11/6/2012 SECOND HOURLY PRACTICE VI **Math 21a, Fall 2012**

Name:

MWF 9 Oliver Knill
MWF 10 Hansheng Diao
MWF 10 Joe Rabinoff
MWF 11 John Hall
MWF 11 Meredith Hegg
MWF 12 Charmaine Sia
TTH 10 Bence Béky
TTH 10 Gijs Heuts
TTH 11:30 Francesco Cavazzani
TTH 11:30 Andrew Cotton-Clay

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, and 5, we need to see **details** of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) T F (F highlighted) The directional derivative $D_u f$ is a vector normal to a level surface of f.

Solution:
The directional derivative is a scalar

2) T F (F highlighted) At a critical point of a function f, the gradient vector has length 1.

Solution:
The gradient vector is the zero vector there.

3) T F (F highlighted) At a critical point (x, y) of a function f, the tangent plane to the graph of f does not exist.

Solution:
The tangent plane is horizontal there.

4) T F (T highlighted) For any point (x, y) which is not a critical point, there is a unit vector $\vec{u}$ for which $D_{\vec{u}} f(x, y)$ is nonzero.

Solution:
Take a vector vector is perpendicular to the gradient, the directional derivative $D_{\vec{u}} f = \nabla f \cdot \vec{u}$ is zero.

5) T F (T highlighted) If $f_{xx}(0,0) = 0, D = f_{xx} f_{yy} - f_{xy}^2 \neq 0$, and $\nabla f(0,0) = \langle 0, 0 \rangle$, then $(0,0)$ is a saddle point.

Solution:
Because $f_{xx} = 0$, we have $D = f_{xx} f_{yy} - f_{xy}^2 = -f_{xy}^2$ which can not be positive. Because $D \neq 0$, we must have $D < 0$. By the second derivative test, the critical point is a saddle point.

6) T F (T highlighted) A continuous function defined on the closed unit disc $x^2 + y^2 \leq 1$ has an absolute maximum inside the disc or on the boundary.

Solution:
The maximum can be either in the interior or at the boundary.

7) | T | F | The function $f(x,y) = x^2 - y^2$ has a neither a local maximum nor a local minimum at $(0,0)$.

Solution:
It is a saddle point.

8) | T | F | If (x,y) is a maximum of $f(x,y)$ under the constraint $g(x,y) = 5$ then it is also a maximum of $f(x,y) + g(x,y)$ under the constraint $g(x,y) = 5$.

Solution:
Indeed, on the constraint curve, the function $f + g$ is just $f + 5$, which has the same maxima and minima as f on that curve.

9) | T | F | The functions $f(x,y)$ and $g(x,y) = (f(x,y))^6$ always have the same critical points.

Solution:
The gradient of g is $6f^5(x,y)\nabla f$. So, the second function has critical points, where f vanishes.

10) | T | F | For $f(x,y,z) = x^2 + y^2 + 2z^2$, the vector $\nabla f(1,1,1)$ is perpendicular to the surface $f(x,y,z) = 4$ at the point $(1,1,1)$.

Solution:
This is a basic property of gradients.

11) | T | F | $f(x,y) = \sqrt{16 - x^2 - y^2}$ has both an absolute maximum and an absolute minimum on its domain of definition.

Solution:
The domain of definition is the disc $x^2 + y^2 \leq 16$. The maximum 4 is in the center the absolute minimum 0 at the boundary.

12) | T | F (highlighted) |

If (x_0, y_0) is a critical point of $f(x, y)$ and $f_{xy}(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point of f.

Solution:
The discriminant $D = f_{xx}f_{yy} - f_{xy}^2$ can be positive. An example is $f(x, y) = 100x^2 + 100y^2 - xy$.

13) | T (highlighted) | F |

If $(1, 1, 1)$ is a maximum of f under the constraints $g(x, y, z) = c, h(x, y, z) = d$, and the Lagrange multipliers satisfy $\lambda = 0, \mu = 0$, then $(1, 1, 1)$ is a critical point of f.

