11/6/2012 SECOND HOURLY PRACTICE I **Math 21a, Fall 2012**

Name:

MWF 9 Oliver Knill
MWF 10 Hansheng Diao
MWF 10 Joe Rabinoff
MWF 11 John Hall
MWF 11 Meredith Hegg
MWF 12 Charmaine Sia
TTH 10 Bence Béky
TTH 10 Gijs Heuts
TTH 11:30 Francesco Cavazzani
TTH 11:30 Andrew Cotton-Clay

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, and 5, we need to see **details** of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) T **F** There is a function $f(x, y)$ for which the linearization at $(0, 0)$ is $L(x, y) = x^2 + y^2$.

Solution:
The linearization is a linear function and not quadratic.

2) **T** F For any two functions f, g and unit vector $\vec{u}$ we have $D_{\vec{u}}(f + g) = D_{\vec{u}}f + D_{\vec{u}}g$.

Solution:
This follows directly from the definition.

3) T **F** $\int_0^2 \int_0^{\sqrt{4-x^2}} (x^2 + y^2)\, dydx = \int_0^2 \int_0^{\pi/2} r^2\, d\theta dr$.

Solution:
The r factor is forgotten.

4) T **F** If we solve $\sin(y) - xy^2 = 0$ for y, then $y' = -y^2/(\cos(y) - 2xy)$.

Solution:
This is an application of the implicit differentiation formula. But the sign is wrong.

5) **T** F If $f(x, 0) = 0$ for all x and $f(0, y) = 0$ for all y, then $g(x, y) = \int_0^x \int_0^y f(s, t)\, dtds$ solves $g_{xy}(x, y) = f(x, y)$.

Solution:
We can have $f(x, y) = x$ for example.

6) **T** F If $|\nabla f| = 1$ at $(0, 0)$, then there exists a direction in which the slope of the graph of f at $(0, 0)$ is 1.

Solution:
It is the direction of the gradient

7) T F

The function $f(x, y) = x^2 + y^2$ satisfies the partial differential equation $f_{xx} f_{yy} - f_{xy}^2 = 4$.

Solution:
Yes, this is a computation.

8) T F

The height of Mount Wachusett is $f(x, y) = 4 - 2x^2 - y^2$. On the trail $x^2 + y^2 = 1$, the point $(1, 0)$ is a maximum.

Solution:
It is a local minimum.

9) T F

Mount Wachusett has height $f(x, y) = 4 - 2x^2 - y^2$. Except at the maximum $(0, 0)$, the gradient vector is perpendicular to the graph of the function.

Solution:
The gradient vector is a vector with two components, not a vector in space.

Solution:
The gradient vector of f is a vector with two components and not in space.

10) T F

If $f_x(a, b) > 0$ and $f_y(a, b) > 0$ then for any unit vector $\vec{u}$ we must have $D_{\vec{u}} f(a, b) > 0$.

Solution:
Take a unit vector $\langle -1, - \rangle$ for example. The directional derivative in this direction is zero.

11) T F

If $f(x, y)$ has two local minima, then f must have at least one local maximum.

Solution:
Take a function like $-x \exp(-x^2 - y^2)$. It has two local minima and a saddle point.

12) T F

If $\vec{r}(t)$ is a curve on the surface $g(x, y, z) = x^2 + y^2 - z^2 = 6$ then $\nabla g(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$.

Solution:
This is a recurring theme. The gradient vector is perpendicular to the surface. This fact is based on the chain rule and the fact that $g(r(t))$ is constant so that $d/dtg(r(t))$ is zero.

13) | T | **F** | If f and g have the same trace $\{x = 5\}$ then $f_x(5, y) = g_x(5, y)$ for all y.

Solution:
We know $f(5, y) = g(5, y)$ but the x derivative can be different. Take $f(x, y) = x - 5$ and $g(x, y) = y(x - 5)$ then $f_x = 1$ and $g_x = y$.

14) | **T** | F | If f and g have the same trace $\{x = 5\}$ then $f_y(5, y) = g_y(5, y)$ for all y.

Solution:
Because $f(5, y) = g(5, y)$, also the derivatives are the same.

15) | **T** | F | The surface area of $\vec{r}_1(u, v) = \langle u\cos(v), u\sin(v), u^2\rangle$ and $\vec{r}_2(u, v) = \langle\sqrt{u}\cos(v), \sqrt{u}\sin(v), u\rangle$ defined on $\{0 \leq u, v \leq 1\}$ are the same.

