11/6/2012 SECOND HOURLY PRACTICE V **Math 21a, Fall 2012**

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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, and 5, we need to see **details** of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (30 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) | T | **F** | $f(x, y)$ and $g(x, y) = f(x^2, y^2)$ have the same critical points.

Solution:
The function g has always $(0, 0)$ as a critical point, even if f has not.

2) | **T** | F | If a function $f(x, y) = ax + by$ has a critical point, then $f(x, y) = 0$ for all (x, y).

Solution:
At a critical point the gradient is $(a, b) = (0, 0)$, which implies $f = 0$.

3) | **T** | F | Given 2 arbitrary points in the plane, there is a function $f(x, y)$ which has these points as critical points and no other critical points.

Solution:
Connect the two points with a line and take this height as the x-axes, centered at the midpoint and with units such that the two points have coordinates (-1,0),(1,0). The function $f(x, y) = -y^2(x^3 - 1)$ has the two points as critical points. One is a local max, the other is a saddle point.

4) | T | **F** | If (x_0, y_0) is the maximum of $f(x, y)$ on the disc $x^2+y^2 \leq 1$ then $x_0^2+y_0^2 < 1$.

Solution:
The maximum could be on the boundary.

5) | T | **F** | There are no functions $f(x, y)$ for which every point on the unit circle is a critical point.

Solution:
There are many rotationally symmetric functions with this property.

6) **T** F

An absolute maximum (x_0, y_0) of $f(x, y)$ is also an absolute maximum of $f(x, y)$ constrained to a curve $g(x, y) = c$ that goes through the point (x_0, y_0).

Solution:
The Lagrange multiplier vanishes in this case.

7) T **F**

If $f(x, y)$ has two local maxima on the plane, then f must have a local minimum on the plane.

Solution:
Look at a camel type surface. It has a saddle between the local maxima.

8) **T** F

There exists a function $f(x, y)$ of two variables which has no critical points at all.

Solution:
True. Every non-constant linear function for example.

9) T **F**

If $f_x(x, y) = f_y(x, y) = 0$ for all (x, y) then $f(x, y) = 0$ for all (x, y) .

Solution:
False, f could be constant.

10) T **F**

$(0, 0)$ is a local maximum of the function $f(x, y) = x^2 - y^2 + x^4 + y^4$.

Solution:
$(0, 0)$ is a saddle point.

11) **T** F

If $f(x, y)$ has a local maximum at the point $(0, 0)$ with discriminant $D > 0$ then $g(x, y) = f(x, y) - x^4 + y^3$ has a local maximum at the point $(0, 0)$ too.

Solution:
Adding $x^4 + y^3$ does not change the first and second derivatives.

12) T **F** Every critical point (x, y) of a function $f(x, y)$ for which the discriminant D is not zero is either a local maximum or a local minimum.

Solution:
The second derivative test give for negative D that we have a saddle point.

13) **T** F If $(0, 0)$ is a critical point of $f(x, y)$ and the discriminant D is zero but $f_{xx}(0, 0) < 0$ then $(0, 0)$ can not be a local minimum.

Solution:
If $f_{xx}(0, 0) < 0$ then on the x-axis the function $g(x) = f(x, 0)$ has a local maximum. This means that there are points close to $(0, 0)$ where the value of f is smaller.

14) **T** F In the second derivative test, one can replace the condition $D > 0, f_{xx} > 0$ with $D > 0, f_{yy} > 0$ to check whether a point is a local minimum.

Solution:
True. If $f_{xx}f_{yy} - f_{xy}^2 > 0$, then f_{xx} and f_{yy} must have the same signs.

15) **T** F The function $f(x, y) = (x^4 - y^4)$ has neither a local maximum nor a local minimum at $(0, 0)$.

Solution:
The function is both smaller and bigger than $f(0, 0)$ for points near $(0, 0)$.

16) **T** F It is possible to find a function of two variables which has no maximum and no minimum.

Solution:
There are many linear functions like that.

17) **T** F The value of the function $f(x, y) = \sqrt{1 + 3x + 5y}$ at $(-0.002, 0.01)$ can by linear approximation be estimated as $1 - (3/2) \cdot 0.002 + (5/2) \cdot 0.01$.

Solution:
Use formula for $L(x, y)$.

18) T F

The function $f(x,y) = e^y x^2 \sin(y^2)$ satisfies the partial differential equation $f_{xxyyyxyy} = 0$.