Solution:
Look at the Lagrange equations. If $\lambda = \mu = 0$, then $\nabla f = (0, 0, 0)$.

14) | T (highlighted) | F |

Suppose f has a maximum value at a point P relative to the constraint $g = 0$. If the Lagrange multiplier $\lambda = 0$, then P is also a critical point for f without the constraint.

Solution:
The Lagrange equations tell that $\nabla f(x, y) = (0, 0)$.

15) | T (highlighted) | F |

At a saddle point, all directional derivatives are zero.

Solution:
Because $\nabla f(x, y) = (0, 0)$ at a saddle point, all directional derivatives $D_{\vec{v}}f = \nabla f \cdot \vec{v}$ are zero.

16) | T | F (highlighted) |

The minimum of $f(x, y)$ under the constraint $g(x, y) = 0$ is always the same as the maximum of $g(x, y)$ under the constraint $f(x, y) = 0$.

Solution:
This can not be true, because the first problem is the same if we replace $g(x, y)$ with $2g(x, y)$, but this will change the value of the maximum of g on the right hand side.

17) | T (highlighted) | F |

At a local maximum (x_0, y_0) of $f(x, y)$, one has $f_{yy}(x_0, y_0) \leq 0$.

Solution:
Indeed, at a local maximum, $f_{yy} \leq 0$.

18) | T | F | It is possible that $f(x,y)$ attains a maximum under the constraint $g(x,y) = 0$ at a point, where $\nabla f \neq \lambda \nabla g$.

Solution:
If $\nabla g = 0$.

19) | T | F | Any Lagrange problem which asks for an extremum of $f(x,y)$ under a constraint $g(x,y) = 0$ has either a maximum or a minimum.

Solution:
Take $f(x,y) = x^3 + y^3$ and $g(x,y) = x - y$.

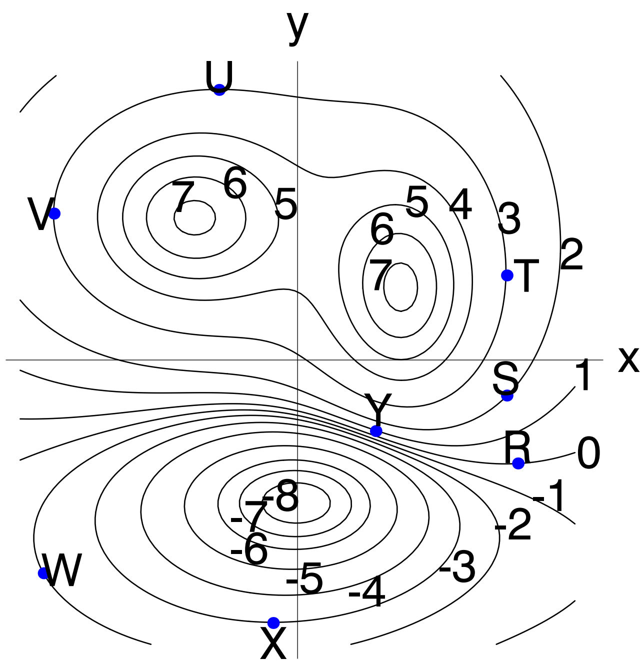
20) | T | F | The function $u(x,y) = \sin(x+y)$ satisfies the PDE $u_{xx} + u_{yy} - 2u_{xy} = 0$.

Solution:
Actually $(\partial_x - \partial_y)f = 0$ and so $(\partial_x - \partial_y)^2 f = 0$.

Problem 2) (10 points) No justifications needed.

a) (4 points) Fill in the boxes. You do not need to give additional explanations.