Solution:
Surface area does not depend on parametrization.

16) | **T** | F | If $\vec{r}(t)$ is a curve on a graph of a function $f(x, y)$, then the velocity vector of r is perpendicular to the vector $\langle f_x, f_y, -1\rangle$.

Solution:
The vector $\langle f_x, f_y, -1\rangle$ is the gradient of the graph of f.

17) | **T** | F | A continuous function $f(x, y)$ on the closed disc $R = \{x^2 + y^2 \leq 51^2\}$ (of course, R is called **"area** 51π**"**) has a global maximum on R.

Solution:
We know that a continuous function has a maximum on a closed bounded domain. Interesting corollary: take for f the probability density that an alien has landed there. Then there is a point where the probability density is largest, proving so that aliens are likely in area 51.

18) T F — Any continuous function $f(x, y)$ has a global minimum and maximum on the curve $y = x^2$.

Solution:
The curve is an unbounded parabola. The function $f(x, y) = x$ for example is unbounded on it.

19) T F — Fubini's theorem assures that $\int_a^b \int_c^d f(x, y)\, dydx = \int_a^b \int_c^d f(x, y)\, dxdy$.

Solution:
The integrals have not been switched.

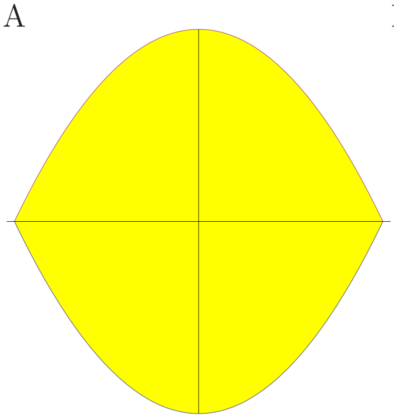
20) T F — $\int\int_R \sin(x + y)\, dxdy = 0$ for $R = \{-\pi \le x \le \pi, -\pi \le y \le \pi\}$.

Solution:
Directly integrate. One can see this also by symmetry. The integral has an interpretation as a volume, with exactly the same amount below than above the plane.

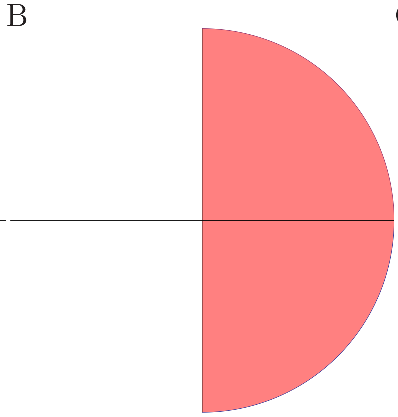
Problem 2) (10 points)

a) (6 points) Match the integration regions with the integrals. Each integral matches exactly one region $A - F$.