Solution:
By Clairots theorem, we can have all three x derivatives at the beginning.

19) T F

If $\vec{r}(t)$ is a curve with unit speed in the plane with $\vec{r}(0) = (0,0)$ and $D_{\vec{r}\,'(0)} f(0,0) = 0$, then $\frac{d}{dt} f(\vec{r}(t)) = 0$ at the time $t = 0$.

Solution:
By the chain rule.

20) T F

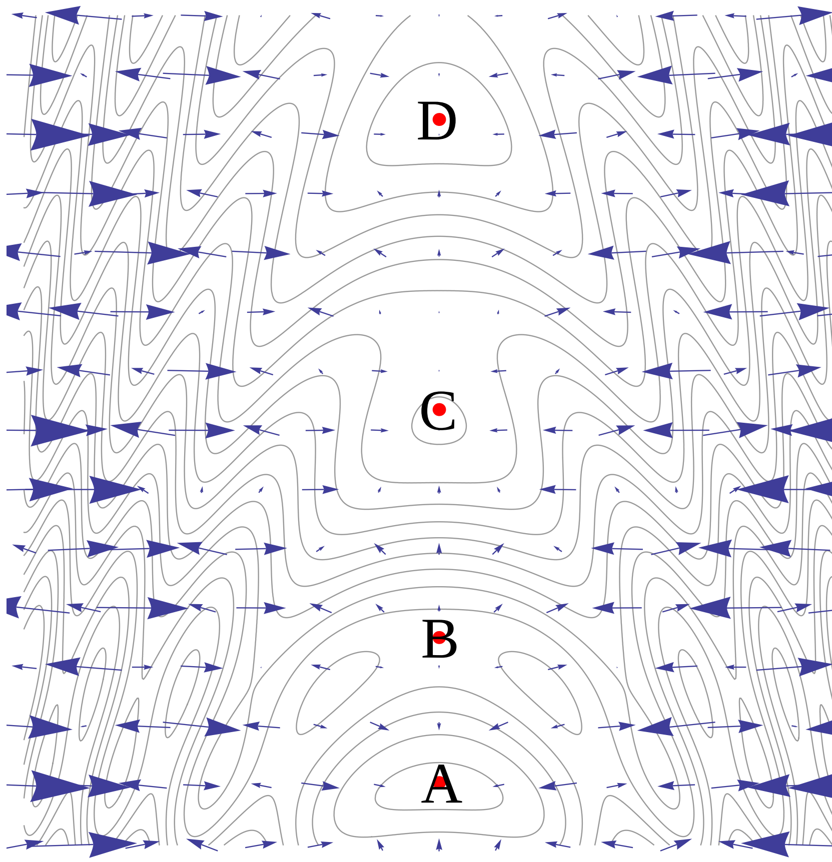
If a function $f(x,y)$ satisfies the partial differential equation $f_x^2 - f_y^2 = 0$, then f is the constant function.

Solution:
We can have $f = x + y$ for example.

Problem 2) (10 points)

a) (5 points) The picture below shows the contour map of a function $f(x, y)$ which has many critical points. Four of them are outlined for you on the y axes and are labeled A, B, C, D and ordered in increasing y value. The picture shows also the gradient vectors. Determine from each of the 4 points whether it is a local maximum, a local minimum or a saddle point. No justification is necessary in this problem.

Point	Max	Min	Saddle
D			
C			
B			
A			



b) (5 points)

Match the integrals with those obtained by changing the order of integration. No justifications are needed. Note that one of the Roman letters I)-V) will not be used, you have to chose four out of five.

Enter I,II,III,IV or V here.	Integral
	$\int_0^1 \int_{1-y}^1 f(x, y)\ dxdy$
	$\int_0^1 \int_y^1 f(x, y)\ dxdy$
	$\int_0^1 \int_0^{1-y} f(x, y)\ dxdy$
	$\int_0^1 \int_0^y f(x, y)\ dxdy$

I) $\int_0^1 \int_0^x f(x, y)\ dydx$
II) $\int_0^1 \int_0^{1-x} f(x, y)\ dydx$

III) $\int_0^1 \int_x^1 f(x,y)\ dydx$
IV) $\int_0^1 \int_0^{x-1} f(x,y)\ dydx$
V) $\int_0^1 \int_{1-x}^1 f(x,y)\ dydx$

Solution:
a) D is a local maximum because the gradient arrows point towards it and gradient arrows point into the direction of maximal increase. Similarly, C and A are local maxima. The point B is a saddle point.
b) Make a picture of the region. Each of the regions is a triangle in the unit square. The first is the upper right half, the second the lower right, the third the lower left, the forth the upper left half. The solution is $\boxed{\text{V,I,II,III}}$.