Chain rule:	$\frac{d}{dt} f(\vec{r}(t)) =$ ☐ $\cdot \vec{r}\,'(t)$
Directional derivative D_v	$D_{\langle 2,3\rangle/\sqrt{13}} f(1,1) = \nabla f(1,1) \cdot$ ☐
Linearization of $f(x,y)$ at $(1,1)$	$L(x,y) =$ ☐ $+ \nabla f(1,1) \cdot (x-1, y-1)$
Equation of tangent line at $(1,1)$	$\nabla f(1,1) \cdot \langle x-1, y-1 \rangle =$ ☐
Critical point $(1,1)$ of f	$\nabla f(1,1) =$ ☐
Lagrange equations	$\nabla f(x,y) =$ ☐ $\nabla g(x,y),\ g(x,y) = c.$
Type I integral	$\int_a^b \int_{c(x)}^{d(x)} f(x,y)$ ☐ .
Type II integral	$\int_c^d \int_{a(y)}^{b(y)} f(x,y)$ ☐ .
Integration in polar coordinates	$\int_a^b \int_{f(\theta)}^{g(\theta)}$ ☐ $f(r\cos(\theta), r\sin(\theta))\ dr d\theta.$
Area	$\int\int_R$ ☐ $dxdy$



b) (2 points) Circle the point at which the magnitude of the gradient vector ∇f is greatest. Mark exactly one point. Justify your answer.

R	S	T	U	V	W	X	Y

c) (2 points) Circle the points at which the partial derivative f_x is strictly positive. Mark any number of points on this question. Justify your answers.

R	S	T	U	V	W	X	Y

d) (2 points) We know that the directional derivative in the direction $(1,1)/\sqrt{2}$ is zero at one of the following points. Which one? Mark exactly one point on this question.

R	S	T	U	V	W	X	Y

Solution:
a) $\nabla f(r(t))$, $\langle 2,3\rangle/\sqrt{13}$, $f(1,1)$, 0, $\langle 0,0\rangle$, λ, $dydx$, $dxdy$,r,1.
b) At the point Y the level curves are closest to each other indicating the steepest place and so the largest gradient.
c) At the points V and Y, the function increases, if we go into the x direction. In the other points, the function decreases, if we go into the x direction.
d) In order to have a zero directional derivative, we need the gradient to be zero or perpendicular into the direction $\vec{v}$. This is the case at the point S.

Problem 3) (10 points)

a) Locate and classify all the critical points of

$$f(x, y) = 3y - y^3 - 3x^2 y \ .$$

Solution:
This is a routine problem. Find the gradient, put it to zero to find the critical points and apply the second derivative test.

x	y	D	f_{xx}	Nature	f
-1	0	-36	0	saddle	0
1	0	-36	0	saddle	0
0	-1	36	6	minimum	-2
0	1	36	-6	maximum * 2	

b) Where on the parameterized surface

$$\vec{r}(x, y) = \langle u, v, w \rangle = \langle xy^3, x^2/2, 3y^2/2 \rangle$$

is the function $g(u, v, w) = u - v - w$ extremal? To investigate this, find all the critical points of the function $f(x, y) = xy^3 - \frac{x^2}{2} - \frac{3y^2}{2}$. For each critical point, specify whether it is a local maximum, a local minimum or a saddle point and show how you know.

Solution:
For $f(x, y) = xy^3 - \frac{x^2}{2} - \frac{3y^2}{2}$ the gradient is $\nabla f(x, y) = \langle y^3 - x, 3xy^2 - 3y \rangle$. It is zero if $3y - 3y^5 = 0$ or $y(1 - y^4) = 0$ which means $y = 0$ or $y = \pm 1$. In the case $y = 0$, we have $x = 0$. In the case $y = 1$, we have $x = 1$, in the case $y = -1$, we have $x = -1$. The critical points are $(0, 0), (1, 1), (-1, -1)$.
The discriminant is $D = f_{xx} f_{yy} - f_{xy}^2 = 3 - 6xy - 9y^4$. The entry f_{xx} is -1 everywhere. Applying the second derivative test gives

Critical point	(0,0)	(1,1)	(-1,-1)
Discriminant	3	-12	-12
f_{xx}	-1	-1	-1
Analysis	max	saddle	saddle

Problem 4) (10 points)

Evaluate the double integral

$$\int_0^4 \int_0^{y^2} \frac{x^4}{4 - \sqrt{x}} \, dx \, dy \ .$$

Solution:
Change the order of integration:

$$\int_0^{16} \int_{\sqrt{x}}^{4} \frac{x^4}{4-\sqrt{x}} \, dydx$$

The inner integral $\int_{\sqrt{x}}^{4} \frac{x^4}{4-\sqrt{x}} \, dx$ can now be computed and gives x^4 so that we end up with

$$\int_0^{16} x^4 \, dx = 16^5/5$$

Problem 5) (10 points)

a) (6 points) Find all critical points of $f(x, y) = 3xe^y - e^{3y} - x^3$ and classify them.

b) (4 points) Does the function have an absolute maximum or absolute minimum? Make sure to justify also this answer.