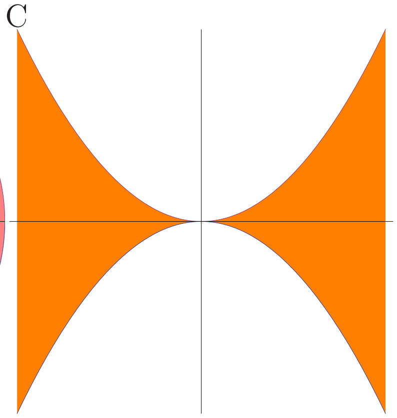
A

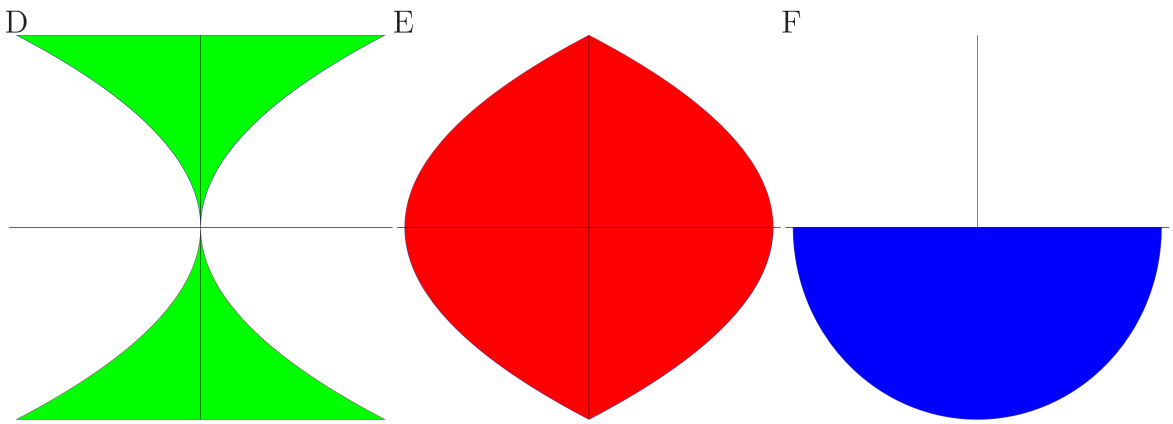


B



C





Enter A-F	Integral
	$\int_{-1}^{1}\int_{-x^2}^{x^2} f(x,y)\; dydx.$
	$\int_{-1}^{1}\int_{-y^2}^{y^2} f(x,y)\; dxdy.$
	$\int_{-1}^{1}\int_{y^2-1}^{1-y^2} f(x,y)\; dxdy.$
	$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} f(x,y)\; dxdy.$
	$\int_{-1}^{1}\int_{x^2-1}^{1-x^2} f(x,y)\; dydx.$
	$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0} f(x,y)\; dydx.$

b) (4 points) Fill in one word names (like "Heat","Wave" etc) for the partial differential equations:

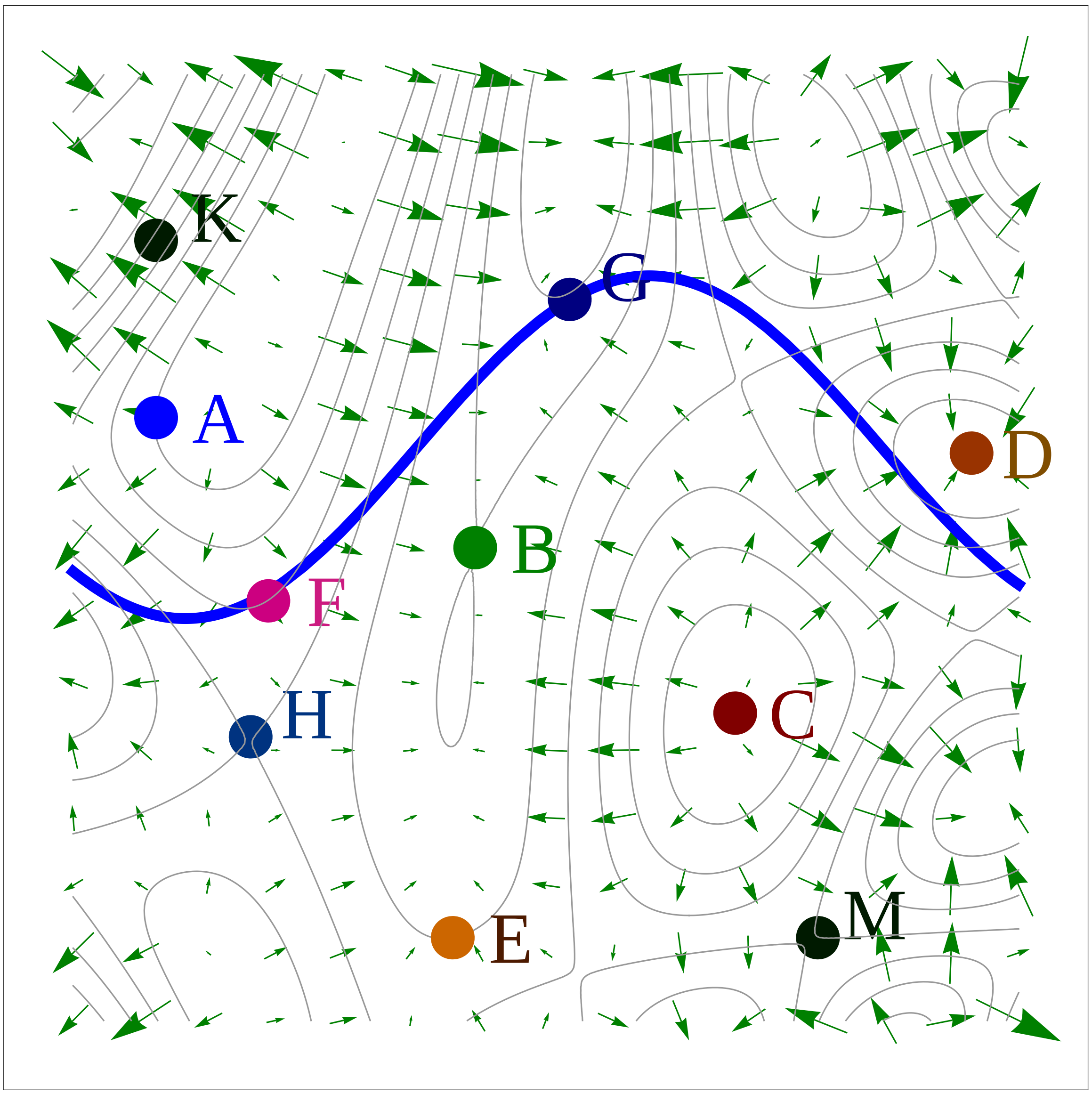
Enter one word	PDE
	$g_x = g_y$
	$g_{xx} = g_{yy}$
	$g_{xx} = -g_{yy}$
	$g_x = g_{yy}$