Problem 3) (10 points)

When Ramanujan, the amazing India born mathematician was sick in the hospital in england and the English mathematician Hardy visited him, Ramanujan asked "whats up?" Hardy answered: "Nothing special. Even the number of the taxi cab was boring: 1729". Ramanujan answered: "No, that is a remarkable number. It is the smallest number, which can be written in two different ways as a sum of two perfect cubes. Indeed $1729 = 1^3 + 12^3 = 9^3 + 10^3$.

a) (5 points) Find the linearization $L(x,y,z)$ of the function $f(x,y,z) = x^3 + y^3 - z^3$ at the point $(9,10,12)$.

b) (5 points) Use the technique of linear approximation to estimate $9.001^3 + 10.02^3 - 12.001^3$. Since we are not all Ramanujans, you can leave the end result as a product and sum of numbers. For example, $234 \cdot 0.001 - 100 \cdot 0.002$ would be an acceptable end result.

Solution:
We make a linear approximation of $f(x,y,z) = x^3 + y^3 - z^3$ at the point $(9,10,12)$. We have $\nabla f(x,y,z) = (3x^2, 3y^2, -3z^2)$ which is $(243, 300, -432)$. a) $L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$.
b) $L(9.001, 10.02, 12.001) = 1 + 243 \cdot 0.001 + 300 \cdot 0.02 - 432 \cdot 0.001$.

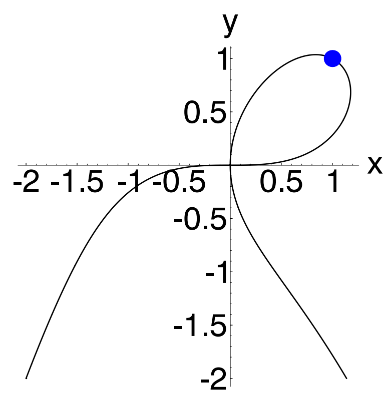
Solution:
$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$
$f(x_0,y_0) = e^{2\log 2} = 4$
$f_x(x_0,y_0) = 8$
$f_y(x_0,y_0) = -4$
$L(x,y) = 4 + 0.001 \cdot 8 - 4 \cdot 0.006 = \boxed{3.984}$.

Problem 4) (10 points)

Consider the equation

$$f(x,y) = 2y^3 + x^2y^2 - 4xy + x^4 = 0$$

It defines a curve, which you can see in the picture. Near the point $x=1, y=1$, the function can be written as a graph $y=y(x)$. Find the slope of that graph at the point $(1,1)$.



Solution:
Use the formula for implicit differentiation which is derived from the chain rule $f_x(x,y(x)) \cdot 1 + f_y(x,y(x)) \cdot y'(x) = 0$. The slope is $y'(x) = -f_x(x,y)/f_y(x,y)_{(x,y)} = (1,1) = -1/2$. An other possibility to solve this problem is to find the equation of the tangent line which is $f_x(1,1)(x-1) + f_y(1,1)(y-1) = 0$ and find the slope m by writing this equation as $y = mx + b$. It gives of course the same result.

Problem 5) (10 points)

a) Find a point on the surface $g(x,y,z) = \frac{1}{x} + \frac{1}{y} + \frac{8}{z} = 1$ which is locally closest to the origin.
b) Is this a global minimum? Hint: look at points $(x,y,z) = (1,-1/n,8/n)$ where n is an integer.

Solution:
This is a Lagrange problem. One wants to minimize $f(x,y,z) = x^2 + y^2 + z^2$ under the constraint $g(x,y,z) = 1$. The Lagrange equations are

$$\begin{aligned} 2x &= \lambda \frac{-1}{x^2} \\ 2y &= \lambda \frac{-1}{y^2} \\ 2z &= \lambda \frac{-8}{z^2} \\ \frac{1}{x} + \frac{1}{y} + \frac{8}{z} &= 1 \end{aligned}$$

The first two equations show $x = y$, the first and third equations show $8/z^3 = 1/x^3$ or $z = 2x$. Plugging this into the last equation gives $2/x + 8/(2x) = 1$ or $x = 6, y = 6, z = 12$. $\boxed{(x,y,z) = (6,6,12)}$.
b) consider the points $(x,y,z) = (1, -1/n, 8/n)$, where n is a large integer, One can check that these points lye on the surface $g(x,y,z) = 1$. Their distance to the origin however decreases to 1 if n goes to infinity. So the point $(6,6,12)$, while a local minimum is not a global minimum.