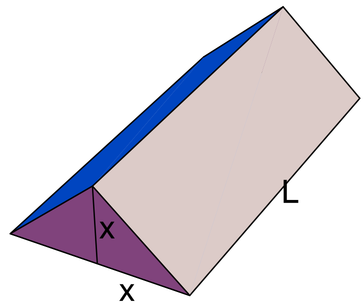
Solution:
a) Lets find the critical points and classify them. Setting the gradient to 0 gives

$$\begin{aligned} f_x &= 3e^y - 3x^2 = 0 \\ f_y &= 3xe^y - 3e^{3y} = 0 \,. \end{aligned}$$

The first equation gives $x = \pm e^{y/2}$. Plugging this into the second gives $y = 0, x = 1$. The case $x = -1$ is not possible. Applying the second derivative test $f_{xx} = -6$ and $D > 0$ shows that $(1, 0)$ is a local maximum.
b) If we look at the function f restricted to the x axes, we have $g(x) = f(x, 0) = 3x-1-x^3$. This goes to $+\infty$ for $x \to -\infty$ and goes to $-\infty$ for $x \to \infty$. We have no global maximum nor a global minimum for f.
Remark: this is a remarkable example. In single variable calculus, sometimes the statement is proven, that if one has a local maximum and no global maximum nor minimum for a function $f(x)$, then there also must exist at least one local minimum. The example here shows that this is not the case for functions for several variables.

Problem 6) (10 points)

We minimize the surface of a roof of height x and width $2x$ and length $L = \sqrt{2}y$ if the volume $V(x, y) = x^2\sqrt{2}y$ of the roof is fixed and equal to $\sqrt{2}$. In other words, you have to minimize $f(x,y) = 2x^2 + 4xy$ under the constraint $g(x,y) = x^2y = 1$. Solve the problem with the Lagrange method.



Solution:
The Lagrange equations

$$\nabla f = \lambda \nabla g, g = 1$$

are

$$\begin{aligned} 4x + 4y &= \lambda 2xy \\ 4x &= \lambda x^2 \\ x^2 y &= 1 \end{aligned}$$

Eliminating λ gives $(4x+4y)/4y = \lambda 2xy/\lambda x^2$ or $y/x + 1 = 2y/x$ so that $1 = y/x$. The only critical point with positive x, y is $(1,1)$. The minimum of f is $f(1,1) = 6$. The minimal surface area is 6.

Problem 7) (10 points)

Find all the critical points of $f(x,y) = \frac{x^5}{5} - \frac{x^2}{2} + \frac{y^3}{3} - y$ and indicate whether they are local maxima, local minima or saddle points.

Solution:
$\nabla f(x,y) = (x^4 - x, (y^2 - 1)) = (0,0)$ so that the critical points are $(0,1), (0,-1), (1,1), (1,-1)$. We have $D = (4x^3 - 1)2y$ and $f_{xx} = 4x^3 - 1$.

Point	D	f_{xx}	type
$(0,1)$	$D = -2$	-	saddle
$(0,-1)$	$D = 2$	-1	local max
$(1,1)$	$D = 6$	3	local min
$(1,-1)$	$D = -6$	-	saddle

Problem 8) (10 points)

The temperature distribution in a room is $T(x, y, z) = x + y + z$. On which point of the parametrized surface

$$\vec{r}(s, t) = \langle x, y, z \rangle = \langle s^2 + t^2, st, 2s - t \rangle$$

is the temperature extremal? Is it a maximum or a minimum?