Solution:
a) C,D,E,B,A,F
b) Transport, Wave, Laplace, Heat.

Problem 3) (10 points)

(10 points) A function $f(x,y)$ of two variables has level curves as shown in the picture. We also see a constraint in the form of a curve $g(x,y) = 0$ which has the shape of the graph of the cos function. The arrows show the gradient. In this problem, each of the 10 letters $A, B, C, D, E, F, G, H, K, M$ appears exactly once.

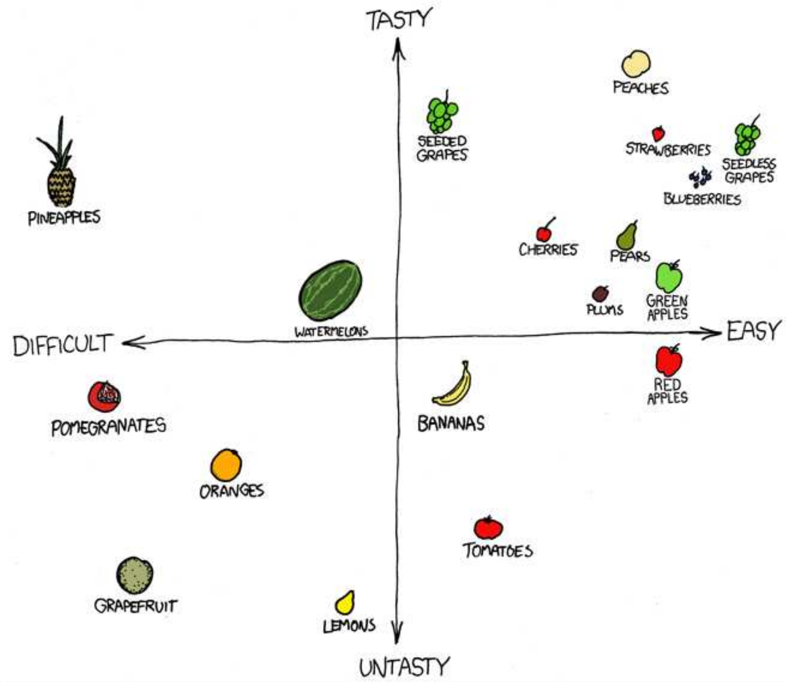
Enter A-P	Description
	a local maximum of $f(x,y)$.
	a local minimum of $f(x,y)$.
	a saddle point of $f(x,y)$ where $f_{xx} < 0$.
	a saddle point of $f(x,y)$ where $f_{xx} > 0$.
	a saddle point of $f(x,y)$ where f_{xx} is close to zero
	a point, where $f_x = 0$ and $f_y \neq 0$
	a point, where $f_y = 0$ and $f_x \neq 0$
	the point, where $\|\nabla f\|$ is largest
	a local maximum of $f(x,y)$ under the constraint $g(x,y) = 0$.
	a local minimum of $f(x,y)$ under the constraint $g(x,y) = 0$.