Problem 6) (10 points)

Find all extrema of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$ on the plane and characterize them. Do you find a absolute maximum or absolute minimum among them?

Solution:
The critical points satisfy $\nabla f(x,y) = (0,0)$ or $(3x^2 - 3, 3y^2 - 12) = (0,0)$. There are 4 critical points $(x,y) = (\pm 1, \pm 2)$. The discriminant is $D = f_{xx} f_{yy} - f_{xy}^2 = 36xy$ and $f_{xx} = 6x$.

point	D	f_{xx}	classification	value
(-1,-2)	72	-6	maximum	38
(-1, 2)	-72	-6	saddle	6
(1, -2)	-72	6	saddle	34
(1, 2)	72	6	minimum	2

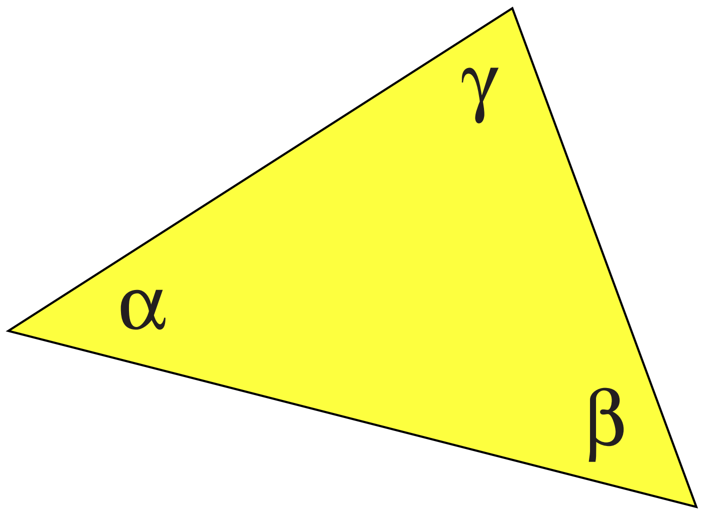
Note that there are no global (= absolute) maxima nor global minima because the function takes arbitrarily large and small values. For $y = 0$ the function is $g(x) = f(x,0) = x^3 - 3x + 20$ which satisfies $\lim_{x \to \pm\infty} g(x) = \pm\infty$.

Problem 7) (10 points)

What is the shape of the triangle with angles α, β, γ for which

$$f(\alpha, \beta, \gamma) = \log(\sin(\alpha)\sin(\beta)\sin(\gamma))$$

is maximal?



Solution:
The Lagrange equations are $\cot(\alpha) = \lambda, \cot(\beta) = \lambda, \cot(\gamma) = \lambda$. Because α, β, γ are all in $[0, \pi]$, we conclude that all are the same. From the last equation follows $\boxed{\alpha = \beta = \gamma = \pi/3}$ and $\sin(\alpha)\sin(\beta)\sin(\gamma) = (\sqrt{3}/2)^3$.

Problem 8) (10 points)

Let $g(x, y)$ be the distance from a point (x, y) to the curve $x^2 + 2y^2 + y^4/10 = 1$. Show that g is a solution of the partial differential equation

$$f_x^2 + f_y^2 = 1$$

outside the curve.

Hint: no computations are needed. The shape of the curve is pretty much irrelevant. What does the PDE say about the gradient ∇f?

Remark: This problem only needs thought. Use it as a "pillow problem" that is think about it before going to sleep. By the way, the PDE is called **eiconal equation**. It describes wave fronts in optics.

Solution:
The level curves of g are curves, for which the distance to the curve is constant. Lets look at the level curves of f, if f is the solution to the PDE. The PDE $|\nabla f|^2 = 1$ tells us that the gradient of f is a unit vector everywhere. This means that the directional derivative in the gradient direction is 1 everywhere on a level curve. This implies that the level curves of f are equidistributed too. The level curves of f and g are the same. Because f and g are both zero at the curve, the two functions are the same.