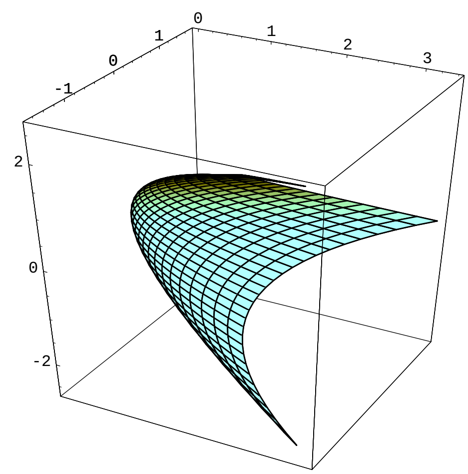
Solution:
We have to extremize

$$f(s, t) = s^2 + t^2 + st + 2s - t \; .$$

The gradient

$$\nabla f(s, t) = \langle 2s + t + 2, 2t + s - 1 \rangle$$

The gradient is zero at $(s, t) = (-5/3, 4/3)$. The discriminant is $D = f_{xx}f_{yy} - f_{xy}^2 = 3$. Since $f_{xx} = 0$, we have a local minimum.



Solution:
The function to extremize is $f(s, t) = s^2 + t^2 + st + 2s - t$. The gradient is $\nabla f(s, t) = \langle 2s + t + 2, 2t + s - 1 \rangle$. The system of equations

$$\begin{aligned} 2s + t + 2 &= 0 \\ 2t + s - 1 &= 0 \end{aligned}$$

has the solution $s = -5/3, t = 4/3$. The discriminant is $D = 2 \cdot 2 - 1 = 3 > 0$ and $f_{ss} = 2 > 0$. So the point $(-5/3, 4/3)$ is a $\boxed{\text{minimum}}$.

Problem 9) (10 points)

A region R in the xy-plane is given in polar coordinates by $r(\theta) \leq \theta^2$ for $\theta \in [0, \pi]$. You see the region in the picture to the right. Evaluate the double integral

$$\iint_R \frac{\cos(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}(\pi - (x^2+y^2)^{1/4})} \, dx \, dy \ .$$

Solution:
The region becomes a triangle in polar coordinates. Setting up the integral with $dA = dr d\theta$ does not work. The integral $\int_0^\pi \int_0^{\theta^2} \frac{\cos(r)}{r(\pi - \sqrt{r})} r dr d\theta$ can not be solved. We have to change the order of integration:

$$\int_0^{\pi^2} \int_{\sqrt{r}}^{\pi} \frac{\cos(r)}{r(\pi - \sqrt{r})} r \, d\theta \, dr$$

Evaluating the inner integral leads to $\int_0^{\pi^2} \cos(r) \, dr = \boxed{\sin(\pi^2)}$.

Problem 10) (10 points)

Suppose $2x + 3y + 2z = 9$ is the tangent plane to the graph of $z = f(x, y)$ at the point $(1, 1, 2)$.

a) Find the linear approximation of $f(1.01, 0.98)$.
b) What is the gradient ∇f at $(1, 1)$?
c) What is the equation $ax + by = d$ of the tangent line at $(1, 1)$?

Solution:
a) The approximation must lie on the tangent plane. Plug in $x = 1.01, y = 0.98$. We get $z = (9 - 2 * 1.01 - 3 * 0.98)/2 = 2.02$.
b) The normal vector to the graph of $f(x, y)$ is the gradient of $g(x, y, z) = z - f(x, y)$ at the point $(1, 1, 2)$ which is $\langle -f_x(1, 1), -f_y(1, 1), 1 \rangle$. We know this is parallel to $\langle 2, 3, 2 \rangle$. Because of the last equation, we know that $\langle -f_x(1, 1), -f_y(1, 1), 1 \rangle = \langle 1, 3/2, 1 \rangle$. Therefore $f_x = -1, f_y = -3/2$.
c) We have $a = f_x, b = f_y$ so that the equation is $-x - 3/2y = d$, Because the point $(1, 1)$ is on the line, we have $-x - 3/2y = -5/2$ or $2x + 3y = 5$.