Solution:
D,C,B,H,M,E,A,K,G,F.

Problem 4) (10 points)

Find and classify all the extrema of the function $f(x,y) = x^5 + y^3 - 5x - 3y$. This function measures "eat temptation" in the x=Easy-y=Tasty plane. Is there a global minimum or global maximum?



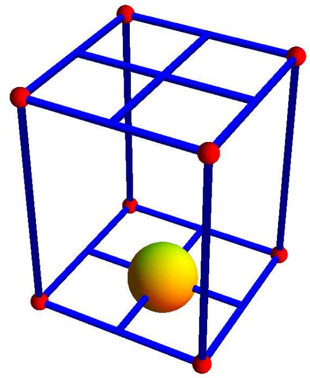
The "Easy-Tasty plane" was introduced in the XKCD cartoon titled "F&#% Grapefruits".

Solution:
Find the critical points $\nabla f(x,y) = \langle 0,0 \rangle$, then find the discriminant D and f_{xx}.

ne]hline point	D	f_{xx}	type
$(-1,-1)$	120	-20	maximum
$(-1,1)$	-120	-20	saddle
$(1,-1)$	-120	20	saddle
$(1,1)$	120	20	minimum

Problem 5) (10 points)

After having watched the latest Disney movie "Tangled", we want to build a hot air balloon with a cuboid mesh of dimension x, y, z which together with the top and bottom fortifications uses wires of total length $g(x, y, z) = 6x + 6y + 4z = 32$. Find the balloon with maximal volume $f(x, y, z) = xyz$.

Solution:
The gradients are $\nabla f(x, y, z) = \langle yz, xz, xy \rangle$ and $\nabla g(x, y, z) = \langle 6, 6, 4 \rangle$. The Lagrange equations are

$$\begin{aligned} yz &= \lambda \cdot 6 \\ xz &= \lambda \cdot 6 \\ xy &= \lambda \cdot 4 \end{aligned}$$

Getting rid of λ and using that x, y, z and so λ must all be positive for having a positive volume gives $x = y = 2z$. Plugging this into the constraints gives $\boxed{x = 16/9, y = 16/9, z = 8/3}$.

Problem 6) (10 points)

a) (8 points) Find the tangent plane to the surface $f(x, y, z) = x^2 - y^2 + z = 6$ at the point $(2, 1, 3)$.

b) (2 points) A curve $\vec{r}(t)$ on that tangent plane of the function $f(x, y, z)$ in a) has constant speed $|\vec{r}\,'| = 1$ and passes through the point $(2, 1, 3)$ at $t = 0$. What is $\frac{d}{dt} f(\vec{r}(t))$ at $t = 0$?

Solution:
a) The gradient is $\nabla f(x,y,z) = \langle 2x, -2y, 1\rangle$. At the point $(2,1,3)$ this is $\langle 4,-2,3\rangle$. The plane has the form $4x-2y+z=d$ where d can now be obtained by plugging in the point, which is $8-2+3=9$. The plane is $\boxed{4x-2y+z=9.}$
b) Since the curve is on the surface and f does not change, we have $d/dt f(\vec{r}(t)) = 0$. We can also see it from the fact that $\vec{r}'(t)$ is perpendicular to the gradient. The answer is $\boxed{0}$.

Problem 7) (10 points)

a) (5 points) Estimate $\sqrt{\sin(0.0004) + 1.001^2}$ using linear approximation.

b) (5 points) We know $f(0,0) = 1$, $D_{\langle \frac{3}{5}, \frac{4}{5}\rangle} f(0,0) = 2$ and $D_{\langle -\frac{4}{5}, \frac{3}{5}\rangle} f(0,0) = -1$. If $L(x,y)$ is the linear approximation to $f(x,y)$ at the point $(0,0)$, find $L(0.06, 0.08)$.

Solution:
a) $f_x = \cos(x)/(2\sqrt{\sin(x)+y^2})$ and $f_y = 2y/(2\sqrt{\sin(x)+y^2})$ at the point $(0,1)$ we get $\nabla f(0,1) = \langle 1/2, 1\rangle$. The linearization is

$$L(x,y) = 1 + \frac{1}{2}\cdot 0.0004 + 1\cdot 0.001 = 1 + 0.0002 + 0.001 = 1.0012\ .$$

The answer is $\boxed{1.0012}$.
b) Write $\nabla f = \langle a, b\rangle$. We get $a\cdot(3/5) + b\cdot(4/5) = 2, (-4/5)\cdot a + (3/5)\cdot b = -1$. We can solve for a, b to get $a = 2, b = 1$. We have therefore the linear approximation

$$L(x,y) = f(0,0) + 2\cdot 0.06 + 1\cdot 0.08 = 1.2\ .$$

The answer is $\boxed{1.2}$.

