is a rectangular floor $ACBD$ of length a and width b. A pole of height c is located at the corner C and perpendicular to the ground surface. The top point P of the pole forms with the corners A and C one triangle and with the corners B and C an other triangle. The total material has a fixed area of $g(a, b, c) = ab + ac/2 + bc/2 = 12$ square meters. For which dimensions a, b, c is the volume $f(a, b, c) = abc/6$ of the tetrahedral protected by this configuration maximal?