Solution:
More explanation: if you move perpendicular to a level curve, then your function changes according to the length of the gradient. If the gradient is large, then the function changes a lot, if the gradient is small, then the function changes little. The given partial differential equation tells that that the gradient of f has length 1. This means that if go along the steepest decent directions all the time, then you descend by the exactly same amount with which you go away from the level curve. So, the level curve of the height function $f(x, y)$ is equal to the level curve of the distance function $g(x, y)$.
While the solution to the problem is difficult to explain, it essentially just asks you to see what are the consequences of having a gradient of length 1 everywhere. If you have a mountain of height f(x,y) for which the gradient is 1 everywhere, it has the property that if you decent always perpendicular to the level curve (the fastest decent), then the height decreases in the same way as the distance you go away (the slope is 1 in that direction, the directional derivative is 1 in the gradient direction).
If you are still confused, look at the one dimensional problem: the function $f(x)$ giving the distance to an interval $[a, b]$ satisfies the differential equation $|f'(x)|^2 = 1$. The reasoning is almost the same and if you can answer this, you will understand the 2 dimensional problem better too.

Problem 9) (10 points)

a) (6 points) Find all critical points of $f(x, y) = 3xe^y - e^{3y} - x^3$ and classify them.

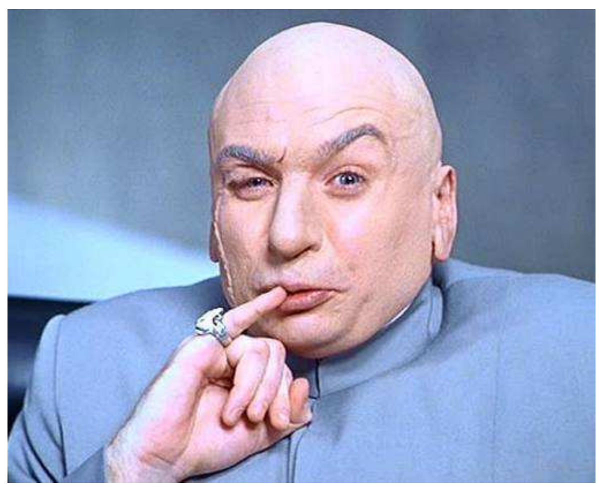
b) (4 points) Does the function have a absolute maximum or absolute minimum? Make sure to justify also this answer.

Solution:
a) Lets find the critical points and classify them. Setting the gradient to 0 gives

$$\begin{aligned} f_x &= 3e^y - 3x^2 = 0 \\ f_y &= 3xe^y - 3e^{3y} = 0 \ . \end{aligned}$$

The first equation gives $x = \pm e^{y/2}$. Plugging it into the second gives $y = 0, x = 1$ Applying the second derivative test $f_{xx} = -6$ and $D > 0$ shows that $(1, 0)$ is a local maximum.
b) If we look at the function f restricted to the x axes, we have $g(x) = f(x, 0) = 3x-1-x^3$. This goes to $+\infty$ for $x \to -\infty$ and goes to $-\infty$ for $x \to \infty$. We have no global maximum nor a global minimum for f.
Remark: this is a remarkable example. In single variable calculus, sometimes the statement is proven, that if one has a local maximum and no global maximum nor minimum for a function $f(x)$, then there also must exist at least one local minimum. The example here shows that this is not the case for functions for several variables.

Problem 10) (10 points)



a) (5 points) Integrate $f(x, y) = x^2 - y^2$ over the unit disk $\{x^2 + y^2 \le 1\}$.

b) (5 points) An evil integral!

$$\int_0^1 \int_0^{\sqrt{1-\theta^2}} r^2 \, dr d\theta \ .$$

Solution:
a) Use polar coordinates:

$$\int_0^1 \int_0^{2\pi} (r^2 \cos^2(\theta) - r^2 \sin^2(\theta))r \; d\theta dr = \int_0^1 r^3 \; dr (\int_0^{2\pi} \cos(2\theta) \; d\theta) = (1/4) \cdot 0 = 0 \; .$$

The final answer is $\boxed{\text{zero}}$.
b) Write it in more convenient coordinates:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \; dydx \; .$$

This is a quarter disc in the x, y plane. Now use polar coordinates. The integral is evil because now, the θ, r have a different meaning. The integral in polar coordinates is

$$\int_0^1 \int_0^{\pi/2} r^2 \sin^2(\theta) r \; d\theta dr$$

which is $\int_0^{\pi/2} \sin^2(\theta) \; d\theta \int_0^1 r^3 \; dr = (\pi/4)(1/4) = \pi/16$. (to compute the first integral, use the double angle formula $(1 - \cos(2\theta))/2 = \sin^2(\theta)$.)