Problem 8) (10 points)

a) (5 points) Find the following double integral

$$\int_0^1 \int_{x^2}^{\sqrt{x}} \frac{\pi\sin(\pi y)}{y^2 - \sqrt{y}}\, dydx\ .$$

b) (5 points) Evaluate the following double integral

$$\int\int_R \frac{\sin(\pi\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}\, dxdy$$

over the region

$$R = \{x^2 + y^2 \leq 1, x > 0\ \} \ .$$

Solution:
a) Make a picture and change the order of integration.

$$\int_0^1 \int_{y^2}^{\sqrt{y}} \frac{\pi \sin(\pi y)}{(y^2 - \sqrt{y})} \, dxdy = \int_0^1 - \sin(\pi y) \, dy = -2 \ .$$

The answer is $\boxed{-2}$.
b) We use polar coordinates:

$$\int_0^1 \int_{-\pi/2}^{\pi/2} \frac{\sin(\pi r)}{r} r \, d\theta \, dr = 2\pi/\pi = 2 \ .$$

The answer is $\boxed{2}$.
Remark. In a), Mathematica 8 can not compute the integral. We have to help it to change the order of integration. This example shows that Mathematica does not have the change of order of integration trick implemented.

Problem 9) (10 points)

a) (8 points) Find the surface area of the surface parametrized as

$$\vec{r}(u,v) = \langle u - v, u + v, (u^2 - v^2)/2 \rangle \ ,$$

where (u, v) is in the unit disc $R = \{u^2 + v^2 \leq 1\ \}$.

b) (2 points) Give a nonzero vector $\vec{n}$ normal to the surface at $\vec{r}(4, 2) = \langle 2, 6, 6 \rangle$.

Solution:
a) We have $r_u = \langle 1,1,u \rangle$ and $r_v = \langle -1,1,-v \rangle$ so that $r_u \times r_v = \langle -u-v, v-u, 2 \rangle$. Its length is $|r_u \times r_v| = \sqrt{2}\sqrt{u^2+v^2+2}$. Use polar coordinates to integrate this over R:

$$\int_R |r_u \times r_v| \, dudv = \int_0^{2\pi} \int_0^1 \sqrt{2(r^2+2)} r \, drd\theta \; .$$

Use substitution $u = r^2 + 2$ to solve the inner integral:

$$\int_0^{2\pi} \frac{\sqrt{2}}{3} (r^2+2)^{3/2}|_0^1 \, d\theta \; .$$

Evaluate this to get the final answer $\boxed{(\pi/3)(6\sqrt{6}-8)}$.
b) We have computed $\vec{r}_u \times \vec{r}_v$ already in a). Just take $(u,v) = (4,2)$ to get $\boxed{\langle -6,-2,2 \rangle}$. Of course any vector parallel to this is also correct.

Problem 10) (10 points)

a) (6 points) Integrate

$$\int_0^{\pi/2} \int_x^{\pi/2} \frac{\cos(y)}{y} \, dydx$$

b) (4 points) Find the moment of inertia

$$\iint_R (x^2+y^2) \, dydx \; ,$$

where R is the ring $1 \le x^2 + y^2 \le 9$.

Solution:
a) To change the order of integration, make a figure. The integration region is the upper left triangle in the square $[0,\pi/2] \times [0,\pi/2]$. We get

$$\int_0^{\pi/2} \int_0^y \cos(y)/y \, dxdy = \int_0^{\pi/2} \cos(y) \, dy = \sin(y)_0^{\pi/2} = 1 \; .$$

The answer is $\boxed{1}$.
b) This is a polar integration problem

$$\int_0 0^{2\pi} \int_1^3 r^2 r \, dr \, d\theta = \int_0^{2\pi} \frac{r^4}{4}|_1^3 \, d\theta = \int_0^{2\pi} 20 \, d\theta = 40\pi \; .$$

The answer is $\boxed{40\pi}$.